ON THE CONGRUENCES OF EISENSTEIN SERIES WITH POLYNOMIAL INDEXES

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Abstract. In this paper, based on Serre's p-adic family of Eisenstein series, we prove a general family of congruences for Eisenstein series G_k in the form

$$\sum_{i=1}^{n} g_i(p)G_{f_i(p)} \equiv g_0(p) \mod p^N,$$

where $f_1(t),\dots,f_n(t) \in \mathbb{Z}[t]$ are non-constant integer polynomials with positive leading coefficients and $g_0(t),\dots,g_n(t) \in \mathbb{Q}(t)$ are rational functions. This generalizes the classical von Staudt–Clausen's and Kummer's congruences of Eisenstein series, and also yields some new congruences.

1. Introduction

1.1. **Motivation.** Let E_k ($k \geq 4$ even) be normalized Eisenstein series of weight k for the modular group $\mathrm{SL}_2(\mathbb{Z})$ given by the following q-expansion:

$$E_k = 1 - \frac{2k}{B_k}\sum_{j=1}^{\infty} \sigma_{k-1}(j)q^j,$$

where $q = e^{2\pi i\tau}$, B_k is the k-th Bernoulli number and $\sigma_{k-1}(j) = \sum_{d|j} d^{k-1}$. E_k can be regarded as formal power series in the indeterminate q. If $f, g \in \mathbb{Q}[[q]]$ are power series and N is a natural number, $f \equiv g \mod N$ means that f and g are both N-integral and the congruence holds coefficientwise [2, page 132].

In what follows, we assume that p is an odd prime.

Several well-known congruences of E_k have been given in [4, page 164, Theorem 7.1]. For example, from von Staudt–Clausen's and Kummer's congruences for Bernoulli numbers, one can easily obtain (see [2,

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Equation (1.3)])

$$E_k \equiv 1 \mod p^r \quad \text{if} \quad k \equiv 0 \mod (p-1)p^{r-1}$$

and

$$E_k \equiv E_l \mod p^r \quad \text{if} \quad k \equiv l \mod (p-1)p^r,$$

for $k, l \geq r+1$, and $(k,p), (l,p)$ are regular. The last condition means that p does not divide (the numerator of) B_k. Since this condition depends only on the residue class of $k \mod p-1$, it holds simultaneously for k and l (see [2, Equation (1.3)]). By using Serre's theory of p-adic moular forms [6] and viewing the coefficients of Eisenstein series as Iwasawa functions [2, Theorem 4.7], Gekeler [2] proved several congruences of the shape $E_{k+l} \equiv E_k \cdot E_l$ modular prime power.

In this paper, we study congruence relations of Serre's normalized Eisenstein series (see [6, page 194])

$$G_k = -\frac{B_k}{2k} + \sum_{j=1}^{\infty} \sigma_{k-1}(j) q^j, \quad k \geq 4 \text{ even}. \tag{1.1}$$

For further deductions, we make a convention that

$$G_k = 0, \quad \text{if } k \in \mathbb{Z} \text{ but } k \text{ is not even greater than } 2. \tag{1.2}$$

Let $f(t) \in \mathbb{Z}[t]$ have positive leading coefficient and satisfy $f(1) = 0$. Then, von Staudt–Clausen's congruence of Bernoulli numbers in polynomial index (see [5, Equation (1.2)]) implies that

$$2pf(p)G_{f(p)} \equiv 1 \mod p \tag{1.3}$$

for every sufficiently large prime p (note that $f(p)$ is even because $f(1) = 0$).

Besides, let $f(t)$, $g(t) \in \mathbb{Z}[t]$ be distinct non-constant polynomials with positive leading coefficient, and suppose that $f(1) = g(1) \neq 0$. Let d be the largest power of t dividing $f(t) - g(t)$. Then, by using Kummer's congruence of Bernoulli numbers in polynomial index (see [5, Equation (1.3)]), one can obtain

$$G_{f(p)} \equiv G_{g(p)} \mod p^{d+1} \tag{1.4}$$

for every sufficiently large odd prime p.

In order to generalize (1.3) and (1.4), we consider the following problem:

Question 1.1. *Given polynomials $f_1(t), \ldots, f_n(t) \in \mathbb{Z}[t]$ with positive leading coefficient, rational functions $g_0(t), g_1(t), \ldots, g_n(t) \in \mathbb{Q}(t)$, and* *a positive integer N, determine whether the congruence*

$$\sum_{i=1}^{n} g_i(p)G_{f_i(p)} \equiv g_0(p) \mod p^N$$

is true for any sufficiently large prime p.

This is inspired by a recent work of Julian Rosen [5]. He investigated a similar problem for Bernoulli numbers [5, Question 1.1], and he also obtained a very general criterion (see [5, Theorem 1.2]). The main tool is a Taylor expansion for the Kubota-Leopoldt's p-adic zeta functions (see [5, Proposition 2.1]). As pointed out in [5, page 1896], the well-known Kummer's and von Staudt–Clausen's congruences of Bernoulli numbers in polynomial index which have been given in [1, Sections 9.5 and 11.4.2] can be deduced from this criterion.

1.2. **Main results.** From now on, let N be a fixed positive integer. Let $f_1(t), \dots, f_n(t) \in \mathbb{Z}[t]$ be non-constant integer polynomials with positive leading coefficients, and let $g_0(t), \dots, g_n(t) \in \mathbb{Q}(t)$ be rational functions. Write v_t for the t-adic valuation on $\mathbb{Q}(t)$, and set

$$M = \min_{i=1,\dots,n}\{v_t(g_i(t))\}.$$

Here, we fix a convention that $v_t(0) = \infty$. For the p-adic valuation v_p, as usual we fix the convention $v_p(0) = \infty$.

We define the following four conditions:

C1:

$$v_t\Big(g_0(t) + \frac{1}{2}\big(1 - \frac{1}{t}\big) \sum_{\substack{i=1 \\ f_i(1)=0}}^{n} g_i(t)f_i(t)^{-1} + \frac{1}{2} \sum_{\substack{i=1 \\ f_i(1) \geq 4 \text{ even}}}^{n} \frac{B_{f_i(1)}}{f_i(1)}(1 - t^{f_i(1)-1})g_i(t)\Big) \geq N;$$

C2: for every even integer $l \leq 2$ and every $0 \leq m \leq N - M - 1$,

$$v_t\left(\sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(t)f_i(t)^m\right) \geq N - m;$$

C3: for every even integer $l \geq 4$ and every $1 \leq m \leq N - M - 1$,

$$v_t\left(\sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(t)(f_i(t)^m - l^m)\right) \geq N - m;$$

C4: for every even integer $l \geq 4$,

$$v_t\left(\sum_{\substack{i=1\\f_i(1)=l}}^{n} g_i(t)\right) \geq N.$$

We remark that if $N - M < 1$, then the condition **C2** automatically holds; and if $N - M \leq 1$, the condition **C3** also holds automatically. Besides, if $f_i(1) \leq 3$ for each $1 \leq i \leq n$, then the conditions **C3** and **C4** hold automatically.

In order to make our main result effective, let P be a positive integer safisfying:

- $P \geq N - M + 3$;
- $P \geq |f_i(1)| + 1$ for each $1 \leq i \leq n$;
- for each $1 \leq i \leq n$ and any integer $j > P$, $f_i(j) > \max\{3, N\}$;
- for each $1 \leq i \leq n$, write $g_i(t) = t^{d_i} h_i(t)$ with $h_i(0) \neq 0$ for some integer $d_i \geq 0$, P is not less than the numerator and denominator of $|h_i(0)| \in \mathbb{Q}$; (Under this condition, for any prime $p > P$ we have $v_p(g_i(p)) = v_t(g_i(t))$ for each $1 \leq i \leq n$; see [5, Proposition 3.1].)
- for each valuated function, say $h(t)$, in the v_t valuation in the conditions **C1**, **C2**, **C3** and **C4** (for instance, in **C4**, $h(t) = \sum_{\substack{i=1\\f_i(1)=l}}^{n} g_i(t)$), write $h(t) = t^d q(t)$ with $q(0) \neq 0$ for some integer $d \geq 0$, P is not less than the numerator and denominator of $|q(0)| \in \mathbb{Q}$. (Under this condition, for any prime $p > P$ we have $v_p(h(p)) = v_t(h(t))$ for each such function $h(t)$; see [5, Proposition 3.1].)

When the initial data $N, f_1, \ldots, f_n, g_0, \ldots, g_n$ are given, the verification of the conditions **C1**, **C2**, **C3** and **C4** is in fact a finite computation, and also it is easy to get an explicit choice for the integer P.

Our main result is the following congruence relation of Eisenstein series G_k.

Theorem 1.2. *The congruence*

$$\sum_{i=1}^{n} g_i(p) G_{f_i(p)} \equiv g_0(p) \mod p^N$$

holds for every odd prime $p > P$ if all the conditions **C1**, **C2**, **C3** *and* **C4** *hold.*

Theorem 1.2 is an analogue of [5, Theorem 1.2] and moreover in an effective manner.

The condition that all the polynomials f_i are non-constant is for simplicity. But it is not essential, that is, for each f_i that is constant, we can move the term $g_i(p)G_{f_i(p)}$ to the right hand side of the congruence in Theorem 1.2.

Remark 1.3. Using Theorem 1.2, we can directly recover the congruences (1.3) and (1.4). For proving (1.3), we choose $N = 1$, $n = 1$, $f_1(t) = f(t)$ satisfying $f(1) = 0$, $g_0(t) = 1$ and $g_1(t) = 2tf(t)$ in Theorem 1.2; while for proving (1.4), we choose $n = 2$, $f_1(t) = f(t)$, $f_2(t) = g(t)$, $g_0(t) = 0$, $g_1(t) = 1$, $g_2(t) = -1$ and $N = v_t(f - g) + 1$ in Theorem 1.2 and notice the condition $f(1) = g(1) \neq 0$.

The following corollary is a generalization of (1.4).

Corollary 1.4. *In Theorem 1.2, choose*

$$N = 1 + \min_{1 \leq i,j \leq n} v_t(f_i - f_j),$$

and assume that $f_1(1) = \ldots = f_n(1) \neq 0$, $g_0 = 0$, $g_1 + \ldots + g_n = 0$. *Then, for any odd prime* $p > P$, *we have*

$$\sum_{i=1}^{n} g_i(p)G_{f_i(p)} \equiv 0 \mod p^N.$$

The following corollary is a direct consequence of Theorem 1.2.

Corollary 1.5. *Assume that* $f_i(1) \leq 3$ *for each* $1 \leq i \leq n$. *Then, The congruence*

$$\sum_{i=1}^{n} g_i(p)G_{f_i(p)} \equiv g_0(p) \mod p^N$$

holds for every odd prime $p > P$ *if the following two conditions hold:*

(1)

$$v_t\Big(g_0(t) + \frac{1}{2}\big(1 - \frac{1}{t}\big) \sum_{\substack{i=1 \\ f_i(1)=0}}^{n} g_i(t)f_i(t)^{-1}\Big) \geq N;$$

(2) *for every even integer* $l \leq 2$ *and every* $0 \leq m \leq N - M - 1$,

$$v_t\left(\sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(t)f_i(t)^m\right) \geq N - m.$$

From Corollary 1.5, one can get some more examples about congruence of Eisenstein series.

Example 1.6. In Corollary 1.5, we choose $n = 2$, $g_0(t) = 0$, $g_1(t) = 1$, $g_2(t) = -1$ and $N = v_t(f_1 - f_2) - 1$ such that $f_1(0)f_2(0) \neq 0$, $f_1(1) = f_2(1) = 0$ and $N \geq 1$, then we have

$$G_{f_1(p)} \equiv G_{f_2(p)} \mod p^N$$

for any sufficiently large prime p.

Example 1.7. In Corollary 1.5, we choose $n \geq 2$, $f_i(t) = a_i(t-1)$ for each $1 \leq i \leq n$, $g_0(t) = \frac{1}{2}(\frac{1}{a_2} + \ldots + \frac{1}{a_n} - \frac{n-1}{a_1})$, $g_1(t) = (n-1)t$, $g_i(t) = -t$ for each $2 \leq i \leq n$, and $N = 2$, then we have for any prime $p > \max\{4, n-1\}$,

$$(n-1)pG_{a_1(p-1)} - \sum_{i=2}^{n} pG_{a_i(p-1)} \equiv \frac{1}{2}(\frac{1}{a_2} + \ldots + \frac{1}{a_n} - \frac{n-1}{a_1}) \mod p^2.$$

In particular, we have for any prime $p > 4$,

$$pG_{a_1(p-1)} - pG_{a_2(p-1)} \equiv \frac{1}{2}(\frac{1}{a_2} - \frac{1}{a_1}) \mod p^2.$$

For the proof of Theorem 1.2, the approach is similar as in [5], but it indeed needs some extra considerations in the setting of Eisenstein series. For example, we need a new ingredient, that is, a Taylor expansion for the non-constant coefficients of p-adic Eisenstein series in Proposition 3.3, which will play a key role in the proof.

Our paper will be organized as follows. In Section 2, we give a brief recall of Serre's p-adic family of Eisenstein series. In Section 3, we prove Theorem 1.2 and Corollary 1.4.

2. p-adic Eisenstein series

In this section, we recall some facts about Serre's p-adic family of Eisenstein series [6]; see also [3]. Recall that p is an odd prime.

Serre's normalized Eisenstein series has been defined in (1.1). We pass to the p-adic limit. Let $X = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$, where $\mathbb{Z}_p$ is the ring of p-adic integer. The integers $\mathbb{Z}$ are embedded into X naturally by $j \mapsto (j, j)$. For $k \in X$ and $j \geq 1$, define

$$\sigma^*_{k-1}(j) = \sum_{\substack{d \mid j \\ (p,d)=1}} d^{k-1}$$

(see [6, page 205], and see [6, page 201] for the definition of d^{k-1}). If k is even (that is, $k \in 2X$), there exists a sequence of even integers

$\{k_i\}_{i=1}^{\infty}$ such that $|k_i| \to \infty$ and $k_i \to k$ when $i \to \infty$. Then, the sequence $G_{k_i} = -\frac{B_{k_i}}{2k_i} + \sum_{j=1}^{\infty} \sigma_{k_i-1}(j)q^j$ has a limit: (see [6, page 206])

$$G_k^* = a_0(G_k^*) + \sum_{j=1}^{\infty} a_j(G_k^*)q^j \tag{2.1}$$

with $a_0(G_k^*) = \frac{1}{2}\zeta^*(1-k)$ by defining $\zeta^*(1-k) = \lim_{i\to\infty} \zeta(1-k_i)$ and $a_j(G_k^*) = \sigma_{k-1}^*(j)$, where $\zeta(s)$ is the Riemann zeta function. The function ζ^* is thus defined on the odd elements of $X \setminus \{1\}$.

Let χ be a Dirichlet character on $\mathbb{Z}_p$, and let $L_p(s,\chi)$ be the p-adic L-function. We have the following result on ζ^*.

Theorem 2.1 (see [6, page 206, Théorème 3])**.** *If $(s,u) \neq 1$ is an odd element of $X = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$, then*

$$\zeta^*(s,u) = L_p(s, \omega^{1-u}),$$

where ω is the Teichmüller character.

For $k = (s,u) \in X$ and u is even, by Theorem 2.1 the coefficients of $G_k^* = G_{s,u}^*$ are given by (see [6, page 245])

$$\begin{aligned} a_0(G_{s,u}^*) &= \frac{1}{2}\zeta^*(1-s, 1-u) = \frac{1}{2}L_p(1-s, \omega^u), \\ a_j(G_{s,u}^*) &= \sum_{\substack{d|j \\ (p,d)=1}} d^{-1}\omega(d)^u \langle d \rangle^s, \end{aligned} \tag{2.2}$$

where $\langle d \rangle = d/\omega(d) \equiv 1 \mod p$.

Thus, the assignment

$$(s,u) \mapsto G_{s,u}^*$$

gives a family of p-adic modular forms parametrized by the group of weights X.

For any even integer $k \geq 4$, we first write

$$G_k = a_0(G_k) + \sum_{j=1}^{\infty} a_j(G_k)q^j, \tag{2.3}$$

where $a_0(G_k) = -\frac{B_k}{2k}$, $a_j(G_k) = \sigma_{k-1}(j)$; and then from (2.2), we have

$$\begin{aligned} a_0(G_k^*) = a_0(G_{k,k}^*) &= \frac{1}{2}\zeta^*(1-k, 1-k) \\ &= \frac{1}{2}L_p(1-k, \omega^k) = -\frac{1-p^{k-1}}{2}\frac{B_k}{k} \\ &= (1-p^{k-1})a_0(G_k), \end{aligned} \tag{2.4}$$

where we also use the relation between L_p and Bernoulli numbers (see, for instance, the first paragraph in the proof of [5, Proposition 2.1]), and

$$a_j(G_k^*) = a_j(G_{k,k}^*) = \sum_{\substack{d|j \\ (p,d)=1}} d^{-1}\omega(d)^k \langle d\rangle^k = \sum_{\substack{d|j \\ (p,d)=1}} d^{k-1}. \tag{2.5}$$

The proof of our main result is based on the following relationship between the p-adic Eisenstein Series G_k^* and the Eisenstein series G_k:

$$G_k \equiv G_k^* \mod p^{k-1}, \qquad k \geq 4 \text{ even}, \tag{2.6}$$

which can be easily deduced from (2.1), (2.3), (2.4) and (2.5).

As in (1.2), we also make a convention that

$$G_k^* = 0, \quad \text{if } k \in \mathbb{Z} \text{ but } k \text{ is not even greater than 2.} \tag{2.7}$$

3. Proofs of the main results

Recall that p is an odd prime. For the proof, we need some preparations. The first one follows from (2.4) and [5, Proposition 2.1] directly.

Proposition 3.1. *Let l be an even residue class modulo $p-1$. Then, there exist coefficients $a_m^{(0)}(p,l) \in \mathbb{Q}_p, m = 0,1,2,\ldots,$ such that for every even integer $k \geq 4$ with $k \equiv l \pmod{p-1}$, there is a convergent p-adic series identity*

$$a_0(G_k^*) = -\frac{1-p^{k-1}}{2}\frac{B_k}{k} = -\frac{1}{2}\sum_{m=0}^{\infty} a_m^{(0)}(p,l)k^{m-1}. \tag{3.1}$$

The coefficients $a_m^{(0)}(p,l)$ satisfy the following conditions:

(1)

$$a_0^{(0)}(p,l) = \begin{cases} 1-\frac{1}{p} & \text{if } l \equiv 0 \mod p-1, \\ 0 & \text{otherwise,} \end{cases}$$

(2) *for all m, p and l,*

$$v_p(a_m^{(0)}(p,l)) \geq \frac{p-2}{p-1}m - 2,$$

(3) *for $p \geq m+2$ and all l,*

$$v_p(a_m^{(0)}(p,l)) \geq m-1.$$

Using Proposition 3.1, we obtain a congruence relation for the coefficient $a_0(G_k^*)$ in polynomial index.

Proposition 3.2. *The congruence*

$$\sum_{i=1}^{n} g_i(p)a_0(G^*_{f_i(p)}) \equiv g_0(p) \mod p^N$$

holds for every prime $p > P$ if the conditions **C1**, **C2** *and* **C3** *hold.*

Proof. We extend the proof of [5, Theorem 1.2] to our case.

Since $p > P$ and noticing the choice of P, we know that $f_i(p) \geq 4$ for each $1 \leq i \leq n$. In view of the convention (2.7), we consider the quantity

$$A^{(0)}(p) = g_0(p) - \sum_{i=1}^{n} g_i(p)a_0(G^*_{f_i(p)}).$$

By Proposition 3.1, we have

$$\begin{aligned} A^{(0)}(p) &= g_0(p) + \sum_{\substack{i=1 \\ f_i(p) \text{ even}}}^{n} g_i(p) \left(\frac{1}{2} \sum_{m=0}^{\infty} a_m^{(0)}(p, f_i(p)) f_i(p)^{m-1} \right) \\ &= g_0(p) + \sum_{\substack{h \in \mathbb{Z}/(p-1)\mathbb{Z} \\ h \text{ even},\ m \geq 0}} \sum_{\substack{i=1 \\ f_i(p) \equiv h \bmod p-1}}^{n} \frac{1}{2} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, h). \end{aligned}$$

Since $f_i(p) \equiv f_i(1) \pmod{p-1}$ for each $1 \leq i \leq n$, we have

$$\begin{aligned} A^{(0)}(p) &= g_0(p) + \sum_{\substack{\text{even } l \in \mathbb{Z} \\ m \geq 0}} \sum_{\substack{i=1 \\ f_i(1)=l}}^{n} \frac{1}{2} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l) \\ &= g_0(p) + \frac{1}{2} \sum_{\substack{l \leq 2 \text{ even} \\ l \neq 0,\, m \geq 0}} \sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l) \\ &\quad + \frac{1}{2} \sum_{m \geq 0} \sum_{\substack{i=1 \\ f_i(1)=0}}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, 0) \\ &\quad + \frac{1}{2} \sum_{\substack{l \geq 4 \text{ even} \\ m \geq 0}} \sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p, l), \end{aligned}$$

which, by Proposition 3.1 (1), becomes

(3.2)
$$A^{(0)}(p)=g_0(p)+\frac{1}{2}\sum_{\substack{l\le 2 \text{ even}\\ l\neq 0,\, m\ge 0}}\sum_{\substack{i=1\\ f_i(1)=l}}^{n} g_i(p)f_i(p)^{m-1}a_m^{(0)}(p,l)$$
$$+\frac{1}{2}\left(1-\frac{1}{p}\right)\sum_{\substack{i=1\\ f_i(1)=0}}^{n} g_i(p)f_i(p)^{-1}+\frac{1}{2}\sum_{m\ge 1}\sum_{\substack{i=1\\ f_i(1)=0}}^{n} g_i(p)f_i(p)^{m-1}a_m^{(0)}(p,0)$$
$$+\frac{1}{2}\sum_{\substack{l\ge 4 \text{ even}\\ m\ge 0}}\sum_{\substack{i=1\\ f_i(1)=l}}^{n} g_i(p)f_i(p)^{m-1}a_m^{(0)}(p,l).$$

Due to the choice of P and $p>P$, we have $p>|f_i(1)|+1$ for each $1\le i\le n$. So, for any even l satisfying $l=f_i(1)$ for some $1\le i\le n$, it can not happen that $l\equiv 0 \pmod{p-1}$, which together with Proposition 3.1 (1) implies that

$$a_0^{(0)}(p,l)=0. \tag{3.3}$$

Thus, from (2.4), (3.1) and (3.3), for any even $l\ge 4$ satisfying $l=f_i(1)$ for some $1\le i\le n$, we have

$$\begin{aligned} a_1^{(0)}(p,l)&=-2a_0(G_l^*)-\sum_{m\ge 2}a_m^{(0)}(p,l)l^{m-1}\\ &=(1-p^{l-1})\frac{B_l}{l}-\sum_{m\ge 2}a_m^{(0)}(p,l)l^{m-1}. \end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we have

(3.5)
$$A^{(0)}(p)=g_0(p)+\frac{1}{2}\sum_{\substack{l\le 2 \text{ even}\\ l\neq 0,\, m\ge 1}}\sum_{\substack{i=1\\ f_i(1)=l}}^{n} g_i(p)f_i(p)^{m-1}a_m^{(0)}(p,l)$$
$$+\frac{1}{2}\left(1-\frac{1}{p}\right)\sum_{\substack{i=1\\ f_i(1)=0}}^{n} g_i(p)f_i(p)^{-1}+\frac{1}{2}\sum_{m\ge 1}\sum_{\substack{i=1\\ f_i(1)=0}}^{n} g_i(p)f_i(p)^{m-1}a_m^{(0)}(p,0)$$
$$+\frac{1}{2}\sum_{l\ge 4 \text{ even}}\sum_{\substack{i=1\\ f_i(1)=l}}^{n} \frac{B_l}{l}(1-p^{l-1})g_i(p)$$
$$+\frac{1}{2}\sum_{\substack{l\ge 4 \text{ even}\\ m\ge 2}}\sum_{\substack{i=1\\ f_i(1)=l}}^{n} g_i(p)(f_i(p)^{m-1}-l^{m-1})a_m^{(0)}(p,l).$$

Under the condition **C1** and noticing the choices of p and P, we have

$$\begin{aligned} g_0(p) &+ \frac{1}{2}\left(1-\frac{1}{p}\right)\sum_{\substack{i=1\\ f_i(1)=0}}^{n} g_i(p)f_i(p)^{-1} \\ &+ \frac{1}{2}\sum_{l\geq 4 \text{ even}}\sum_{\substack{i=1\\ f_i(1)=l}}^{n}\frac{B_l}{l}(1-p^{l-1})g_i(p)\equiv 0 \mod p^N. \end{aligned} \tag{3.6}$$

For every even integer $l \leq 2$ satisfying $l = f_i(1)$ for some $1 \leq i \leq n$, under the condition **C2** and due to the choices of p and P, for any $1 \leq m \leq N-M$ we have

$$v_p\Big(\sum_{\substack{i=1\\ f_i(1)=l}}^{n} g_i(p)f_i(p)^{m-1}\Big) = v_p\Big(\sum_{\substack{i=1\\ f_i(1)=l}}^{n} g_i(t)f_i(t)^{m-1}\Big) \geq N-(m-1);$$

and by Proposition 3.1 (3) and noticing $p \geq N-M+2 \geq m+2$ due to the choice of P, we have

$$v_p(a_m^{(0)}(p,l)) \geq m-1;$$

and so we obtain

$$v_p\Big(\sum_{\substack{i=1\\ f_i(1)=l}}^{n} g_i(p)f_i(p)^{m-1}a_m^{(0)}(p,l)\Big) \geq N, \quad 1 \leq m \leq N-M. \tag{3.7}$$

If $m \geq N-M+1$ and $p \geq m+2$, then by Proposition 3.1 (3), we have $v_p(a_m^{(0)}(p,l)) \geq m-1$, which together with $v_p(g_i(p)) = v_t(g_i(t)) \geq M$ for each $1 \leq i \leq n$ (due to the choices of p and P) implies that for $m \geq N-M+1$ and $p \geq m+2$, for some j with $f_j(1) = l$,

$$\begin{aligned} v_p\Big(\sum_{\substack{i=1\\ f_i(1)=l}}^{n} g_i(p)f_i(p)^{m-1}a_m^{(0)}(p,l)\Big) &\geq v_p(g_j(p)) + v_p(a_m^{(0)}(p,l)) \\ &\geq M+m-1 \geq N. \end{aligned} \tag{3.8}$$

If $m \geq N-M+1$ and $p \leq m+1$, then by Proposition 3.1 (2) and noticing $v_p(g_i(p)) \geq M$ and $p > P \geq N-M+3$, we obtain

$$\begin{aligned} v_p\Big(\sum_{\substack{i=1\\ f_i(1)=l}}^{n} g_i(p)f_i(p)^{m-1}a_m^{(0)}(p,l)\Big) &\geq M + v_p(a_m^{(0)}(p,l)) \\ &\geq M + \frac{p-2}{p-1}m - 2 \geq M+p-4 \geq N. \end{aligned} \tag{3.9}$$

Thus, under the condition **C2** and combining (3.7), (3.8) with (3.9), for every even integer $l \leq 2$ satisfying $l = f_i(1)$ for some $1 \leq i \leq n$ and any $m \geq 1$, we have

$$\sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p) f_i(p)^{m-1} a_m^{(0)}(p,l) \equiv 0 \mod p^N. \tag{3.10}$$

As the above, under the condition **C3** and noticing the choices of p and P, for every even $l \geq 4$ and $m \geq 2$, we have

$$\sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p)(f_i(p)^{m-1} - l^{m-1}) a_m^{(0)}(p,l) \equiv 0 \mod p^N. \tag{3.11}$$

Finally, by (3.5), (3.6), (3.10) and (3.11) we conclude that $A^{(0)}(p) \equiv 0 \mod p^N$ for any prime $p > P$. This completes the proof. □

As an analogue of Proposition 3.1, we obtain a convergent p-adic series identity for each coefficient $a_j(G_k^*), j \geq 1$. The approach here is different from the one in [5, Proposition 2.1].

Proposition 3.3. *Let p be an odd prime, l an even residue class modulo $p-1$, and j a positive integer. Then, there exist coefficients $a_m^{(j)}(p,l) \in \mathbb{Q}_p, m = 0, 1, 2, \ldots,$ such that for every even integer $k \geq 4$ with $k \equiv l \pmod{p-1}$, there is a convergent p-adic series identity*

$$a_j(G_k^*) = \sum_{m=0}^{\infty} a_m^{(j)}(p,l) k^m.$$

The coefficients $a_m^{(j)}(p,l)$ satisfy the following conditions:

(1) *for all m, p, l,*

$$v_p(a_m^{(j)}(p,l)) \geq \frac{p-2}{p-1} m,$$

(2) *for $p \geq m+2$ and all l,*

$$v_p(a_m^{(j)}(p,l)) \geq m.$$

Proof. For $(s,u) \in X$ and u is even, by (2.2), we have

$$a_j(G_{s,u}^*) = \sum_{\substack{d \mid j \\ (p,d)=1}} d^{-1} \omega(d)^u \langle d \rangle^s. \tag{3.12}$$

Write $\langle d \rangle = 1 + pq_d$ with $q_d \in \mathbb{Z}_p$, we have

$$\langle d \rangle^s = \sum_{m=0}^{\infty} \binom{s}{m} p^m q_d^m.$$

Substituting the above into (3.12), we have

$$\begin{aligned} a_j(G^*_{s,u}) &= \sum_{\substack{d\mid j\\(p,d)=1}} d^{-1}\omega(d)^u \sum_{m=0}^{\infty}\binom{s}{m}p^m q_d^m \\ &= \sum_{m=0}^{\infty}\binom{s}{m}p^m \sum_{\substack{d\mid j\\(p,d)=1}} q_d^m d^{-1}\omega(d)^u. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} a_j(G^*_k) = a_j(G^*_{k,k}) &= \sum_{m=0}^{\infty}\binom{k}{m}p^m \sum_{\substack{d\mid j\\(p,d)=1}} q_d^m d^{-1}\omega(d)^k \\ &= \sum_{m=0}^{\infty}\binom{k}{m}p^m \sum_{\substack{d\mid j\\(p,d)=1}} q_d^m d^{-1}\omega(d)^l, \end{aligned} \tag{3.13}$$

where the last equality comes from the fact that $\omega(a)^k = \omega(a)^l$ if $k \equiv l \pmod{p-1}$.

For each $1 \le m \le k$, we have

$$\begin{aligned} \binom{k}{m} &= \frac{k(k-1)\cdots(k-m+1)}{m!} \\ &= \frac{1}{m!}(k^m + b_{m,m-1}k^{m-1} + \cdots + b_{m,1}k) \end{aligned} \tag{3.14}$$

for some integers $b_{m,1},\ldots,b_{m,m-1} \in \mathbb{Z}$ dependings only on m.

Substituting (3.14) into (3.13), we obtain

$$a_j(G^*_k) = \sum_{m=0}^{\infty} a_m^{(j)}(p,l)k^m$$

for some $a_m^{(j)}(p,l) \in \mathbb{Q}_p$ satisfying

$$\begin{aligned} v_p(a_m^{(j)}(p,l)) &\ge \min\{v_p(p^m/m!), v_p(p^{m+1}/(m+1)!),\ldots\} \\ &\ge m - \frac{m}{p-1}, \end{aligned}$$

where the number of terms in the min function is finite and the last inequality follows from the fact that $v_p(m!) \le m/(p-1)$. This gives the conclusion (1) of the proposition. The conclusion (2) (in the case $p \ge m+2$) follows from (1) directly by noticing $v_p(a_m^{(j)}(p,l)) \in \mathbb{Z}$. □

Applying Proposition 3.3, we can also obtain a congruence relation for the coefficient $a_j(G^*_k)$ in polynomial index.

Proposition 3.4. *For any integer $j \geq 1$, the congruence*

$$\sum_{i=1}^{n} g_i(p) a_j(G^*_{f_i(p)}) \equiv 0 \mod p^N$$

holds for every prime $p > P$ if the conditions **C2**, **C3** *and* **C4** *hold.*

Proof. We apply the same strategy as in the proof of Proposition 3.2.

Since $p > P$ and noticing the choice of P, we know that $f_i(p) \geq 4$ for each $1 \leq i \leq n$. In view of the convention (2.7), we consider the quantity

$$A^{(j)}(p) = \sum_{i=1}^{n} g_i(p) a_j(G^*_{f_i(p)}).$$

By Proposition 3.3, we have

$$\begin{aligned} A^{(j)}(p) &= \sum_{\substack{i=1 \\ f_i(p) \text{ even}}}^{n} g_i(p) \sum_{m=0}^{\infty} a_m^{(j)}(p, f_i(p)) f_i(p)^m \\ &= \sum_{\substack{h \in \mathbb{Z}/(p-1)\mathbb{Z} \\ h \text{ even}, \ m \geq 0}} \sum_{\substack{i=1 \\ f_i(p) \equiv h \text{ mod } p-1}}^{n} g_i(p) f_i(p)^m a_m^{(j)}(p, h). \end{aligned}$$

Since $f_i(p) \equiv f_i(1) \mod p-1$, we have

$$\begin{aligned} A^{(j)}(p) &= \sum_{\substack{\text{even } l \in \mathbb{Z} \\ m \geq 0}} \sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p) f_i(p)^m a_m^{(j)}(p, l) \\ &= \sum_{\substack{l \leq 2 \text{ even} \\ m \geq 0}} \sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p) f_i(p)^m a_m^{(j)}(p, l) \qquad (3.15) \\ &\quad + \sum_{\substack{l \geq 4 \text{ even} \\ m \geq 0}} \sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p) f_i(p)^m a_m^{(j)}(p, l). \end{aligned}$$

For any even integer $l \geq 4$ satisfying $l = f_i(1)$ for some $1 \leq i \leq n$, by Proposition 3.3 we have

$$a_0^{(j)}(p, l) = a_j(G^*_l) - \sum_{m=1}^{\infty} a_m^{(j)}(p, l) l^m.$$

Substituting the above equation into (3.15), we have

$$A^{(j)}(p) = \sum_{\substack{l\le 2 \text{ even} \\ m\ge 0}} \sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p) f_i(p)^m a_m^{(j)}(p,l)$$
$$+ \sum_{l\ge 4 \text{ even}} a_j(G_l^*) \sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p) \tag{3.16}$$
$$+ \sum_{\substack{l\ge 4 \text{ even} \\ m\ge 1}} \sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p)(f_i(p)^m - l^m) a_m^{(j)}(p,l).$$

As in the proof of Proposition 3.2, under the condition **C2** and the choices of p and P and using Proposition 3.3, for every even integer $l \le 2$ and $m \ge 0$, we obtain

$$\sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p) f_i(p)^m a_m^{(j)}(p,l) \equiv 0 \mod p^N. \tag{3.17}$$

Similarly, under the condition **C3**, for every even integer $l \ge 4$ and $m \ge 1$, we have

$$\sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p)(f_i(p)^m - l^m) a_m^{(j)}(p,l) \equiv 0 \mod p^N. \tag{3.18}$$

Also, under the condition **C4** and noticing $a_j(G_l^*) \in \mathbb{Z}_p$ by (2.5), for every even integer $l \ge 4$ we have

$$a_j(G_l^*) \sum_{\substack{i=1 \\ f_i(1)=l}}^{n} g_i(p) \equiv 0 \mod p^N. \tag{3.19}$$

Finally, by (3.16), (3.17), (3.18) and (3.19), we conclude that $A^{(j)}(p) \equiv 0 \mod p^N$ for any prime $p > P$. This completes the proof. □

We are now at the point to prove Theorem 1.2.

Proof of Theorem 1.2. Since $p > P$ and noticing the choice of P, we have $f_i(p) > N$ for each $1 \le i \le n$. Thus, by (2.6), for any $1 \le i \le n$ with even $f_i(p)$ we have

$$G_{f_i(p)} \equiv G^*_{f_i(p)} \mod p^N. \tag{3.20}$$

Otherwise if $f_i(p)$ is odd, then by the conventions (1.2) and (2.7), we have $G_{f_i(p)} = G^*_{f_i(p)} = 0$, and so (3.20) still holds. On the other hand, by Propositions 3.2 and 3.4, we directly obtain

$$\sum_{i=1}^{n} g_i(p) G^*_{f_i(p)} \equiv g_0(p) \mod p^N \tag{3.21}$$

for every prime $p > P$ if all the conditions **C1**, **C2**, **C3** and **C4** hold. The desired result now follows from (3.20) and (3.21). □

Finally, we prove Corollary 1.4.

Proof of Corollary 1.4. First, by assumption, it is easy to see that the conditions **C1** and **C4** hold.

Since $f_1(1) = \ldots = f_n(1)$ and $g_1 + \ldots + g_n = 0$, for verifying the conditions **C2** and **C3**, it suffices to show that for any $m \geq 1$,

$$v_t\Big(\sum_{i=1}^{n} g_i(t) f_i(t)^m\Big) \geq N - 1.$$

Now, we first prove the case $m = 1$. Let $d = \min_{1\leq i,j\leq n} v_t(f_i - f_j)$. Then, $N = d + 1$. The case $d = 0$ is trivial. Assume $d \geq 1$ and write $F(t) = \sum_{i=1}^{n} g_i(t) f_i(t)$. To prove $v_t(F) \geq d$, it suffices to show that $F^{(k)}(0) = 0$ for any $0 \leq k \leq d-1$, where $F^{(k)}$ denotes the k-th derivative of F. Note that

$$F^{(k)}(t) = \sum_{j=0}^{k} \binom{k}{j} \sum_{i=1}^{n} g_i^{(j)}(t) f_i^{(k-j)}(t).$$

Since $0 \leq k \leq d-1$, by definition we have $f_1^{(k-j)}(0) = \ldots = f_n^{(k-j)}(0)$, and so

$$\sum_{i=1}^{n} g_i^{(j)}(0) f_i^{(k-j)}(0) = f_1^{(k-j)}(0) \sum_{i=1}^{n} g_i^{(j)}(0) = 0,$$

where we use the assumption $g_1 + \ldots + g_n = 0$. Hence, we have $F^{(k)}(0) = 0$ for any $0 \leq k \leq d-1$. This completes the proof of the case $m = 1$.

For $m \geq 2$, we have

$$\min_{1\leq i,j\leq n} v_t(f_i^m - f_j^m) \geq \min_{1\leq i,j\leq n} v_t(f_i - f_j) = d.$$

Hence, applying the same argument as the above, we obtain

$$v_t\Big(\sum_{i=1}^{n} g_i(t) f_i(t)^m\Big) \geq N - 1.$$

The desired result now follows. □

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