

A NOTE ON A SMOOTH PROJECTIVE SURFACE WITH PICARD NUMBER 2

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ABSTRACT. We characterize the integral Zariski decomposition of a smooth projective surface with Picard number 2 to partially solve a problem of B. Harbourne, P. Pokora, and H. Tutaj-Gasinska [Electron. Res. Announc. Math. Sci. 22 (2015), 103–108].

1. INTRODUCTION

In this note we work over the field \mathbb{C} of complex numbers. By a *negative curve* on a surface we will always mean a reduced, irreducible curve with negative self-intersection. By a $(-k)$ -curve, we mean a negative curve C with $C^2 = -k < 0$.

The bounded negativity conjecture is one of the most intriguing problems in the theory of projective surfaces and can be formulated as follows.

Conjecture 1.1. [B.etc.13, Conjecture 1.1] *For each smooth complex projective surface X there exists a number $b(X) \geq 0$ such that $C^2 \geq -b(X)$ for every negative curve $C \subseteq X$.*

Let us say that a smooth projective surface X has

$$b(X) > 0$$

if there is at least one negative curve on X .

In [BPS17], T. Bauer, P. Pokora and D. Schmitz established the following theorem.

Theorem 1.2. [BPS17, Theorem] *For a smooth projective surface X over an algebraically closed field the following two statements are equivalent:*

- (1) X has bounded Zariski denominators.
- (2) X satisfies Conjecture 1.1.

Let us say that a smooth projective surface X has

$$d(X) = 1$$

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if every *pseudo-effective divisor* D (cf. [Laz04, Definition 2.2.25]) on X has an integral Zariski decomposition (cf. Definition 2.2). An interesting criterion for surfaces to have bounded Zariski denominators was given in [BPS17] as follows.

Proposition 1.3. [HPT15, Proposition 1.2] *Let X be a smooth projective surface such that for every curve C one has $C^2 \geq -1$. Then $d(X) = 1$.*

The above proposition introduces a converse question:

Question 1.4. [HPT15, Question] *Let X be a smooth projective surface with $d(X) = 1$. Is every negative curve then a (-1)-curve?*

In [HPT15], the authors disproved Question 1.4 by giving a K3 surface X with $d(X) = 1$, Picard number $\rho(X) = 2$ and two (-2)-rational curves (cf. Claim 2.12). However, for a smooth projective surface X with $|\Delta(X)| = 1$, sometimes the answer for Question 1.4 is affirmative, where $\Delta(X)$ is the determinant of the intersection form on the Néron-Severi lattice of X . They end by giving the following problem.

Problem 1.5. [HPT15, Problem 2.3] *Classify all algebraic surfaces with $d(X) = 1$.*

To solve Problem 1.5 partially, for the case when $\rho(X) = 2$, we give our main theorem as follows.

Theorem 1.6. *Let X be a smooth projective surface with Picard number 2. If $b(X) > 0$ and $d(X) = 1$, then the following statements hold.*

- (1) *X has at most two negative curves.*
- (2) *If X has two negative curves, then X must be one of the following types: K3 surface, surface of general type, or one point blow-up of either an abelian surface or a K3 surface with Picard number 1.*
- (3) *For every negative curve C and every another curve D on X , the intersection number, $(C \cdot D)$ is divisible by the self-intersection number C^2 , i.e., $C^2 | (C \cdot D)$.*
- (4) *If the Kodaira dimension $\kappa(X) = -\infty$, then X is a ruled surface with invariant $e = 1$ or one point blow up of \mathbb{P}^2 .*
- (5) *If $\kappa(X) = 0$ and the canonical divisor K_X is nef, then X is a K3 surface admitting an intersection form on the Néron-Severi lattice of X which is*

$$\begin{pmatrix} a & b \\ b & -2 \end{pmatrix}$$

where $a \in \{0, -2\}$ and $b + a \in 2\mathbf{Z}_{>0}$.

- (6) *If $\kappa(X) = 1$, then X has exactly one negative curve C and every singular fibre is irreducible. In particular, if every fibre is of type mI_0 , then the genus $g(C) \geq 2$. Here, mI_0 is one type in Kodaira's table of singular fibres (cf. [BHPV04, V.7. Table 3]).*

It is well-known that the following SHGH conjecture implies Nagata's conjecture (cf. [Nag59, p.772]), which is motivated by Hilbert's 14-th problem.

Conjecture 1.7. (cf. [C.etc.13, Conjectures 1.1, 2.3]) *Let X be a composite of blow-ups of \mathbb{P}^2 at points p_1, \dots, p_n in very general position. Then, every negative curve on X is a (-1)-rational curve.*

Finally, we note two corresponding results of Conjecture 1.7 as follows.

Proposition 1.8. (cf. [BPS17, Theorems 2.2, 2.3]) *Let X be a composite of blow-ups of \mathbb{P}^2 at n distinct points. Then, $b(X) = 1$ if and only if $d(X) = 1$.*

Here, a smooth projective surface X has $b(X) = 1$ if every negative curve C on X is a (-1)-curve. By Proposition 1.8 and Lemma 2.3, we obtain the following result.

Proposition 1.9. *Let X be a composite of blow-ups of \mathbb{P}^2 at points p_1, \dots, p_n in very general position. If there is a negative curve C and another curve D on X such that the intersection matrix of C and D is not negative definite and $C^2 \nmid (C \cdot D)$, then Conjecture 1.7 fails.*

2. THE PROOF OF THEOREM 1.6

In this section, we divide our proof of Theorem 1.6 into some steps.

Notation 2.1. [Fuj79, 1.6] Let C_1, \dots, C_q be prime divisors. By $V(C_1, \dots, C_q)$ we denote the \mathbf{Q} -vector space of \mathbf{Q} -divisors generated by C_1, \dots, C_q . $I(C_1, \dots, C_q)$ denotes the quadratic form on $V(C_1, \dots, C_q)$ defined by the self-intersection number.

Definition 2.2. (Fujita-Zariski decomposition [Zar62, Fuj79]) Let X be a smooth projective surface and D a pseudo-effective divisor on X . Then D can be written uniquely as a sum

$$D = P + N$$

of \mathbf{Q} -divisors such that

- (1) P is nef;
- (2) $N = \sum_{i=1}^q a_i C_i$ is effective with $I(C_1, \dots, C_q)$ negative definite if $N \neq 0$;
- (3) $P \cdot C_i = 0$ for every component C_i of N .

In particular, X is said to satisfy $d(X) = 1$ if every pseudo-effective divisor D has an integral Zariski decomposition $D = P + N$, i.e., P and N are integral divisors.

Lemma 2.3. *Let X be a smooth projective surface with $b(X) > 0$ and $d(X) = 1$. Suppose $I(C_1, C_2)$ is not negative definite. Then, for every negative curve C_1 and every another curve C_2 , $C_1^2 \mid (C_1 \cdot C_2)$.*

Proof. Let $D(m_1, m_2) := m_1 C_1 + m_2 C_2$ with $m_1, m_2 > 0$. If $D(m_1, m_2) \cdot C_1 < 0$ and $D(m_1, m_2) \cdot C_2 < 0$, then by [Fuj79, Lemma 1.10], $I(C_1, C_2)$ is negative definite. Therefore, $D(m_1, m_2) \cdot C_1 < 0$ implies that $D(m_1, m_2) \cdot C_2 \geq 0$.

If $C_1 \cdot C_2 = 0$, then $C_1^2 | (C_1 \cdot C_2)$, where $C_2^2 \geq 0$. Hence, we have completed the proof.

Now suppose $C_1 \cdot C_2 > 0$. Then, there are infinitely many coprime positive integer number pairs (m_1, m_2) such that

$$D(m_1, m_2) \cdot C_1 < 0, \text{ i.e., } \frac{m_2}{m_1} < \frac{-C_1^2}{(C_1 \cdot C_2)},$$

since there are infinitely many prime integers. Therefore, we have the following Zariski decomposition:

$$D(m_1, m_2) = m_2 \left(\frac{(C_1 \cdot C_2)}{-C_1^2} C_1 + C_2 \right) + \left(m_1 - m_2 \frac{(C_1 \cdot C_2)}{-C_1^2} \right) C_1.$$

Note that $-C_1^2$ has only finitely many prime divisors, there exists a positive integer m_2 such that $(m_2, -C_1^2) = 1$. Since $d(X) = 1$, $D(m_1, m_2)$ has an integral Zariski decomposition. Hence, $C_1^2 | (C_1 \cdot C_2)$. \square

By Lemma 2.3, we can answer the following question in some sense which was posed in [B.etc.13].

Question 2.4. [B.etc.13, Question 4.5] Is there for each $g > 1$ a surface X with infinitely many (-1) -curves of genus g ?

Proposition 2.5. *Let $f : X \rightarrow B$ be a relatively minimal elliptic fibration of a smooth projective surface X with the Kodaira dimension $\kappa(X) = 2$ over a smooth base curve B of genus $g \geq 2$. If $d(X) = 1$ and X has infinitely many sections, then X has infinitely many (-1) -curves of genus $g \geq 2$ and $q(X) = p_g(X)$. Here, $q(X)$ is the irregularity of X , $p_g(X)$ is the geometric genus of X .*

Proof. Since there exists a section C on X , X has no multiple fibres. In this case, by the well-known result of Kodaira (cf. [BHPV04, Corollary V.12.3]), K_X is a sum of a specific choice of $2g(B) - 2 + \chi(\mathcal{O}_X)$ fibres of the elliptic fibration. By [Bea96, Theorem X.4] and the adjunction formula, $-C^2 = \chi(\mathcal{O}_X) > 0$. If $d(X) = 1$, then applying Lemma 2.3 to $C_2 = C$, we obtain $C^2 = -1$ and $q(X) = p_g(X)$. \square

Proposition 2.6. *Every smooth projective surface with Picard number 2 satisfies Conjecture 1.1.*

Indeed, Proposition 2.6 follows from the following claim immediately.

Claim 2.7. If C_1, C_2 are two negative curves on a smooth projective surface X with $\rho(X) = 2$, then

$$\overline{NE}(X) = \mathbf{R}_{\geq 0}[C_1] + \mathbf{R}_{\geq 0}[C_2]$$

and C_i ($i = 1, 2$) are the only two negative curves.

Proof. By [KM98, Lemma 1.22], C_1, C_2 are both extremal curves in the closed Mori cone $\overline{NE}(X)$ which has only two extremal rays since $\rho(X) = 2$. Thus, the first part of Claim 2.7 follows. Moreover, if C_3 is another negative curve (except for C_1, C_2), then the class $[C_3]$ is also extremal. Since $\rho(X) = 2$, $C_3 \equiv a_i C_i$ for $i = 1$ or 2 with $a_i \in \mathbf{Q}_+$. Thus, $0 \leq C_i \cdot C_3 = a_i C_i^2 < 0$, a contradiction. \square

By Lemma 2.3, for the case when $\rho(X) = 2$, we have the following result.

Claim 2.8. Let X be a smooth projective surface with $\rho(X) = 2$. If $b(X) > 0$ and $d(X) = 1$, then for every negative curve C and every another curve D on X , $C^2 | (C \cdot D)$.

It is well-known that the smooth projective surfaces satisfy the minimal model conjecture (cf. [KM98, BCHM10]) as follows.

Lemma 2.9. *Let X be a smooth projective surface. If the canonical divisor K_X is pseudo-effective, then the Kodaira dimension $\kappa(X) \geq 0$.*

Claim 2.10. Let X be a smooth projective surface with $\rho(X) = 2$. If $\kappa(X) = -\infty$, $b(X) > 0$ and $d(X) = 1$, then X is a ruled surface with invariant $e = 1$ or one point blow-up of \mathbb{P}^2 .

Proof. Let S be a relatively minimal model of X . A smooth projective surface S is relatively minimal if it has no (-1) -rational curves. By the classification of relatively minimal surfaces (cf. [Har77, BHPV04, KM98]), it must be one of the following cases: a surface with nef canonical divisor, a ruled surface or \mathbb{P}^2 . Since $\kappa(X) = -\infty$, by Lemma 2.9, K_S is not nef. Therefore, S is either a ruled surface or \mathbb{P}^2 . As a result, $\rho(X) = 2$ implies that X is either a ruled surface or one point blow-up of \mathbb{P}^2 .

Now suppose X is ruled. Let $\pi : X \rightarrow C$ be a ruled surface over a curve C with invariant e , let $C_0 \subseteq X$ be a suitable section, and let f be a fibre. Then, we have the following (cf. [Har77, Propositions V.2.3 and V.2.9]):

$$\text{Pic } X \simeq \mathbf{Z}C_0 \oplus \pi^* \text{Pic } C, C_0 \cdot f = 1, f^2 = 0, C_0^2 = -e.$$

Let $D = aC_0 + bf$ be a curve on X . By [Har77, Proposition V.2.20], $D^2 < 0$ if and only if $D = C_0$ and $e > 0$. Since $d(X) = 1$, applying Claim 2.8 to a fibre f , we obtain $e = 1$. \square

Claim 2.11. Let X be a smooth projective surface with $\rho(X) = 2$. If X has two negative curves, then X must be one of the following types: K3 surface, surface of general type, or one point blow-up of either an abelian surface or a K3 surface with Picard number 1.

Proof. Suppose X has two negative curves C_1, C_2 . By Claim 2.10, if $\kappa(X) = -\infty$, then X has at most one negative curve. Thus, $\kappa(X) \geq 0$, i.e., there exists a positive integral number m such that $h^0(X, \mathcal{O}_X(mK_X)) \geq 0$. Therefore, K_X is a \mathbf{Q} -effective divisor. As a result, by Claim 2.7, we have the following result:

$$K_X \in \overline{NE}(X) = \mathbf{R}_{\geq 0}[C_1] + \mathbf{R}_{\geq 0}[C_2], \text{ i.e., } K_X \equiv a_1 C_1 + a_2 C_2, a_1, a_2 \geq 0.$$

Hence, we have three cases as follows.

- (1) $a_1, a_2 > 0$. Then K_X is an interior point of $\overline{NE}(X)$, and by [Lit82, Lemma 10.5] or [Laz04, Theorem 2.2.26], K_X is big. Thus, X is a surface of general type.
- (2) $a_1 = a_2 = 0$. Then $K_X \equiv 0$, i.e., X is minimal. By Enriques Kodaira classification (cf. [Har77, Theorem V.6.3]), X has the following cases: K3 surface, Enriques surface, abelian surface, hyperelliptic surface, where in the latter two cases, X has no any rational curves. By the genus formula, every negative curve C on X is a (-2)-rational curve. As a result, X is either a K3 surface or an Enriques surface. Moreover, since an Enriques surface X has $\rho(X) = 10$ by [BHPV04, Proposition VIII.15.2], X is a K3 surface.
- (3) $a_1 > 0, a_2 = 0$. Then $K_X \equiv a_1 C_1$. Since K_X is a \mathbb{Q} -effective divisor, there exists an effective divisor D such that $K_X \sim_{\mathbb{Q}} D$. Therefore, we can find an effective divisor $D' \neq C_1$ such that

$$a'_1 C_1 + D' = D \equiv a_1 C_1,$$

where $a'_1 \geq 0$, and D' and C_1 have no common components. Then, $D' \equiv (a_1 - a'_1)C_1$.

If $a_1 = a'_1$, then $K_X \sim_{\mathbb{Q}} a_1 C_1$ with $a_1 > 0$. By the genus formula, C_1 is a (-1)-rational curve. In this case, $\kappa(X) = \kappa(X, C) = 0$. By Castelnuovo's contractibility criterion (cf. [Har77, Theorem V.5.7] or [Bea96, Theorem II.17]), X is a one point blow-up of either an abelian surface or a K3 surface with Picard number 1.

If $a_1 > a'_1$, then $D' \cdot C_1 = (a_1 - a'_1)C_1^2 < 0$, a contradiction.

If $a_1 < a'_1$, on the one hand $D' + (a'_1 - a_1)C_1 \equiv 0$ with $a'_1 - a_1 \geq 0$; on the other hand, there is an ample divisor H on X such that $(D' + (a'_1 - a_1)C_1) \cdot H = 0$. Since the restriction of an ample divisor to a curve is still ample, $D' + (a'_1 - a_1)C_1 = 0$, i.e., $D = a_1 C_1, a_1 = a'_1$, a contradiction.

□

Theorem A of [HPT15] is a special case of the following Claim 2.12.

Claim 2.12. Let X be a smooth projective surface with $\rho(X) = 2$. If $\kappa(X) = 0, b(X) > 0, d(X) = 1$ and K_X is nef, then X is a K3 surface admitting the intersection form on the Néron-Severi lattice of X , which is

$$\begin{pmatrix} a & b \\ b & -2 \end{pmatrix}$$

where $a \in \{0, -2\}$ and $b + a \in 2\mathbb{Z}_{>0}$.

Proof. Since $\kappa(X) = 0$ and K_X is nef, $K_X \equiv 0$. By the genus formula, every negative curve on X is a (-2)-rational curve. Note that abelian surfaces and hyperelliptic surfaces have no rational curves. Then by [Har77, Theorem V.6.3] and [BHPV04, Proposition VIII.15.2], we know that X is a K3 surface. In [Kov94], the author showed that $\overline{NE}(X) = \mathbb{R}_{\geq 0}[C_1] + \mathbb{R}_{\geq 0}[C_2]$, where either $C_1^2 = C_2^2 = -2$ or $C_1^2 = 0$ and $C_2^2 = -2$. Since $d(X) = 1$,

applying Claim 2.8 to a negative curve C_i , $C_i^2 | (C_1 \cdot C_2)$. By the Hodge index theorem, $(C_1 \cdot C_2)^2 - C_1^2 \cdot C_2^2 > 0$. Finally, the desired result holds by using [Kov94, Corollary 1.4]. \square

The following lemma is well known.

Lemma 2.13. [BHPV04, Proposition III.11.4] *Let $p : X \rightarrow B$ be an elliptic fibration from a smooth projective surface X to a curve B . If every fibre is of type mI_0 , then $c_2(X) = 0$.*

Claim 2.14. Let X be a smooth projective surface with $\rho(X) = 2$. If $\kappa(X) = 1$ and $b(X) > 0$, then X has exactly one negative curve C and every singular fibre is irreducible. In particular, if every fibre is of type mI_0 , then $g(C) \geq 2$.

Proof. Since $\kappa(X) = 1$, $\rho(X) = 2$ and $\kappa(X)$ is a birational invariant, K_X is nef. By [Bea96, Proposition IX.2], we have $K_X^2 = 0$ and there is a surjective morphism $p : X \rightarrow B$ over a smooth curve B , whose general fibre F is an elliptic curve. Suppose $F = \sum_{i=1}^r m_i C_i$ with $m_i \in \mathbf{Z}_{>0}$, $r \geq 2$ is a singular fibre. Then by Zariski's Lemma (cf. [BHPV04, Lemma III.8.2]),

$$(F - m_1 C_1)^2 < 0, C_1^2 < 0.$$

Therefore, X has at least two negative curves, a contradiction (cf. Claim 2.10). As a result, every singular fibre is irreducible and X has exactly one negative curve C since $b(X) > 0$. Moreover, if every fibre is of type mI_0 , then by Lemma 2.13, we have $c_2(X) = 0$. Hence, by [B.etc.13, Theorem 2.4], we have the following inequality:

$$0 < -C^2 \leq 2g(C) - 2.$$

Thus, $g(C) \geq 2$. \square

Proof of Theorem 1.6. By Claims 2.7, 2.8, 2.10 to 2.12 and 2.14, we have completed the proof of Theorem 1.6. \square

We end by asking the following two questions.

Question 2.15. Is there a positive constant l such that $b(X) \leq l$ for any smooth projective surface X with $\rho(X) = 2$ and $d(X) = 1$?

Question 2.16. Let X be a smooth projective surface with Picard number $\rho(X) \geq 3$ and $d(X) = 1$. Take some negative curves C_1, \dots, C_k with $k \geq 2$ on X such that $I(C_1, \dots, C_k)$ is negative definite. Is the determinant $\det(C_i \cdot C_j)_{1 \leq i, j \leq k}$ equal to $(-1)^k$?

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