NON-INVARIANCE OF WEAK APPROXIMATION PROPERTIES UNDER EXTENSION OF THE GROUND FIELD

YONGQI LIANG

Abstract. For rational points on algebraic varieties defined over a number field K, we study the behavior of the property of weak approximation with Brauer–Manin obstruction under extension of the ground field. We construct K-varieties accompanied with a quadratic extension $L|K$ such that the property holds over K (conditional on a conjecture) while fails over L. The result is unconditional when $K=\mathbb{Q}$ or K is one of several quadratic number fields. Over $\mathbb{Q}$, we give an explicit example.

1. Introduction

Historically, Chinese remainder theorem has been extended to weak approximation on algebraic groups and on more general algebraic varieties. Let K be a number field. We consider smooth proper algebraic varieties X defined over K. *Weak approximation* means that the image of the diagonal embedding of rational points $X(K)\subset\prod_v X(K_v)$ is dense with respect to the product topology of v-adic topologies. Analytic methods (e.g. Hardy–Littlewood circle method) and cohomological obstructions have been developed to the study of weak approximation.

In 1970s, Manin defined the so-called Brauer–Manin pairing between $\prod_v X(K_v)$ and the cohomological Brauer group $\mathrm{Br}(X)=\mathrm{H}^2_{\text{ét}}(X,\mathbb{G}_m)$. The *Brauer–Manin set* $[\prod_v X(K_v)]^{\mathrm{Br}}$ is the subset of families of local points that are orthogonal to the Brauer group. The inclusions $X(K)\subset[\prod_v X(K_v)]^{\mathrm{Br}}\subset\prod_v X(K_v)$ obstruct weak approximation if the second inclusion is strict. We say that X verifies *weak approximation with Brauer–Manin obstruction* if $X(K)$ is dense in the Brauer–Manin set. Further obstructions to weak approximation have also been studied, for example the étale-Brauer–Manin or equivalently the descent obstruction. Similarly, strong approximation properties with respect to the adelic topology on not necessarily proper varieties are also considered in the literature.

The following natural question is of general interest for a property P on a variety. In this paper we will discuss arithmetic cases in which the property P is one of the following:

Key words and phrases. weak approximation, Brauer–Manin obstruction, extension of the ground field.

MSC 2010 : 11G35 14G05 14G25 14J20.

- weak approximation;
- weak approximation with Brauer–Manin obstruction;
- weak approximation with étale-Brauer–Manin obstruction.

Question 1.1. *If X verifies a property P, does $X \otimes_K L$ verify the same property under any finite extension L of the ground field K?*

To the knowledge of the author, for weak approximation properties it has not yet been seriously discussed in the literature.

For *strong* approximation with Brauer–Manin obstruction, in a collaboration of Y. Cao, F. Xu and the author [CLX, the paragraph after Corollary 8.2], we found examples of punctured Abelian varieties to give a negative answer to the question over $K = \mathbb{Q}$, and over an arbitrary number field conditionally on the finiteness of the Tate–Shafarevich group of certain Abelian varieties. However, if the concerned Tate–Shafarevich groups are finite, such examples can **not** be served as a negative answer to the question for *weak* approximation with Brauer–Manin obstruction according to the Cassels–Tate exact sequence for Abelian varieties and the purity theorem for the Brauer group. Nevertheless, we may expect the answer to be negative.

Leading to a negative answer to the question for weak approximation with Brauer–Manin obstruction (off ∞), it will not be easy to find examples within the following families of smooth proper varieties:

- curves;
- rational surfaces, or even rationally connected varieties.

Indeed, respectively by V. Scharaschkin [Sch99], A. Skorobogatov [Sko01, §6.2], M. Stoll [Sto07, Conjecture 9.1], by J.-L. Colliot-Thélène and J.-J. Sansuc [CT03, page 174], the Brauer–Manin obstruction is suggested or conjectured to be the only obstruction to weak approximation over *arbitrary* number fields for varieties in each family above. A. Skorobogatov [Sko99], B. Poonen [Poo10], and many others [HS14, CTPS16, Sme17] found varieties which do not verify weak approximation with Brauer–Manin (or further) obstruction(s). But they did not consider the (non-)invariance of such properties under extension of the ground field.

Concerning weak approximation properties, to avoid the case of less interest, varieties appeared in this paper admit at least one rational point over the ground field.

Main results. The purpose of this paper is to construct examples to answer negatively to the question. First of all, making use of the Brauer–Manin obstruction, we construct Châtelet surfaces over an arbitrary number field K such that they verify weak approximation over K but fail weak approximation over a quadratic extension L of K, cf. Theorem 3.1. Secondly, using such Châtelet surfaces with the idea of B. Poonen [Poo10], we construct varieties that verify weak approximation with Brauer–Manin obstruction (or further obstructions) over K but fail the same property over L.

Theorem 1.2 (Theorem 4.5)**.** *Let K be a number field. There exists a 3-fold X over K and a quadratic extension L of K such that*

- $X(\boldsymbol{A}_K)^{\mathrm{Br}} \neq \emptyset$, and if Conjecture 4.3 is assumed to hold over K then X_K verifies weak approximation with Brauer–Manin obstruction off ∞_K;
- X_L does not verify weak approximation with Brauer–Manin obstruction off ∞_L.

For a general K, the first conclusion is conditional. When $K = \mathbb{Q}$ or some quadratic number fields ($\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{3})$ etcetera) our result is unconditional. In particular, we give an explicit example over $\mathbb{Q}$ in §4.5. It is given by gluing affine pieces, one of which is defined by the equations

$$\left\{ \begin{array}{rcl} y^2 - 17z^2 & = & 137\left[(1+y'^2)^2(x^4+10x^2-155) - x'^2y'^2(x^4-155)\right] \\ y'^2 & = & x'^3 - 4x' \end{array} \right.$$

in $(x, y, z, x', y') \in \mathbb{A}^5$. It satisfies unconditionally the conclusions of Theorem 1.2 with $L = \mathbb{Q}(\sqrt{5})$.

2. Preliminaries

In this paper, K is always a number field. For completeness, we recall some results in class field theory. All statements in this section are well known.

We denote by Ω_K the set of places of K and by $\infty_K \subset \Omega_K$ the subset of archimedean places. For $v \in \Omega_K$, K_v is the completion of K. The ring of adèles (resp. adèles without archimedean components) is denoted by $\mathbf{A}_K$ (resp. $\mathbf{A}_K^\infty$). The natural projection is denoted by $pr^\infty : \mathbf{A}_K \to \mathbf{A}_K^\infty$.

If an element a in the ring of integer $\mathcal{O}_K$ generates a prime ideal, we will denote the prime ideal and the associated valuation respectively by $\mathfrak{p}_a$ and v_a.

Applying the Chebotarev density theorem and global class field theory to a ray class field, we obtain the following generalised Dirichlet's theorem on arithmetic progressions.

Proposition 2.1. *Let $\mathfrak{a}_i \subset \mathcal{O}_K (i = 1, \ldots, s)$ be ideals that are pairwise prime to each other. Let $x_i \in \mathcal{O}_K$ be an element that is invertible in $\mathcal{O}_K/\mathfrak{a}_i$. Then there exists a principal prime ideal $\mathfrak{p} = (p) \subset \mathcal{O}_K$ such that*

- *$p \equiv x_i \mod \mathfrak{a}_i$ for all i;*
- *p is positive and sufficiently large with respect to all real places of K.*

In fact, the Dirichlet density of such principal prime ideals is positive.

Proof. Let $\mathfrak{m}_0 = \prod_i \mathfrak{a}_i = \bigcap_i \mathfrak{a}_i$. According to the Chinese Remainder Theorem, there exists an element $x_0 \in \mathcal{O}_K$ invertible in $\mathcal{O}_K/\mathfrak{m}_0$ such that $x_0 \equiv x_i \mod \mathfrak{a}_i$ for all i. Furthermore, one can require that x_0 is positive and sufficiently large with respect to all real places of K.

Let $\mathfrak{m} = \mathfrak{m}_0\mathfrak{m}_\infty$ be a modulus of K, where $\mathfrak{m}_\infty$ is the product of all real places of K. Let $K_\mathfrak{m}$ be the ray class field of K of modulus $\mathfrak{m}$. Let $I^\mathfrak{m}$ be the group of fractional ideals of K that are prime to $\mathfrak{m}_0$. Let $\psi : I^\mathfrak{m} \to \mathrm{Gal}(K_\mathfrak{m}|K)$ be the Artin reciprocity map. Define $K_{\mathfrak{m},1}$ to be the set of all elements $a \in K^*$ such that $v_\mathfrak{p}(a-1) \geq v_\mathfrak{p}(\mathfrak{m}_0)$ for all places $\mathfrak{p}|\mathfrak{m}_0$ and such that a is positive with respect to all real places of k. The ray class group $C_{\mathfrak{m}}$ is the cokernel of $i: K_{\mathfrak{m},1} \to I^{\mathfrak{m}}$ which associates to an element $a \in K_{\mathfrak{m},1}$ the principal ideal (a). Global class field theory says that ψ is surjective with kernel $\mathrm{Ker}(\psi) = i(K_{\mathfrak{m},1})$, cf. [Neu99, Chapter VI]. According to the Chebotarev density theorem [Neu99, Theorem VII.13.4], there exists a prime ideal $\mathfrak{p} \nmid \mathfrak{m}_0$ mapping to $\psi((x_0)) \in \mathrm{Gal}(K_{\mathfrak{m}}|K)$. Then $\mathfrak{p} = (x_0) \cdot (x) \in I^{\mathfrak{m}}$ for a certain $x \in K_{\mathfrak{m},1}$. One checks that $p = x_0 \cdot x$ satisfies the conditions. Such prime ideals form a subset of Ω_K of positive Dirichlet density. □

Lemma 2.2. *Let v be an odd place of K. For $s, t \in \mathcal{O}_K$ with $v(s) = 0$, then the Hilbert symbol $(s,t)_v = -1$ if and only if $v(t)$ is odd and s is not a square modulo v.*

Proof. It follows from [Neu99, Proposition V.3.4]. □

Lemma 2.3 (Quadratic reciprocity law)**.** *Let $s, t \in \mathcal{O}_K$ be elements generating odd prime ideals. Assume that either $s \equiv 1 \mod 8\mathcal{O}_K$ or $t \equiv 1 \mod 8\mathcal{O}_K$ and assume that for each real place either s or t is positive. Then s is a square modulo $\mathfrak{p}_t$ if and only if t is a square modulo $\mathfrak{p}_s$.*

Proof. By Lemma 2.2, s is a square modulo $\mathfrak{p}_t$ if and only if $(s,t)_{v_t} = 1$. And t is a square modulo $\mathfrak{p}_s$ if and only if $(s,t)_{v_s} = 1$. It suffices to show that $(s,t)_{v_s} = (s,t)_{v_t}$. For any real place v we have $(s,t)_v = 1$ by hypothesis. For any $v|2$, either s or t is a square in K_v by Hensel's lemma, hence $(s,t)_v = 1$. For any finite place $v \nmid 2st$ we have $(s,t)_v = 1$ by Lemma 2.2. The statement follows from the product formula $\prod_{v \in \Omega_K} (s,t)_v = 1$. □

3. Examples for non-invariance of weak approximation

V. A. Iskovskikh [Isk71] showed that the Châtelet surface over $\mathbb{Q}$ given by

$$y^2 + z^2 = (x^2 - 2)(3 - x^2)$$

violates the Hasse principle. Summarising many others' work, A. Skorobogatov [Sko01, page 145] studied variations of such a counterexample to Hasse principle over $\mathbb{Q}$. B. Poonen [Poo09, Proposition 5.1] further generalised the argument to constructions over any number field K. A fortiori these examples do not verify weak approximation over the ground field K. In this section, we prove the following statement.

Theorem 3.1. *Let K be a number field. There exists a Châtelet surface V over K and a quadratic extension L of K such that*

- *V_K verifies weak approximation and $V(\boldsymbol{A}_K) \neq \emptyset$;*
- *V_L does not verify weak approximation.*

Remark 3.2. For the second conclusion above, we will prove a slightly stronger statement: V_L does not verify weak approximation off ∞_L, cf. Proposition 3.4, which will be needed in the next section.

Let $V_0 \subset \mathbb{A}^3$ be the affine surface over K defined by the equation

$$(\star) \qquad y^2 - az^2 = b(x^4 + 2cx^2 + d)$$

with $a, b, c, d \in \mathcal{O}_K$. We define a Châtelet K-surface V as a smooth compactification of V_0 over K. As the property of weak approximation with Brauer–Manin obstruction is birationally invariant between smooth proper geometrically rational varieties [CTPS16, Proposition 6.1], the forthcoming discussion will not depend on the choice of the compactification. We are going to choose the parameters a, b, c and d to ensure that $D = c^2 - d$ is not a square in K and that V verifies the theorem with $L = K(\sqrt{D})$ a quadratic extension of K, cf. Propositions 3.3 and 3.4.

Choice of the parameters

Applying Proposition 2.1, we can choose sequentially $a, b, c, e \in \mathcal{O}_K$ generating different prime ideals such that

(1) $a \equiv 1 \mod 8\mathcal{O}_K$ and $a > 0$ with respect to all real places;
(2) $b \equiv 1 \mod 2a\mathcal{O}_K$;
(3) $c \equiv 1 \mod 2\mathcal{O}_K$ and c is not a square modulo $\mathfrak{p}_a$;
(4) e is not a square modulo $\mathfrak{p}_a$.

Set $d = ce$. Then $D = c^2 - d = c(c - e)$ is not a square in K since $v_c(D) = 1$. We find that

- a is a square modulo $\mathfrak{p}_b$ by Lemma 2.3;
- a is not a square modulo $\mathfrak{p}_c$ by Lemma 2.3;
- $bd = bce$ is a square modulo $\mathfrak{p}_a$ by (2)(3)(4).

Proposition 3.3. *V_K verifies weak approximation and V_K admits rational points locally everywhere, i.e.* $\overline{V(K)} = V(\boldsymbol{A}_K) \neq \emptyset$.

Proof. By Eisenstein's criterion, the polynomial $b(x^4+2cx^2+d)$ is irreducible over the local field $K_{\mathfrak{p}_c}$, hence irreducible over K. Then $\mathrm{Br}(V_K)/\mathrm{Br}(K) = 0$ and V_K satisfies weak approximation [CTSSD87, Theorem 8.11].

It remains to show that $V_0(K_v) \neq \emptyset$ for all v.

As bd is a square modulo $\mathfrak{p}_a$, hence it is a square in K_{v_a} by Hensel's lemma. By setting $z = x = 0$ we get a K_{v_a}-point on V_0.

For any $v|2$ or $v = v_b$, a is a square in K_v by Hensel's lemma, V_0 is then K_v-rational. V_0 is also K_v-rational for all real v since a is positive.

For any $v \nmid 2ab$, if a is a square in K_v then V_0 is K_v-rational. Suppose that a is not a square in K_v. Consider $x \in K_v$ such that $v(x) < 0$, we find that $v(b(x^4 + 2cx^2 + d)) = 4v(x)$ is always even. As the extension $K_v(\sqrt{a})|K_v$ is unramified, $b(x^4 + 2cx^2 + d)$ is a norm by [Neu99, Corollary V.1.2], hence V_0 admits a K_v-rational point.

In summary $V(\mathbf{A}_K) \neq \emptyset$. □

Proposition 3.4. *V_L does not verify weak approximation off ∞_L, i.e. $V(L)$ is not dense in $V(\boldsymbol{A}_L^\infty)$.*

Proof. Over $L = K(\sqrt{D})$, the polynomial $x^4 + 2cx^2 + d$ factorizes as $(x^2 + c + \sqrt{D})(x^2 + c - \sqrt{D})$. The affine surface $V_{0,L}$ is defined by

$$y^2 - az^2 = b(x^2 + c + \sqrt{D})(x^2 + c - \sqrt{D}).$$

Consider $A \in \mathrm{Br}(L(V))$ the class of the quaternion algebra $(a, x^2 + c + \sqrt{D})$. As classes of quaternion algebras are of order 2 in the Brauer group, A equals to the class of $(a, 1 + \frac{c+\sqrt{D}}{x^2})$. It equals also to the class of $(a, b(x^2 + c - \sqrt{D}))$ since $b(x^2+c-\sqrt{D})(x^2+c+\sqrt{D}) = y^2 - az^2$ lies in the image of the norm map $N_{L(V)(\sqrt{a})|L(V)}$. These three representations show that the residue of A vanishes at every point of codimension 1 of V_L. Whence $A \in \mathrm{Br}(V_L)$. Indeed, A generates $\mathrm{Br}(V_L)/\mathrm{Br}(L) \simeq \mathbb{Z}/2\mathbb{Z}$ [Sko01, Proposition 7.1.2], but this will not help our argument.

Since $c|D$ but $c^2 \nmid D$, $X^2 - D \in K_{\mathfrak{p}_c}[X]$ is an Eisenstein polynomial. The prime $\mathfrak{p}_c$ is ramified in the quadratic extension $L|K$. Denote by $\mathfrak{P}$ the prime ideal above $\mathfrak{p}_c$. To show that weak approximation does not hold on V_L, according to the Brauer–Manin pairing, it suffices to show that the evaluation of A on $V_L(L_{\mathfrak{P}})$ takes both 0 and $1/2 \in \mathbb{Q}/\mathbb{Z}$ as values. This is to say that there exist different points $(x, y, z) \in V_L(L_{\mathfrak{P}})$ such that the Hilbert symbol $(a, x^2 + c + \sqrt{D})_{\mathfrak{P}}$ can take both 1 and -1 as values. As $\mathcal{O}_L/\mathfrak{P} = \mathcal{O}_K/\mathfrak{p}_c$, a is not a square modulo $\mathfrak{P}$. According to Lemma 2.2, $(a, x^2 + c + \sqrt{D})_{\mathfrak{P}} = 1$ (resp. -1) if and only if $v_{\mathfrak{P}}(x^2 + c + \sqrt{D})$ is even (resp. odd). Remember that $L_{\mathfrak{P}}(\sqrt{a})|L_{\mathfrak{P}}$ is unramified, V_L admits $L_{\mathfrak{P}}$-points of coordinate x if $v_{\mathfrak{P}}(b(x^4 + 2cx^2 + d))$ is even [Neu99, Corollary V.1.2]. Firstly, if $v_{\mathfrak{P}}(x) < 0$, then both $v_{\mathfrak{P}}(b(x^4 + 2cx^2 + d)) = v_{\mathfrak{P}}(x^4)$ and $v_{\mathfrak{P}}(x^2 + c + \sqrt{D}) = v_{\mathfrak{P}}(x^2)$ are even. Therefore there exist $L_{\mathfrak{P}}$-points (x, y, z) such that $(a, x^2 + c + \sqrt{D})_{\mathfrak{P}} = 1$. Secondly, as $v_{\mathfrak{P}}(c) = 2$ implies that $v_{\mathfrak{P}}(\sqrt{D}) = v_{\mathfrak{P}}(\sqrt{c(c-e)}) = 1$, we find that $v_{\mathfrak{P}}(c^2 + c + \sqrt{D}) = 1$ is odd and $v_{\mathfrak{P}}(b(c^4 + 2c \cdot c^2 + d)) = 2$ is even. Hence there exist $L_{\mathfrak{P}}$-points $(x, y, z) = (c, y, z)$ such that $(a, x^2 + c + \sqrt{D})_{\mathfrak{P}} = -1$. Therefore $\overline{V(L)} \subset V(\mathbf{A}_L)^{\mathrm{Br}(V_L)} \subsetneq V(\mathbf{A}_L)$, weak approximation fails on V_L.

Moreover, the pull-back of the class A to $\mathrm{Br}(V_{L_w})$ is trivial for any archimedean place w of L since a is a square in L_w. Thus archimedean places make no contribution to the Brauer–Manin pairing with A. Whence $pr^{\infty}(V(\mathbf{A}_L)^{\mathrm{Br}(V_L)}) \subsetneq V(\mathbf{A}_L^{\infty})$, weak approximation off ∞_L also fails on V_L. □

4. Examples for non-invariance of weak approximation with Brauer–Manin obstruction

The aim of this section is to adapt Poonen's construction [Poo10] to find a K-variety X such that it verifies weak approximation with Brauer–Manin obstruction off ∞_K but X_L fails the same property. In general, the conclusion over K is conditional on Conjecture 4.3. When $K = \mathbb{Q}$ we will give an explicit unconditional example. We will also show that over several quadratic number field K the result is unconditional.

Over an arbitrary number field F, Poonen [Poo10] constructed a fibration X in Châtelet surfaces over a curve possessing only finitely many rational points such that X violates the Hasse principle with étale-Brauer–Manin obstruction over the ground field. First of all, we recall his construction over F. Eventually F will be the quadratic extension L of K in the previous section. We are able to make the construction be defined over the smaller field K. Moreover, we will show that the arithmetic of the fibration over K behaves in contrast to that over L if a key fiber is chosen to be one of the Châtelet surfaces descripted in the previous section.

4.1. **Construction of Poonen.**
Start with a Châtelet surface Y_∞ defined over F by the equation

$$y^2 - az^2 = P_\infty(x)$$

with $a \in F^*$ and $P_\infty(x) \in F[x]$ a separable polynomial of degree 4. At the end, we will tell Poonen's choice of Y_∞ and its consequences. In the next paragraph, we will specify F and a different suitable Y_∞ for our purpose.

Take another separable polynomial $P_0(x)$ of degree 4 and relatively prime to P_∞. Let $\tilde{P}_\infty(w,x)$ and $\tilde{P}_0(w,x)$ be their homogenizations. Take a section

$$s' = u^2\tilde{P}_\infty(w,x) + v^2\tilde{P}_0(w,x) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,2)^{\otimes 2})$$

where $\mathbb{P}^1 \times \mathbb{P}^1$ has homogeneous coordinates $(u : v, w : x)$. Let Z' be the zero locus of s' and let $R \subset \mathbb{P}^1$ be the (finite) branch locus of the first projection $Z' \subset \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{pr^1} \mathbb{P}^1$. Then $\infty = (1 : 0) \notin R$ since P_∞ is separable.

Let $\alpha' : Y \to \mathbb{P}^1 \times \mathbb{P}^1$ be the conic bundle given by $y^2 - az^2 = s'$. As a closed subvariety, Y lies inside the projective plane bundle $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1 \times \mathbb{P}^1$, where $\mathcal{E} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1,2)$ is a 3-dimensional vector sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. Denote by $\beta' : Y \to \mathbb{P}^1$ the composition of α' and the first projection pr^1. Then the fiber $\beta'^{-1}(\infty)$ is exactly the Châtelet surface Y_∞ at the beginning.

Choose a smooth projective curve C over F such that $C(F)$ is finite and non-empty. Take a dominant morphism $\gamma : C \to \mathbb{P}^1$ which is étale above R and which maps all rational points to $\infty \in \mathbb{P}^1$. The étaleness is to ensure that $Z = Z' \times_{\mathbb{P}^1} C$ is a smooth curve. Finally, take $\beta : X \to C$ to be the pull-back of $\beta' : Y \to \mathbb{P}^1$ by γ. Poonen showed that X is a smooth proper geometrically integral variety over F verifying the following property.

Proposition 4.1 ([Poo10, Theorem 7.2])**.** *The Brauer–Manin set* $X(\boldsymbol{A}_F)^{\mathrm{Br}}$ *contains* $Y_\infty(\boldsymbol{A}_F) \times C(F)$*.*

In the situation of [Poo10], the constant a and the polynomial $P_\infty(x)$ are carefully chosen such that $Y_\infty(\mathbf{A}_F) \neq \emptyset$ but $Y_\infty(F) = \emptyset$. It follows immediately that $X(F) = \emptyset$ but both $X(\mathbf{A}_F)^{\mathrm{Br}}$ and $X(\mathbf{A}_F)^{\mathrm{et,Br}}$ are non-empty.

4.2. **Brauer–Manin obstruction over** L**.**
For our purpose, we specify F and Y_∞ now.

Proposition 4.2. *If $F = L$ and Y_∞ is the Châtelet surface V_L constructed in §3, then X_L does not verify weak approximation with Brauer–Manin obstruction off ∞_L, i.e. $X(L)$ is not dense in $pr^\infty(X(\boldsymbol{A}_L)^{\mathrm{Br}}) \subset X(\boldsymbol{A}_L^\infty)$.*

Proof. Suppose, for the sake of contradiction, that X_L verifies weak approximation with Brauer–Manin obstruction off ∞_L. Take arbitrary family $(y_w) \in V_L(\mathbf{A}_L^\infty)$. As V_L admits rational points at all archimedean places, we complete the family into $(y_w)_{w\in\Omega_L} \in V_L(\mathbf{A}_L)$. Take a point $P \in C(L)$, then

$$(x_w)_{w\in\Omega_L} = ((y_w)_{w\in\Omega_L}, P) \in Y_\infty(\mathbf{A}_L) \times C(L) \subset X(\mathbf{A}_L)^{\mathrm{Br}}$$

by Proposition 4.1. Let $S \subset \Omega_L \setminus \infty_L$ be any non-empty finite set of non-archimedean places. Then $(x_w)_{w\in S}$ can be approximated by a rational points $x \in X(L)$. Therefore $(\beta(x_w))_{w\in S} = (P)_{w\in S}$ can be approximated by $\beta(x) \in C(L)$. As $C(L)$ is finite thus discrete in $\prod_{w\in S} C(L_w)$, we find that $\beta(x) = P$. Whence $x \in \beta^{-1}(P) \simeq Y_\infty = V_L$ can approximate $(y_w)_{w\in S}$, which contradicts to Propsition 3.4. □

4.3. **Brauer–Manin obstruction over K.**
For $F = L$, we explain how to make Poonen's construction be defined over the subfield K. And we will study its arithmetic over K.

As constructed in §3, $Y_\infty = V_L$ is actually obtained from the base change of a K-variety V to L. If we also take $P_0(x)$ with coefficients in K, then the fibration $\beta' : Y \to \mathbb{P}^1$ is defined over K. Now we need to find a smooth projective K-curve C such that not only $C(K)$ but also $C(L)$ is finite (and non-empty). According to Faltings' theorem, it is always the case if C is of genus ≥ 2. But it seems unclear whether there exist such elliptic curves.

By Riemann–Roch theorem, once $C(L)$ is a finite set of closed points of C, we can choose a rational function $\phi \in K(C)^* \setminus K^*$ such that $C(L)$ is contained in the set of poles of ϕ. This give us a morphism $\gamma : C \to \mathbb{P}^1$ defined over K mapping $C(L)$ to $\infty \in \mathbb{P}^1$. Composing with a linear automorphism of $\mathbb{P}^1$ if necessary, we may require that γ is étale above the finite set $R \subset \mathbb{P}^1$. Finally the pul-back $\beta : X \to C$ is also defined over K.

The following conjecture was stated by M. Stoll [Sto07, Conjecture 9.1]. Similar questions were raised even earlier, independently by V. Scharaschkin [Sch99] and A. Skorobogatov [Sko01, §6.2]. The conjecture holds for $C = \mathbb{P}^1$, and according to Cassels–Tate exact sequence it holds for elliptic curves $C = E$ provided that the Tate–Shafarevich group $Ш(E, K)$ is finite.

Conjecture 4.3 (Stoll)**.** *Let C be a smooth projective curve defined over a number field K. Then C verifies weak approximation with Brauer–Manin obstruction off ∞_K, i.e. $C(K)$ is dense in $pr^\infty(C(\boldsymbol{A}_K)^{\mathrm{Br}}) \subset C(\boldsymbol{A}_K^\infty)$.*

Proposition 4.4. *The Brauer–Manin set $X(\boldsymbol{A}_K)^{\mathrm{Br}}$ is non-empty. Assuming Conjecture 4.3, the variety X_K verifies weak approximation with Brauer–Manin obstruction off ∞_K, i.e. $X(K)$ is dense in $pr^\infty(X(\boldsymbol{A}_K)^{\mathrm{Br}}) \subset X(\boldsymbol{A}_K^\infty)$.*

Proof. As the fibration X is defined over K, we apply Proposition 4.1 with $F = K$ to find that $X(\mathbf{A}_K)^{\mathrm{Br}}$ contains $Y_\infty(\mathbf{A}_K) \times C(K)$ which is non-empty by Proposition 3.3.

For any $(x_v)_{v\in\Omega_K} \in X(\mathbf{A}_K)^{\mathrm{Br}}$, we have $(\beta(x_v))_{v\in\Omega_K} \in C(\mathbf{A}_K)^{\mathrm{Br}}$ by functoriality. As $C(K)$ is finite and dense in $pr^\infty(C(\mathbf{A}_K)^{\mathrm{Br}}) \subset C(\mathbf{A}_K^\infty)$. There exists a rational point $P \in C(K)$ such that for any place $v \in \Omega_K \setminus \infty_K$ we have $P = \beta(x_v)$. In other words, x_v lies on the fiber $\beta^{-1}(P) \simeq V_K$ for $v \notin \infty_K$. The fiber $\beta^{-1}(P)$ admits rational points at every archimedean place by the construction of V_K. According to Proposition 3.3, there exists a K-rational point of $\beta^{-1}(P) \subset X$ approximating $(x_v)_{v\notin\infty_K}$ as desired. □

4.4. **The main result.**
To summarise, we obtain the following result.

Theorem 4.5. *Let K be a number field. There exists a 3-fold X over K and a quadratic extension L of K such that*

- $X(\boldsymbol{A}_K)^{\mathrm{Br}} \neq \emptyset$, and if Conjecture 4.3 is assumed to hold over K then X_K verifies weak approximation with Brauer–Manin obstruction off ∞_K;

- X_L does not verify weak approximation with Brauer–Manin obstruction off ∞_L.

Remark 4.6. The statement of Theorem 4.5 remains valid if we replace the Brauer–Manin obstruction by the étale-Brauer–Manin obstruction, or equivalently the (iterated) descent obstruction according to C. Demarche [Dem09], A. Skorobogatov [Sko09], and Y. Cao [Cao]. Indeed, in the proof we only need to apply [Poo10, Theorem 8.2] instead of Proposition 4.1.

4.5. **An unconditional explicit example over $\mathbb{Q}$.**
For $K = \mathbb{Q}$, we give an unconditional example following the constructions in §3 and §4. The example is given by gluing affine pieces defined by explicit equations.

4.5.1. *Châtelet surfaces.* The Châtelet surface $V_{/\mathbb{Q}}$ is given by

$$y^2 - 17z^2 = P_\infty(x) = 137(x^4 + 10x^2 - 155)$$

with $a = 17$, $b = 137$, $c = 5$, $e = -31$, $d = ce = -155$. Then $D = c^2 - d = 180$ and $L = K(\sqrt{D}) = \mathbb{Q}(\sqrt{5})$. The variety $V_\mathbb{Q}$ and the extension $L|\mathbb{Q}$ satisfy Theorem 3.1. The fiber Y_∞ that appears later will be birational to V. And Y_0 will be birational to the Châtelet surface defined by

$$y^2 - 17z^2 = P_0(x) = -137(x^4 - 155).$$

4.5.2. *Conic bundle.* Let $s' \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,4))$ be the section

$$\begin{aligned} s' &= u^2\tilde{P}_\infty(w,x) + v^2\tilde{P}_0(w,x) \\ &= 137\left[u^2(x^4 + 10x^2w^2 - 155w^4) - v^2(x^4 - 155w^4)\right], \end{aligned}$$

where $\tilde{P}_\infty(w,x)$ and $\tilde{P}_0(w,x)$ are homogenizations of $P_\infty(x)$ and $P_0(x)$ and $(u : v, w : x) \in \mathbb{P}^1 \times \mathbb{P}^1$ are homogeneous coordinates. Let $\mathcal{E} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1,2)$ be a vector sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. The closed subvariety Y, defined by the "equation" $y^2 - 17z^2 = s'$, of the total space of the projective plane bundle $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1 \times \mathbb{P}^1$ is a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. We will explain the word "equation" later. The fiber over ∞ (respectively 0) of the composition $\beta' = pr^1 \circ \alpha' : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the Châtelet surface Y_∞ (respectively Y_0) above.

4.5.3. *Degenerate locus and ramification.* The degenerate locus $Z' \subset \mathbb{P}^1 \times \mathbb{P}^1$ of the conic bundle is defined by

$$s' = 137\left[u^2(x^4 + 10x^2w^2 - 155w^4) - v^2(x^4 - 155w^4)\right] = 0.$$

According to Jacobian criterion, it is smooth since $P_0(x)$ and $P_\infty(x)$ are separable and coprime to each other. Then Y is also smooth. Z' projects onto $\mathbb{P}^1$ via the first projection. The branch locus $R \subset \mathbb{P}^1$ is a closed subscheme of dimension 0. It consists of 6 geometric points ± 1 and $\pm\frac{\sqrt{31 \pm \sqrt{-155}}}{6}$. In particular, $\infty \notin R$.

4.5.4. *Base curve.* Let $C = E$ be the elliptic curve over $\mathbb{Q}$ defined by

$$y'^2 z' = x'^3 - 4x'z'^2$$

with homogeneous coordinates $(x' : y' : z') \in \mathbb{P}^2$. The CM elliptic curve E and its quadratic twist $E^{(5)}$ by $L|\mathbb{Q}$ both have L-functions that are non-vanishing at the complex 1. According to K. Rubin [Rub87], the Tate–Shafarevich group $Ш(E, \mathbb{Q})$ is finite. Stoll's Conjecture 4.3 thus holds for $E_{/\mathbb{Q}}$. According to J. Coates and A. Wiles [CW77], the Mordell–Weil groups $E(\mathbb{Q})$ and $E^{(5)}(\mathbb{Q})$ are both finite, whence so is $E(L)$. Indeed, $E(L) = E(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} = \{(0:1:0), (0:0:1), (\pm 2:0:1)\}$.

4.5.5. *Base change morphism.* We are defining a non-constant morphism $\gamma : E \to \mathbb{P}^1$ étale over $R \subset \mathbb{P}^1$ such that rational points in $E(L)$ are all mapped to ∞.

Define γ by $(x' : y' : z') \mapsto (y'^2 + z'^2 : x'y')$. It extends to $\mathbb{P}^2 \setminus D \to \mathbb{P}^1$ where D is a 0-dimensional closed subscheme consisting of 3 geometric points $(1:0:0)$ and $(0 : \pm\sqrt{-1} : 1)$. It maps all points in $E(L)$ to ∞.

According to Jacobian criterion, one can check by hand that γ is étale over $\pm 1 \in R$. It is much more complicated, but one can check with the help of a computer that γ is also étale over $\pm\frac{\sqrt{31\pm\sqrt{-155}}}{6} \in R$. Practically, it suffices to check only one of the four since they are conjugate by the Galois action. With some tricks in the explicit calculation, finally, the étaleness reduces to decide whether two polynomials of degree 6 and 12 with integral coefficients (11 decimal digits) are coprime to each other.

4.5.6. *Châtelet surface bundle.* As $\gamma : E \to \mathbb{P}^1$ is étale over the branch locus R of $Z' \to \mathbb{P}^1$, the fiber product $Z = E \times_{\mathbb{P}^1} Z'$ is then smooth. Our $\mathbb{Q}$-variety X is defined to be the pull-back of Y by γ. Viewed as a conic bundle over $E \times \mathbb{P}^1$, the smooth variety Z is its degenerate locus. Hence X is also smooth by Jacobian criterion since the coefficients of y^2 and z^2 are non-zero constants.

The variety X is also the restriction of the pull-back $\mathcal{X}$ of Y by

$$(\gamma, 1) : (\mathbb{P}^2 \setminus D) \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$$

to $E \times \mathbb{P}^1 \subset (\mathbb{P}^2 \setminus D) \times \mathbb{P}^1$. It is defined explicitly by the following "equations".

$$\left\{ \begin{array}{rcl} y^2 - 17z^2 & = & 137t^2 \left[(y'^2 + z'^2)^2 (x^4 + 10x^2w^2 - 155w^4) - x'^2y'^2(x^4 - 155w^4) \right] \\ y'^2z' & = & x'^3 - 4x'z'^2 \end{array} \right. ,$$

The second is the equation of $E \subset \mathbb{P}^2$ with homogeneous coordinates $(x' : y' : z')$.

The first "equation" needs more explanation. It is the "equation" of $\mathcal{X}$ viewed as a subvariety of $\mathbb{P}(\mathcal{E}')$. Here $\mathbb{P}(\mathcal{E}') \to (\mathbb{P}^2 \setminus D) \times \mathbb{P}^1$ is the projective bundle associated to $\mathcal{E}' = (\gamma, 1)^*\mathcal{E}$. The vector sheaf $\mathcal{E}'$ equals to $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2, 2)$ if we identify $\mathrm{Pic}(\mathbb{P}^2 \setminus D)$ with $\mathrm{Pic}(\mathbb{P}^2)$. If one denotes by s the section

$$137 \left[(y'^2 + z'^2)^2 (x^4 + 10x^2w^2 - 155w^4) - x'^2y'^2(x^4 - 155w^4) \right] \in \Gamma(\mathbb{P}^2 \setminus D, \mathcal{O}(2, 2)^{\otimes 2})$$

where $(x' : y' : z', w : x)$ are coordinates of $(\mathbb{P}^2 \setminus D) \times \mathbb{P}^1$, then $\mathcal{X}$ is the zero locus in $\mathbb{P}(\mathcal{E}')$ of

$$1 \oplus -17 \oplus -s \in \Gamma(\mathbb{P}^2 \setminus D, \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2, 2)^{\otimes 2}) \subset \Gamma(\mathbb{P}^2 \setminus D, \mathrm{Sym}^2(\mathcal{E}')).$$

So $\mathcal{X}$ is given by the first "equation" with homogeneous "coordinates" $(y : z : t)$ of the fibers of the projective plane bundle.

We make precise what "equation" means. Restricted to one of the affine subsets of $(\mathbb{P}^2 \setminus D) \times \mathbb{P}^1$ given by non-vanishing of one of the coordinates of each factor, the vector sheaf $\mathcal{E}'$ is trivialised. Only on such a trivial fibration, the coordinates $(y : z : t)$ of the fiber $\mathbb{P}^2$ make sense. For example, restricted to the affine $(\mathbb{A}^2 \setminus D) \times \mathbb{A}^1$ given by $w \neq 0$ and $z' \neq 0$. We get equations

$$\left\{ \begin{array}{rcl} y^2 - 17z^2 & = & 137t^2 \left[(1 + y'^2)^2 (x^4 + 10x^2 - 155) - x'^2y'^2(x^4 - 155) \right] \\ y'^2 & = & x'^3 - 4x' \end{array} \right. ,$$

describing a Zariski open dense subset of X lying inside $(\mathbb{A}^2 \setminus D) \times \mathbb{A}^1 \times \mathbb{P}^2$ by dehomogenization. Then X is given by gluing 6 similar explicit quasi-projective varieties via obvious isomorphisms on their intersections. From such equations, we get the affine piece presented in the introduction via a further dehomogenization by taking $t = 1$.

By the way, $(x', y', x,\ y : z : t) = (0, 0, 1,\ 48 : 36 : 1)$ is an explicit $\mathbb{Q}$-rational point on X, on which the existence of rational points is also clear from the construction.

4.6. **Quadratic number fields.**
Though the conclusion over K of Theorem 4.5 is conditional in general, it is not difficult to prove unconditional results over a specific quadratic field with the help of the results on elliptic curves by B. Gross and D. Zagier [GZ86], V. A. Kolyvagin [Kol90], and many others.

The following lemma should be well known, we state it here for the convenience of the reader.

Lemma 4.7. *Let E be an elliptic curve defined over a number field k. Let $k(\sqrt{a})$ be a quadratic extension of k. Denote by $E^{(a)}$ the quadratic twist of E.*

- *If both $Ш(E,k)$ and $Ш(E^{(a)},k)$ are finite, then $Ш(E,k(\sqrt{a}))$ is also finite.*
- *If both $E(k)$ and $E^{(a)}(k)$ are finite, then $E(k(\sqrt{a}))$ is also finite.*

Proof. It follows from the fact that $\mathrm{Res}_{k(\sqrt{a})|k}E_{k(\sqrt{a})}$ and $E\times E^{(a)}$ are isogenous [Mil72, page 185, Example 1] and the fact that the finiteness of the Tate–Shafarevich group depends only on the isogenous class [Mil06, Lemma I.7.1(b)]. □

To the knowledge of the author, the only proved case of the following Bunyakovsky's Conjecture 4.8 is Dirichlet's theorem on arithmetic progressions. As a generalisation of the conjecture to several polynomials, Schinzel's hypothesis (H) has turned out to be a crucial point to the question on Brauer–Manin obstruction, particularly for the fibration method, see the work of Colliot-Thélène and Sansuc [CTS82] and many other subsequent papers. Bunyakovsky's conjecture also plays a role in our constructions but in a different way.

Conjecture 4.8 (Bunyakovsky)**.** *Let $f(x)\in\mathbb{Z}[x]$ be an irreducible polynomial of positive degree and of positive leading coefficient. Suppose that there does not exist a prime number dividing $f(n)$ for all $n\in\mathbb{N}$. Then $f(n)$ has infinitely many prime values.*

Start with a quadratic number field K. We want to prove the existence of a K-variety X and a quadratic extension L of K verifying *unconditionally* the conclusions of Theorem 4.5.

According to the proof of Theorem 4.5, it suffices to show the existence of parameters $a,b,c,e\in\mathcal{O}_K$ satisfying the conditions in §3 and an elliptic curve defined over K such that Stoll's Conjecture 4.3 holds for E and such that $E(K(\sqrt{c^2-ce}))$ is finite.

To fix notation, let $K=\mathbb{Q}(\sqrt{\delta_0})$ with δ_0 a square-free integer. The ring of integers $\mathcal{O}_K=\mathbb{Z}[x]/(\varphi)$ where φ is a degree 2 monic polynomial.

First, we need to find an elliptic curve E over $\mathbb{Q}$ such that both E and its quadratic twist $E^{(\delta_0)}$ have L-functions that are non-vanishing at $s=1$. Thanks to Kolyvagin [Kol90], Gross and Zagier [GZ86], the Tate–Shafarevich groups and the Mordell–Weil groups of these two curves are finite. Then $Ш(E,K)$ and $E(K)$ are finite by Lemma 4.7. Stoll's Conjecture 4.3 holds for E_K.

Second, we choose a square-free rational integer δ such that

- at least one prime factor p of δ is odd and inert in K;
- both $E^{(\delta)}$ and $E^{(\delta\delta_0)}$ are of analytic rank 0.

The first condition is equivalent to the irreducibility of φ modulo p which can be easily tested by quadratic reciprocity. If one believes the rank part of the BSD conjecture, the second condition is requiring that $E^{(\delta)}$ and $E^{(\delta\delta_0)}$ are of Mordell–Weil rank 0. According to B. Mazur and K. Rubin [MR10, Corollary 1.9], there are many quadratic twists of E that have Mordell–Weil rank 0, but now we need to twist simultaneously E and $E^{(\delta_0)}$ by a certain δ. In practice, we often find such an integer δ satisfying these two conditions even in the set of prime numbers. Then $E(L)$ is finite by Lemma 4.7 where $L = K(\sqrt{\delta})$.

Finally, it remains to show the existence of a, b, c, e satisfying the conditions in §3 and such that $L = K(\sqrt{\delta}) = K(\sqrt{c^2 - ce})$. Define $f \in \mathbb{Z}[x]$ to be $f(x) = \frac{-\delta}{c}x^2 + c$ by taking $c = p$ or $-p$ such that $-\delta/c$ is positive. This polynomial is of content 1 and is irreducible by Eisenstein's criterion. If a natural prime l divides $f(n)$ for all $n \in \mathbb{N}$, then $l \leq \deg(f) = 2$ by [CTS82, Lemma 2]. But $l | f(l)$ implies that $l | c = \pm p$, which is impossible since p is odd. Therefore $f(x)$ verifies the assumption of Bunyakovsky's conjecture, which predicts the existence of many natural numbers n such that $e = f(n)$ is a prime larger than p. If, moreover, one of such e's is inert in K, we claim that the existence of the desired variety X is ensured. Indeed, $L = K(\sqrt{\delta}) = K(\sqrt{c^2 - ce})$ since $\delta n^2 = c(c - e)$. And c, e generate different prime ideals in $\mathcal{O}_K$. By Proposition 2.1 and Lemma 2.3, we can complete them to parameters $a, b, c, e \in \mathcal{O}_K$ satisfying all the conditions in §3.

With this strategy, we prove the existence in Theorem 4.5 with unconditional conclusion over quadratic number fields K in the following table (not an exhaustive list). In practice, even if in a very unlucky situation: δ does not have a prime factor that is inert in K, one still has chance to find appropriate parameters, see the last line of the table.

K	δ	c	e	n	E
$\mathbb{Q}(\sqrt{3})$	11	-11	5	4	$y^2 = x^3 - x$
$\mathbb{Q}(\sqrt{-3})$	-11	11	47	6	$y^2 = x^3 - x$
$\mathbb{Q}(\sqrt{-19})$	-3	3	67	8	$y^2 = x^3 - x$
$\mathbb{Q}(\sqrt{-5})$	13	-13	131	12	$y^2 = x^3 - 4x$
$\mathbb{Q}(\sqrt{-1})$	5	$2 + \sqrt{-1}$	$-6 + 5\sqrt{-1}$	2	$y^2 = x^3 - 4x$

5. Remarks on related questions

5.1. **Remarks on the other direction.** In the other direction to answer negatively to Question 1.1, we also seek examples that fail a certain weak approximation property over the ground field but verify the property over a certain finite extension. It turns out that existing examples feed our needs.

Over an arbitrary number field K, due to Poonen [Poo09, Proposition 5.1] there exists a Châtelet surface V which violates the Hasse principle (and explained by the Brauer–Manin obstruction), a fortiori it fails weak approximation. Suppose that V is defined by

$$y^2 - az^2 = P(x).$$

Then $a \in K^*$ must not be a square and the polynomial $P(x) \in K[x]$ must be reducible over K [CTSSD87, Theorem 8.11]. Let L be the quadratic extension $K(\sqrt{a})$ of K. Then V is a rational variety over L, it verifies weak approximation over L.

Using V as Y_∞ in the construction of §4.1 with $F = K$, Poonen show that X_K violates Hasse principle with (étale-)Brauer–Manin obstruction over K, a fortiori it fails weak approximation with (étale-)Brauer–Manin obstruction off ∞_K. Let L be as above. If the base curve C admits only finitely many L-rational points which are all mapped to $\infty \in \mathbb{P}^1$ by γ, then one can argue as in Proposition 4.4 to show that X_L verifies weak approximation with (étale-)Brauer–Manin obstruction off ∞_L assuming Conjecture 4.3 over L. As explained in §4.5, for $K = \mathbb{Q}$ the last statement is unconditional for a certain elliptic curve.

Note that in these examples neither V nor X possesses K-rational points. One may ask for examples possessing K-rational points. Indeed, it is even easier, it suffices to define the Châtelet surface V by

$$y^2 - az^2 = -(x^2 + b)(x^2 - b)$$

with $a, b \in \mathcal{O}_K$ generating different prime ideals such that $a > 0$ with respect to all real places. Then $(x, y, z) = (0, b, 0)$ is a rational point on V and hence $X(K)$ is non-empty. We know that X_K fails the Brauer–Manin obstruction off ∞ as long as V_K does not verify weak approximation off ∞_K. To show the latter, the key argument is similar to that in Proposition 3.4: if $A \in \mathrm{Br}(K(V))$ is the class of the quaternion algebra $(a, x^2 + b)$, then the evaluation of A on $V(K_{v_b})$ takes both 0 and $1/2 \in \mathbb{Q}/\mathbb{Z}$ as values (for the value $1/2$, one considers $x = b$). It is also clear that over the extension $L = K(\sqrt{a})$, V_L is a rational variety and X_L satisfies weak approximation with Brauer–Manin obstruction if $C(L)$ is finite and mapped to ∞.

5.2. **Over global function fields.** Poonen's construction also works over odd characteristic global function fields and B. Viray extended the result to characteristic 2 [Vir12]. Moreover, if C is a smooth projective curve of genus at least 2 whose Jacobian J satisfies some mild conditions, Conjecture 4.3 for C has been proved by B. Poonen and J. F. Voloch [PV10] even without finiteness assumption of the Tate–Shafarevich group of J. As explained to the author by Voloch, his paper [Vol95] showed that those conditions on J are satisfied for many curves. Poonen can explicitly construct such genus 2 curves in odd characteristic case using the method in [HLP00, §3.2]. We may expect our main result to hold unconditionally over a global function field (at least in odd characteristic case). However, the proof of Proposition 3.3 uses the fact that the Brauer–Manin obstruction is the only obstruction to weak approximation for Châtelet surfaces. This result was stated in [CTSSD87, Theorem 8.11] only for number fields. It is not clear if the analog holds over odd characteristic global function fields, or even more difficult in characteristic 2.

5.3. **A related question.** Though the answer to Question 1.1 is negative, we may still ask the following question.

Question 5.1. *Under which finite extensions, weak approximation properties are invariant?*

The question is easy for the families of varieties X_K in §4 and $Y_{\infty,K}$ in §4.3 (or V_K in §3). Over any finite extension K' that is linearly disjoint from $K[x]/(P_\infty(x))$ over K, the polynomial $P_\infty(x)$ is still irreducible. Then the varieties $V_{K'} = Y_{\infty,K'}$ and $X_{K'}$ preserve the same property as on the ground field (Propositions 3.3 and 4.4). But it seems hopeless to answer to the question in general.

Acknowledgements. The author would like to thank B. Poonen, B. Viray, and J. F. Voloch for having explained their work to the author.

References

[Cao] Y. Cao. Sous-groupe de Brauer invariant et obstruction de descente itérée. Preprint available at arXiv 1704.05425.

[CLX] Y. Cao, Y. Liang, and F. Xu. Arithmetic purity of strong approximation for homogeneous spaces. Preprint available at arXiv 1701.07259.

[CT03] J.-L. Colliot-Thélène. Points rationnels sur les fibrations. In K.J. Böröczky, J. Kollár, and T. Szamuely, editors, *Higher Dimensional Varieties and Rational Points*, volume 12 of *Bolyai Society Mathematical Series*, pages 171–221. Springer-Verlag, 2003.

[CTPS16] J.-L. Colliot-Thélène, A. Pál, and A. Skorobogatov. Pathologies of the Brauer-Manin obstruction. *Mathematische Zeitschrift*, 282:799–817, 2016.

[CTS82] J.-L. Colliot-Thélène and J.-J. Sansuc. Sur le principe de Hasse et l'approximation faible, et sur une hypothèse de Schinzel. *Acta Arithmetica*, 41(1):33–53, 1982.

[CTSSD87] J.-L. Colliot-Thélène, J.-J. Sansuc, and Sir Peter Swinnerton-Dyer. Intersections of two quadrics and Châtelet surfaces II. *J. reine angew. Math.*, 374:72–168, 1987.

[CW77] J. Coates and A. Wiles. On the conjecture of Birch and Swinnerton-Dyer. *Invent. Math.*, 39(3):223–251, 1977.

[Dem09] C. Demarche. Obstruction de descente et obstruction de Brauer-Manin étale. *Algebra and Number Theory*, 3(2):237–254, 2009.

[GZ86] B. Gross and D. Zagier. Heegner points and derivatives of L-series. *Invent. Math.*, 84(2):225–320, 1986.

[HLP00] E. W. Howe, F. Leprévost, and B. Poonen. Large torsion subgroups of split Jacobians of curves of genus two or three. *Forum Mathematicum*, 12(3):315–364, 2000.

[HS14] Y. Harpaz and A.N. Skorobogatov. Singular curves and the étale Brauer-Manin obstruction for surfaces. *Annales Scientifiques de l'École Normale Supérieure*, 47:765–778, 2014.

[Isk71] V. A. Iskovskikh. A counterexample to the Hasse principle for systems of two quadratic forms in five variables. *Mat. Zametki*, 10:253–257, 1971.

[Kol90] V.A. Kolyvagin. Euler systems. In *The Grothendieck Festschrift*, volume 87 of *Progr. Math.*, pages 435–483. Birkhäuser Boston, 1990.

[Mil72] J.S. Milne. On the arithmetic of abelian varieties. *Invent. Math.*, 17:177–190, 1972.

[Mil06] J.S. Milne. *Arithmetic Duality Theorems*. BookSurge, LLC, second edition, 2006.

[MR10] B. Mazur and K. Rubin. Ranks of twists of elliptic curves and Hilbert's tenth problem. *Invent. Math.*, 181(3):541–575, 2010.

[Neu99] J. Neukirch. *Algebraic Number Theory*. Grundlehren der mathematischen Wissenschaften. Springer, 1999.

[Poo09] B. Poonen. Existence of rational points on smooth projective varieties. *J. Eur. Math. Soc.*, 11(3):529–543, 2009.

[Poo10] B. Poonen. Insufficiency of the Brauer-Manin obstruction applied to étale covers. *Ann. of Math.*, 171(3):2157–2169, 2010.

[PV10] B. Poonen and J. F. Voloch. The Brauer-Manin obstruction for subvarieties of abelian varieties over function fields. *Annals of Math.*, 171(1):511–532, 2010.

[Rub87] K. Rubin. Tate-Shafarevich groups and L-functions of elliptic curves with complex multiplication. *Invent. Math.*, 89(3):527–559, 1987.

[Sch99] V. Scharaschkin. *Local-global problems and the Brauer-Manin obstruction*. PhD thesis, University of Michigan, 1999.

[Sko99] A.N. Skorobogatov. Beyond the Manin obstruction. *Invent. Math.*, 135:399–424, 1999.

[Sko01] A.N. Skorobogatov. *Torsors and Rational Points*. Cambridge University Press, 2001.

[Sko09] A. Skorobogatov. Descent obstruction is equivalent to étale Brauer–Manin obstruction. *Math. Ann.*, 344:501–510, 2009.

[Sme17] A. Smeets. Insufficiency of the étale Brauer-Manin obstruction: towards a simply connected example. *American Journal of Mathematics*, 139(2):417–431, 2017.

[Sto07] M. Stoll. Finite descent obstructions and rational points on curves. *Algebra & Number Theory*, 1:349–391, 2007.

[Vir12] B. Viray. Failure of the Hasse principle for Châtelet surfaces in characteristic 2. *J. Théorie de nombres de Bordeaux*, 24(1):231–236, 2012.

[Vol95] J. F. Voloch. Diophantine approximation on abelian varieties in characteristic p. *Amer. J. Math.*, 117(4):1089–1095, 1995.