# Independence of Artin L-functions

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**Abstract**

Let $K/\mathbb{Q}$ be a finite Galois extension. Let $\chi_1,\ldots,\chi_r$ be $r\geq 1$ distinct characters of the Galois group with the associated Artin L-functions $L(s,\chi_1),\ldots,L(s,\chi_r)$. Let $m\geq 0$. We prove that the derivatives $L^{(k)}(s,\chi_j)$, $1\leq j\leq r$, $0\leq k\leq m$, are linearly independent over the field of meromorphic functions of order $<1$. From this it follows that the L-functions corresponding to the irreducible characters are algebraically independent over the field of meromorphic functions of order $<1$.

*Key words:* Artin L-function; linear independence; algebraic independence; arithmetic functions; Dirichlet series

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## Introduction

Let $K/\mathbb{Q}$ be a finite Galois extension. For the character $\chi$ of a representation of the Galois group $G:=\mathrm{Gal}(K/\mathbb{Q})$ on a finite dimensional complex vector space, let $L(s,\chi,K/\mathbb{Q})$ be the corresponding Artin $L$-function ([2], P. 296). It was proved in [6, Theorem 1] that the derivatives of any order of Artin $L$-functions to finitely many distinct characters of $G$ are linearly independent over $\mathbb{C}$. A more general result concerning linear independence over $\mathbb{C}$ of functions in a class which contains the Artin $L$-functions was proved in [4]. In [5] it was proved the independence of any family of suitable $L$-functions (including Artin $L$-functions) with respect to a ring generated by a slow varying function.

For $\varepsilon>0$ let

$$\mathcal{F}_\varepsilon=\{f:(1+\varepsilon,+\infty)\to\mathbb{C}\},$$

$$\mathcal{B}_\varepsilon = \{f \in \mathcal{F}_\varepsilon \;:\; \forall a > 0 \lim_{\sigma \to +\infty} e^{-a\sigma}|f(\sigma)| = 0\},$$

$$\mathcal{V}_\varepsilon = \{f \in \mathcal{F}_\varepsilon \;: \exists a > 0 \lim_{\sigma \to +\infty} e^{a\sigma}|f(\sigma)| = 0\}.$$

The main result of this paper is

**Theorem 7.** *Let $K/\mathbb{Q}$ be a finite Galois extension, and let $\chi_1, \dots, \chi_r$ be distinct characters of the Galois group with the associated Artin L-functions $L(s, \chi_1), \dots, L(s, \chi_r)$. Let $\varepsilon > 0$. Let $\mathcal{A}_\varepsilon \subset \mathcal{B}_\varepsilon$ be a $\mathbb{C}$-vector space with*

$$\mathcal{A}_\varepsilon \cap \mathcal{V}_\varepsilon = \{0\}.$$

*Let $m \geq 0$. If the functions $G_{jk}(\sigma) \in \mathcal{A}_\varepsilon$ satisfy*

$$\sum_{j=1}^{r} \sum_{k=0}^{m} G_{jk}(\sigma) L^{(k)}(\sigma, \chi_j) = 0, \; \sigma > 1 + \varepsilon,$$

*then $G_{jk} = 0$, $1 \leq j \leq r$, $0 \leq k \leq m$.*

As a consequence we have

**Corollary 8.** *The functions $L^{(k)}(s, \chi_j)$, $1 \leq j \leq r$, $0 \leq k \leq m$ are linearly independent over the field of meromorphic functions of order $< 1$.*

Since the Artin L-functions are meromorphic of order 1 this result is best possible when we look for linear dependence with coefficients meromorphic functions. This extends the main result of [6]. Let $\chi_1, \dots, \chi_h$ be the irreducible characters of the Galois group $Gal(K/\mathbb{Q})$. In [6, Corollary 4] it was proved that the Artin L-functions $L(s, \chi_1), \dots, L(s, \chi_h)$ are algebraically independent over $\mathbb{C}$. This extended Artin's result [1, Satz 5, P. 106] that $L(s, \chi_1), \dots$ , $L(s, \chi_h)$ are multiplicatively independent. In Corollary 9 we prove that $L(s, \chi_1), \dots, L(s, \chi_h)$ are algebraically independent over the field of meromorphic functions of order $< 1$.

## 1. Arithmetic Functions and Dirichlet Series

In this section we present some properties of Dirichlet series which are needed for the proof of the results on Artin L-functions in section 2.

**Lemma 1.** *Let $(a_n(\sigma))_{n \geq 1}$ be a sequence of functions in $(\mathcal{F}_\varepsilon \setminus \mathcal{V}_\varepsilon) \cup \{0\}$ such that there exists a function $M : (1 + \varepsilon, +\infty) \to [0, +\infty)$ with the properties:*

*(i)* $n^{-\varepsilon}|a_n(\sigma)| \leq M(\sigma)$, $\forall n \geq 1, \sigma > 1 + \varepsilon$,

*(ii) For all $a > 0$ it holds that* $\lim_{\sigma \to +\infty} e^{-a\sigma} M(\sigma) = 0$.

*Then*

*(1) The series of functions* $F(\sigma) := \sum_{n=1}^{\infty} \frac{a_n(\sigma)}{n^\sigma}$ *is absolutely convergent on* $(1+\varepsilon, +\infty)$.

*(2) If* $F(\sigma)$ *is identically zero then* $a_n(\sigma)$ *is identically zero for any* $n \geq 1$.

*Proof.* (1) Let $\sigma > 1+\varepsilon$. By $(i)$, it holds that

$$\left|\frac{a_n(\sigma)}{\sigma^n}\right| \leq \frac{M(\sigma)}{n^{\sigma-\varepsilon}}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^{\sigma-\varepsilon}}$ is convergent, it follows that the series $\sum_{n=1}^{\infty} \frac{a_n(\sigma)}{n^\sigma}$ is absolutely convergent.

(2) We show inductively that $a_n$ is identically zero for any $n \geq 1$. Suppose that $k = 1$ or that $k > 1$ and, for induction, $a_1, \ldots, a_{k-1}$ are identically zero. For $\sigma > 1+\varepsilon$ we have that

$$|a_k(\sigma)| = \left|k^\sigma \sum_{n=k+1}^{\infty} \frac{a_n(\sigma)}{n^\sigma}\right| \leq \sum_{n=k+1}^{\infty} \frac{|a_n(\sigma)| k^\sigma}{n^\sigma} \leq$$

$$\leq M(\sigma) \sum_{n=k+1}^{\infty} \frac{k^\sigma}{n^{\sigma-\varepsilon}} = k^\varepsilon M(\sigma) \sum_{n=k+1}^{\infty} \frac{1}{(\frac{n}{k})^{\sigma-\varepsilon}}. \tag{1}$$

We choose $\delta > \varepsilon$. Let $\sigma \geq 1+\delta$. From (1) it follows that

$$|a_k(\sigma)| \leq k^\varepsilon M(\sigma) \sum_{n=k+1}^{\infty} \frac{1}{(\frac{n}{k})^{\frac{\sigma+1-\varepsilon}{2}}} \frac{1}{(\frac{n}{k})^{\frac{\sigma-1-\varepsilon}{2}}} \leq$$

$$\leq k^\varepsilon M(\sigma) \left(\sum_{n=k+1}^{\infty} \frac{1}{(\frac{n}{k})^{1+\frac{\delta-\varepsilon}{2}}}\right) \frac{1}{(\frac{k+1}{k})^{\frac{\sigma-1-\varepsilon}{2}}}. \tag{2}$$

Let $0 < a < \frac{1}{2}(\log(k+1) - \log k)$. From (2) it follows that

$$e^{a\sigma}|a_k(\sigma)| \leq k^{\frac{\varepsilon-1}{2}} (k+1)^{\frac{1+\varepsilon}{2}} \left(\sum_{n=k+1}^{\infty} \frac{1}{\left(\frac{n}{k}\right)^{1+\frac{\delta-\varepsilon}{2}}}\right) e^{(a-\frac{\log(k+1)-\log k}{2})\sigma} M(\sigma). \tag{3}$$

From (3) and hypothesis $(ii)$ it follows that

$$\lim_{\sigma \to +\infty} e^{a\sigma}|a_k(\sigma)| = 0,$$

so $a_k \in \mathcal{V}_\varepsilon$. Since, by hypothesis, $a_k \in (\mathcal{F}_\varepsilon \backslash \mathcal{V}_\varepsilon) \cup \{0\}$ it follows that $a_k = 0$. □

Let $\varepsilon > 0$. If $f(n)$ is an arithmetic function of order $O(n^\varepsilon)$, i.e. there exists $C > 0$ such that $|f(n)| \leq Cn^\varepsilon$ for every $n \geq 1$ , then the associated Dirichlet series

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

defines a holomorphic function in the half plane $\mathrm{Re}\,(s) > 1 + \varepsilon$. The *k-th arithmetic derivative* of $f(n)$ is

$$f^{(k)}(n) = (-1)^k f(n) \log^k n.$$

It holds that the *k-th derivative* of $F(s)$ is the Dirichlet series

$$F^{(k)}(s) = \sum_{n=1}^{\infty} \frac{f^{(k)}(n)}{n^s}.$$

**Theorem 2.** *Let $\varepsilon > 0$. Let $f_1(n), \dots, f_r(n)$ be arithmetic functions of order $O(n^\varepsilon)$ linearly independent over $\mathbb{C}$ with the associated Dirichlet series $F_1(s), \dots, F_r(s)$. Let $\mathcal{A}_\varepsilon \subset \mathcal{B}_\varepsilon$ be a $\mathbb{C}$-vector space with*

$$\mathcal{A}_\varepsilon \cap \mathcal{V}_\varepsilon = \{0\}.$$

*Let $G_1(\sigma), \dots, G_r(\sigma) \in \mathcal{A}_\varepsilon$ such that*

$$\sum_{j=1}^{r} G_j(\sigma) F_j(\sigma) = 0, \ \sigma > 1 + \varepsilon. \tag{4}$$

*Then $G_1 = \cdots = G_r = 0$.*

*Proof.* Let $\sigma > 1 + \varepsilon$. From (4) it follows that

$$\sum_{n=1}^{\infty} \left( \sum_{j=1}^{r} G_j(\sigma) f_j(n) \right) n^{-\sigma} = 0. \tag{5}$$

For $n \geq 1$ let

$$a_n(\sigma) := \sum_{j=1}^{r} G_j(\sigma) f_j(n). \tag{6}$$

The functions $a_n(\sigma)$ are contained in the space $\mathcal{A}_\varepsilon$. Since the functions $f_j(n)$ have order $O(n^\varepsilon)$, there exists a constant $C > 0$ such that

$$\max\{|f_j(n)| \ : \ 1 \leq j \leq r\} \leq Cn^\varepsilon, \ \forall n \geq 1. \tag{7}$$

Let

$$M(\sigma) := C\sum_{j=1}^{r} |G_j(\sigma)|. \tag{8}$$

From (6), (7) and (8) it follows that

$$|a_n(\sigma)| \leq M(\sigma)n^{\varepsilon},\ \forall n \geq 1. \tag{9}$$

Also, since $G_j \in \mathcal{B}_\varepsilon$, from (6) and (9) it follows that

$$\lim_{\sigma\to+\infty} e^{-a\sigma}M(\sigma) = 0,\ \forall a > 0,$$

hence the conditions of the Lemma 1 are satisfied. From (5) and Lemma 1 it follows that

$$a_n(\sigma) = 0,\ \forall n \geq 1. \tag{10}$$

Since the functions $f_1(n), \dots, f_r(n)$ are linearly independent over $\mathbb{C}$, from (6) and (10) it follows that $G_j(\sigma) = 0$, $1 \leq j \leq r$, as required. □

An arithmetic function $f(n)$ is called *multiplicative*, if $f(1) = 1$ and

$$f(nm) = f(n)f(m),\ n, m \in \mathbb{N} \text{ with } \gcd(n, m) = 1.$$

Two multiplicative arithmetical functions $f(n)$ and $g(n)$ are called *equivalent* (see [3]) if $f(p^j) = g(p^j)$ for all integers $j \geq 1$ and all but finitely many primes $p$. Let $e(n)$ be the identity function, defined by $e(1) = 1$ and $e(n) = 0$ for $n \geq 2$. We recall the following result of Kaczorowski, Molteni and Perelli [4].

**Lemma 3.** *([4, Lemma 1]) Let $f_1(n), \dots, f_r(n)$ be multiplicative functions such that $e(n), f_1(n), \dots, f_r(n)$ are pairwise non-equivalent, and let $m$ be a non-negative integer. Then the functions*

$$f_1^{(0)}(n), \dots, f_1^{(m)}(n), f_2^{(0)}(n), \dots, f_2^{(m)}(n), \dots, f_r^{(0)}(n), \dots, f_r^{(m)}(n)$$

*are linearly independent over $\mathbb{C}$.*

**Corollary 4.** *Let $\varepsilon > 0$. Let $f_1(n), \dots, f_r(n)$ be multiplicative functions of order $O(n^\varepsilon)$ such that $e(n), f_1(n), \dots, f_r(n)$ are pairwise non-equivalent. Let*

$$F_j(s) = \sum_{n=1}^{\infty} \frac{f_j(n)}{n^s},\ j = 1, \dots, r,$$

*be the associated Dirichlet series. Let $\mathcal{A}_\varepsilon \subset \mathcal{B}_\varepsilon$ be a $\mathbb{C}$-vector space with*

$$\mathcal{A}_\varepsilon \cap \mathcal{V}_\varepsilon = \{0\}.$$

*Let $m \geq 0$. If the functions $G_{jk}(\sigma) \in \mathcal{A}_\varepsilon$ satisfy*

$$\sum_{j=1}^{r}\sum_{k=0}^{m} G_{jk}(\sigma)F_j^{(k)}(\sigma) = 0,\ \sigma > 1+\varepsilon,$$

*then $G_{jk} = 0$, $1 \leq j \leq r$, $0 \leq k \leq m$.*

*Proof.* If follows from Lemma 3 and Theorem 2. □

**Proposition 5.** *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function of order $\rho < 1$ which is not identically zero. Let $\varepsilon > 0$. Then $f|_{(1+\varepsilon,+\infty)} \in \mathcal{B}_\varepsilon \setminus \mathcal{V}_\varepsilon$.*

*Proof.* Let $\rho < \lambda < 1$. There exists a constant $C > 0$ such that

$$|f(s)| \leq Ce^{|s|^\lambda},\ s \in \mathbb{C}.$$

Let $a > 0$ and $\sigma > 1 + \epsilon$. We have that

$$e^{-a\sigma}|f(\sigma)| \leq Ce^{-a\sigma+\sigma^\lambda}$$

so

$$\lim_{\sigma\to+\infty} e^{-a\sigma}|f(\sigma)| = 0,$$

hence $f|_{(1+\varepsilon,+\infty)} \in \mathcal{B}_\varepsilon$. If $f$ is polynomial then

$$\lim_{\sigma\to+\infty} e^{a\sigma}|f(\sigma)| = +\infty,\ \forall a > 0,$$

so $f|_{(1+\varepsilon,+\infty)} \notin \mathcal{V}_\varepsilon$. If $f$ is not polynomial, then by Hadamard's Theorem, there exists $A \neq 0$ such that

$$f(s) = As^m \prod_{n=1}^{\infty}(1 - \frac{s}{s_n}), s \in \mathbb{C}, \tag{11}$$

where $m$ is the multiplicity of $s_0 = 0$ as zero of $f$, and $s_1, s_2, \ldots$ are the non-zero zeros of $f$. Let

$$E(s) := \prod_{n=1}^{\infty}(1 - \frac{s}{s_n}).$$

It is known (see [7, Ch. 5, Corollary 5.4]) that there exists a stricly increasing sequence $(r_k)_{k\geq 1}$ of positive numbers with $\lim_{k\to+\infty} r_k = +\infty$ and a constant $B > 0$ such that

$$|E(r_k)| \geq e^{-Br_k^\lambda},\ \forall k \geq 1. \tag{12}$$

Let $a > 0$. From (11) and (12) it follows that

$$e^{ar_k}|f(r_k)| = e^{ar_k}|A|r_k^m|E(r_k)| \geq |A|r_k^m e^{ar_k - Br_k^\lambda} \to +\infty,$$

hence $f|_{(1+\varepsilon,+\infty)} \notin \mathcal{V}_\varepsilon$. □

**Corollary 6.** *With the assumptions of Corollary 4, the holomorphic functions $F_j^{(k)}(s)$, $1 \leq j \leq r$, $0 \leq k \leq m$ are linearly independent over the field of meromorphic functions of order $< 1$.*

*Proof.* Suppose that there exists a linear combination

$$\sum_{j=1}^{r} \sum_{k=0}^{m} Q_{jk}(s) F_j^{(k)}(s) = 0, \ \operatorname{Re} s > 1 + \epsilon,$$

where $Q_{jk}$ are meromorphic functions of order $< 1$. It is known that a meromorphic function of order $< 1$ is a quotient of entire functions of order $< 1$. So we may suppose that $Q_{jk}$ are entire functions of order $< 1$. Let $\sigma > 1 + \epsilon$. We have that

$$\sum_{j=1}^{r} \sum_{k=0}^{m} Q_{jk}(\sigma) F_j^{(k)}(\sigma) = 0.$$

From Proposition 5 and Corollary 4 it follows that

$$Q_{jk}(\sigma) = 0, \ \sigma > 1 + \epsilon,$$

hence

$$Q_{jk}(s) = 0, \ s \in \mathbb{C},$$

by the identity principle for holomorphic functions. ☐

# 2. Artin L-functions

Let $K/\mathbb{Q}$ be a finite Galois extension. It was proved in [6, Theorem 1] that the derivatives of any order of Artin $L$-functions to finitely many distinct characters of the Galois group are linearly independent over $\mathbb{C}$. In our main result we extend this:

**Theorem 7.** *Let $K/\mathbb{Q}$ be a finite Galois extension, and let $\chi_1, \ldots, \chi_r$ be distinct characters of the Galois group with the associated Artin L-functions $L(s, \chi_1), \ldots, L(s, \chi_r)$. Let $\varepsilon > 0$. Let $\mathcal{A}_\varepsilon \subset \mathcal{B}_\varepsilon$ be a $\mathbb{C}$-vector space with*

$$\mathcal{A}_\varepsilon \cap \mathcal{V}_\varepsilon = \{0\}.$$

*Let $m \geq 0$. If the functions $G_{jk}(\sigma) \in \mathcal{A}_\varepsilon$ satisfy*

$$\sum_{j=1}^{r} \sum_{k=0}^{m} G_{jk}(\sigma) L^{(k)}(\sigma, \chi_j) = 0, \ \sigma > 1 + \varepsilon,$$

*then $G_{jk} = 0$, $1 \leq j \leq r$, $0 \leq k \leq m$.*

*Proof.* Let

$$L(s,\chi_j)=\sum_{n=1}^{\infty}\frac{f_j(n)}{n^s},\ j=1,\dots,r$$

be the Dirichlet series expansion of $L(s,\chi_j)$ in the half-plane $\operatorname{Re} s>1$. Since in $\operatorname{Re} s>1$ the function $L(s,\chi_j)$ is defined by an Euler product, the arithmetic function $f_j(n)$ is multiplicative. It is well known that for Artin L-functions, the function $f_j(n)$ is $O(n^\delta)$ for any $\delta>0$, hence $O(n^\varepsilon)$. We show that $e(n), f_1(n),\dots,f_r(n)$ are pairwise non-equivalent. For a prime number $p$ which is not ramified in $K$, the value $f_j(p)$ equals the value of the character $\chi_j$ on the Frobenius class associated to $p$. For distinct characters $\chi_j$ and $\chi_k$ there are, by Chebotarev's density theorem, infinitely many primes $p$ such that $f_j(p)\neq f_k(p)$, so the arithmetic functions $f_j$ and $f_k$ are not equivalent in the sense of section 1. Also, $e(p)=0$ for any prime $p$, while there exist infinitely many non-ramified primes $p$ with $f_j(p)=\chi_j(1)\neq 0$, hence the arithmetic functions $e$ and $f_j$ are not equivalent. We apply Corollary 4. □

**Corollary 8.** *Let $K/\mathbb{Q}$ be a finite Galois extension. Let $\chi_1,\dots,\chi_r$ be $r\geq 1$ distinct characters of the Galois group with the associated Artin L-functions $L(s,\chi_1),\dots,L(s,\chi_r)$. Let $m\geq 0$. The meromorphic functions $L^{(k)}(s,\chi_j)$, $1\leq j\leq r$, $0\leq k\leq m$ are linearly independent over the field of meromorphic functions of order $<1$.*

*Proof.* Apply Theorem 7 and Corollary 6. □

Let $\chi_1,\dots,\chi_h$ be the irreducible characters of the Galois group. In [6, Corollary 4] it was proved that the Artin L-functions $L(s,\chi_1),\dots,L(s,\chi_h)$ are algebraically independent over $\mathbb{C}$. This extended Artin's result [1, Satz 5, P. 106] that $L(s,\chi_1),\dots,L(s,\chi_h)$ are multiplicatively independent. Now we can prove more:

**Corollary 9.** *Let $K/\mathbb{Q}$ be a finite Galois extension, and let $\chi_1,\dots,\chi_h$ be the irreducible characters of the Galois group. Then the Artin L-functions $L(s,\chi_1),\dots,L(s,\chi_h)$ are algebraically independent over the field of meromorphic functions of order $<1$.*

*Proof.* This follows from Corollary 8 and the fundamental property

$$L(s,\chi_1)^{n_1}\cdots L(s,\chi_h)^{n_h}=L(s,n_1\chi_1+\cdots+n_h\chi_h).$$

□

**Corollary 10.** *Let $K_1/\mathbb{Q}, \ldots, K_r/\mathbb{Q}$ be $r \geq 1$ distinct finite Galois extensions with the Dedekind zeta-functions $\zeta_{K_1}, \ldots, \zeta_{K_r}$. Let $m \geq 0$. The functions*

$$\zeta_{K_1}^{(0)}, \ldots, \zeta_{K_1}^{(m)}, \zeta_{K_2}^{(0)}, \ldots, \zeta_{K_2}^{(m)}, \ldots, \zeta_{K_r}^{(0)}, \ldots, \zeta_{K_r}^{(m)}$$

*are linearly independent over the field of meromorphic functions of order $< 1$.*

*Proof.* As in the proof of [6, Corollary 5], it holds that

$$\zeta_{K_j}(s) = L(s, \chi_j, K/\mathbb{Q}),\ j = 1, \ldots, r,$$

where $K := K_1 \cdots K_r$ is the compositum of the fields $K_1, \ldots, K_r$ and $\chi_1, \ldots, \chi_r$ are distinct characters of the Galois group of $K$. We apply Corollary 8. □

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