# SOME MULTIDIMENSIONAL INTEGRALS IN NUMBER THEORY AND CONNECTIONS WITH THE PAINLEVÉ V EQUATION

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ABSTRACT. We study piecewise polynomial functions $\gamma_k(c)$ that appear in the asymptotics of averages of the divisor sum in short intervals. Specifically, we express these polynomials as the inverse Fourier transform of a Hankel determinant that satisfies a Painlevé V equation. We prove that $\gamma_k(c)$ is very smooth at its transition points, and also determine the asymptotics of $\gamma_k(c)$ in a large neighbourhood of $k = c/2$. Finally, we consider the coefficients that appear in the asymptotics of elliptic Aliquot cycles.

## 1. Introduction

**Asymptotics of the mean square of sums of the $k$-th divisor function over short intervals.** Let $d_k(n)$ be the $k$-th divisor numbers, i.e. the Dirichlet coefficients of the $k$-th power of the Riemann zeta function:

$$\zeta(s)^k = \sum_1^\infty \frac{d_k(n)}{n^s}, \qquad \Re s > 1. \tag{1.1}$$

The Dirichlet coefficient $d_k(n)$ is equal to the number of ways of writing $n$ as a product of $k$ factors. Define

$$S_k(X) = \sum_{n \le X} d_k(n). \tag{1.2}$$

Let $XP_{k-1}(\log X)$ be the residue, at $s = 1$ of $\zeta(s)^k X^s/s$, with $P_{k-1}(\log X)$ being a polynomial in $\log X$ of degree $k-1$. Then

$$S_k(X) = XP_{k-1}(\log X) + \Delta_k(X), \tag{1.3}$$

with $\Delta_k(X)$ denoting the remainder term.

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The $k$ divisor problem states that the true order of magnitude for $\Delta_k$ is:

$$\Delta_k(X) = O\left(X^{(k-1)/2k+\epsilon}\right). \tag{1.4}$$

When $k = 2$, the traditional Dirichlet divisor problem is

$$D_2(X) = X \log X + (2\gamma - 1)X + \Delta_2(X), \tag{1.5}$$

with a conjectured remainder

$$\Delta_2(X) = O\left(X^{1/4+\epsilon}\right). \tag{1.6}$$

The estimate for the remainder term $\Delta_k(X)$ is based on expected cancellation in Voronoi-type formulas for $\Delta_k(X)$ and also on estimates, due to Cramér [3] ($k = 2$) and Tong [10] ($k > 2$), for the mean square of $\Delta_k$.

Let

$$\Delta_k(x; H) = \Delta_k(x + H) - \Delta_k(x) \tag{1.7}$$

be the remainder term for sums of $d_k$ over the interval $[x, x + H]$.

Define

$$a_k = \prod_p \left\{(1 - \frac{1}{p})^{k^2} \sum_{j=0}^{\infty} \left(\frac{\Gamma(k+j)}{\Gamma(k)j!}\right)^2 \frac{1}{p^j}\right\}. \tag{1.8}$$

Keating, Rodgers, Roditty-Gershon, and Rudnick conjectured [7]:

**Conjecture 1.1.** *If $0 < \alpha < 1 - \frac{1}{k}$ is fixed, then for $H = X^\alpha$,*

$$\frac{1}{X} \int_X^{2X} \Big(\Delta_k(x, H)\Big)^2 dx \sim a_k \mathcal{P}_k(\alpha) H (\log X)^{k^2-1}, \quad X \to \infty \tag{1.9}$$

*where $P_k(\alpha)$ is given by*

$$\mathcal{P}_k(\alpha) = (1 - \alpha)^{k^2-1} \gamma_k(\frac{1}{1-\alpha}). \tag{1.10}$$

*Here*

$$\gamma_k(c) = \frac{1}{k!\, G(1+k)^2} \int_{[0,1]^k} \delta(t_1 + \ldots + t_k - c) \prod_{i<j} (t_i - t_j)^2 \, dt_1 \ldots dt_k, \tag{1.11}$$

*$G$ is the Barnes $G$-function, so that for positive integers $k$, $G(1+k) = 1! \cdot 2! \cdot 3! \cdots (k-1)!$.*

For $1 - \frac{1}{k-1} < \alpha < 1 - \frac{1}{k}$, the conjecture is consistent with a theorem of Lester [8].

Let $U$ be an $N \times N$ matrix. The *secular coefficients* $\mathrm{Sc}_j(U)$ are the coefficients of the characteristic polynomial of $U$:

$$\det(I + xU) = \sum_{j=0}^{N} \mathrm{Sc}_j(U) x^j \tag{1.12}$$

Thus $\mathrm{Sc}_0(U) = 1$, $\mathrm{Sc}_1(U) = \operatorname{tr} U$, $\mathrm{Sc}_N(U) = \det U$. The secular coefficients are the elementary symmetric functions in the eigenvalues of $U$.

Define the matrix integrals, with respect to Haar measure, over the group $U(N)$ of $N \times N$ unitary matrices:

$$I_k(m; N) := \int_{U(N)} \Big| \sum_{\substack{j_1+\cdots+j_k=m \\ 0\le j_1,\ldots,j_k\le N}} \mathrm{Sc}_{j_1}(U) \ldots \mathrm{Sc}_{j_k}(U) \Big|^2 dU \, . \tag{1.13}$$

**Theorem 1.1** (KR$^3$)**.** *Let $c := m/N$. Then for $c \in [0, k]$,*

$$I_k(m; N) = \gamma_k(c) N^{k^2-1} + O_k(N^{k^2-2}), \tag{1.14}$$

*with*

$$\gamma_k(c) = \frac{1}{k!\, G(1+k)^2} \int_{[0,1]^k} \delta(t_1 + \ldots + t_k - c) \prod_{i<j} (t_i - t_j)^2 \, dt_1 \ldots dt_k, \tag{1.15}$$

KR$^3$ also proved the matrix integral satisfies a functional equation $I_k(m; N) = I_k(kN - m; N)$, from which it follows that

$$\gamma_k(c) = \gamma_k(k - c), \tag{1.16}$$

and also that

**Theorem 1.2** (KR$^3$)**.**

$$\gamma_k(c) = \sum_{0\le \ell < c} \binom{k}{\ell}^2 (c - \ell)^{(k-\ell)^2+\ell^2-1} g_{k,\ell}(c - \ell) \tag{1.17}$$

*where $g_{k,\ell}(c - \ell)$ are (complicated) polynomials in $c - \ell$.*

For a fixed $k$, $\gamma_k(c)$ is a piecewise polynomial function of $c$. Specifically, it is a fixed polynomial for $r \le c < r + 1$ ($r$ integer), and each time the value of $c$ passes through an integer it becomes a different polynomial.

For example,

$$\gamma_2(c) = \frac{1}{2!} \int_{\substack{0\le t_1\le 1 \\ 0\le c-t_1\le 1}} (t_1 - (c - t_1))^2 \, dt_1 = \begin{cases} \frac{c^3}{3!}, & 0 \le c \le 1 \\ \frac{(2-c)^3}{3!}, & 1 \le c \le 2 \end{cases} \tag{1.18}$$

and

$$\gamma_3(c) = \begin{cases} \frac{1}{8!}c^8, & 0 < c < 1 \\ \frac{1}{8!}(3-c)^8, & 2 < c < 3 \end{cases} \tag{1.19}$$

while for $1 < c < 2$ we get

$$\gamma_3(c) = \frac{1}{8!}\Big( -2c^8 + 24c^7 - 252c^6 + 1512c^5 - 4830c^4 + 8568c^3 - 8484c^2 + 4392c - 927\Big). \tag{1.20}$$

## 2. Relationship to a Hankel determinant

Our starting point is to derive an expression for $\gamma_k(c)$ as the Fourier transform of a Hankel determinant. In (1.11), we substitute for the Dirac delta function:

$$\delta(x) = \int_{-\infty}^{\infty} \exp(2\pi i x y)dy. \tag{2.1}$$

One can be rigorous by writing $\delta(x)$ as the limit of a highly peaked Gaussian, i.e. as the inverse Fourier transform of a highly spread out Gaussian, but for convenience we proceed as above.

Thus

$$\gamma_k(c) = \frac{1}{k!\,G(1+k)^2}\int_{-\infty}^{\infty}\exp(2\pi i u c)\int_{[0,1]^k}\exp\left(-2\pi i u\sum t_j\right) \times \prod_{i<j}(t_i-t_j)^2\,dt_1\dots dt_k du. \tag{2.2}$$

We also note a more symmetric form of the above by substituting $t_j = x_j + 1/2$, so that

$$\gamma_k(c) = \frac{1}{k!\,G(1+k)^2}\int_{-\infty}^{\infty}\exp(2\pi i u(c-k/2))\int_{[-1/2,1/2]^k}\exp\left(-2\pi i u\sum x_j\right) \times \prod_{i<j}(x_i-x_j)^2\,dt_1\dots dx_k du. \tag{2.3}$$

We will prove the following two formulas for $\gamma_k(c)$.

**Theorem 2.1.**

$$\gamma_k(c) = \frac{1}{G(1+k)^2(2\pi i)^{k(k-1)}}\int_{-\infty}^{\infty}\exp(2\pi i u c)\det_{k\times k}\left(f^{(i+j-2)}(u)\right)du \tag{2.4}$$

*where* $f(u) = \int_0^1 \exp(-2\pi i u t)dt = (1-\exp(-2\pi i u))/(2\pi i u)$. *The determinant is a Hankel determinant.*

*A similar, but more symmetric, identity is:*

$$\gamma_k(c) = \frac{1}{G(1+k)^2(2\pi i)^{k(k-1)}} \int_{-\infty}^{\infty} \exp(2\pi i u(c-k/2)) \det_{k\times k}\left(h^{(i+j-2)}(u)\right) du \tag{2.5}$$

*where* $h(u) = \int_{-1/2}^{1/2} \exp(-2\pi i u x)dx = \sin(\pi u)/(\pi u)$.

Our proof will use the Andreief identity:

**Lemma 2.2** (Andreief)**.** *Let* $A_k(t), B_k(t), r(t)$ *be integrable functions on the interval* $[a,b]$. *Then*

$$\frac{1}{N!}\int_{[a,b]^N} \prod_{j=1}^{N} r(t_j) \det_{N\times N}(A_k(t_j)) \det_{N\times N}(B_k(t_j))\, dt_1 \dots dt_N \tag{2.6}$$

$$= \det_{N\times N}\left(\int_a^b r(t)A_j(t)B_k(t)dt\right). \tag{2.7}$$

*Proof of Theorem 2.1.* To prove the first identity in 2.1, apply Andreief's identity to equation (2.2), with $A$ and $B$ two Vandermonde determinants, and $r(t) = \exp(-2\pi i u t)$, to get:

$$\gamma_k(c) = \frac{1}{G(1+k)^2} \int_{-\infty}^{\infty} \exp(2\pi i u c) \det_{k\times k}\left(\int_0^1 \exp(-2\pi i u t)t^{i+j-2}dt\right) du \tag{2.8}$$

The entries of the matrix can be expressed as derivatives, with respect to $u$, of $\int_0^1 \exp(-2\pi i u t)dt$, and we can then correct for the extra powers of $-2\pi i u$ by dividing the $l$-th row by $(-2\pi i u)^{l-1}$ and the $j$-th column by $(-2\pi i u)^{j-1}$, thus by $(-2\pi i u)^{k(k-1)}$ in total (and then dropping the $-1$ since $k(k-1)$ is even).

Using the second form (2.3), we similarly have (2.5) where $h(u) = \int_{-1/2}^{1/2} \exp(-2\pi i u x)dx = \sin(\pi u)/(\pi u)$. □

Some of the basic properties of $\gamma_k(c)$ can be read from (2.4). For example, the inverse Fourier transform of $f^{(j)}$ is equal to $(-2\pi i)^j c^j$ on the interval $(0,1)$ and 0 outside this interval. Expanding the determinant as a permutation sum, each summand thus has inverse Fourier transform a convolution of such terms, and is thus supported on $c \in (0,k)$.

It also shows that $\gamma_k(c)$ is a polynomial in $c$ on each interval $[j, j+1]$, $0 \le j \le k-1$ of degree at most $k^2-1$, because the $i,j$ entry has inverse Fourier Transform a polynomial in $c$ on $(0,1)$ of degree $i+j-2$. Multiply out the determinant as a permutation sum. Each summand, when integrated with respect to $c$, is the inverse Fourier transform of a product of $k$ functions, and hence consists of $k-1$ convolutions of the individual inverse Fourier transforms. Each convolution increases the degree of the polynomial by 1. Hence, each permutation $\sigma$ has its resulting degree bounded by $(k-1)+\sum_{i=1}^{k}(i+\sigma_i-2)=k^2-1$.

We can thus use (2.4) to compute the polynomials $\gamma_k(c)$ by evaluating it at $\geq k^2$ rational values of $c$, say, in each unit interval and interpolating. In this manner, we determined the polynomials $\gamma_k(c)$ listed in Table 1 and 2.

In the symmetric form (2.5), one also sees that $\gamma_k(c)=\gamma_k(c-k)$, by substituting $-u$ for $u$, and using the fact that the determinant in that formula is an even function of $u$.

Setting

$$g(t)=\int_0^1 \exp(-tx)dx, \tag{2.9}$$

so that

$$g^{(n)}(t)=\int_0^1 (-x)^n \exp(-tx)dx, \tag{2.10}$$

and letting

$$D_k(t)=\det_{k\times k}(g^{(i+j-2)}(t)), \tag{2.11}$$

we have that (2.4) can be written as

$$\gamma_k(c)=\frac{1}{G(k+1)^2}\int_{-\infty}^{\infty}\exp(2\pi icu)D_k(2\pi iu)du. \tag{2.12}$$

$D_k(t)$ also satisfies a Painlevé V equation. This is proven in more generality in a paper of Basor, Chen and Ehrhardt [1] (4.38 of that paper, with $a=0$, $b=t$, $\alpha=0$). Specifically, the following holds.

**Theorem 2.3.** *Let*

$$H_k(t)=t\frac{D_k'(t)}{D_k(t)}+k^2. \tag{2.13}$$

*Then*

$$(tH_k''(t))^2 = \\ (H_k(t)+(2k-t)H_k'(t))^2-4(H_k'(t))^2(k^2-H_k(t)+tH_k'(t)). \tag{2.14}$$

Another interesting feature, is that, while $\gamma_k(c)$ is given by a different polynomial on each $[j,j+1]$, $0\leq j\leq k-1$, $\gamma_k(c)$ can be differentiated $j^2+(k-j)^2-2$ times at $c=j$, i.e. is very smooth.

**Theorem 2.4.** *Let $j$ be an integer and $0<j<k$. Define*

$$\nu(c,k)=c^2+(k-c)^2. \tag{2.15}$$

*Then $\gamma_k(c)$ is $(\nu(j,k)-2)$-times differentiable at $c=j$.*

Note that $\nu(c,k)$ reaches its minimum at $c = \lfloor \frac{k+1}{2} \rfloor$, in which case

$$\nu\left(\left\lfloor \frac{k+1}{2} \right\rfloor, k\right) = \left\lfloor \frac{k^2+1}{2} \right\rfloor. \tag{2.16}$$

Thus, we have

**Corollary 2.5.** *The function $\gamma_k(c)$ is $(\lfloor \frac{k^2+1}{2} \rfloor - 2)$-times differentiable for all $0 < c < k$.*

The following lemma is essentially proved in Section 4 of [6].

**Lemma 2.6.** *Let*

$$I_k(u) = \frac{1}{k!} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i u \sum_j t_j} \prod_{j<\ell} (t_j - t_\ell)^2 dt_1 \cdots dt_k. \tag{2.17}$$

*Then*

$$I_k(u) = \sum_{c=0}^{k} e^{i\pi u(k-2c)} \left( \frac{a(c,k)}{u^{\nu(c,k)}} + O\left( \frac{1}{u^{\nu(c,k)+1}} \right) \right) \tag{2.18}$$

*where*

$$\nu(c,k) = c^2 + (k-c)^2 \tag{2.19}$$

*and*

$$a(c,k) = (-1)^c \, (2\pi i)^{-\nu(c,k)} \, G(c+1)^2 \, G(k-c+1)^2. \tag{2.20}$$

Note that $I_k$ above is essentially the inner multidimensional integral in the expression (2.3) for $\gamma_k$.

**Lemma 2.7.** *We have*

$$\gamma_2(c) = \frac{1}{(2\pi i)^2} \int_{-\infty}^{\infty} e^{2\pi i u(c-1)} \left( -\frac{1}{u^2} + \frac{\sin(\pi u)^2}{\pi^2 u^4} \right) du \tag{2.21}$$

$$\tag{2.22}$$

$$= \begin{cases} \dfrac{c^3}{3!}, & \text{if } 0 \le c \le 1, \\[2ex] \dfrac{(2-c)^3}{3!}, & \text{if } 1 \le c \le 2. \end{cases} \tag{2.23}$$

*In particular, $\gamma_2(c)$ is not differentiable at $c = 1$.*

*Proof of Theorem 2.4.* Substituting (2.17) into equation (2.3),

$$\gamma_k(c) = \frac{1}{G(1+k)^2} \int_{-\infty}^{\infty} e^{2\pi i u(c-\frac{k}{2})} I_k(u) du. \tag{2.24}$$

Moreover, from its multi-integral definition we see that $I_k(u)$ is continuous for all real $u$. In particular, $I_k(u)$ is bounded near the origin.

Therefore, to prove that $\gamma_k(c)$ is $(\nu(j,k)-2)$-times differentiable at $c=j$, it suffices to show that

$$J_k(c) := \int_{|u|>1} e^{2\pi i u(c-\frac{k}{2})} I_k(u) du \tag{2.25}$$

is $(\nu(j,k)-2)$-times differentiable at $c=j$.

By Lemma 2.6,

$$\begin{aligned} J_k(c) &= \int_{|u|>1} e^{2\pi i u(c-\frac{k}{2})} \cdot \sum_{\ell=0}^{k} e^{i\pi u(k-2\ell)} \left( \frac{a(\ell,k)}{u^{\nu(\ell,k)}} + O\left(\frac{1}{u^{\nu(\ell,k)+1}}\right)\right) du \\ &= \sum_{\ell=0}^{k} \int_{|u|>1} e^{2\pi i u(c-\ell)} \cdot \left( \frac{a(\ell,k)}{u^{\nu(\ell,k)}} + O\left(\frac{1}{u^{\nu(\ell,k)+1}}\right)\right) du. \end{aligned}$$

We show that for each $\ell$,

$$J_{\ell,k}(c) := \int_{|u|>1} e^{2\pi i u(c-\ell)} \cdot \left( \frac{a(\ell,k)}{u^{\nu(\ell,k)}} + O\left(\frac{1}{u^{\nu(\ell,k)+1}}\right)\right) du \tag{2.26}$$

is $(\nu(j,k)-2)$-times differentiable at $c=j$.

*Case 1*: $\ell = j$. In this case, we observe that, for $n = 1, 2, \ldots, \nu(j,k)-2$ , the integrals

$$\begin{aligned} &\int_{|u|>1} \frac{\partial^n}{\partial c^n} \left[ e^{2\pi i u(c-j)} \cdot \left( \frac{a(j,k)}{u^{\nu(j,k)}} + O\left(\frac{1}{u^{\nu(j,k)+1}}\right)\right)\right] du \\ &= \int_{|u|>1} e^{2\pi i u(c-j)} \cdot (2\pi i u)^n \left( \frac{a(j,k)}{u^{\nu(j,k)}} + O\left(\frac{1}{u^{\nu(j,k)+1}}\right)\right) du \\ &\ll \int_{|u|>1} u^n \left( \frac{a(j,k)}{u^{\nu(j,k)}} + O\left(\frac{1}{u^{\nu(j,k)+1}}\right)\right) du \end{aligned}$$

are uniformly convergent in $c$. Therefore, $J_{j,k}$ is $(\nu(j,k)-2)$-times differentiable at $c=j$ and, in addition,

$$\frac{d^n}{dc^n} J_{j,k}(c) = \int_{|u|>1} e^{2\pi i u(c-j)} \cdot (2\pi i u)^n \left( \frac{a(j,k)}{u^{\nu(j,k)}} + O\left(\frac{1}{u^{\nu(j,k)+1}}\right)\right) du \tag{2.27}$$

for $n = 1, 2, \ldots, \nu(j,k)-2$ .

*Case 2*: $\ell \neq j$. In this case, we show that $J_{\ell,k}(c)$ is in fact $C^\infty$ at $c=j$. To prove this, it suffices to show that

$$\int_{|c|>1} e^{2\pi i u\delta} \frac{du}{u} \tag{2.28}$$

is $C^\infty$ at $\delta \neq 0$.

Using integration by parts repeatedly we see that

$$\int_{|c|>1} e^{2\pi iu\delta} \frac{du}{u} = \frac{m!}{(2\pi i\delta)^m} \int_{|c|>1} e^{2\pi iu\delta} \frac{du}{u^{m+1}} + O_m(\delta^{-1} + \delta^{-m}) \quad (2.29)$$

for any $m \in \mathbb{N}$ and real $\delta \neq 0$, where the Big-$O$ term is a $C^\infty$ function for $\delta \neq 0$. Also, by uniform convergence (see a similar argument in Case 1)

$$\frac{m!}{(2\pi i\delta)^m} \int_{|c|>1} e^{2\pi iu\delta} \frac{du}{u^{m+1}} \quad (2.30)$$

is $(m-1)$-times differentiable at $\delta \neq 0$. It follows that

$$\int_{|c|>1} e^{2\pi iu\delta} \frac{du}{u} \quad (2.31)$$

is $(m-1)$-times differentiable at $\delta \neq 0$. Since $m$ is arbitrary, we have

$$\int_{|c|>1} e^{2\pi iu\delta} \frac{du}{u} \quad (2.32)$$

is $C^\infty$ at $\delta \neq 0$.

Combining Case 1 and Case 2 we obtain that

$$J_k(c) := \int_{|u|>1} e^{2\pi iu(c-\frac{k}{2})} I_k(u) du \quad (2.33)$$

is $(\nu(j,k)-2)$-times differentiable at $c = j$, and therefore, so is $\gamma_k(c)$.

Lastly, we show that

$$\left(\frac{d}{dc}\right)^{\nu(j,k)-2} \gamma_k(c) \quad (2.34)$$

is not differentiable at $c = j$. It suffices to show that

$$\left(\frac{d}{dc}\right)^{\nu(j,k)-2} J_{j,k} \quad (2.35)$$

is not differentiable at $c = j$. By equation (2.27) we have

$$\left(\frac{d}{dc}\right)^{\nu(j,k)-2} J_{j,k} = \int_{|u|>1} e^{2\pi iu(c-j)} \cdot (2\pi iu)^{\nu(j,k)-2} \left(\frac{a(j,k)}{u^{\nu(j,k)}} + O\left(\frac{1}{u^{\nu(j,k)+1}}\right)\right) du.$$

Again, by the uniform convergence argument we see that

$$\int_{|u|>1} e^{2\pi iu(c-j)} \cdot (2\pi iu)^{\nu(j,k)-2} \cdot O\left(\frac{1}{u^{\nu(j,k)+1}}\right) du$$

is differentiable at $c = j$. Therefore, it remains to show that

$$\int_{|u|>1} e^{2\pi iu(c-j)} \cdot (2\pi iu)^{\nu(j,k)-2} \cdot \frac{a(j,k)}{u^{\nu(j,k)}} du$$

is not differentiable at $c = j$, or equivalently,

$$\int_{|u|>1} e^{2\pi iu(c-1)} \cdot \frac{du}{u^2}$$

is not differentiable at $c = 1$.

It follows from Lemma 2.7 that

$$\int_{|u|>1} e^{2\pi iu(c-1)} \left(-\frac{1}{u^2} + \frac{\sin(\pi u)^2}{\pi^2 u^4}\right) du$$

is not differentiable at $c = 1$. Since

$$\int_{|u|>1} e^{2\pi iu(c-1)} \cdot \frac{\sin(\pi u)^2}{\pi^2 u^4} du$$

is differentiable at $c = 1$, we see that

$$\int_{|u|>1} e^{2\pi iu(c-1)} \cdot \frac{du}{u^2}$$

is not differentiable at $c = 1$. This ends our proof of Theorem 2.4. □

The highly smooth nature of $\gamma_k(c)$ was first observed empirically by Conrey in the related problem of determining the asymptotics of the second moment of Dirichlet polynomials whose coefficients are $k$-th divisor numbers. Specifically, he defines

$$M_k(c) = \lim_{T\to\infty} \frac{(k^2)!}{a_k T(\log T)^{k^2}} \int_0^T \left| \sum_{n=1}^{N} \frac{d_k(n)}{n^{1/2+it}} \right|^2 dt$$

for integer values of $k$ and $N = T^c$ with $c > 0$, and determined $M_k(c)$ for $k \leq 4$ (conjecturally for $k = 3, 4$). By comparing Conrey's tables (personal communication) for $M_k(c)$ with our tables for $\gamma_k(c)$, it appears to be the case that the derivative of $M_k(c)$ is equal to $(k^2)!\gamma_k(c)$. Bettin [2] has proven the analogous smoothness for the polynomials $M_k(c)$.

## 3. Expansion for $\log D_k(t)$ and the limiting behaviour of $\gamma_k(c)$

Notice that

$$g^{(n)}(0) = \int_0^1 (-x)^n dx = (-1)^n/(n+1). \tag{3.1}$$

Thus, pulling out powers of $-1$ from the determinant, of which there are an even number, we have $D_k(0) = \det_{k\times k}(1/(i+j-1))$, which is a special case of the Cauchy determinant and thus

$$D_k(0) = G(k+1)^4/G(2k+1). \tag{3.2}$$

Now, $D_k(t)$ satisfies the Toda equation [9]:

$$\frac{D_{k-1}(t)D_{k+1}(t)}{D_k(t)^2} = \frac{D_k''(t)}{D_k(t)} - \frac{(D_k'(t))^2}{D_k(t)^2} = (\log(D_k(t)))'' \tag{3.3}$$

This follows from a recursion of Dodgson (aka Lewis Carroll) for computing determinants [5]. Define $c_m(k)$ by:

$$D_k(t) = D_k(0)\exp\left(\sum_1^\infty \frac{c_m(k)}{m}t^m\right). \tag{3.4}$$

Take the log derivative of the lhs and rhs of the above identity, substitute the series for $\log(D_k(t))$, and clear the denominator of the rhs. Comparing coefficients gives the recursion, for $M > 2$:

$$\begin{aligned} c_M(k) = \frac{1}{(M-1)(M-2)} &\sum_{m=0}^{M-3}(m+1)c_{m+2}(k) \\ &\times (c_{M-m-2}(k-1) + c_{M-m-2}(k+1) - 2c_{M-m-2}(k)) \end{aligned} \tag{3.5}$$

This recursion determines the coefficients $c_M(k)$ in terms of $c_1(k), \ldots, c_{M-2}(k)$.

To get $c_1(k)$:

$$c_1(k) = D_k'(0)/D_k(0). \tag{3.6}$$

One can differentiate $D_k(t)$ by using the product rule to get a sum of determinants where we differentiate the $i$-th row. However, because the entries of $D_k(t)$ are derivatives, differentiating the $i$-th row produces a row that matches the one below it, and the determinant vanishes. Thus, only the last of these terms, where we differentiate the last row, survives. However, that determinant is also a Cauchy determinant with $i, j$ entry $(-1)^{i+j-1}/(i+j-1)$ as before, except for the last row where the entry is $(-1)^{i+j}/(i+j)$.

Using the formula for the Cauchy determinant, a lot of cancellation occurs and we get

$$c_1(k) = -k/2. \tag{3.7}$$

To determine $c_2(k)$, substitute $t = 0$ into identity (3.3). On the lhs:

$$\begin{aligned} &D_{k-1}(0)D_{k+1}(0)/D_k(0)^2 \\ &\quad = G(k)^4G(k+2)^4G(2k+1)^2/(G(2k-1)G(2k+3)G(k+1)^8) \\ &\quad = k^2/(4(4k^2-1)). \end{aligned} \tag{3.8}$$

On the rhs, the constant term of $(\log(D_k(t)))''$ is $c_2(k)$, so

$$c_2(k) = k^2/(4(4k^2-1)). \tag{3.9}$$

The recursion, along with the initial two terms determine all the $c_m(k)$'s. For example, $c_3(k) = 0$, and

$$c_4(k) = \frac{k^2}{16\left(4k^2-1\right)^2\left(4k^2-9\right)}. \tag{3.10}$$

We can apply the above to determine the asymptotic expansion of $\gamma_k(c)$ in a large neighbourhood of $k/2$. To do so, isolate the $m = 1, 2$ terms from the series (3.4), substitute into (2.12) with $t = 2\pi i u$, and compose the series for exp with that of the terms $m \geq 3$ of (3.4), to get that the integrand of (2.12) equals:

$$\exp\left(-\frac{(k\pi u)^2}{2(4k^2-1)} + 2\pi i(c-k/2)u\right)\left(1 + \frac{k^2(\pi u)^4}{4\left(4k^2-1\right)^2\left(4k^2-9\right)} + \dots\right). \tag{3.11}$$

One can obtain more terms, if desired, from the recursion for $c_M(k)$. We thus have the following asymptotic expansion:

**Theorem 3.1.** *Let* $b_k = 8(1 - 1/(4k^2))$ *and* $c = k/2 + o(k)$. *Then*

$$\begin{aligned}\gamma_k(c) \sim \frac{G(k+1)^2}{G(2k+1)}\sqrt{\frac{b_k}{\pi}}\exp(-b_k(c-k/2)^2) \\ \times\left(1 + \frac{1}{4k^2-9}\left(\frac{64(c-k/2)^4 - 24(c-k/2)^2 + 3/4}{k^2}\right.\right. \\ \left.\left. - 2\frac{(c-k/2)^2(16(c-k/2)^2-3)}{k^4} + 4\frac{(c-k/2)^4}{k^6}\right) + \dots\right).\end{aligned} \tag{3.12}$$

*i.e. Gaussian near the centre.*

## 4. Elliptic aliquot cycles

The basic method used to pass from (1.11) to equation (2.2) can be used in the context of elliptic aliquot cycles.

Let $\boldsymbol{p} = (p_1, \dots, p_d)$ be a $d$-tuple of distinct primes. Let $\alpha(\boldsymbol{p})$ be the probability of choosing random and independently $d$ elliptic curves $E_1, \dots, E_d$ over $\mathbb{F}_{p_1}, \dots, \mathbb{F}_{p_d}$, respectively, with the property that $|E(\mathbb{F}_{p_j})| = p_{j+1}$, for $j \in \{1, \dots, d\}$. Here, $p_{d+1} = p_1$. We are choosing the curves $E_j$ uniformly from the set of isomorphism classes of elliptic curves over $\mathbb{F}_p$.

David, Koukoulopoulos, and Smith [4] gave an asymptotic for the average of $\alpha(\boldsymbol{p})$ over the set

$$\mathcal{P}_d(x) = \{(p_1, \dots, p_d) : p_1 \leq x\}. \tag{4.1}$$

(Hasse's bound implies that $\alpha(\boldsymbol{p}) = 0$ unless $|p_{j+1} - p_j - 1| < 2\sqrt{p_j}$ for $1 \leq j \leq d$).

**Theorem 4.1** (DKS)**.** *For any fixed $A > 0$,*

$$\sum_{\boldsymbol{p}\in\mathcal{P}_d(x)} \alpha(\boldsymbol{p}) = C^{(d)}_{aliquot} \int_2^x \frac{du}{2\sqrt{u}(\log u)^d} + O_A\left(\frac{\sqrt{x}}{(\log x)^A}\right) \sim C^{(d)}_{aliquot} \frac{\sqrt{x}}{(\log x)^d},$$

*where*

$$C^{(d)}_{aliquot} := I^{(d)}_{aliquot} \cdot \prod_{\ell} \frac{\ell^d \cdot \# \left\{ \boldsymbol{\sigma} \in GL_2(\mathbb{Z}/\ell\mathbb{Z})^d : \begin{array}{l} \det(\sigma_j) + 1 - \operatorname{tr}(\sigma_j) \equiv \det(\sigma_{j+1})(\ell) \\ \textit{for } 1 \leq j \leq d, \textit{ where } \sigma_{d+1} = \sigma_1 \end{array} \right\}}{|GL_2(\mathbb{Z}/\ell\mathbb{Z})|^d}$$

*with*

$$I^{(d)}_{aliquot} := \frac{2^d}{\pi^d} \int_{\substack{|t_j|\leq 1 \ (1\leq j\leq d-1) \\ |t_1+\cdots+t_{d-1}|\leq 1}} \cdots \int \sqrt{1 - (t_1 + \cdots + t_{d-1})^2} \prod_{j=1}^{d-1} \sqrt{1 - t_j^2} \, dt_1 \cdots dt_{d-1}.$$

Let

$$I(d) := \int_{\substack{|t_j|\leq 1 \ (1\leq j\leq d-1) \\ |t_1+\cdots+t_{d-1}|\leq 1}} \cdots \int \sqrt{1 - (t_1 + \cdots + t_{d-1})^2} \prod_{j=1}^{d-1} \sqrt{1 - t_j^2} \, dt_1 \cdots dt_{d-1}. \tag{4.2}$$

$I(1) = 1$, $I(2) = 4/3$. One might wonder if $I(d)$ persists in being rational. We will show, for $d = 3$, that this seems unlikely.

Replacing the Dirac delta function by the integral in (2.1), we have

$$I(d) = \int_{-\infty}^{\infty} \int_{[-1,1]^d} \prod_1^d (1 - t_j^2)^{1/2} \exp\left(2\pi i y \sum t_j\right) dt_1 \cdots dt_d dy \tag{4.3}$$

But

$$\int_{-1}^{1} (1 - t^2)^{1/2} \exp(2\pi i y t) dt = J_1(2\pi y)/(2y), \tag{4.4}$$

($J$-Bessel function on the rhs). Separating the integral, we get

$$I(d) = \int_{-\infty}^{\infty} \left(\frac{J_1(2\pi y)}{(2y)}\right)^d dy, \tag{4.5}$$

i.e. a one dimensional integral.

This formula can be used to efficiently evaluate $I(d)$ for, say, $d = 3, 4, \dots$, for example with Poisson summation.

Let $f \in L^1(\mathbb{R})$ and let

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i y t} dt. \tag{4.6}$$

denote its Fourier transform. The Poisson summation formula asserts, for, say, $f$ continuous, that

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \tag{4.7}$$

provided the rhs converges absolutely and that $\sum f(n+v)$ converges uniformly in $v$ on compact sets.

Let $\Delta > 0$. By a change of variable

$$\Delta \sum_{n=-\infty}^{\infty} f(n\Delta) = \sum_{n=-\infty}^{\infty} \hat{f}(n/\Delta) = \hat{f}(0) + \sum_{n \neq 0} \hat{f}(n/\Delta), \tag{4.8}$$

so that

$$\int_{-\infty}^{\infty} f(t) dt - \Delta \sum_{n=-\infty}^{\infty} f(n\Delta) = -\sum_{n \neq 0} \hat{f}(n/\Delta) \tag{4.9}$$

tells us how closely the Riemann sum $\Delta \sum_{n=-\infty}^{\infty} f(n\Delta)$ approximates the integral $\int_{-\infty}^{\infty} f(t) dt$.

Apply, with

$$f(y) = \left( \frac{J_1(2\pi y)}{(2y)} \right)^d. \tag{4.10}$$

Note that

$$\int_{-\infty}^{\infty} \frac{J_1(2\pi y)}{(2y)} \exp(-2\pi i u x) dx = \begin{cases} (1-u^2)^{1/2}, & |u| \le 1, \\ 0, & \text{otherwise}. \end{cases} \tag{4.11}$$

Therefore, the Fourier transform of $\left( \frac{J_1(2\pi y)}{(2y)} \right)^d$, being the $d$-fold convolution of $(1-u^2)^{1/2}$ with itself, is supported in $|u| \le d$.

Hence, in the Poisson sum method, any choice of $\Delta \ge 1/d$ gives *no* remainder in the Poisson formula (i.e. 0 contribution from them $|n| \ge 1$ terms). Thus, taking $\Delta = 1/d$ gives:

$$I(d) = \int_{-\infty}^{\infty} \left( \frac{J_1(2\pi y)}{(2y)} \right)^d dy = \frac{1}{d} \sum_{-\infty}^{\infty} \left( \frac{J_1(2\pi n/d)}{(2n/d)} \right)^d. \tag{4.12}$$

Furthermore, $J_1(z) \sim \sqrt{\frac{2}{\pi z}} \cos(z - 3\pi/4)$, hence the sum on the right has terms that are $\ll (2\pi)^{-d} (n/d)^{-3d/2}$. Thus with $d = 3$, the first million terms of the sum gives more than twenty digits accuracy.

One can accelerate the convergence of the sum further using the asymptotics of the $J$-Bessel function, and algorithms for the evaluation of the polylogarithm $\mathrm{Li}_s(z) = \sum_1^\infty z^n/n^s$. Or one can cheat and just use a blackbox like Maple to evaluate (4.5), with $d = 3$:

$$\begin{aligned} I(3) = 1.&7053570421915038354985956872898996791331386909 \\ &78905906671361698193311920077975595946790 11\ldots \end{aligned} \tag{4.13}$$

Let $A_n/B_n$ be the $n-th$ convergent of the continued fraction of the real number $\alpha$. If $p, q \in \mathbb{Z}$ satisfies:

$$|\alpha - p/q| < |\alpha - A_n/B_n| \tag{4.14}$$

then $q > B_n$. Therefore, computing the continued fraction for $I(3)$, the 85-th convergent is:

$$\frac{1470392795121179245920559749163297354942844442 8}{862219909815261328804882569946071642372157646 7} \tag{4.15}$$

(and $|I(3) - A_{85}/B_{85}| \neq 0$. With given precision, there is a limit to how many convergents we can meaningfully use).

Thus, if $I(3)$ is rational, then it has denominator at least $10^{45}$. It would not be too difficult to increase the denominator to hundreds or thousands of digits (millions of digits with some effort), assuming $I(3)$ is irrational.

Maple's identify command did not turn up any obvious expressions for $I(3)$ in terms of algebraic numbers and known constants.

One can also determine the behaviour of $I(d)$ for large $d$. Writing

$$\left(\frac{J_1(2\pi y)}{(2y)}\right)^d = \left(\frac{\pi}{2}\right)^d \exp\left(d \log(J_1(2\pi y)/(\pi y))\right), \tag{4.16}$$

expanding $J_1$ in its Maclaurin series, and pulling out the $y^2$ term, the above becomes

$$\begin{aligned} &\left(\frac{\pi}{2}\right)^d \exp\left(-\frac{d\pi^2 y^2}{2}\right) \\ &\qquad \times \exp\left(-\frac{d\pi^4 y^4}{24} - \frac{d\pi^6 y^6}{144} - \frac{d\pi^8 y^8}{720} - \frac{13 d\pi^{10} y^{10}}{43200} + \ldots\right). \end{aligned} \tag{4.17}$$

Taking the Maclaurin series of the latter exponential (truncated with remainder term), we thus get the asymptotic expansion

$$\begin{aligned} I(d) &= \int_{-\infty}^{\infty} \left( \frac{J_1(2\pi y)}{(2y)} \right)^d dy \\ &\sim \left(\frac{\pi}{2}\right)^{d-1/2} \frac{1}{d^{1/2}} \left( 1 - \frac{1}{8d} - \frac{5}{384d^2} + \frac{7}{3072d^3} + \frac{3829}{491520d^4} + \dots \right). \end{aligned} \tag{4.18}$$

| $k$ | $j$ | $(k^2-1)!\gamma_k(c)$ |
|---|---|---|
| 2 | 0 | $c^3$ |
| | 1 | $(2-c)^3$ |
| 3 | 0 | $c^8$ |
| | 1 | $-2c^8+24c^7-252c^6+1512c^5-4830c^4$ $+8568c^3-8484c^2+4392c-927$ |
| | 2 | $(c-3)^8$ |
| 4 | 0 | $c^{15}$ |
| | 1 | $-3c^{15}+60c^{14}-1680c^{13}+29120c^{12}-294840c^{11}+1873872c^{10}-7927920c^9$ $+23268960c^8-48674340c^7+73653580c^6-80912832c^5+63969360c^4$ $-35497280c^3+13131720c^2-2910240c+292464$ |
| | 2 | $3c^{15}-120c^{14}+3360c^{13}-58240c^{12}+644280c^{11}-4948944c^{10}+28428400c^9$ $-128700000c^8+470398500c^7-1381480100c^6+3179336160c^5-5531176560c^4$ $+6950332480c^3-5910494520c^2+3031004640c-705916304$ |
| | 3 | $(4-c)^{15}$ |
| 5 | 0 | $c^{24}$ |
| | 1 | $-4c^{24}+120c^{23}-6900c^{22}+253000c^{21}-5578650c^{20}+79695000c^{19}$ $-785367660c^{18}+5598232200c^{17}-29915282925c^{16}+123134189200c^{15}$ $-398517412920c^{14}+1029946456560c^{13}-2149736416100c^{12}+3651921075600c^{11}$ $-5072249298600c^{10}+5768661885360c^9-5363308269495c^8+4055447662200c^7$ $-2470634081300c^6+1194550480200c^5-447845361810c^4+125530048600c^3$ $-24758793900c^2+3065085000c-179192775$ |
| | 2 | $6c^{24}-360c^{23}+20700c^{22}-759000c^{21}+17798550c^{20}-292215000c^{19}$ $+3673797820c^{18}-38235839400c^{17}+347123925225c^{16}-2790376974000c^{15}$ $+19589544660840c^{14}-117507788504400c^{13}+592028782736300c^{12}$ $-2479096272534000c^{11}+8573537591434200c^{10}-24367026171730000c^9$ $+56603181050415945c^8-106665764409131400c^7+161304132700472300c^6$ $-192656070655587000c^5+177464649282553710c^4-121528934511474600c^3$ $+58223870087874900c^2-17407730744067000c+2443806916000825$ |
| | 3 | $-4c^{24}+360c^{23}-20700c^{22}+759000c^{21}-18861150c^{20}+345345000c^{19}$ $-4991492660c^{18}+59676982200c^{17}-604502001675c^{16}+5220961534800c^{15}$ $-38343917872920c^{14}+238359873297840c^{13}-1250073382257700c^{12}$ $+5522495132708400c^{11}-20539021982760600c^{10}+64263112978594640c^9$ $-168820549421134545c^8+370693368908418600c^7-674525363862958300c^6$ $+1002229415508043800c^5-1187187920423969310c^4+1078975874367012600c^3$ $-706068990841773900c^2+295689680026989000c-59394510856327775$ |
| | 4 | $(5-c)^{24}$ |

TABLE 1. The polynomials $(k^2-1)!\gamma_k(c)$ for $k \le 5$ and $j \le c \le j+1$.

| $k$ | $j$ | $(k^2-1)!\gamma_k(c)$ |
|---|---|---|
| 6 | 0 | $c^{35}$ |
| 6 | 1 | $-5c^{35}+210c^{34}-21420c^{33}+1413720c^{32}-56862960c^{31}+1501747632c^{30}$ |
| | | $-27736558080c^{29}+375954464160c^{28}-3881009646360c^{27}+31410293440680c^{26}$ |
| | | $-203947162827408c^{25}+1082230579684800c^{24}-4764220775823600c^{23}$ |
| | | $+17613096754503600c^{22}-55229306110228800c^{21}+148080133608311520c^{20}$ |
| | | $-341689133815514100c^{19}+682008750903872700c^{18}-1182119446613536200c^{17}$ |
| | | $+1784232273468783600c^{16}-2349159980084905680c^{15}+2699953776702032400c^{14}$ |
| | | $-2707997790067516800c^{13}+2366932574161864800c^{12}-1798264701411305400c^{11}$ |
| | | $+1182907170763213896c^{10}-670007069282572560c^{9}+324322366699605120c^{8}$ |
| | | $-132818300667235920c^{7}+45395326648924560c^{6}-12709759385961792c^{5}$ |
| | | $+2839179794146080c^{4}-486611119673910c^{3}+60083734292610c^{2}$ |
| | | $-4757721939180c+181451828088$ |
| | 2 | $10c^{35}-840c^{34}+85680c^{33}-5654880c^{32}+238447440c^{31}-7029581328c^{30}$ |
| | | $+158939827200c^{29}-3010298623200c^{28}+51174168784200c^{27}-802885194480600c^{26}$ |
| | | $+11485501718811120c^{25}-145954772087342400c^{24}+1615205663712622800c^{23}$ |
| | | $-15414821245929142800c^{22}+126507768912420350400c^{21}-893399034384858022560c^{20}$ |
| | | $+5440022414523749814300c^{19}-28627456041998656712100c^{18}+130462364245768533732600c^{17}$ |
| | | $-515683796529615245254800c^{16}+1769595318452023551221040c^{15}-5272695333575690900655600c^{14}$ |
| | | $+13632520546818627517123200c^{13}-30536223709478278133815200c^{12}+59100950810144250579990600c^{11}$ |
| | | $-98447935269887910573290424c^{10}+140369638227928515300288240c^{9}-170046927222112798851396480c^{8}$ |
| | | $+173284197564689124463669680c^{7}-146552294343347207749027440c^{6}+100980418141793007531096768c^{5}$ |
| | | $-55222971916535322127277280c^{4}+23052485974924851589246410c^{3}-6898544814307888233994110c^{2}$ |
| | | $+1317633501288006725436180c-120657836168926671721608$ |

TABLE 2. $(k^2-1)!\gamma_k(c)$ for $k=6$ and $j \le c \le j+1$, $j = 0,1,2$. The polynomials for $j=3,4,5$ can be determined from the above using $\gamma_k(c)=\gamma_k(k-c)$.

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