A new bound on Erdős distinct distances problem in the plane over prime fields

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Abstract

In this paper we obtain a new lower bound on the Erdős distinct distances problem in the plane over prime fields. More precisely, we show that for any set $A \subset \mathbb{F}_p^2$ with $|A| \le p^{7/6}$ and $p \equiv 3 \mod 4$, the number of distinct distances determined by pairs of points in A satisfies

$$|\Delta(A)| \gtrsim |A|^{\frac{1}{2}+\frac{149}{4214}}.$$

Our result gives a new lower bound of $|\Delta(A)|$ in the range $|A| \le p^{1+\frac{149}{4065}}$.

The main tools in our method are the energy of a set on a paraboloid due to Rudnev and Shkredov, a point-line incidence bound given by Stevens and de Zeeuw, and a lower bound on the number of distinct distances between a line and a set in $\mathbb{F}_p^2$. The latter is the new feature that allows us to improve the previous bound due Stevens and de Zeeuw.

1 Introduction

The celebrated Erdős distinct distances problem asks for the minimum number of distinct distances determined by a set of n points in the plane over the real numbers. The breakthrough work of Guth and Katz [6] shows that a set of n points in $\mathbb{R}^2$ determines at least $Cn/\log(n)$ distinct distances. The same problem can be considered in the setting of finite fields.

Let $\mathbb{F}_p$ be the prime field of order p. The "distance" formula between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{F}_p^2$ is defined by

$$||x - y|| := (x_1 - y_1)^2 + (x_2 - y_2)^2.$$

While this is not a distance in the traditional sense, the definition above is a reasonable analog of the Euclidean distance in that it is invariant under orthogonal transformations.

For $A \subset \mathbb{F}_p^2$, let

$$\Delta(A) = \{||x - y|| : x, y \in A\}$$

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and let $|\Delta(A)|$ denote its size. It has been shown in a remarkable paper of Bourgain, Katz, and Tao [3] that if $|A| = p^\alpha$, $0 < \alpha < 2$, then we have

$$|\Delta(A)| \geq |A|^{\frac{1}{2}+\varepsilon},$$

for some $\varepsilon = \varepsilon(\alpha) > 0$.

This result has been quantified and improved over time. The recent work of Stevens and De Zeeuw [11] shows that

$$|\Delta(A)| \geq |A|^{\frac{1}{2}+\frac{1}{30}} = |A|^{\frac{8}{15}}, \tag{1}$$

under the condition $|A| \ll p^{\frac{15}{11}}$.

Here and throughout, $X \ll Y$ means that there exists $c_1 > 0$, independent of p, such that $X \leq c_1 Y$, $X \gtrsim Y$ means $X \gg (\log Y)^{-c_2} Y$ for some positive constant c_2, and $X \sim Y$ means that $c_3 X \leq Y \leq c_4 X$ for some positive constants c_3 and c_4.

For the case of large sets, Iosevich and Rudnev [5] used Fourier analytic methods to prove that for $A \subset \mathbb{F}_q^d$, where q is not necessarily prime, with $|A| \geq 4q^{\frac{d+1}{2}}$, we have $\Delta(A) = \mathbb{F}_q$. It was shown in [7] that the threshold $q^{\frac{d+1}{2}}$ cannot in general be improved when d is odd, even if we wish to recover a positive proportion of all the distances in $\mathbb{F}_q$. In prime fields, the question is open in dimension 3 and higher. In two dimensions, Chapman, Erdogan, Koh, Hart and Iosevich ([4]) proved that if $|A| \geq p^{\frac{4}{3}}$, p prime, then $|\Delta(A)| \gg p$. In particular, their proof shows that if $Cp \leq |A| \leq p^{4/3}$ for a sufficiently large $C > 0$, then

$$|\Delta(A)| \gg \frac{|A|^{3/2}}{p}. \tag{2}$$

The 4/3 threshold was extended to all (not necessarily prime) fields by Bennett, Hart, Iosevich, Pakianathan and Rudnev ([2]). We refer the reader to [5, 7] for further details.

The main purpose of this paper is to improve the exponent $\frac{1}{2} + \frac{1}{30} = \frac{8}{15}$ on the magnitude of $\Delta(A)$ when A is a relatively small set in $\mathbb{F}_p^2$ with $p \equiv 3 \mod 4$. The main tools in our arguments are the energy of a set on a paraboloid due to Rudnev and Shkredov, a point-line incidence bound given by Stevens and de Zeeuw, and a lower bound on the number of distinct distances between a line and a set in $\mathbb{F}_p^2$. The following is our main result.

Theorem 1.1. *Let $\mathbb{F}_p$ be a prime field of order p with $p \equiv 3 \mod 4$. For $A \subset \mathbb{F}_p^2$ with $|A| \ll p^{\frac{7}{6}}$, we have*

$$|\Delta(A)| \gtrsim |A|^{\frac{1128}{2107}} = |A|^{\frac{1}{2}+\frac{149}{4214}}.$$

Remark 1.1. *The Stevens-de Zeeuw exponent in (1) is .533..., whereas our exponent is .535358.... Thus our result is better than that of the Stevens-de Zeeuw in the range $|A| \ll p^{7/6}$. On the other hand, our result is superior to (2) in the range $|A| \leq p^{\frac{1}{\frac{3}{2}-\frac{1128}{2107}}} = p^{\frac{4214}{4065}}$. In conclusion, Theorem 1.1 improves the currently known distance results in the range $|A| \ll p^{\frac{4214}{4065}}$.*

Remark 1.2. *While our improvement over the Steven-de Zeeuw estimate is small, we introduce a new idea, namely the count for the number of distances between a line and a set. This should lead to further improvements in the exponent in the future.*

The rest of the paper is devoted to prove Theorem 1.1, and we always assume that $p \equiv 3 \mod 4$.

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2 Proof of Theorem 1.1

To prove Theorem 1.1 we make use of the following lemmas. The first lemma is a point-line incidence bound due to Stevens and De Zeeuw in [11].

Lemma 2.1 ([11])**.** *Let P be a set of m points in $\mathbb{F}_p^2$ and L be a set of n lines in $\mathbb{F}_p^2$. Suppose that $m^{7/8} \le n \le m^{8/7}$ and $m^{-2}n^{13} \ll p^{15}$. Then we have*

$$I(P,L) = \#\{(p,\ell); p \in P, \ell \in L\} \ll m^{11/15}n^{11/15}.$$

Let P be a paraboloid in $\mathbb{F}_p^3$. For $Q \subset P$, let $E(Q)$ be the additive energy of the set Q, namely, the number of tuples $(a,b,c,d) \in Q^4$ such that $a-b=c-d$. Using Pach and Sharir's argument in [9] and Lemma 2.1, Rudnev and Shkredov [8] derived an upper bound of $E(Q)$ as follows.

Lemma 2.2 ([8])**.** *Let P be a paraboloid in $\mathbb{F}_p^3$. For $Q \subset P$ with $|Q| \ll p^{26/21}$, we have*

$$E(Q) \ll |Q|^{17/7}.$$

In the following theorem, we give a lower bound on the number of distinct distances between a set on a line and an arbitrary set in $\mathbb{F}_p^2$. This will be a crucial step in the proof of Theorem 1.1. The precise statement is as follows.

Theorem 2.3. *Let l be a line in $\mathbb{F}_p^2$, P_1 be a set of points on l, and P_2 be an arbitrary set in $\mathbb{F}_p^2$. Suppose that $|P_1|^{\frac{4}{7}} < |P_2| \ll p^{\frac{7}{6}}$. Then the number of distinct distances between P_1 and P_2, denoted by $|\Delta(P_1,P_2)|$, satisfies*

$$|\Delta(P_1,P_2)| \gtrsim \min\left\{|P_1|^{\frac{4}{11}}|P_2|^{\frac{4}{11}},\ |P_1||P_2|^{1/8}, |P_2|^{7/8}, |P_1|^{-1}|P_2|^{8/7}\right\}.$$

We will provide a detailed proof of Theorem 2.3 in Section 3. The following is a direct consequence from Theorem 2.3.

Corollary 2.4. *Let $A \subset \mathbb{F}_p^2$ with $|A| \ll p^{7/6}$. Suppose there is a line containing at least $|A|^{\frac{7}{15}+\epsilon}$ points from A. Then we have*

$$|\Delta(A)| \gtrsim \min\{|A|^{\frac{8}{15}+\frac{4\epsilon}{11}}, |A|^{\frac{8}{15}+\frac{1}{7}-\epsilon}\}.$$

The above corollary shows that the exponent 8/15 in (1) due to Stevens and De Zeeuw is improved when A contains many points on a line.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1: Let $\epsilon > 0$ be a parameter chosen at the end of the proof. Throughout the proof, we assume that that

$$\frac{16}{15} + 2\epsilon < \frac{8}{7}, \tag{3}$$

which is equivalent with $\epsilon < 4/105$. If there is a line containing at least $|A|^{7/15+\epsilon}$ points from A, then we obtain by Corollary 2.4 that

$$|\Delta(A)| \gg \min\{|A|^{\frac{8}{15}+\frac{4\epsilon}{11}}, |A|^{\frac{8}{15}+\frac{1}{7}-\epsilon}\}. \tag{4}$$

Now we assume that there is no line supporting more than $|A|^{7/15+\epsilon}$ points from A.

For any line l in $\mathbb{F}_p^2$ defined by the equation $ax + by - c = 0$, the vector (a, b, c) is called a vector of parameters of l.

We first start with counting the number of triples $(z, x, y) \in A^3$ such that $||z-x|| = ||z-y||$, where $z = (a, b)$, $x = (x_1, x_2), y = (y_1, y_2)$.

It follows from the equation $||z - x|| = ||z - y||$ that

$$(-2a)(x_1 - y_1) + (-2b)(x_2 - y_2) + (x_1^2 + x_2^2) - (y_1^2 + y_2^2) = 0.$$

This equation defines a line in $\mathbb{F}_p^2$ with the parameters

$$(x_1, x_2, x_1^2 + x_2^2) - (y_1, y_2, y_1^2 + y_2^2) = (x_1 - y_1, x_2 - y_2, x_1^2 + x_2^2 - y_1^2 - y_2^2).$$

Let L be the set of these lines. It is clear that L can be a multi-set.

Let Q be the set of points of the form $(x, y, x^2 + y^2)$ with $(x, y) \in A$. We have Q is a set on the paraboloid $z = x^2 + y^2$ and $|Q| = |A|$.

Notice that the number of triples $(z, x, y) \in A^3$ with the property $||z-x|| = ||z-y||$ is equivalent to the number of incidences between lines in L and points in $-2A := \{(-2a_1, -2a_2) : (a_1, a_2) \in A\}$.

For each line l in L, let $f(l)$ be the size of $l \cap (-2A)$, and $m(l)$ be the multiplicity of l. Let L_1 be the set of distinct lines in L.

Thus, we have

$$\begin{aligned} I(-2A, L) &= \sum_{l \in L_1} f(l)m(l) \\ &= \sum_{l \in L_1, f(l) \leq |A|^{7/15-\epsilon}} f(l)m(l) + \sum_{l \in L_1, |A|^{7/15-\epsilon} \leq f(l) \leq |A|^{7/15+\epsilon}} f(l)m(l) \\ &= I_1 + I_2. \end{aligned}$$

We now bound I_1 and I_2 as follows.

One can check that the size of L is bounded by $|A|^2$, which implies that

$$I_1 \leq |A|^{\frac{37}{15}-\epsilon}.$$

Let L_2 be the set of distinct lines l in L_1 such that $|A|^{\frac{7}{15}-\epsilon} \leq f(l) \leq |A|^{\frac{7}{15}+\epsilon}$.

To bound I_2, we consider the following two cases:

Case 1**:** Suppose

$$\sum_{l\in L_2} m(l) \leq |A|^{2-\frac{15\epsilon}{11}}.$$

We see that

$$I_2 = \sum_{l\in L_2} f(l)m(l) \leq |A|^{\frac{37}{15}-\frac{4\epsilon}{11}},$$

since any line in L_2 contains at most $|A|^{7/15+\epsilon}$ points. Thus in this case we obtain that

$$I(-2A, L) = I_1 + I_2 \leq |A|^{\frac{37}{15}-\epsilon} + |A|^{\frac{37}{15}-\frac{4\epsilon}{11}} \ll |A|^{\frac{37}{15}-\frac{4\epsilon}{11}}. \tag{5}$$

Now, for each $t \in \mathbb{F}_p$, let $\nu(t)$ denote the number of pairs $(x, y) \in A^2$ such that $\|x - y\| = t$. We have

$$\nu^2(t) = \left(\sum_{x,y\in A:\|x-y\|=t} 1\right)^2 = \left(\sum_{x\in A} 1 \times (\sum_{y\in A:\|x-y\|=t} 1)\right)^2.$$

By the Cauchy-Schwarz inequality,

$$\nu^2(t) \leq |A| \sum_{x\in A}\left(\sum_{y\in A:\|x-y\|=t} 1\right)^2 = |A| \sum_{x,y,z\in A:\|x-y\|=t=\|x-z\|} 1.$$

Summing over $t \in \mathbb{F}_p$, we obtain

$$\sum_{t\in\mathbb{F}_p} \nu^2(t) \leq |A| \sum_{x,y,z\in A:||x-y||=\|x-z\|} 1.$$

By the Cauchy-Schwarz inequality and the above inequality, we get

$$\frac{|A|^4}{|\Delta(A)|} \leq \sum_{t\in\mathbb{F}_p} \nu^2(t) \leq |A| \#\{(x, y, z) \in A^3; ||x - y|| = ||x - z||\} \ll |A| I(-2A, L).$$

Combining the above inequality with (5), we obtain

$$|\Delta(A)| \gg |A|^{\frac{8}{15}+\frac{4\epsilon}{11}}. \tag{6}$$

Case 2**:** Suppose

$$\sum_{l\in L_2} m(l) \geq |A|^{2-\frac{15\epsilon}{11}}.$$

By the Cauchy-Schwarz inequality and Theorem 2.2, we have

$$\#\{(a - b, ||a|| - ||b||) \colon a, b \in A, (a - b, ||a|| - ||b||) \text{ is a vector of parameters of a line in } L_2\} \tag{7}$$

$$\gg \frac{\left(\sum_{l\in L_2} m(l)\right)^2}{E(Q)}$$

$$\gg |A|^{\frac{11}{7}-\frac{30\epsilon}{11}}.$$

In the next step, we are going to show that

$$|L_2| \le |A|^{1+\frac{15\epsilon}{4}}.$$

Indeed, since each line in L_2 contains at least $|A|^{7/15-\epsilon}$ points, the size of L_2 is at most $|A|^{16/15+2\epsilon} \ll |A|^{8/7}$. The last inequality follows from our assumption (3). Hence, we are able to apply Theorem 2.1 so that we have

$$|A|^{\frac{7}{15}-\epsilon}|L_2| \le I(-2A, L_2) \le |A|^{11/15}|L_2|^{11/15},$$

which gives us that

$$|L_2| \ll |A|^{1+\frac{15\epsilon}{4}}. \tag{8}$$

For each line $l \in L_2$, let $m'(l)$ be the number of distinct vectors $(a-b, ||a||-||b||)$ with $(a,b) \in A^2$ such that $(a-b, ||a||-||b||)$ is a vector of parameters of l.

It follows from (7) and (8) that there exists $l \in L_2$ such that

$$m'(l) \gg |A|^{\frac{4}{7}-\frac{30\epsilon}{11}-\frac{15\epsilon}{4}}. \tag{9}$$

We now claim that $|\Delta(A)| \gg m'(l)$. Indeed, suppose that l is determined by $m'(l)$ distinct vectors $(a_1-b_1, ||a_1||-||b_1||), \ldots, (a_{m'(l)}-b_{m'(l)}, ||a_{m'(l)}||-||b_{m'(l)}||)$. Then we have

$$\begin{aligned}
&(a_2-b_2, ||a_2||-||b_2||) = \lambda_2 \cdot (a_1-b_1, ||a_1||-||b_1||),\\
&(a_3-b_3, ||a_3||-||b_3||) = \lambda_3 \cdot (a_1-b_1, ||a_1||-||b_1||),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(a_{m'(l)}-b_{m'(l)}, ||a_{m'(l)}||-||b_{m'(l)}||) = \lambda_{m'(l)} \cdot (a_1-b_1, ||a_1||-||b_1||),
\end{aligned}$$

for some $\lambda_2, \ldots, \lambda_{m'(l)} \in \mathbb{F}_p$. Since the vectors $(a_1-b_1, ||a_1||-||b_1||), \ldots,$ and $(a_{m'(l)}-b_{m'(l)}, ||a_{m'(l)}||-||b_{m'(l)}||)$ are distinct, we have $\lambda_2, \ldots, \lambda_{m'(l)}$ are distinct. On the other hand, we also have

$$||a_2-b_2|| = \lambda_2^2 \cdot ||a_1-b_1||, \ldots, ||a_{m'(l)}-b_{m'(l)}|| = \lambda_{m'(l)}^2 \cdot ||a_1-b_1||,$$

which gives us $|\Delta(A)| \ge \frac{m'(l)-1}{2}$, and the claim is proved.

Hence, it follows from the equation (9) that

$$|\Delta(A)| \gg |A|^{\frac{4}{7}-\frac{30\epsilon}{11}-\frac{15\epsilon}{4}}. \tag{10}$$

By (6) of **Case 1** and (10) of **Case 2**, it follows that if no line contains more than $|A|^{\frac{7}{15}+\epsilon}$ points in A, then

$$|\Delta(A)| \gg \min\left\{|A|^{\frac{8}{15}+\frac{4\epsilon}{11}},\ |A|^{\frac{4}{7}-\frac{30\epsilon}{11}-\frac{15\epsilon}{4}}\right\}.$$

Finally, combining this fact with (4) yields that

$$|\Delta(A)| \gg \min\left\{|A|^{\frac{8}{15}+\frac{4\epsilon}{11}},\ |A|^{\frac{8}{15}+\frac{1}{7}-\epsilon},\ |A|^{\frac{4}{7}-\frac{30\epsilon}{11}-\frac{15\epsilon}{4}}\right\}.$$

To deduce the desirable result, we consier the common solutions (ϵ, δ) to the system of the following three inequalities:

$$\frac{8}{15}+\frac{4\epsilon}{11} \ge \delta,\quad \frac{8}{15}+\frac{1}{7}-\epsilon \ge \delta,\quad \frac{4}{7}-\frac{30\epsilon}{11}-\frac{15\epsilon}{4} \ge \delta.$$

By a direct computation, we can obtain the largest $\delta = \frac{1128}{2107}$ for $\epsilon = \frac{176}{31605}$. Thus, choosing $\epsilon = \frac{176}{31605}$ gives

$$|\Delta(A)| \gg |A|^{\delta} = |A|^{\frac{1128}{2107}},$$

which completes the proof. □

3 Distances between a set on a line and an arbitrary set in $\mathbb{F}_p^2$

In this section, we will prove Theorem 2.3. We first start with an observation as follows: if

$$|\Delta(P_1, P_2)| \gg \min\left\{|P_2|^{8/7}|P_1|^{-1}, |P_2|^{7/8}\right\},$$

then we are done. So WLOG, we assume that

$$|\Delta(P_1, P_2)| \ll \min\left\{|P_2|^{8/7}|P_1|^{-1}, |P_2|^{7/8}\right\}. \tag{11}$$

Hence, to prove Theorem 2.3, it is sufficient to show that

$$|\Delta(P_1, P_2)| \gtrsim \min\left\{|P_1|^{\frac{4}{11}}|P_2|^{\frac{4}{11}},\ |P_1||P_2|^{1/8}\right\}.$$

Since the distance function is preserved under translations and rotations, we can assume that the line is vertical passing through the origin, i.e. $P_1 \subset \{0\} \times \mathbb{F}_p$. For the simplicity, we identify each point in P_1 with its second coordinate. The following lemma on a point-line incidence bound is known as a direct application of the Kővari–Sós–Turán theorem in [1].

Lemma 3.1. *Let P be a set of m points in $\mathbb{F}_p^2$ and L be a set of n lines in $\mathbb{F}_p^2$. We have*

$$I(P, L) \leq \min\left\{m^{1/2}n + m, n^{1/2}m + n\right\}.$$

For $x \in P_1$ and $P_2 \subset \mathbb{F}_p^2$, we define

$$\mathcal{E}(P_2, x) := \#\left\{((a, b), (c, d)) \in P_2^2 \colon a^2 + (b - x)^2 = c^2 + (d - x)^2\right\},$$

as the number of pairs of points in P_2 with the same distance to $x \in P_1$. In the next lemma, we will give an upper bound for $\sum_{x \in P_1} \mathcal{E}(P_2, x)$.

Lemma 3.2. *Let P_1, P_2 be sets as in Theorem 2.3. Suppose that $|P_1|^{4/7} < |P_2|$ and $|P_2| \ll p^{7/6}$. Then we have*

$$\sum_{x \in P_1} \mathcal{E}(P_2, x) \lesssim |P_1|^{7/11}|P_2|^{18/11} + |P_2|^{15/8}.$$

Proof. For $x \in P_1$ and $\lambda \in \mathbb{F}_p$, let $r_{P_2}(x, \lambda)$ be the number of points (a, b) in P_2 such that $a^2 + (b - x)^2 = \lambda$. Then we have

$$T := \sum_{x \in P_1} \mathcal{E}(P_2, x) = \sum_{(x, \lambda) \in P_1 \times \mathbb{F}_p} r_{P_2}(x, \lambda)^2.$$

Let $t = \frac{|P_2|^{7/11}}{|P_1|^{4/11}} > 1$, and let R_t be the number of pairs $(x, \lambda) \in P_1 \times \mathbb{F}_p$ such that $r_{P_2}(x, \lambda) \geq t$. We have

$$T = \sum_{(x,\lambda)\notin R_t} r_{P_2}(x, \lambda)^2 + \sum_{(x,\lambda)\in R_t} r_{P_2}(x, \lambda)^2 = I + II.$$

Since $\sum_{(x,\lambda)\notin R_t} r_{P_2}(x, \lambda) \leq |P_1||P_2|$ and $r_{P_2}(x, \lambda) < t$ for any pair $(x, \lambda) \notin R_t$, we have

$$I \leq t|P_1||P_2| = |P_1|^{7/11}|P_2|^{18/11}.$$

In the next step, we will bound II.

From the equation $\lambda = a^2 + (b - x)^2$, we have

$$a^2 + b^2 = 2bx - x^2 + \lambda.$$

Let P be the set of points $(b, a^2 + b^2)$ with $(a, b) \in P_2$, and L be the set of lines defined by $y = 2ux - u^2 + v$ with $(u, v) \in R_t$. We have $|L| = |R_t|$ and $|P| \sim |P_2|$.

With these definitions, we observe that II can be viewed as the number of pairs of points in P on lines in L.

We partition L into at most $\log(|P|)$ sets of lines L_i as follows:

$$L_i = \{l \in L \colon 2^i t \leq |l \cap P| < 2^{i+1}t\},$$

and let $II(L_i)$ denote the number of pairs of points in P on lines in L_i.

For each i, we now consider the following cases:

Case 1: $|P|^{1/2} < |L_i| \leq |P|^{7/8}$. It follows from Lemma 3.1 that

$$2^i t|L_i| \leq I(P, L_i) \leq |P|^{1/2}|L_i| + |P| \ll |P|^{1/2}|L_i|,$$

which leads to that $2^i t \leq |P|^{1/2}$. Thus

$$II(L_i) \ll |L_i| \left(|P|^{1/2}\right)^2 \ll |P|^{15/8} \sim |P_2|^{15/8}.$$

Case 2: $|P|^{7/8} \leq |L_i| \leq |P|^{8/7}$. It follows from Lemma 2.1 that

$$2^i t|L_i| \leq I(P, L_i) \leq |L_i|^{11/15}|P|^{11/15}.$$

This implies that

$$|L_i| \leq \frac{|P|^{11/4}}{(2^i t)^{15/4}}.$$

In this case, we have

$$II(L_i) \leq \frac{|P|^{11/4}}{(2^i t)^{15/4}} \cdot 2^{2i+2}t^2 \ll \frac{|P|^{11/4}}{(2^i t)^{7/4}} \sim \frac{|P_2|^{11/4}}{(2^i t)^{7/4}}.$$

One can check that the condition $m^{-2}n^{13} \ll p^{15}$ in the Theorem 2.1 is satisfied once $|P| \leq p^{7/6}$.

Case 3: $|L_i| \leq |P|^{1/2}$. Applying Lemma 3.1 again, we obtain

$$2^i t|L_i| \leq I(P, L_i) \leq |P|^{1/2}|L_i| + |P| \ll |P|. \tag{12}$$

If $2^i t \geq |P|^{7/8}$, then there is at least one line in L which has at least $|P|^{7/8}$ points from P, which follows that there exists $(x, \lambda) \in R_t$ such that the circle centered at $(0, x)$ of radius λ contains at least $|P|^{7/8} \sim |P_2|^{7/8}$ points from P_2. This implies that

$$|\Delta(P_1, P_2)| \gg |P_2|^{7/8},$$

which contradicts to our assumption (11).

Thus, we can assume that $2^i t \ll |P|^{7/8}$. With this condition, we have

$$II(L_i) \ll |L_i|(2^i t)^2 \ll 2^i t \cdot (|L_i|(2^i t)) \ll |P|^{15/8} \sim |P_2|^{15/8},$$

where we have used the inequality (12) in the last step.

Case 4**:** $|L_i| \geq |P|^{8/7}$. In this case, by the pigeon-hole principle, there is a point x in P_1 that determines at least $|P_2|^{8/7}/|P_1|$ lines, and each of these lines contains at least one point from P. This implies that

$$|\Delta(P_1, P_2)| \gg |P_2|^{8/7}|P_1|^{-1},$$

which contradicts to our assumption (11).

Putting these cases together, and taking the sum over all i, we obtain

$$T \lesssim |P_1|^{7/11}|P_2|^{18/11} + |P_2|^{15/8}.$$

This completes the proof of the lemma. □

We are ready to prove Theorem 2.3.

Proof of Theorem 2.3: As in the beginning of this section, if

$$|\Delta(P_1, P_2)| \gg \min\left\{|P_2|^{8/7}|P_1|^{-1}, |P_2|^{7/8}\right\},$$

then we are done. Thus, we might assume that

$$|\Delta(P_1, P_2)| \ll \min\left\{|P_2|^{8/7}|P_1|^{-1}, |P_2|^{7/8}\right\}.$$

Let N be the number of quadruples $(p_1, p_2, p_1', p_2') \in P_1 \times P_2 \times P_1 \times P_2$ such that

$$||p_1 - p_2|| = ||p_1' - p_2'||.$$

Let T be the number of triples $(p_1, p_2, p_2') \in P_1 \times P_2 \times P_2$ such that $||p_1 - p_2|| = ||p_1 - p_2'||$. As in the proof of Lemma 3.2, we have

$$T \lesssim |P_1|^{7/11}|P_2|^{18/11} + |P_2|^{15/8}.$$

By the Cauchy-Schwarz inequality, we have

$$N \ll |P_1|T \lesssim |P_1|^{18/11}|P_2|^{18/11} + |P_1||P_2|^{15/8}.$$

By the Cauchy-Schwarz inequality again, one can show that $\frac{|P_1|^2|P_2|^2}{|\Delta(P_1,P_2)|} \leq N$. Thus we have

$$|\Delta(P_1, P_2)| \gtrsim \min\{|P_1|^{4/11}|P_2|^{4/11},\ |P_1||P_2|^{1/8}\}.$$

This ends the proof of the theorem. □

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