ON HUH'S CONJECTURES FOR THE POLAR DEGREE

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ABSTRACT. We prove a precise version of a general conjecture on the polar degree stated by June Huh. We confirm Huh's conjectural list of all projective hypersurfaces with isolated singularities and polar degree equal to 2.

1. INTRODUCTION

Let $f \in \mathbb{C}[x_0, \ldots, x_n]$, $n \geq 2$, be a homogeneous polynomial of degree $d \geq 2$ and $V := \{f = 0\} \subset \mathbb{P}^n$. The *polar degree* of f is defined as the topological degree of the gradient mapping

$$(1) \qquad \operatorname{grad} f : \mathbb{P}^n \setminus \operatorname{Sing} V \to \mathbb{P}^n.$$

It only depends on the reduced structure of V and thus can be denoted by $\operatorname{pol}(V)$. This fact has been conjectured by Dolgachev [Do] and proved by Dimca and Papadima [DP] as a consequence of the following interpretation:

$$(2) \qquad \operatorname{pol}(V) = \operatorname{rank} H_{n-1}(V \setminus H)$$

where $H \subset \mathbb{P}^n$ is a general hyperplane. More than 160 years ago, Otto Hesse had claimed [Hes1, Hes2] that $\operatorname{pol}(V) = 0$ if and only if the hypersurface V is a cone[1], but this has been disproved by a nice example due to Gordan and Noether [GN], see [Huh] for a discussion of this example and other remarks. More recently Dolgachev [Do] considered hypersurfaces with $\operatorname{pol}(V) = 1$ (so-called *homaloidal*) and classified such plane curves, cf the list below, see also [Di, FM]. We refer to [Huh] for some recent papers about various questions around the polar degree.

In this note we focus on the polar degree of hypersurfaces with isolated singularities. The formula:

$$(3) \qquad \operatorname{pol}(V) = (d-1)^n - \sum_{p \in \operatorname{Sing} V} \mu(V, p)$$

proved by Dimca and Papadima [DP] in terms of the degree d and the Milnor numbers of V at its singular points, implies in particular that the quadratic hypersurface is the only smooth V which is homaloidal. In order to bound $\operatorname{pol}(V)$ from below, Huh [Huh] used the theory of slicing by *pencils with isolated singularities in the axis* introduced and developed in [Ti]. He found a key inequality in a more general setting relating $\operatorname{pol}(V)$

[1]Gordan and Noether [GN] showed that this claim is true modulo a birational transformation.

to the sectional Milnor number $\mu^{\langle n-2 \rangle}(V, p)$ of any chosen singular point $p \in \operatorname{Sing} V$ of a hypersurface V with isolated singularities and[2] $\operatorname{pol}(V) > 0$, that we shall call *Huh's inequality*:

$$(4) \qquad \operatorname{pol}(V) \geq \mu^{\langle n-2 \rangle}(V, p).$$

With this bound at hand, Huh could prove Dimca and Papadima's [DP] conjectural list of all homaloidal hypersurfaces with isolated singularities [Huh, Theorem 4]:

A projective hypersurface $V \subset \mathbb{P}^n$ with only isolated singularities has polar degree 1 if and only if it is one of the following, after a linear change of homogeneous coordinates:

(i) $(n \geq 2, d = 2)$ *a smooth quadric:*
$$f = x_0^2 + \cdots + x_n^2 = 0.$$

(ii) $(n = 2, d = 3)$ *the union of three non-concurrent lines:*
$$f = x_0 x_1 x_2 = 0, \qquad (3A_1).$$

(iii) $(n = 2, d = 3)$ *the union of a smooth conic and one of its tangents:*
$$f = x_0(x_1^2 + x_0 x_2) = 0, \qquad (A_3).$$

This list contains the plane curves found by Dolgachev [Do].

June Huh conjectured a general finiteness principle for projective hypersurfaces with isolated singularities and fixed polar number $\operatorname{pol}(V) = k$. In case $k = 2$, he conjectured that such hypersurfaces are only those in the following list:

Conjecture 1.1. [Huh, Conjecture 20] *A projective hypersurface $V \subset \mathbb{P}^n$ with only isolated singularities has polar degree 2 if and only if it is one of the following, after a linear change of homogeneous coordinates:*

(a) $(n = 3, d = 3)$ *a normal cubic surface containing a single line:*
$$f = x_0 x_1^2 + x_1 x_2^2 + x_1 x_3^2 + x_2^3 = 0, \qquad (E_6).$$

(b) $(n = 3, d = 3)$ *a normal cubic surface containing two lines:*
$$f = x_0 x_1 x_2 + x_0 x_3^2 + x_1^3 = 0, \qquad (A_5, A_1).$$

(c) $(n = 3, d = 3)$ *a normal cubic surface containing three lines and three binodes:*
$$f = x_0 x_1 x_2 + x_3^3 = 0, \qquad (A_2, A_2, A_2).$$

(d) $(n = 2, d = 5)$ *two smooth conics meeting at a single point and their common tangent:*
$$f = x_0(x_1^2 + x_0 x_2)(x_1^2 + x_0 x_2 + x_0^2) = 0, \qquad (J_{2,4}).$$

(e) $(n = 2, d = 4)$ *two smooth conics meeting at a single point:*
$$f = (x_1^2 + x_0 x_2)(x_1^2 + x_0 x_2 + x_0^2) = 0, \qquad (A_7).$$

(f) $(n = 2, d = 4)$ *a smooth conic, a tangent and a line passing through the tangency point:*
$$f = x_0(x_0 + x_1)(x_1^2 + x_0 x_2) = 0, \qquad (D_6, A_1).$$

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[2]it is explicitly stated in *loc.cit.* that the inequality does not apply to V which is a cone with apex at p; indeed, being a cone implies $\operatorname{pol}(V) = 0$.

(g) $(n = 2, d = 4)$ *a smooth conic and two tangent lines:*
$$f = x_0 x_2 (x_1^2 + x_0 x_2) = 0, \qquad (A_1, A_3, A_3).$$

(h) $(n = 2, d = 4)$ *three concurrent lines and a line not meeting the center point:*
$$f = x_0 x_1 x_2 (x_0 + x_1) = 0, \qquad (D_4, A_1, A_1, A_1).$$

(i) $(n = 2, d = 4)$ *a cuspidal cubic and its tangent at the cusp:*
$$f = x_0 (x_1^3 + x_0^2 x_2) = 0, \qquad (E_7).$$

(j) $(n = 2, d = 4)$ *a cuspidal cubic and its tangent at the smooth flex point:*
$$f = x_2 (x_1^3 + x_0^2 x_2) = 0, \qquad (A_2, A_5).$$

(k) $(n = 2, d = 3)$ *a cuspidal cubic:*
$$f = x_1^3 + x_0^2 x_2 = 0, \qquad (A_2).$$

(l) $(n = 2, d = 3)$ *a smooth conic and a secant line:*
$$f = x_1 (x_1^2 + x_0 x_2) = 0, \qquad (A_1, A_1).$$

Huh showed his conjecture for plane curves, cubic and quartic surfaces [Huh]. Fasarella and Medeiros [FM] classified plane curves with $\mathrm{pol}(V) = 2$ as a consequence of formula (2) and the computation of Euler characteristics.

We first prove Huh's conjecture for $\mathrm{pol}(V) = 2$, then Huh's finiteness conjecture for $\mathrm{pol}(V) = k$. More precisely:

Theorem 1.2. *The hypersurfaces $V \in \mathbb{P}^n$ with isolated singularities and of polar degree 2 are only those in Huh's list.*
In particular there are no such hypersurfaces for $n > 3$.

The main idea is to consider the affine hypersurface $V \setminus H$, for some general hyperplane H not passing through any singularity, and to deform it into the cone $V_{n,d} \setminus H$ defined by $f_{n,d} := x_1^d + \cdots + x_n^d = 0$. This induces an embedding of the direct sums of the Milnor lattices of the singularities of V into the Milnor lattice of the isolated singularity of $f_{n,d}$. We apply to this deformation the semicontinuity of the spectrum proved by Varchenko [Va] and Steenbrink [St3] (Section 3).

June Huh formulated the following general principle for any $k > 2$:

Conjecture 1.3. [Huh, page 1545] *There is no projective hypersurface $V \subset \mathbb{P}^n$ of polar degree k with only isolated singular points, for sufficiently large n and $d = \deg V$.*

We answer positively to Huh's general conjecture in the following precise way:

Theorem 1.4 (Finiteness Theorem)**.** *For any integer $k \geq 2$, let K_k denote the set of pairs of integers (n, d) with $n \geq 2$ and $d \geq 3$, such that there exists a projective hypersurface V in $\mathbb{P}^n$ of degree d with isolated singularities and $\mathrm{pol}(V) = k$.*
Then K_k is finite for any $k \geq 2$.

By Theorem 1.2 we are showing that $K_2 = \{(2,3),(2,4),(2,5),(3,3)\}$ and that this is a sharp equality. To prove Theorem 1.4 for $k \geq 3$, we show that any $(n,d) \in K_k$ satisfies the inequalities $n < \max(k, 5 + \log_2 k)$ by Theorem 4.1, and $d < \max\{2 + \ell_{n,k}, (n + \ell_{n,k})(k + 2)/(n-1)\}$ by Theorem 4.5, where $\ell_{n,k} := \min \{\ell \in \mathbb{N} \,|\, \binom{n+\ell}{n} > k\}$.

2. Spectrum and the semicontinuity argument

We recall here the properties of the spectrum that we shall use in the proofs.

2.1. **Properties.** Let X be a complex manifold of dimension n and $f : X \to \mathbb{C}$ a holomorphic function. Let $x \in X$ be an isolated critical point of f with Milnor fibre $X_{f,x}$. The group $\tilde{H}^{n-1}(X_{f,x})$ is free abelian of finite rank equal to the Milnor number $\mu(f,x)$. Let T denote the monodromy operator and let T_s be its semisimple part. Then $\tilde{H}^{n-1}(X_{f,x})$ underlies a mixed Hodge structure with Hodge filtration $F^\bullet = (0 \subset F^{n-1} \subset F^{n-2} \subset \cdots \subset F^0 = \tilde{H}^{n-1}(X_{f,x}, \mathbb{C}))$, preserved by T_s. By the monodromy theorem, the eigenvalues of T_s are roots of unity.

The *spectrum* $\mathrm{Sp}(f,x)$ is an element of the group ring $\mathbb{Z}[\mathbb{Q}]$:

$$\mathrm{Sp}(f,x) = \sum_\alpha n_\alpha(\alpha)$$

a finite sum with $n_\alpha \in \mathbb{N}$. Here n_α is defined as the multiplicity of $\exp(-2\pi i\alpha)$ as an eigenvalue of T_s acting on F^p/F^{p+1}, where $p = \lfloor n - 1 - \alpha \rfloor$. The spectrum has the following properties.

Range: $n_\alpha \neq 0 \Rightarrow -1 < \alpha < n - 1$.
Symmetry: If $\alpha + \beta = n - 2$ then $n_\alpha = n_\beta$. See [AGV, Corollary to Lemma 13.14] for these two properties.
Stability: If $g : X \times \mathbb{C}$ is defined by $g(z,t) = f(z) + t^2$, and $x \in X$ with $\mathrm{Sp}(f,x) = \sum_\alpha n_\alpha(\alpha)$, then $\mathrm{Sp}(g,(x,0)) = \sum_\alpha n_\alpha(\alpha + \frac{1}{2})$. See [AGV, Corollary 1 to Theorem 13.7].
μ-constant deformation invariance: The spectrum is constant in any deformation of isolated hypersurface singularity germs with constant Milnor number, see [St3].
Semicontinuity: Let $(Y_0, 0)$, where $Y_0 := \{g = 0\} \subset \mathbb{C}^n$, be the germ of a hypersurface with isolated singularity. Let g_s be a good representative of a deformation of g such that $Y_s := \{g_s = 0\}$ has isolated singular points $\mathrm{Sing}(Y_s) = \{p_1, \ldots, p_r\}$ which tend to the origin 0 when $s \to 0$. Let $g_i : (\mathbb{C}^n, p_i) \to (\mathbb{C}, 0)$ be a local equation for Y_s at p_i, for $i = 1, \ldots, r$. Then for each $a \in \mathbb{R}$ one has:

$$\sum_{i=1}^r \deg_{]a,a+1[} \mathrm{Sp}(g_i, p_i) \leq \deg_{]a,a+1[} \mathrm{Sp}(g, 0)$$

in case we have a deformation of lower weight of a quasi-homogeneous function germ g, and

$$\sum_{i=1}^r \deg_{]a,a+1]} \mathrm{Sp}(g_i, p_i) \leq \deg_{]a,a+1]} \mathrm{Sp}(g, 0)$$

in general.

Here for any $A \subset \mathbb{R}$ the function $\deg_A : \mathbb{Z}[\mathbb{Q}] \to \mathbb{Z}$ is defined by $\deg_A(\sum_\alpha n_\alpha(\alpha)) := \sum_{\alpha \in A} n_\alpha$. The first inequality is a special case of a result of Varchenko [Va], and the second inequality follows from [St3, Theorem 2.4].

2.2. **Deformation to** $f_{n,d}$. Let $H := \{x_0 = 0\} \subset \mathbb{P}^n$ be a generic hyperplane with respect to our hypersurface $V := \{f = 0\}$ and write $f = f_d + x_0 f_{d-1} + \cdots + x_0^d f_0$ where $f_i \in \mathbb{C}[x_1, \ldots, x_n]$ is homogeneous of degree i. We consider the 1-parameter family of polynomials $g_s(1, x_1, \cdots, x_n) := f_d + s f_{d-1} + \cdots + s^d f_0$ on $\mathbb{C}^n = \mathbb{P}^n \setminus H$, see e.g. [Br]. Then f_d is a general homogeneous polynomial which, as germ at the origin $0 \in \mathbb{C}^n$, is topologically equivalent to the polynomial $f_{n,d} := \sum_{i=1}^n x_i^d$.

Therefore the family g_s describes a deformation of lower weight of $g_1 = f_{|\mathbb{C}^n}$ to $g_0 = f_d$, hence a deformation from the hypersurface $V \setminus H$ to the hypersurface $\{f_d = 0\}$, which is topological equivalent to the hypersurface $\{f_{n,d} = 0\}$. We may thus apply the semicontinuity of the spectrum and, if the local singularities (V, p_i) of V are defined by $f_i = 0$, we get:

$$(5) \qquad \sum_{i=1}^r \deg_{]a,a+1[} \mathrm{Sp}(f_i, p_i) \leq \deg_{]a,a+1[} \mathrm{Sp}(f_{n,d}, 0)$$

and

$$(6) \qquad \sum_{i=1}^r \deg_{]a,a+1]} \mathrm{Sp}(f_i, p_i) \leq \deg_{]a,a+1]} \mathrm{Sp}(f_{n,d}, 0).$$

2.3. **Spectra of special singularities.** As Huh showed in [Huh, Lemma 19] by using his bound (4) and Arnold's classification results [AGV], the singularities which may occur on a projective hypersurface of dimension $n-1$ of polar degree 1 or 2 are within the following list:

type	equation	Milnor number
A_k with $k \geq 1$	$x_1^{k+1} + q_{n-1}$	k
D_k with $k \geq 4$	$x_1^2 x_2 + x_2^{k-1} + q_{n-2}$	k
E_{6k} with $k \geq 1$	$x_1^3 + x_2^{3k+1} + q_{n-2}$	$6k$
E_{6k+1} with $k \geq 1$	$x_1^3 + x_1 x_2^{2k+1} + q_{n-2}$	$6k+1$
E_{6k+2} with $k \geq 1$	$x_1^3 + x_2^{3k+2} + q_{n-2}$	$6k+2$
$J_{k,i}$ with $k \geq 2, \ i \geq 0$	$x_1^3 + x_1^2 x_2^k + x_2^{3k+i} + q_{n-2}$	$6k-2+i$

where $q_{n-1-j} := x_{j+2}^2 + \cdots + x_n^2$, and in each case we have exhibited one of the possible equations in the μ-class of the singularity. Note that all types except $J_{k,i}$ with $i > 0$ are represented by a weighted homogeneous function. It was shown in [St1, Example 5.11] for $f \in \mathbb{C}[x_1, x_2]$ with an isolated singular point at 0 and weighted homogeneous with weights w_1 and w_2, that:

$$\sum n_\alpha t^{\alpha+1} = \frac{t^{w_1} - t}{1 - t^{w_1}} \cdot \frac{t^{w_2} - t}{1 - t^{w_2}}$$

The weights are as follows:

Type	A_k	D_k	E_{6r}	E_{6r+1}	E_{6r+2}	$J_{k,0}$
w_1	$\frac{1}{k+1}$	$\frac{k-2}{2k-2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
w_2	$\frac{1}{2}$	$\frac{1}{k-1}$	$\frac{1}{3r+1}$	$\frac{2}{6r+3}$	$\frac{1}{3r+2}$	$\frac{1}{3k}$

and the spectra of the corresponding curve singularities are:

A_k	$\sum_{j=1}^{k} \left(-\frac{1}{2} + \frac{j}{k+1}\right)$
D_k	$(0) + \sum_{j=1}^{k-1} \left(-\frac{1}{2} + \frac{2j-1}{2k-2}\right)$
E_{6k}	$\sum_{j=1}^{3k} \left(-\frac{2}{3} + \frac{j}{3k+1}\right) + \sum_{j=1}^{3k} \left(-\frac{1}{3} + \frac{j}{3k+1}\right)$
E_{6k+1}	$(0) + \sum_{i=1}^{2} \sum_{j=1}^{3k} \left(-\frac{i}{3} + \frac{2j}{6k+3}\right)$
E_{6k+2}	$\sum_{j=1}^{3k+1} \left(-\frac{2}{3} + \frac{j}{3k+2}\right) + \sum_{j=1}^{3k+1} \left(-\frac{1}{3} + \frac{j}{3k+2}\right)$
$J_{k,0}$	$\sum_{j=1}^{3k-1} \left(-\frac{2}{3} + \frac{j}{3k}\right) + \sum_{j=1}^{3k-1} \left(-\frac{1}{3} + \frac{j}{3k}\right).$

If $i > 0$ then the singularity $J_{k,i}$ is still nondegenerate with respect to its Newton diagram, so its spectrum can be computed as in [St1, Sect. 5.15]. See also [AGV, Sect. 13.3.4]. To describe the result, we only list the negative spectral numbers (again in the case of the corresponding curve singularities). These numbers occur in two groups, those from the first group having denominator $3k$ and those from the second having denominator $6k + 2i$. The numerators occurring depend on the parities of k and i. The numerators of the first group are:

k even	$-2k + 1, -2k + 2, \ldots, -3k/2, -k + 1, -k + 2, \ldots, -1$
k odd	$-2k + 1, -2k + 2, \ldots, (-3k - 1)/2, -k + 1, -k + 2, \ldots, -1$

and the numerators of the second group are the integers ℓ in the interval $(-(3k + i), 0)$ with the same parity as i.

Observe that the spectral numbers in the second group are all greater than $-\frac{1}{2}$.

2.4. **Spectrum of $f_{n,d}$.** The singularity $f_{n,d} = \sum_{i=1}^{n} x_i^d$ has Milnor number $(d - 1)^n$ and spectrum $\sum_{k \in \mathbb{Z}} u_k(\frac{k}{d})$, where:

$$u_k = \sharp\{(a_1, \ldots, a_n) \in \mathbb{N}^n \mid 1 \leq a_j \leq d - 1 \text{ and } \sum_j a_j = k + d\}.$$

3. HYPERSURFACES OF POLAR DEGREE 2

Theorem 3.1. *There is no projective hypersurface $V \subset \mathbb{P}^n$ with isolated singularities, of polar degree $\mathrm{pol}(V) = 2$, and dimension and degree at least three (i.e. $n \geq 4$ and $d \geq 3$).*

Proof. Suppose that there exists a hypersurface $V \subset \mathbb{P}^n$ with isolated singularities of degree d and polar degree 2. By [Huh, Lemma 19] we know that the singularities of V are of type $A, D, E, J_{*,*}$. It follows that the singularity $f_{n,d}$ deforms to those singularities in one fibre, whose Milnor numbers add up to $(d - 1)^n - 2$, which is just two less than the Milnor number of $f_{n,d}$. The singularity $f_{n,d}$ has as its two smallest spectral numbers $-1 + \frac{n}{d}$ with multiplicity 1, and $-1 + \frac{n+1}{d}$ with multiplicity n.

Lemma 3.2. *Suppose $d \geq 3$ and $n \geq 3$. Then either $n = 3$ and $d \leq 4$, or $d = 3$ and $4 \leq n \leq 5$.*

Proof. All singularities of type $A, D, E, J_{*,*}$ in dimension $n - 1$ have a smallest spectral number which is greater than $-\frac{2}{3} + \frac{n-2}{2} = \frac{n-1}{2} - \frac{7}{6}$. Suppose that the second spectral number $\frac{n+1}{d} - 1$ of $f_{n,d}$ is smaller or equal to $t_n := \frac{n-1}{2} - \frac{7}{6}$. As this spectral number has multiplicity equal to n, it follows that $\deg_{]-\infty,t_n]} \mathrm{Sp}(\tilde{f}_{n,d}) \geq n + 1$. Therefore:

$$\deg_{]t_n,\infty[} \mathrm{Sp}(f_{n,d}) < (d-1)^n - n - 1 < (d-1)^n - 2 = \sum_i \deg_{]t_n,\infty[} \mathrm{Sp}(f_i, p_i)$$

since $n \geq 2$, and the last equality is due to our assumption $\mathrm{pol}(V) = 2$. This contradicts the semicontinuity of the spectrum (6).

So the second spectral number $\frac{n+1}{d} - 1$ of $f_{n,d}$ must be greater than $\frac{n-1}{2} - \frac{7}{6}$. But from $\frac{n+1}{d} - 1 > \frac{n-1}{2} - \frac{7}{6}$ it follows that $(3(n-1) - 1)(d - 2) < 14$.

If $n = 3$, this implies $d \leq 4$. If $n \geq 4$, this implies that $3(n-1) - 1 > 8$, so $d = 3$ is the only solution, and therefore $3(n-1) - 1 < 14$, thus $n \leq 5$. □

We continue the proof of Theorem 3.1 with the remaining cases after the reduction by Lemma 3.2.

The case $n = 5, d = 3$. The spectrum of $f_{5,3}$ is $\left(\frac{2}{3}\right) + 5\,(1) + 10\left(\frac{4}{3}\right) + 10\left(\frac{5}{3}\right) + 5\,(2) + \left(\frac{7}{3}\right)$, therefore its degree with respect to the open interval $]1, 2[$ is 20. For our cubic fourfold V with finite singular set $\{p_1, \ldots, p_s\}$ and $\mathrm{pol}(V) = 2$ we have $\sum_i \mu(V, p_i) = 2^5 - 2 = 30$.

The semicontinuity of the spectrum requires that $\sum_i \deg_{]1,2[} \mathrm{Sp}(f_i, p_i) \leq 20$, thus using the symmetry of the spectrum with respect to $3/2$ we get:

$$\sum_i \deg_{]-\infty,1]} \mathrm{Sp}(f_i, p_i) \geq \frac{1}{2}(30 - 20) = 5.$$

But the semicontinuity also gives:

$$\sum_i \deg_{]-\infty,1[} \mathrm{Sp}(f_i, p_i) \leq \deg_{]-\infty,1[} \mathrm{Sp}(f_{5,3}) = 1.$$

It follows that the spectral number 1 in $\sum_i \mathrm{Sp}(f_i, p_i)$ occurs with multiplicity $n_1 \geq 4$. One checks easily that there do not exist singularities in our list satisfying $\sum_i \mu(f_i, p_i) \leq 30$ and $n_1 \geq 4$. Indeed, the only singularities from our list which have a spectral number equal to 1 are $J_{2,0}$ and $J_{4,0}$, with Milnor numbers 10 and 22, respectively, thus the total multiplicity n_1 cannot be more than 3.

The case $n = 4, d = 3$. The spectrum of $f_{4,3}$ is $\left(\frac{1}{3}\right) + 4\left(\frac{2}{3}\right) + 6\,(1) + 4\left(\frac{4}{3}\right) + \left(\frac{5}{3}\right)$. For all curve singularity germs g in our table one checks that their smallest spectral number is greater than $-\frac{2}{3}$ and

$$\deg_{]-\infty,-\frac{1}{3}]} \mathrm{Sp}(g, 0) \leq \mu(g, 0)/4.$$

Hence for a cubic threefold V of polar degree 2 with singular set $\{p_1, \ldots, p_s\}$, considering the shift of $+1$ of the spectral numbers, we have:

$$\sum_i \deg_{]-\infty,\frac{2}{3}]} \mathrm{Sp}(f_i, p_i) \leq \sum_i \mu(f_i, p_i)/4 = \frac{7}{2}.$$

Using that the smallest spectral number of each possible singularity (f_i, p_i) is greater than $\frac{1}{3}$, and hence by symmetry the greatest spectral number is smaller than $\frac{5}{3}$, we then get:

$$\sum_i \deg_{]\frac{2}{3},\frac{5}{3}[} \mathrm{Sp}(f_i, p_i) = \sum_i \deg_{]\frac{2}{3},\infty[} \mathrm{Sp}(f_i, p_i) \geq 14 - \frac{7}{2} > 10$$

whereas $\deg_{]\frac{2}{3},\frac{5}{3}[} \mathrm{Sp}(f_{4,3}) = 10$. This contradicts the semicontinuity of the spectrum.

$\square$

3.1. **Proof of Theorem 1.2.** By the above Theorem 3.1 and Lemma 3.2 we have reduced the proof of Theorem 1.2 to the cases $n \leq 3$ and $d \leq 4$. The projective surfaces of degree $d = 3$ have been classified by Bruce and Wall [BW] and, as noticed by Huh [Huh, proof of Prop. 21], the case $\mathrm{pol}(V) = 2$ can be extracted and yields a part of Huh's list.

The proof for quartic surfaces is also due to Huh [Huh, proof of Prop. 21], it is nontrivial and it uses a whole bunch of classical results. An alternate proof can be made by using semicontinuity.

The case $n = 2$ of plane curves with $\mathrm{pol}(V) = d$ has been treated in [FM], as we have mentioned in the Introduction, and yields the corresponding part of Huh's list.

Let us remark also that the case $d = 2$ and general n is excluded from the list since $\mathrm{pol}(V)$ can be at most 1, by formula (3).

This completes the proof of our Theorem 1.2.

4. HUH'S GENERAL CONJECTURE: PROOF OF THEOREM 1.4

For polar degree $k > 2$ we are able to find bounds for the dimension $n - 1$ and degree d of V. Our Theorem 1.4 answers Huh's Conjecture 1.3 in the more concrete terms of Theorem 4.1 and 4.5 below.

4.1. **Corank and spectrum.**

Theorem 4.1. *Let $k \geq 3$. There is no projective hypersurface $V \subset \mathbb{P}^n$ of polar degree k with only isolated singular points, for $n \geq \max\{k, 5 + 3\log_2 k\}$ and degree $d \geq 3$.*

Proof. Let $\{p_1, \ldots, p_s\}$ be the set of singular points of a projective hypersurface V of $\mathbb{P}^n$ of polar degree $k \geq 3$. By [Huh, Theorem 2], one has $k \geq \mu^{(n-2)}(V, p_i)$. This is the Milnor number of the slice germ $(V \cap H, p_i)$ for some general hyperplane H through p_i. If f_i is a local equation of V at p_i, then the restriction $f_{i|H}$ is a local equation for $(V \cap H, p_i)$.

Let us denote $r_i := \mathrm{corank} f_{i|H}$, where $0 \leq r_i \leq n - 1$. We first show:

Lemma 4.2. *For any $i \in \{1, \ldots, s\}$, we have:*
  (a) *$r_i \leq \log_2 k$.*
  (b) *The smallest spectral number $\alpha_{1,i}$ of the hypersurface singularity (V, p_i) is bounded from below by $\frac{n - r_i - 3}{2}$.*

Proof. We shall use the following two standard facts concerning the corank. Let $h : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be some function germ with isolated singularity.
(1). If $r = \mathrm{corank}\, h$ then $h = g(x_1, \ldots, x_r) + x_{r+1}^2 + \cdots + x_m^2$ with $\mathrm{ord}_0 g \geq 3$ and therefore (by using the Sebastiani-Thom formula) we get $\mu(h) = \mu(g) \geq \mu(x_1^3 + \cdots + x_r^3) = 2^r$.

(2). If the restriction $h_{|\{x_1=0\}}$ has isolated singularity and has corank p, then corank $h \leq p+1$. Indeed one has $h_{|\{x_1=0\}} = \hat{h}(x_2,\ldots,x_{p+1}) + x_{p+2}^2 + \cdots + x_m^2$ and thus $h = x_1 h_1(x_1,\ldots,x_{p+1}) + x_1 h_2(x_{p+2},\ldots,x_m) + x_{p+2}^2 + \cdots + x_m^2$ is right-equivalent, modulo $\mathfrak{m}^3$, to $x_1 \tilde{h}_1(x_1,\ldots,x_{p+1}) + x_{p+2}^2 + \cdots + x_m^2$ where $x_1 \tilde{h}_1(x_1,\ldots,x_{p+1})$ is either of order ≥ 3 or it is the sum of x_1^2 and some function of order ≥ 3.

(a). Using fact (1), we get that the Milnor number $\mu(f_{i|H}, p_i)$ is bounded from below by 2^{r_i}. Then, from the above inequalities, we get:

$$r_i \leq \log_2 \mu^{(n-2)}(V, p_i) \leq \log_2 k.$$

(b). By fact (2), after slicing with any hyperplane such that the local singularity is still an isolated singularity of the function restricted to the slice, the corank can drop by at most one. We thus have:

$$\text{corank } f_i \leq r_i + 1 \leq 1 + \log_2 \mu^{(n-2)}(V, p_i) \leq 1 + \log_2 k.$$

Therefore we may write in some well chosen system of local coordinates:

$$f_i(x_1,\ldots,x_n) = g(x_1,\ldots,x_{r_i+1}) + x_{r_i+2}^2 + \cdots + x_n^2,$$

where $g \in m^3$ is a germ of a function in $r_i + 1$ variables with trivial 2-jet. By using the properties of the spectrum, we get that the spectrum of g is in the interval $]-1, r_i[$ and that by adding $n - r_i - 1$ squares to g, since the spectrum shifts by $+\frac{1}{2}$ for each square, the spectrum of f_i is in the interval:

$$\left] -1 + \frac{n - r_i - 1}{2}, r_i + \frac{n - r_i - 1}{2} \right[.$$

and we have $n - r_i - 1 > 0$ for any i, since $r_i \leq \log_2 k \leq \log_2 n < n - 1$ by our hypotheses $n \geq k \geq 3$. $\qquad\square$

If the second spectral number of $f_{n,d}$, which is $\frac{n+1}{d} - 1$ and has multiplicity n, is smaller than $\beta := \min_i \frac{n - r_i - 3}{2}$, then:

$$\deg_{[\beta,\infty[} \text{Sp}(f_{n,d}) \leq (d-1)^n - n - 1.$$

On the other hand we have:

$$\deg_{[\beta,\infty[} \sum_i \text{Sp}(f_i, p_i) = \sum_i \mu(f_i, p_i) = (d-1)^n - k.$$

By the semicontinuity of the spectrum we get:

$$(d-1)^n - k \leq (d-1)^n - n - 1,$$

or $n + 1 \leq k$, which contradicts the hypothesis $n \geq k$.

We therefore must have $-1 + \frac{n+1}{d} \geq \beta$, and since $r_i \leq \log_2 k$ for any i, we get $\beta \geq \frac{n - \log_2 k - 3}{2}$. This amounts to the inequality:

$$d \leq \frac{2(n+1)}{n - 1 - \log_2 k}.$$

Since the last fraction is less than 3 if $n > 5 + 3 \log_2 k$, our proof of Theorem 4.1 is complete. $\qquad\square$

4.2. **An upper bound for the degree d.** The first spectral number of g at the point x will be denoted by $\alpha_1(g, x)$. We continue to use the notation $f_{n,d}$ for a generic homogeneous polynomial of degree d in n variables, which is deformation-equivalent to $\sum_{i=1}^{n} x_i^d$.

Lemma 4.3. *Let* $\mathrm{pol}(V) = k \geq 2$, *where* $V = \{f = 0\} \subset \mathbb{P}^n$ *has isolated singularities. Then:*

$$\alpha_1(f, p) > -1 + \frac{n-1}{k+2}$$

for any $p \in \mathrm{Sing}\, V := \{p_1, \ldots, p_s\}$.

Proof. One has Huh's inequality [Huh, Theorem 2]: $\mu^{(n-2)}(V, p_i) \leq k$, for any i. In the notations of the proof of Theorem 4.1, this amounts to $\mu(f_{i|H}, p_i) \leq k$, where H is some general hyperplane through p_i. We consider the composition of $f_{i|H}$ with the translate of the point p_i at the origin and denote the result by $\tilde{f}_{i|H}$. It follows that the function germ $\tilde{f}_{i|H}$ is $k + 1$-determined, which means that it is deformation-equivalent to its $(k + 1)$-jet $g_i := \mathrm{j}^{k+1}(\tilde{f}_{i|H})$, which is a polynomial of degree $k + 1$. We then apply the spectrum semi-continuity to the deformation:

$$h_i := sg_i + (1 - s)f_{n-1,k+2}$$

and get $\alpha_1(f_{i|H}, p_i) = \alpha_1(g_i, 0) \geq \alpha_1(f_{n-1,k+2}, 0) = -1 + \frac{n-1}{k+2}$. By the next Lemma 4.4 one has $\alpha_1(f_{i|H}, p_i) < \alpha_1(f_{i|}, p_i)$, and our claim follows by chaining these inequalities. $\qquad\square$

Lemma 4.4. *Let* $h : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be an isolated hypersurface singularity with smallest spectral number* α_1. *Let* H *be a general hyperplane through* 0 *in* $\mathbb{C}^n$ *and let* α_1' *be the smallest spectral number of the restriction* $h_{|H}$. *Then* $\alpha_1 > \alpha_1'$.

Proof. Since h is finitely determined, we may assume without loss of generality that h is a polynomial, that H is defined by $z_1 = 0$ and that

$$h(z_1, \ldots, z_n) = \sum_{m=0}^{N-1} z_1^m h_m(z_2, \ldots, z_n) + z_1^N$$

with $\deg h_m \leq N - m$ and $N \gg 1$. Then by:

$$h_t(z_1, \ldots, z_n) := \sum_{m=0}^{N-1} (tz_1)^m h_m(z_2, \ldots, z_n) + z_1^N$$

we obtain h as a deformation of the singularity $z_1^N + h_0(z_2, \ldots, z_n)$. By the Thom-Sebastiani theorem for the spectrum, this singularity has its smallest spectral number equal to $\alpha_1' + \frac{1}{N}$ and by the semicontinuity of the spectrum we get: $\alpha_1 \geq \alpha_1' + \frac{1}{N} > \alpha_1'$. $\quad\square$

Let $n, k \geq 2$ be fixed. Let v_j denote the multiplicity of the $(j + 1)$st spectral number $-1 + \frac{n+j}{d}$ of the germ $f_{n,d}$, for some $d \geq 2$. In particular $v_0 = 1$ and $v_1 = n$. In general these numbers depend both on n and d. However, for $j \leq d - 2$ we have $v_j = \binom{n+j-1}{n-1}$, see the general formula for the spectrum of $f_{n,d}$ in §2.4. We define

$$(7) \qquad \ell_{n,k} := \min\left\{ \ell \in \mathbb{N} \,\middle|\, \binom{n+\ell}{n} > k \right\}.$$

The following estimation of the upper bound for the degree d completes now the proof of Theorem 1.4:

Theorem 4.5. *Let* $n, k \geq 2$. *Any hypersurface* $V \subset \mathbb{P}^n$ *with isolated singularities and* $\mathrm{pol}(V) = k$ *has degree*
$$d < \max\{2 + \ell_{n,k}, (n + \ell_{n,k})(k+2)/(n-1)\}.$$

Proof. In case $d \geq 2 + \ell_{n,k}$, by the definition (7), the expression for v_j given above, and by taking into account the identity $\sum_{j=0}^{\ell} \binom{n+j-1}{n-1} = \binom{n+\ell}{n}$ for any $\ell \geq 0$, we have:

$$(8) \qquad \sum_{j=0}^{\ell_{n,k}} v_j = \binom{n + \ell_{n,k}}{n} > k.$$

Let us then denote $\gamma_{n,k} := -1 + \frac{n-1}{k+2}$. We will show that $\gamma_{n,k} < -1 + \frac{n+\ell_{n,k}}{d}$, as follows.

Assume by contradiction that $\gamma_{n,k} \geq -1 + \frac{n+\ell_{n,k}}{d}$. Since V has singular points $p_1, \ldots, p_s$ with total Milnor number $\mu = \sum_{i=1}^{s} \mu_i$, by Lemma 4.3 we get:

$$\deg_{]\gamma_{n,k}, \infty[} \sum_i \mathrm{Sp}(f_i, p_i) = \mu = (d-1)^n - k.$$

On the other hand, by our assumption we have:

$$\deg_{]\gamma_{n,k}, \infty[} \mathrm{Sp}(f_{n,d}) \leq (d-1)^n - (1 + v_1 + \ldots + v_{\ell_{n,k}}).$$

Applying the semicontinuity of the spectrum we get the inequality:

$$(d-1)^n - k \leq (d-1)^n - (1 + v_1 + \ldots + v_{\ell_{n,k}})$$

which contradicts (8).

We have thus shown $\gamma_{n,k} < -1 + \frac{n+\ell_{n,k}}{d}$. This amounts to the inequality $d < (n + \ell_{n,k})(k+2)/(n-1)$, which concludes the proof of our theorem. $\square$

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