

The Toda system and solution to the $\mathcal{N}=2$ SUSY Yang-Mills theory

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We briefly review the place of the Toda closed chain and the Toda field theory in solution to the $\mathcal{N}=2$ supersymmetric Yang-Mills (SYM) theory. The classical and quantum aspects of the correspondence are mentioned and the role of branes as degrees of freedom is emphasized.

I. INTRODUCTION

The family of Hamiltonian systems invented by Toda 50 years ago [1] serves in theoretical physics as the sample model for integrable finite-dimensional systems and 2d field theories in many occasions. Sometimes it provides the idealized system to be perturbed in some way to fit the problem under consideration, sometimes it describes the physical systems exactly when it has enough symmetry. The Toda system is uniquely attributed to the group-like phase spaces. For the open finite-dimensional Toda chains the finite-dimensional groups are relevant while the affine groups provide the phase space for the closed Toda chain. The Toda field theories also can be considered as the simple system at the Kac-Moody related phase spaces. The classical and quantum aspects of the Toda systems are reviewed in [2].

The classical closed finite-dimensional Toda chain has appeared in the interesting way in Seiberg-Witten solution to the 4D $\mathcal{N}=2$ pure SYM theory [3, 4]. It has been recognized for the $SU(2)$ SYM theory in [5] and the guess about the solution of $SU(N)$ SYM theory was based at correspondence with $SU(N)$ closed Toda chain [6]. Later two more integrable systems which have Toda closed chain as a proper limit were identified in SYM theory with some additional matter. Namely the $\mathcal{N}=2^*$ $SU(N)$ SYM with adjoint matter corresponds to the $SU(N)$ elliptic Calogero model [7] while the $SU(N)$ SYM with fundamental matter [8, 9] to the particular spin chains. The limit to the closed Toda chain corresponds to the decoupling of the heavy matter in the asymptotically free theory when the scale in the pure YM theory emerges from the dimensional transmutation. If the 4D SYM gets lifted to the $(4+1)$ dimensional theory with one compact dimension the closed Toda chains gets lifted to the relativistic closed Toda chain [10]. The review of the early development of these issues can be found in [11].

In spite of the great effectiveness of the integrability approach for derivation of low-energy effective actions for SYM the origin of the Toda-like classical Hamiltonian systems and their generalizations for the theories with matter remained obscure for a while. In particular it was not clear precisely what are degrees of freedom in these systems and why these systems are not quantized. Two major achievements allowed to fill these gaps in understanding. First, the introduction of the Ω -background allowed to derive the Seiberg-Witten solution microscopically from the instanton ensemble [12, 13]. Later it was shown [14–16] that in the particular limit of Ω -background $\epsilon_1 = 0, \epsilon_2 \neq 0$ the integrable system gets quantized and the Planck constant gets identified with ϵ_2 .

Secondly the AGT correspondence between the $\mathcal{N}=2$ SYM theories and Liouville for $SU(2)$ SYM and Toda field theories for $SU(N)$ SYM has been formulated [17, 18]. The Nekrasov instanton partition functions were identified with the particular conformal blocks in Liouville and Toda field theories. One can add into the corresponding conformal block the additional operator degenerated at the second level which induces the second order differential equation of the Schrodinger type for the conformal block. It is solution to this equation for the conformal block with insertion of

degenerate operator coincides with the wave function of the finite-dimensional Hamiltonian system. That is the coordinates of the Toda chain or Calogero model are the positions of the full surface defects in the internal space. The very approach providing the derivation of the conformal block and therefore the Nekrasov partition function via the insertion of the additional degenerate operator is the particular realization of the Zamolodchikov's monodromy method [19]. This method allows to evaluate the conformal block itself from the monodromy of the solutions to the decoupling equation.

We will very briefly review in the simple terms the ideas and steps which have lead to the current understanding of the place for the Toda integrable systems in the rich world of $\mathcal{N}=2$ SUSY YM systems.

II. SUMMARY ON THE SEIBERG-WITTEN SOLUTION

The Seiberg-Witten solution to the $\mathcal{N}=2$ SYM theory [3, 4] provides the answer for the low-energy effective action and the spectrum of the stable BPS particles. To get the low-energy effective action it is necessary to take into account a one-loop contribution and the sum up the infinite instanton series. The latter problem looked hopeless however the combination of the deep physical guesses and some mathematical tools provided the answer without the microscopic evaluation of the instanton sum.

Let us describe several steps which lead to the explicit solution [3, 4]

- Holomorphy.

The Lagrangian involves the potential term for the scalar fields ϕ of the form

$$V(\phi) = \text{Tr}[\phi, \phi^+]^2, \quad (1)$$

where the trace is taken over adjoint representation of the gauge group $SU(N)$. It gives rise to the valleys in the theory, when $[\phi, \phi^+] = 0$. The vacuum energy vanishes along the valleys, hence, the supersymmetry remains unbroken. One may always choose the v.e.v.'s of the scalar field to lie in the Cartan subalgebra $\phi = \text{diag}(a_1, \dots, a_n)$. These parameters a_i can not serve, however, good order parameters, since there is still a residual Weyl symmetry which changes a_i but leave the same vacuum state and one should consider the set of the gauge invariant order parameters $u_k = \langle \text{Tr} \phi^k \rangle$ that fix the vacuum state unambiguously. Hence we obtain a moduli space parametrized by the vacuum expectation values of the scalars, which is known as the Coulomb branch of the whole moduli space of the theory.

The choice of the point on the Coulomb branch is equivalent to the choice of the vacuum state and simultaneously it yields the scale which the coupling is frozen on. At the generic point of the moduli space, the theory becomes effectively Abelian after the condensation of the scalar. As soon as the scalar field acquires the vacuum expectation value, the standard Higgs mechanism works and there emerge heavy gauge bosons at large values of the vacuum condensate.

The initial action of $\mathcal{N}=2$ theory written in $\mathcal{N}=2$ superfield Ψ has the simple structure

$$S(\Psi) = \text{Im} \tau \int \text{Tr} \Psi^2 \quad \tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \quad (2)$$

where g is coupling constant and θ - the standard coefficient in front of topological charge. The $\mathcal{N}=2$ supersymmetry implies that the low energy effective action gets renormalized only

by holomorphic contributions so that it is ultimately given by a single function known as prepotential $S_{eff}(\Psi) = \text{Im} \int \mathcal{F}(\Psi)$. The prepotential is a holomorphic function of moduli u_k except for possible singular points at the values of moduli where additional massless states can appear disturbing the low energy behavior.

Thus, the problem effectively reduces to the determination of single holomorphic function. If one manage to fix its behavior nearby singularities, the function can be unambiguously restored. One of the singularities, corresponding to large values of v.e.v.'s, i.e. to the perturbative limit is under control. All other singularities are treated with the use of duality and of the non-renormalization theorems for the central charges of the SUSY algebra. A combination of these two ideas allows one to predict the spectrum of the stable BPS states which become massless in the deep non-perturbative region.

- Duality.

The duality transformation can be easily defined in the finite $\mathcal{N} = 4$ SUSY theory just as the modular transformations generated by $\tau \rightarrow \tau^{-1}$ and $\tau \rightarrow \tau + 1$. The complexified coupling constant plays the role of modulus of the auxiliary elliptic curve. This makes a strong hint that the duality can be related with a modular space of some Riemann surfaces, where the modular group acts.

- Non-perturbative RG flows and auxiliary Riemann surface

In the asymptotically free theory, one has to match the duality with the renormalization group. This is non-trivial, since now τ depends on the scale which is involved into the duality transformation. The duality acts on the moduli space of vacua and this moduli space is associated with the moduli space of the auxiliary Riemann surface, where the modular transformations act.

At the next step, one has to find out proper variables whose modular properties fit the field theory interpretation. These variables are the integrals of a meromorphic 1-form dS over the cycles on the Riemann surface, a_i and $a_{D,i}$

$$a_i = \oint_{A_i} dS, \quad a_{D,i} = \oint_{B_i} dS, \quad (3)$$

(where $i, j = 1, \dots, N - 1$ for the gauge group $SU(N)$).

These integrals play the two-fold role in the Seiberg-Witten approach. First of all, one may calculate the prepotential \mathcal{F} and, therefore, the low energy effective action through the identification of a_D and $\frac{\partial \mathcal{F}}{\partial a}$ with a defined as a function of moduli by formula (3). Then, using the property of the differential dS that its variations w.r.t. moduli are holomorphic one may also calculate the matrix of coupling constants

$$T_{ij}(u) = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}, \quad (4)$$

The second role of formula (3) is that, as was shown in [3, 4] these integrals define the spectrum of the stable states in the theory which saturate the Bogomolny-Prasad-Sommerfeld (BPS) limit. For instance, the formula for the BPS spectrum in the $SU(2)$ theory reads as

$$M_{n,m}(u) = |na(u) + ma_D(u)|, \quad (5)$$

where the quantum numbers n, m correspond to the “electric” and “magnetic” states.

The column $(a_i, a_{D,i})$ transforms under the action of the modular group as a section of the linear bundle. Its global behavior, in particular, the structure of the singularities is uniquely determined by the monodromy data. As we discussed earlier, the duality transformation connects different singular points. In particular, it interchanges “electric”, a_i and “magnetic”, $a_{D,i}$ variables which describe the perturbative degrees of freedom at the strong and weak coupling regimes respectively. Manifest calculations involving the Riemann surface allow one to analyze the monodromy properties of dual variables when moving in the space of the order parameters. For instance, in the simplest $SU(2)$ case, on the u -plane of the single order parameter there are three singular points, where some cycle shrinks and the magnetic and electric variables mix when encircling these points.

Summarizing, the information about the Seiberg-Witten solution is encoded in the Riemann surface bundled over the moduli space of vacua and the meromorphic differentials on this Riemann surface encoding the spectrum of BPS states and prepotential. It was observed in [5] that these data are standard ingredients of the integrable holomorphic many-body systems.

III. SW SOLUTION AND CLASSICAL HOLOMORPHIC SYSTEMS

The complexified periodic Toda chain is a one-dimensional system of N non-relativistic particles interacting with the following potential

$$V(x_1, \dots, x_N) = \Lambda^2 \sum_{i=1}^N e^{x_i - x_{i+1}}, \quad x_{N+1} = x_1$$

where the coordinates $x_i \in \mathbf{C}/(2\pi\mathbf{Z})$ while the momenta $p_i \in \mathbf{C}$. The Riemann surface and meromorphic differentials involved into the solution for pure $SU(N)$ SYM theory are intrinsic objects for the N -body periodic Toda chain [5, 6, 23] whose equations of motion read

$$\frac{\partial q_i}{\partial t} = p_i \quad \frac{\partial p_i}{\partial t} = e^{q_{i+1} - q_i} - e^{q_i - q_{i-1}} \quad (6)$$

where the periodic boundary conditions $p_{i+N} = p_i$, $q_{i+N} = q_i$ imposed. In the Toda system there are exactly N conservation laws constructed from the Lax operator. The Lax operator is represented by the $N \times N$ matrix depending on dynamical variables

$$\mathcal{L}^{TC}(w) = \begin{pmatrix} p_1 & e^{\frac{1}{2}(q_2 - q_1)} & 0 & w e^{\frac{1}{2}(q_1 - q_N)} \\ e^{\frac{1}{2}(q_2 - q_1)} & p_2 & e^{\frac{1}{2}(q_3 - q_2)} & \dots & 0 \\ 0 & e^{\frac{1}{2}(q_3 - q_2)} & -p_3 & & 0 \\ & & \dots & & \\ \frac{1}{w} e^{\frac{1}{2}(q_1 - q_N)} & 0 & 0 & & p_N \end{pmatrix} \quad (7)$$

The characteristic equation for the Lax matrix

$$\mathcal{P}(\lambda, w) = \det_{N \times N} (\mathcal{L}^{TC}(w) - \lambda) = 0 \quad (8)$$

generates the conservation laws and determines the spectral curve. If one restores the dependence on Λ_{QCD} in this spectral curve, it takes the form

$$w + \frac{\Lambda_{QCD}^{2N}}{w} = 2P_N(\lambda) \quad (9)$$

where $P_N(\lambda) = \det(\phi - \lambda)$, $\phi = \text{diag}(a_1, \dots, a_N)$. The spectral curve exactly coincides with the Riemann surface introduced in the context of $SU(N)$ gauge theory. It can be also put into the hyperelliptic form

$$2Y \equiv w - \frac{1}{w}, \quad Y^2 = P_N^2(\lambda) - 1 \quad (10)$$

The integrals of motion parameterize the moduli space of the complex structures of the hyperelliptic surfaces of genus $N - 1$, which is the moduli space of vacua in physical theory.

After having constructed the Riemann surface and the moduli space describing the $4d$ pure gauge theory, we turn to the third crucial ingredient of solution that comes from integrable systems, the generating differential dS . This differential is in essence the ‘‘shorten’’ action pdq . Indeed, in order to construct action variables, a_i one needs to integrate the differential $\tilde{dS} = \sum_i p_i dq_i$ over $N - 1$ non-contractable cycles in the Liouville torus which is nothing but the level submanifold of the phase space, i.e. the submanifold defined by values of all $N - 1$ integrals of motion fixed. On this submanifold, the momenta p_i are functions of the coordinates, q_i . The Liouville torus is just the Jacobian corresponding to the spectral curve.

In the general case of a g -parameter family of complex curves of genus g , the Seiberg-Witten differential dS is characterized by the property $\delta dS = \sum_{i=1}^g \delta u_i dv_i$, where $dv_i(z)$ are g holomorphic 1-differentials on the curves, while δu_i are variations of g moduli along the base. In the associated integrable system, u_i are integrals of motion and π_i , some g points on the curve are momenta. The symplectic structure is

$$\sum_{i=1}^g da_i \wedge dp_i^{Jac} = \sum_{i,k=1}^g du_i \wedge dv_i(\pi_k) \quad (11)$$

The vector of the angle variables,

$$p_i^{Jac} = \sum_{k=1}^g \int^{\pi_k} d\omega_i \quad (12)$$

is a point of the Jacobian, and the Jacobi map identifies this with the g -th power of the curve, $Jac \cong \mathcal{C}^{\otimes g}$. Here $d\omega_i$ are *canonical* holomorphic differentials, $dv_i = \sum_{j=1}^g d\omega_j \oint_{A_j} dv_i$.

In $SU(2)$ case, the spectral curve is

$$w + \frac{1}{w} = 2(\lambda^2 - u) \quad (13)$$

while $u = p^2 - \cosh q$. Therefore, one can write for the action variable

$$a = \oint pdq = \oint \sqrt{u - \cosh q} dq = \oint \lambda \frac{dw}{w}, \quad dS = \lambda \frac{dw}{w} \quad (14)$$

where we made the change of variable $w = e^q$ and used equation (13).

One can easily check that the derivatives of dS with respect to moduli are holomorphic, up to total derivatives. Say, if one parameterizes $P_N(\lambda) = -\lambda^N + s_{N-2}\lambda^{N_c-2} + \dots$ and note that $dS = \lambda dw/w = \lambda dP_N(\lambda)/Y = P_N(\lambda)d\lambda/Y + \text{total derivatives}$, then

$$\frac{\partial dS}{\partial s_k} = \frac{\lambda^k d\lambda}{Y} \quad (15)$$

and these differentials are holomorphic if $k \leq N - 2$. Thus, $N - 1$ moduli gives rise to $N - 1$ holomorphic differentials which perfectly fits the genus of the curve (we use here that there is no modulus s_{N-1}).

If more complicated SYM theory with different matter content is considered the periodic Toda chain gets substituted by more general Hamiltonian system but the scheme is generic. The spectral curve in each case gets identified with the SW curve while the "pdq" differential gets identified with the SW differential. At the classical level this relation looks like a curious coincidence however later the deep reason behind the correspondence between the integrable systems and SYM theories has been recognized.

IV. NEKRASOV PARTITION FUNCTION AND QUANTIZATION OF INTEGRABLE SYSTEM

A. Instanton partition function

First, let us recall the definition of the Nekrasov partition function [12, 13] of a four dimensional gauge theory in the Ω -background $\mathbf{C}_{\epsilon, \hbar}^2$ where ϵ, \hbar are two equivariant parameters for the torus action on the instanton moduli space. In this Section we adopt $\epsilon_1 = \epsilon, \epsilon_2 = \hbar$ to fit with QM notation. The Ω -background was initially introduced to regularize the integrals over the instanton moduli space however it turned out that it provides a lot of additional information about the instanton ensembles and serves as crucial ingredient for the explanation of the origin of the integrable systems.

Let \mathbf{a} denote a set of complex scalars which parametrize the moduli space of vacua, \mathbf{m} is set of masses for fundamental matter multiplets and $\Lambda^{2N} = \exp(2\pi i\tau)$ is a generated mass scale that counts instantons. The full Nekrasov partition function consists of perturbative and non-perturbative contributions

$$Z(\mathbf{a}, \mathbf{m}, \Lambda; \epsilon, \hbar) = Z^{pert.}(\mathbf{a}, \mathbf{m}, \Lambda; \epsilon, \hbar) \times Z^{inst.}(\mathbf{a}, \mathbf{m}, \Lambda; \epsilon, \hbar) \quad (16)$$

The non-perturbative part of the partition function is obtained by the equivariant localization on the instanton moduli space and is defined as follows. Let

$$\mathcal{V}_\lambda = \sum_{i=1}^N \sum_{(r,s) \in \lambda_i} e^{a_i + (r-1)\epsilon + (s-1)\hbar}, \quad \mathcal{W} = \sum_{i=1}^N e^{a_i}, \quad \mathcal{M} = \sum_{i=1}^{N_f} e^{m_i} \quad (17)$$

$$\mathcal{T}_\lambda = -\mathcal{M}\mathcal{V}_\lambda^* + \mathcal{W}\mathcal{V}_\lambda^* + e^{\epsilon+\hbar}\mathcal{V}_\lambda\mathcal{W}^* - (1-e^\epsilon)(1-e^\hbar)\mathcal{V}_\lambda\mathcal{V}_\lambda^* \quad (18)$$

which appear as the characters of the natural bundles on the instanton moduli space at a fixed point, parametrized by a set $\{\lambda_i\}$ of N Young diagrams. The collection of fixed points is defined as follows. Consider the contribution from the instanton charge K and introduce partition $K = \sum_1^N k_i$

where $k_i \geq 0$. Introduce λ_l - the partition of k_l and consider the distribution of K boxes among the collection of N Young tableau $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$. We denote $|\lambda_l| = k_l$ the number of boxes in l 'th diagram hence $\sum_l |\lambda_l| = K$. The fixed points from the l 'th Young tableau which contribute the partition function are located at

$$\phi_l = a_l + \epsilon(r-1) + \hbar(s-1) \quad (19)$$

where the pair (r, s) corresponds to the box in the l 'th tableau.

The star-operation inverts the weights of all character:

$$\left(\sum_a e^{w_a} \right)^* = \sum_a e^{-w_a} \quad (20)$$

The instanton partition function can be written as

$$Z^{inst.}(\mathbf{a}, \mathbf{m}, \Lambda; \epsilon, \hbar) = \sum_{\{\lambda\}} \Lambda^{2N|\lambda|} e(\mathcal{T}_\lambda) \quad (21)$$

where

$$e\left(\sum_a e^{w_a} - \sum_b e^{w_b}\right) = \frac{\prod_b w_b}{\prod_a w_a} \quad (22)$$

is a symbol that converts the sum of characters into the product of weights.

The perturbative part is more subtle due to ambiguity of the boundary conditions at infinity [40]. It can be written as

$$Z^{pert.}(\mathbf{a}, \mathbf{m}, \Lambda; \epsilon, \hbar) = \Lambda^{-\frac{N}{\epsilon\hbar} \sum_{i=1}^N a_i^2} e\left(\frac{e^{\epsilon+\hbar}(\mathcal{M}\mathcal{W}^* - \mathcal{W}\mathcal{W}^*)}{(1-e^\epsilon)(1-e^\hbar)}\right) \quad (23)$$

but since the character now has infinitely many terms a proper regularization is needed. Physically the first multiplier comes from the tree level contribution, while the first and the second terms in the character come from the one-loop contribution of the matter multiplets and W -bosons, respectively.

B. Nekrasov-Shatashvili limit

The low-energy description of undeformed four dimensional gauge theory is characterized by the prepotential, which can be obtained as the limit of the deformed partition function

$$\mathcal{F}(\mathbf{a}, \mathbf{m}, \Lambda) = \lim_{\epsilon, \hbar \rightarrow 0} \epsilon \hbar \log Z(\mathbf{a}, \mathbf{m}, \Lambda; \epsilon, \hbar) \quad (24)$$

In [16] a refinement of the correspondence with integrable systems was proposed. It was argued that the effective two dimensional theory, which appears in the limit $\epsilon \rightarrow 0$, is related to a corresponding quantized algebraic integrable system, and \hbar plays the role of the quantization parameter.

More precisely we can consider a four dimensional gauge theory on $\mathbf{C} \times \mathbf{D}_\hbar$ where \mathbf{D}_\hbar is the cigar-like geometry of [31] with the Ω -deformation turned on. With an appropriate twist this geometry breaks half of the supersymmetries. Upon choosing the boundary conditions on $\mathbf{C} \times \partial\mathbf{D}_\hbar = \mathbf{C} \times S^1$

which preserve the remaining supersymmetries, we can reduce our theory to a two-dimensional $\mathcal{N} = (2, 2)$ theory on \mathbf{C} , the low energy description of which is characterized by the effective twisted superpotential

$$\mathcal{W}(\mathbf{a}, \mathbf{m}, \Lambda; \hbar) = \lim_{\epsilon \rightarrow 0} \epsilon \log Z(\mathbf{a}, \mathbf{m}, \Lambda; \hbar) \quad (25)$$

For the perturbative contribution we have

$$\mathcal{W}^{pert.}(\mathbf{a}, \mathbf{m}, \Lambda; \hbar) = \lim_{\epsilon \rightarrow 0} \epsilon \log Z^{pert.}(\mathbf{a}, \mathbf{m}, \Lambda; \epsilon, \hbar) = \quad (26)$$

$$-\frac{1}{2\hbar} \log \left(\frac{\Lambda^{2N}}{\hbar^{2N}} \right) \sum_{i=1}^N a_i^2 + \sum_{i,j=1}^N \varpi_{\hbar}(a_i - a_j) - \sum_{i=1}^N \sum_{a=1}^{N_f} \varpi_{\hbar}(a_i - m_a)$$

where $\varpi_{\hbar}(x)$ obeys

$$\frac{d}{dx} \varpi_{\hbar}(x) = \log \Gamma \left(1 + \frac{x}{\hbar} \right) = const. - \sum_{n=1}^{\infty} \log \left(\frac{x + n\hbar}{\hbar} \right) \quad (27)$$

The one-loop contribution $\varpi_{\hbar}(m)$ has a simple intuitive explanation as a contribution of infinite number of angular momentum modes with mass parameters $(m+n\hbar)$ for n -th mode into the effective twisted superpotential, which are chiral multiplets in the effective two dimensional theory. Indeed, after integrating out a single chiral multiplet we get

$$\Delta \mathcal{W}_n = -(m + n\hbar) \left[\log \left(\frac{m + n\hbar}{\hbar} \right) - 1 \right] \quad (28)$$

and after summing over n we get $\varpi_{\hbar}(m)$.

There are two natural boundary conditions [31] which require

$$Type A : \quad \frac{a_i^D}{\hbar} = \frac{\partial \mathcal{W}(\mathbf{a}, \mathbf{m}, \Lambda; \hbar)}{\partial a_i} \in \mathbf{Z} \quad (29)$$

$$Type B : \quad \frac{a_i}{\hbar} \in \mathbf{Z} + \frac{\theta_i}{2\pi}, \quad \theta \in [0, 2\pi) \quad (30)$$

Type *A* corresponds to Neumann-type boundary condition for gauge fields leading to a dynamical vector multiplet in two dimensions. On the contrary type *B* corresponds to Dirichlet boundary conditions, fixing the holonomy along the boundary of the cigar parametrized by θ_i and freezing gauge degrees of freedom. The choice of this boundary conditions specifies the quantization of an algebraic integrable system.

The Bethe/gauge correspondence states that the vacua of the effective two dimensional theory are in one-to-one correspondence with the eigenstates of the Hamiltonians of the quantum integrable system. Moreover, the expectation values of the topologically protected chiral observables, which are traces of the adjoint scalars in a vector multiplet $Tr \phi^k$ of a four dimensional theory, coincide with the eigenvalues of the Hamiltonians \mathcal{H}_k on this states:

$$\langle vac | \mathcal{H}_k | vac \rangle \longleftrightarrow \langle Tr \phi^k \rangle_{vac}. \quad (31)$$

We are interested in the quantization of this Hamiltonian system. The Hamiltonians and the momenta are now promoted to differential operators, acting on wave functions $\psi(x_1, \dots, x_N)$ with $p_i = \hbar \partial_i$. There are two natural quantizations, corresponding to type *A* and type *B* boundary conditions [16]. Type *A* quantization corresponds to $x_i \in \mathbf{R}$ and $\psi(x_1 - \bar{x}, \dots, x_N - \bar{x}) \in L^2(\mathbf{R}^{N-1})$ where $\bar{x} = \sum_i^N x_i/N$ is the center of mass mode, which decouples in a trivial way. This condition leads to discrete unambiguous spectrum that corresponds to the set of vacua in the gauge theory, provided that the type *A* boundary condition $a_D/\hbar \in \mathbf{Z}$ is satisfied.

Type *B* quantization corresponds to $x_i \in i\mathbf{R}/2\pi\mathbf{Z}$ and quasi-periodic non-singular wave functions

$$\psi(x_1, \dots, x_a + 2\pi i, \dots, x_N) = e^{i\theta_a} \psi(x_1, \dots, x_N) \quad (32)$$

The quasi-periodicity parameters $\theta_a \in [0, 2\pi)$ are also known as Bloch-phases. In the special case of $N = 2$, after the decoupling of the center of mass mode, the equation on the wave function coincided with Mathieu equation

$$-\hbar^2 \psi''(x) + 8\Lambda^2 \cos(2x)\psi(x) = 8u\psi(x) \quad (33)$$

where $u = \frac{1}{4}\langle Tr\phi^2 \rangle$. For real Λ and \hbar it describes a particle moving in a periodic cosine potential. At fixed θ -parameters the spectrum is discrete. However as we vary them the spectrum consists of peculiar structure of bands and gaps. In particular for small Λ and when we sit near the edge of some band, the spectrum has exponentially small in $\sim 1/\hbar$ splitting of the eigenvalues. More precisely if $\theta_1 = \dots = \theta_{k_1}; \theta_{k_1+1} = \dots = \theta_{k_2}; \dots; \theta_{k_m+1} = \dots = \theta_{k_N}$ and are all integers, then instead of naive $\frac{N!}{k_1!k_2!\dots k_N!}$ -fold degeneracy as for $\Lambda = 0$ we have non-degenerate spectrum due to quantum reflection on the potential.

This second type of quantization appears to be more mysterious from the gauge theory point of view. When all θ -parameters are zero a_a/\hbar are forced to be equal to integer, and when $a_{ab}/\hbar \in \mathbf{Z} \setminus \{0\}$ some perturbative *W*-boson modes naively become massless that is clear from the logarithmic singularities in their perturbative contribution into the effective twisted superpotential. Recently new interesting aspects of Bethe/gauge correspondence were elaborated in [32].

Let us emphasize that the Bethe/gauge correspondence yields the connection between the Nekrasov partition function in the NS limit and the ingredients of the quantum integrable systems. However it does not tell what are degrees of freedom in this system. To answer this question we have to add the second crucial ingredient - AGT correspondence [17, 18]

V. AGT CORRESPONDENCE

Let us briefly remind that according the AGT correspondence the Nekrasov partition function for *SU*(2) is identified with the particular conformal block in the Liouville theory whose type depends on the matter content on the gauge theory side. The central charge c in the Liouville theory is expressed in terms of the parameters of the Ω -deformation as follows

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}, \quad b^2 = \frac{\epsilon_2}{\epsilon_1} \quad (34)$$

hence the NS limit $\epsilon_2 \rightarrow 0$ on the gauge theory side corresponds to the classical $c \rightarrow \infty$ limit in the Liouville theory. The dimensions of the degenerate operators in the classical limit behave as

$$h_{s,1} = -\frac{s-1}{2} + O(1/c) \quad h_{1,r} = -\frac{r^2-1}{24}c + O(c^0) \quad (35)$$

The operators are naturally classified at large c limit according to behavior of their conformal dimensions Δ_i . The operators whose dimensions are proportional to c are called heavy while ones whose dimensions are $O(1)$ are called light operators. It is natural to introduce the classical dimensions δ_i for the heavy operators defined as

$$\Delta_i = b^{-2}\delta_i \quad (36)$$

According to AGT correspondence the 4-point spherical Liouville conformal block is identified with the total Nekrasov partition function for $SU(2)$ $N_f = 4$ theory

$$\mathcal{F}_a(\epsilon_1, \epsilon_2, m_i, q) = Z_{tot}(a, \epsilon_1, \epsilon_2, m_i, \tau) \quad (37)$$

$$Z_{tot} = Z_{cl}Z_{1-loop}Z_{inst} \quad (38)$$

where the factors correspond to the classical, one-loop and instanton contribution to the partition function. The coordinate at the Coulomb branch a corresponds to the intermediate conformal dimension in the conformal block, masses m_i yield the corresponding conformal dimensions of insertions δ_i and the complexified coupling constant τ in SYM theory gets mapped into the conformal cross-ratio in the 4-point spherical conformal block [17]. The 4-point correlator in the Liouville theory is expressed in terms the Nekrasov partition function as follows [17]

$$\langle V(0)V(\infty)V(1)V(q) \rangle \propto \int da a^2 |Z_{tot}(a)|^2 \quad (39)$$

In the classical NS limit $\epsilon_2 \rightarrow 0$ the twisted superpotential gets identified with the classical conformal block .

$$Z_{inst}(a, \epsilon_1, \epsilon_2, \tau) \rightarrow \exp(b^{-2}W(a, m_i, \hbar, \tau)) \quad (40)$$

$$\mathcal{F}_a(\epsilon_1, \epsilon_2, \delta_i, q) \rightarrow \exp(b^{-2}f_{\delta_{in}}(\delta_i, \hbar, q)), \quad \delta_{in} = b^{-2}\left(\frac{1}{4} - \frac{a^2}{\hbar^2}\right) \quad (41)$$

Since we have exact coincidence of the twisted superpotential and the classical conformal block the naive poles in the twisted superpotential correspond to the naive poles in the intermediate dimension plane in the classical Liouville conformal block.

Similarly the torus one-point classical conformal block corresponds to twisted superpotential in $N = 2^*$ theory which depends on the external and intermediate dimensions Δ, Δ_{in} . To have the unified picture it is useful to represent the one-point torus conformal block $\mathcal{F}_{c, \Delta_{in}}(\lambda, q)$ as the spherical conformal block with four insertions $\mathcal{F}_{c, \Delta_{in}}^{sp}[\lambda_1, \lambda_2, \lambda_3, \lambda_4](q)$. The explicit mapping of parameters under the map goes in arbitrary Ω -background as follows

$$\mathcal{F}_{c, \Delta_{in}}(\lambda, q) = \mathcal{F}_{c, \Delta_{in}}^{sp}\left[\frac{\lambda}{2} - \frac{1}{2b}, \frac{\lambda}{2} + \frac{1}{2b}, \frac{b}{2}, \frac{b}{2}\right](q) \quad (42)$$

where conformal dimensions Δ_i are equal to

$$\Delta_i = \frac{1}{4}(Q^2 - \lambda_i^2) \quad (43)$$

The modulus of the torus q gets mapped into the position of the insertion x on the sphere as

$$q(x) = \exp\left(-\frac{\pi K(1-x)}{K(x)}\right) \quad K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}} \quad (44)$$

At the classical $b \rightarrow 0$ limit the corresponding classical 4-point spherical conformal block representation

$$f_{\delta_{in}}^{sp}[\delta_1, \delta_2, \delta_3, \delta_4](q) = f_{\frac{1}{4} - \frac{a^2}{\hbar}}^{sp} \left[\frac{1}{4} \left(1 - \frac{m^2}{\hbar} \right), \frac{1}{4} \left(1 - \frac{(m + \hbar)^2}{\hbar} \right), \frac{1}{4}, \frac{1}{4} \right](q) \quad (45)$$

where m is the adjoint mass in $N = 2^*$ theory. All of insertions correspond to the heavy operators and the intermediate classical dimension is heavy as well.

If we insert the additional light $\Psi_{2,1}(z)$ operator in 1-point torus conformal block and consider the 2-point classical conformal block the Lamé equation can be identified [41]. The decoupling equation for the 2-point block can be brought into the conventional QM form with Lamé potential

$$\left(\frac{d^2}{dz^2} + k\wp(z|\tau) + E \right) \Psi(z, \tau, E) = 0 \quad (46)$$

where τ is the modulus of the elliptic curve and $k = \frac{m}{\hbar} \left(\frac{m}{\hbar} + 1 \right)$. The energy in the \wp potential is related with the classical conformal block via the properly normalized quantum Matone relation [30]

$$\frac{E}{4\pi^2} = - \left(\frac{a}{\hbar} \right)^2 - \frac{k}{12} (1 - 2E_2(\tau)) + \hbar^{-1} q \frac{d}{dq} W^{N=2^*}(q, a, m, \hbar) \quad (47)$$

where E_2 is the Eisenstein series and a is the Bloch phase.

To get the Mathieu equation one has to take the Inozemtsev limit in the decoupling equation for the degenerate irregular conformal block. The wave function of two-particle Toda system is represented as the matrix element of the degenerate chiral vertex operator between two Gaiotto states [46].

$$\langle \Delta_1, \Lambda^2 | V_+(z) | \Delta_2, \Lambda^2 \rangle \rightarrow z^{\Delta_1 - \Delta_2 - \Delta_2} \phi\left(\frac{\Lambda}{\hbar}, z\right) \exp\left(\frac{1}{b^2} f_\delta\left(\frac{\Lambda}{\hbar}\right)\right) \quad (48)$$

The function

$$\psi\left(\frac{\Lambda}{\hbar}, z\right) = z^r \phi\left(\frac{\Lambda}{\hbar}, z\right), \quad \delta = 1/4 - r^2 \quad (49)$$

obeys the Mathieu equation with the energy

$$E = 4r^2 - \Lambda \frac{\partial}{\partial \Lambda} f_\delta\left(\frac{\Lambda}{\hbar}\right) \quad (50)$$

Given the solutions to the decoupling equations we can evaluate the monodromy matrix. The Zamolodchikov's method [19] allows to extract the conformal block itself from the monodromy data.

Concluding this Section note that the QM enjoys several exponentially small non-perturbative phenomena like the energy splitting or the small gaps well above the barrier. The natural question concerns the interpretation of these phenomena at the SYM side. The whole picture has not been developed yet however a few results have to be mentioned. It was demonstrated in [38] that to reproduce the gaps above the barrier one has to perform the peculiar resummation of the non-perturbative contributions first applied in this context in [29]. Another approach based of the summing over the complex saddles has been suggested in [39].

VI. RG FLOWS AND WHITHAM DYNAMICS

The prepotential which sums up the perturbative and instanton contributions depends on the vacuum moduli \vec{a} and the parameters of the the Lagrangian. For the finite SYM theory the useful parameter is the bare coupling constant τ while in the asymptotically free theories the corresponding parameter is the $\log\Lambda$. The dependence of the prepotential on these parameters is covered by the non-perturbative renormalization group (RG) dynamics which fits with the integrable systems in the interesting way.

It turned out [5, 22–25] that on the top of the holomorphic integrable system like closed Toda chain and elliptic Calogero system there is the second Hamiltonian system which is example of the Whitham Hamiltonian system. Such systems are related to the perturbation of the finite-gap solutions parametrized by the Riemann surfaces. The phase space of the Whitham system relevant fro SW solution is finite-dimensional and is equipped with by the Poisson bracket

$$\{a_i, a_{D,j}\} = \delta_{ij} \quad (51)$$

The time variable for the Toda chain is identified with the $t = \log\Lambda$ and the Hamiltonian gets identified with $H = \langle vac|Tr\phi^2|vac \rangle$. The Whitham equations read as

$$a_{D,i} = \frac{\partial\mathcal{F}}{\partial a_i} \quad E = -\frac{\partial\mathcal{F}}{\log\Lambda} \quad (52)$$

therefore the prepotential plays the role as the action $S(x)$ for the Hamiltonian dynamics. Let us emphasize that the Hamilton-Jacobi equation in the Whitham system plays the role of the anomaly equation at the gauge theory side. Note that the Hamiltonian equation in the Whitham dynamics

$$\frac{da_{D,i}}{d\log\Lambda} = \frac{dH}{da_i} \quad (53)$$

plays an interesting meaning in the first Hamiltonian system like Toda chain [28]. It gets identified with the so-called P/NP relation [26, 27] which connects the perturbative and non-perturbative contributions in the corresponding quantum mechanical system.

In the Ω -deformed case in the Nekrasov-Shatashvili limit the Whitham system remains to be classical and only Hamiltonians get deformed [30]. If both deformation parameters are included the Whitham dynamics gets quantized [40] and the product $\epsilon_1\epsilon_2$ plays the role of the Planck constant for the Whitham dynamics. Since the prepotential according to AGT correspondence gets identified with the conformal block in Liouville and Toda theory the Whitham equation coincides with the KZ equation for the dependence of conformal block on the moduli space of the complex structures of the punctured Riemann surfaces.

It is possible to derive the Whitham hierarchy for the SYM theory perturbed by the terms $T_k Tr\Phi^k$ when T_k plays the role of higher times in the Whitham hierarchy and the prepotential plays the role of the semi-classical tau-function [5, 22, 24, 25]. The prepotential in the theory without the Ω -deformation obeys the WDVV equations [47].

VII. BRANES AS DEGREES OF FREEDOM IN INTEGRABLE SYSTEMS

Let us briefly describe the geometrical aspects of the correspondence. The key players in the integrable dynamics behind SYM are D-branes which carry the abelian gauge field on their

worldvolume [42]. The gauge theories with the prescribed gauge group matter content can be manufactured from the set of branes of different dimensions [43] in particular the SW solution can be derived via M5 brane wrapped around the particular Riemann surface [44, 45]. The set of T and S dualities allows to transform one brane picture into another. Generically the coordinates of the branes in the directions transverse to their world-volume in 10d play the role of the vacuum moduli or coupling constants. The back reaction of one brane into another due to the non-vanishing tensions amounts to their nontrivial profiles which corresponds to the particular solution to the RG equations.

It is not surprise that integrable systems of Toda and Calogero type are intimately related with the brane configurations. The clear cut reason behind this relation is as follows. It was found in [20] that the elliptic Calogero model is the particular example of the Hitchin system [21] on the cotangent bundle to the moduli space of the holomorphic G -valued bundles over the Riemann surface Σ with n marked points $T^*M_{(G,\Sigma,n)}$. The elliptic Calogero system corresponds to the genus one surface with one marked point. The Hitchin system as the system on the D-branes has been described in [7] and the separation of variables in the Hitchin system which allows to represent the phase space in terms of D0 branes as Hilbert space of points on $T^*\Sigma$ has been found in [37].

In spite of this well justified relation between Calogero and Toda system and theories on the D-brane worldvolumes it took some time to understand what type of branes should be considered in SYM to explain the hidden integrability. It was argued in [9, 36] that integrability deals with the artificially added defect probe branes however the type of these branes has not been clarified. The answer to this question has been found when the AGT correspondence has been formulated. It turned out that in the Liouville or Toda theory one has to add the particular surface operator introduced in [48].

In the Liouville and Toda theory this defect corresponds to the operator which is degenerate at the second level $\Psi_{2,1}$ [35] and therefore obeys the second order differential equation which we have discussed in the previous Section. It is this equation which plays the role of Schrodinger equation for the integrable models. For instance the decoupling equation for the two-point function on the torus in the Liouville theory obeys the Schrodinger equation for the elliptic Calogero model [41]. One can recognize the interplay between the surface operators and the integrable systems at the classical level as well [34].

Hence the steps are as follows: first one adds the additional brane - degenerate operator, investigate the monodromy properties of the solution to the decoupling equation and extract from monodromy the expression for the conformal block without the insertion according to the recipe [19]. There is some analogue of this procedure in the superconductivity. One can consider the one-particle excitation at the top of the condensate. This excitation does not ruin the condensate ground state and investigating its properties one can extract the important information concerning its ground state.

Summarizing we could claim that the degrees of freedom in the finite-dimensional system are coordinates of particular branes in the internal space. A dynamics itself reflects the consistency of the whole brane configuration with the non-perturbative RG flows while quantization is provided by the AGT correspondence for the Nekrasov partition function.

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