# ON ABHYANKAR'S LEMMA ABOUT RAMIFICATION INDICES

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ABSTRACT. We provide a simple proof of the fact that the ramification index of the *compositum* of two finite extensions of local fields is equal to the least common multiple of the ramification indices when at least one of the extensions is tamely ramified.

## 1. Introduction

Let L/K be an extension of number fields and assume that L is the *compositum* of two sub-extensions  $K_1$  and  $K_2$ . Let  $\mathfrak{q}$  be a prime ideal of L and let  $\mathfrak{p} = \mathfrak{q} \cap K$ ,  $\mathfrak{p}_1 = \mathfrak{q} \cap K_1$ ,  $\mathfrak{p}_2 = \mathfrak{q} \cap K_2$ . We denote by  $e(\mathfrak{q}/\mathfrak{p})$  the ramification index of  $\mathfrak{q}$  in the extension L/K. Then, by multiplicativity of the ramification indices, that is,

$$e(\mathfrak{q}/\mathfrak{p}) = e(\mathfrak{q}/\mathfrak{p}_i) \times e(\mathfrak{p}_i/\mathfrak{p}) \quad (i = 1, 2),$$

we obviously have:

(1) 
$$\operatorname{lcm}\left\{e(\mathfrak{p}_1/\mathfrak{p}), e(\mathfrak{p}_2/\mathfrak{p})\right\} \mid e(\mathfrak{q}/\mathfrak{p}).$$

On the other hand, if one of the extensions  $K_i/K$  is normal, for instance if  $K_1/K$  is normal, the extension  $L/K_2$  is normal and the following morphism is injective:

$$\rho: \sigma \in \operatorname{Gal}(L/K_2) \mapsto \sigma_{|K_1|} \in \operatorname{Gal}(K_1/K)$$
.

Recall that, as the residue fields are perfect, the ramification index  $e(\mathfrak{q}/\mathfrak{p}_2)$  is equal to the order of the inertia group  $\mathcal{I}_{\mathfrak{q}}(L/K_2)$  of  $\mathfrak{q}$  in the extension  $L/K_2$ , that is,

$$\mathcal{I}_{\mathfrak{g}}(L/K_2) = \{ \sigma \in \operatorname{Gal}(L/K_2) \mid \forall x \in \mathcal{O}_L \ \sigma(x) - x \in \mathfrak{q} \}.$$

Now, the image by  $\rho$  of  $\mathcal{I}_{\mathfrak{q}}(L/K_2)$  is clearly contained in the inertia group  $\mathcal{I}_{\mathfrak{p}_1}(K_1/K)$ . Thus,  $e(\mathfrak{q}/\mathfrak{p}_2)$  divides  $e(\mathfrak{p}_1/\mathfrak{p})$ , and hence,

(2) 
$$e(\mathfrak{q}/\mathfrak{p}) \mid e(\mathfrak{p}_1/\mathfrak{p}) \times e(\mathfrak{p}_2/\mathfrak{p})$$
.

Formula (2) may be false in general (see Remark 2.3 below).

There is another well known result about ramification indices of *composita*, namely Abhyankar's lemma. This result is generally known in the following form:

**Proposition 1.1.** (Narkiewicz [2, p. 229]) If  $K, K_1, K_2$  are local fields such that  $K_1/K$  is tame,  $K_2/K$  is finite and  $e(K_1/K)$  divides  $e(K_2/K)$ , then  $K_1K_2/K_2$  is unramified.

Roughly speaking, one may kill tame ramification by taking an extension of the base field (see also [1, p. 279]). In fact, one finds in [4] a stronger formulation, but it is stated only for function fields:

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**Proposition 1.2.** (Stichtenoth [4, Th. 3.9.1]) Let L/K be a finite separable extension of function fields. Suppose that  $L = K_1K_2$  is the compositum of two intermediate fields  $K \subseteq K_1, K_2 \subseteq L$ . Let Q be a place of L extension of a place P of K and set  $P_i := Q \cap K_i$  for i = 1, 2. Assume that at least one of the extensions  $P_1/P$  or  $P_2/P$  is tame. Then

$$e(Q/P) = lcm\{e(P_1/P), e(P_2/P)\}.$$

Since we did not find in the literature such a statement with respect to number fields (although it probably exists somewhere hidden under an indirect formulation), we provide here a simple proof of this generalized result (proof which in some sense is close to that of Proposition 1.2).

## 2. Abhyankar's lemma

**Theorem 2.1.** Let A be a Dedekind domain with quotient field K. Let L/K be a finite separable extension of fields. Assume that L is the compositum of two subfields  $K_1$  and  $K_2$ . Denote by B,  $B_1$  and  $B_2$  the integral closures of A in L,  $K_1$  and  $K_2$ . Let  $\mathfrak{p}$  be a maximal ideal of A whose residue field  $A/\mathfrak{p}$  is perfect with characteristic p (which may be 0). Finally, let  $\mathfrak{q}$  be a maximal ideal of B lying over  $\mathfrak{p}$  and let  $\mathfrak{p}_i = \mathfrak{q} \cap A_i$  for i = 1, 2.

If at least one of the extensions  $K_i/K$  is tamely ramified in  $\mathfrak{p}_i$  (that is, if one of the integers  $e(\mathfrak{p}_i/\mathfrak{p})$  is not divisible by p), then one has the equality:

(3) 
$$e(\mathfrak{q}/\mathfrak{p}) = \operatorname{lcm} \left\{ e(\mathfrak{p}_1/\mathfrak{p}), e(\mathfrak{p}_2/\mathfrak{p}) \right\}.$$

Note that, if the characteristic of  $A/\mathfrak{p}$  is 0, then the ramification is always tame. Of course, Propositions 1.1 and 1.2 are consequences of Theorem 2.1.

*Proof.* Let L' be the normal closure of L over K, let B' be the integral closure of B in L' and let  $\mathfrak{q}'$  be a maximal ideal of B' lying over  $\mathfrak{q}$ . One knows that  $e(\mathfrak{q}'/\mathfrak{p}) = |G_0|$  where  $G_0$  denotes the inertia group of  $\mathfrak{q}'$  in the extension L'/K. Moreover, denoting by  $\pi \in B'$  a uniformizer with respect to  $\mathfrak{q}'$ , we have a group homomorphism:

$$s \in G_0 \mapsto s(\pi)/\pi \mod \mathfrak{q}' \in (B'/\mathfrak{q}')^*$$

whose kernel is the subgroup:

$$G_1 = \{ s \in \operatorname{Gal}(L'/K) \mid \forall x \in B' \ s(x) - x \in \mathfrak{q}^{2} \}.$$

Thus,  $G_1 \triangleleft G_0$ . The injectivity of the morphism  $G_0/G_1 \rightarrow (B'/\mathfrak{q}')^*$  shows that the group  $G_0/G_1$  is cyclic and that its order is prime to the characteristic p of  $B'/\mathfrak{q}'$ . Moreover, one knows also that, if p=0, then  $G_1=\{1\}$  and, if p>0, then  $G_1$  is a p-group (cf., for instance, [3, IV, §2]). Finally,  $G_0$  is a semidirect product of a cyclic group of order prime to p with the normal p-group  $G_1$ .

If E is a field between K and L', the analogs of the groups  $G_j$  for j=0,1 with respect to  $\mathfrak{q}'$  in the extension L'/E are clearly the groups  $G_j \cap \operatorname{Gal}(L'/E)$ . Let  $\Gamma_1 = \operatorname{Gal}(L'/K_1)$ ,  $\Gamma_2 = \operatorname{Gal}(L'/K_2)$  and  $\Gamma = \operatorname{Gal}(L'/L)$ . One has  $\Gamma = \Gamma_1 \cap \Gamma_2$  since  $L = K_1K_2$ . Then, by multiplicativity, one has:

$$e(\mathfrak{q}/\mathfrak{p}) = \frac{e(\mathfrak{q}'/\mathfrak{p})}{e(\mathfrak{q}'/\mathfrak{q})} = \frac{|G_0|}{|G_0 \cap \Gamma|} \text{ and } e(\mathfrak{p}_i/\mathfrak{p}) = \frac{e(\mathfrak{q}'/\mathfrak{p})}{e(\mathfrak{q}'/\mathfrak{p}_i)} = \frac{|G_0|}{|G_0 \cap \Gamma_i|} \ (i = 1, 2).$$

Let  $m = \operatorname{lcm} \{ e(\mathfrak{p}_1/\mathfrak{p}), e(\mathfrak{p}_2/\mathfrak{p}) \}$ , then  $m = \operatorname{lcm} \{ \frac{|G_0|}{|G_0 \cap \Gamma_1|}, \frac{|G_0|}{|G_0 \cap \Gamma_2|} \}$ . Now, let  $d = \operatorname{gcd} \{ |G_0 \cap \Gamma_1|, |G_0 \cap \Gamma_2| \}$ , then  $m \times d = |G_0|$ . Finally,  $\frac{e(\mathfrak{q}/\mathfrak{p})}{m} = \frac{d}{|G_0 \cap \Gamma|}$ . Thus, we have to prove that  $d = |G_0 \cap \Gamma|$ .

 $1^{st}$  case. If p=0,  $G_0$  is cyclic. Since in a cyclic group the order of a subgroup which is the intersection of two subgroups is the gcd of the orders of these two subgroups, it follows from the equality  $\Gamma = \Gamma_1 \cap \Gamma_2$  that  $|G_0 \cap \Gamma| = d$ .

 $2^{nd}$  case. If  $p \neq 0$ , by hypothesis p does not divide one of the integers  $e(\mathfrak{p}_i/\mathfrak{p}) = \frac{|G_0|}{|G_0 \cap \Gamma_i|}$ . Assume that p does not divide  $\frac{|G_0|}{|G_0 \cap \Gamma_1|}$ . As  $G_1$  is the only p-Sylow subgroup of  $G_0$ ,  $G_1$  is contained in  $\Gamma_1$ , and hence,  $G_1 = G_1 \cap \Gamma_1$  and  $G_1 \cap \Gamma = G_1 \cap \Gamma_2$ . Thus, trivially, we have:

$$(4) |G_1 \cap \Gamma| = \gcd\{ |G_1 \cap \Gamma_1|, |G_1 \cap \Gamma_2| \}.$$

Moreover, let  $\pi: G_0 \to G_0/G_1$  be the canonical morphism. Clearly, we have:

$$|G_0 \cap \Gamma| = |\pi(G_0 \cap \Gamma)| \times |G_1 \cap \Gamma|$$
 and  $|G_0 \cap \Gamma_i| = |\pi(G_0 \cap \Gamma_i)| \times |G_1 \cap \Gamma_i|$   $(i = 1, 2)$ .

The containment  $\pi(G_0 \cap \Gamma) \subseteq \pi(G_0 \cap \Gamma_1) \cap \pi(G_0 \cap \Gamma_2)$  is obvious. Let us prove the reverse inclusion. Let  $x_i \in G_0 \cap \Gamma_i$  (i = 1, 2) such that  $\pi(x_1) = \pi(x_2)$ , then  $x_2 = g_1x_1$  for some  $g_1 \in G_1 \subseteq \Gamma_1$ , and hence,  $x_2 \in G_0 \cap \Gamma$ . From the equality  $\pi(G_0 \cap \Gamma) = \pi(G_0 \cap \Gamma_1) \cap \pi(G_0 \cap \Gamma_2)$  and the fact that the group  $G_0/G_1$  is cyclic, we deduce:

(5) 
$$|\pi(G_0 \cap \Gamma)| = \gcd\{ |\pi(G_0 \cap \Gamma_1)|, |\pi(G_0 \cap \Gamma_2)| \}.$$

Since  $|G_0|/|G_1|$  and  $|G_1|$  are coprime it follows by multiplicativity from (4) and (5) that:

$$|G_0 \cap \Gamma| = \gcd\{|G_0 \cap \Gamma_1|, |G_0 \cap \Gamma_2|\} = d.$$

**Corollary 2.2.** With the same notations as in Theorem 2.1, if  $e(\mathfrak{p}_1/\mathfrak{p})$  and  $e(\mathfrak{p}_2,\mathfrak{p})$  are coprime, then

$$e(\mathfrak{q}/\mathfrak{p}) = e(\mathfrak{p}_1/\mathfrak{p}) \times e(\mathfrak{p}_2/\mathfrak{p}).$$

Remark 2.3. When  $K_1/K$  and  $K_2/K$  are both widely ramified, not only  $e(\mathfrak{q}/\mathfrak{p})$  may be strictly greater than the least common multiple of  $e(\mathfrak{p}_1/\mathfrak{p})$  and  $e(\mathfrak{p}_2/\mathfrak{p})$ , but it may happen that it does not divide the product  $e(\mathfrak{p}_1/\mathfrak{p}) \times e(\mathfrak{p}_2/\mathfrak{p})$ .

For instance, let  $K = \mathbb{Q}$ ,  $K_0 = \mathbb{Q}(j)$ ,  $K_1 = \mathbb{Q}(\sqrt[3]{3})$ ,  $K_2 = \mathbb{Q}(j\sqrt[3]{3})$ , and  $L = \mathbb{Q}(j,\sqrt[3]{3}) = K_0K_1 = K_1K_2$  where  $j = \exp(2i\pi/3)$ . Then, if  $\mathfrak{q}$  is a (in fact, the) prime ideal of L lying over  $\mathfrak{p} = 3\mathbb{Z}$ , then we have  $e(\mathfrak{p}_0/\mathfrak{p}) = 2$ ,  $e(\mathfrak{p}_1/\mathfrak{p}) = e(\mathfrak{p}_2/\mathfrak{p}) = 3$ . It follows from formula (1) that  $e(\mathfrak{q}/\mathfrak{p}) = 6$  which does not divide  $e(\mathfrak{p}_1/\mathfrak{p}) \times e(\mathfrak{p}_2/\mathfrak{p}) = 9$ .

#### References

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