CATEGORICAL SMOOTH COMPACTIFICATIONS AND GENERALIZED HODGE-TO-DE RHAM DEGENERATION

ALEXANDER I. EFIMOV

ABSTRACT. We disprove two (unpublished) conjectures of Kontsevich which state generalized versions of categorical Hodge-to-de Rham degeneration for smooth and for proper DG categories (but not smooth and proper, in which case degeneration is proved by Kaledin [Ka]). In particular, we show that there exists a minimal 10-dimensional A_{∞} -algebra over a field of characteristic zero, for which the supertrace of μ_3 on the second argument is non-zero.

As a byproduct, we obtain an example of a homotopically finitely presented DG category (over a field of characteristic zero) that does not have a smooth categorical compactification, giving a negative answer to a question of Toën. This can be interpreted as a lack of resolution of singularities in the noncommutative setup.

We also obtain an example of a proper DG category which does not admit a categorical resolution of singularities in the terminology of [KL] (that is, it cannot be embedded into a smooth and proper DG category).

Contents

0. Introduction	2
1. Preliminaries on DG categories and A_{∞} -algebras	3
1.1. DG categories	3
1.2. A_{∞} -algebras and A_{∞} -(bi)modules	5
2. Preliminaries on the Hochschild complex, pairings and copairings	9
3. A counterexample to the generalized degeneration conjecture	12
4. A counterexample to Conjecture 0.3	13
5. A counterexample to Conjecture 0.2	15
References	20

The author is partially supported by Laboratory of Mirror Symmetry NRU HSE, RF government grant, ag. N 14.641.31.0001.

0. Introduction

Given a smooth algebraic variety X over a field of characteristic zero, we have the Hodge-to-de Rham spectral sequence $E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow H_{DR}^{p+q}(X)$. It is classically known that when X is additionally proper, this spectral sequence degenerates at E_1 , that is, all differentials vanish. This follows from the classical Hodge theory for compact Kähler manifolds, and can be also proved algebraically [DI].

We recall the following fundamental result of Kaledin [Ka], see also [M] for a different proof.

Theorem 0.1. [Ka, Theorem 5.4] Let A be a smooth and proper DG algebra. Then the Hochschild-to-cyclic spectral sequence degenerates, so that we have an isomorphism $HP_{\bullet}(A) = HH_{\bullet}(A)((u))$.

Here u denotes a variable of degree 2.

When applied to $\operatorname{Perf}(A) \simeq \operatorname{Perf}(X)$ for smooth and proper variety X, Theorem 0.1 gives exactly the classical Hodge-to-de Rham degeneration.

In this paper we study some generalizations of Hodge-to-de-Rham degeneration to DG categories which are not smooth and proper.

Recall that for a proper DG algebra B one has a pairing on $HH_{\bullet}(B) \otimes HH_{\bullet}(B^{op}) \rightarrow$ k, introduced by Shklyarov [S]. Kontsevich [Ko] proposed the following generalization of Theorem 0.1.

Conjecture 0.2. Let B be a proper DG algebra. Then the composition map

$$(0.1) (HH_{\bullet}(B) \otimes HC_{\bullet}(B^{op}))[1] \xrightarrow{\mathrm{id} \otimes \delta} HH_{\bullet}(B) \otimes HH_{\bullet}(B^{op}) \to \mathsf{k}$$

is zero.

Kontsevich also proposed a "dual" version of Conjecture 0.2 for smooth DG algebras.

Conjecture 0.3. Let A be a smooth DG algebra. Then the composition

$$K_0(A \otimes A^{op}) \xrightarrow{\operatorname{ch}} (HH_{\bullet}(A) \otimes HH_{\bullet}(A^{op})_0 \xrightarrow{\operatorname{id} \otimes \delta} (HH_{\bullet}(A) \otimes HC_{\bullet}^-(A^{op}))_1$$

vanishes on the class [A] of the diagonal bimodule.

The motivation for Conjectures 0.2 and 0.3 is explained in Propositions 4.1 and 5.1 below. Here we mention that the results of [KL] imply that Conjecture 0.2 holds for proper DG algebras of algebra-geometric origin: that is, for DG algebras of the form $B = \mathbf{R} \operatorname{End}(\mathcal{F})$, where $\mathcal{F} \in \operatorname{Perf}_Z(X)$ is a perfect complex on a separated scheme X of finite type over k, supported on a *proper* closed subscheme $Z \subset X$. Similarly, the (weak version of) results of

[E2] imply that Conjecture 0.3 holds for smooth DG algebras of the form $\mathbf{R} \operatorname{End}(\mathcal{G})$, where $\mathcal{G} \in D^b_{coh}(X)$ is a generator of the category $D^b_{coh}(X)$.

There is a closely related question formulated by B. Toën [To1].

Question 0.4. Is it true that any homotopically finitely presented DG category \mathcal{B} is quasi-equivalent to a quotient \mathcal{A}/\mathcal{S} , where \mathcal{A} is smooth and proper, and $\mathcal{S} \subset \mathcal{A}$ is a full subcategory?

Such a quotient presentation of \mathcal{B} is called a smooth categorical compactification.

In this paper we disprove both Conjectures 0.2 and 0.3. As an application, we give a negative answer to Question 0.4.

The starting point for our counterexamples is to disprove the main conjecture of [E1], see Section 3. A counterexample to Conjecture 0.3 is obtained in Section 4. It is deduced from the results of Section 3 by some trick.

Finally, a counterexample to Conjecture 0.2 is obtained in Section 5. It is deduced from our new result on nilpotent elements in the cohomology of a DG algebra (Theorem 5.3), which is of independent interest. In particular, we obtain an example of a proper DG algebra B such that the DG category Perf(B) cannot be fully faithfully embedded into a saturated DG category. That is, it does not have a categorical resolution of singularities in the terminology of [KL].

Section 5 can be read independently from Sections 3 and 4.

Acknowledgements. I am grateful to Dmitry Kaledin, Maxim Kontsevich and Bertrand Toën for useful discussions.

1. Preliminaries on DG categories and A_{∞} -algebras

1.1. **DG** categories. For the introduction on DG categories, we refer the reader to [Ke1]. The references for DG quotients are [Dr, Ke2]. For the model structures on DG categories we refer to [Tab1, Tab2], and for a general introduction on model categories we refer to [Ho]. Everything will be considered over some base field k.

Mostly we will consider DG categories up to a quasi-equivalence. By a functor between DG categories we sometimes mean a quasi-functor. In some cases it is convenient for us to choose a concrete DG model or a concrete DG functor. By a commutative diagram of functors we usually mean the commutative diagram in the homotopy category $Ho(dgcat_k)$. Finally, we denote by $Ho_M(dgcat_k)$ the Morita homotopy category of DG categories (with inverted Morita equivalences).

All modules are assumed to be right unless otherwise stated. For a small DG category \mathcal{C} and a \mathcal{C} -module M, we denote by M^{\vee} the \mathcal{C}^{op} -module $\operatorname{Hom}_{\mathcal{C}}(M,\mathcal{C})$. We denote by M^* the \mathcal{C}^{op} -module $\operatorname{Hom}_{\mathcal{C}}(M,\mathbb{k})$.

Given a small DG category \mathcal{C} , we denote by $D(\mathcal{C})$ its derived category of DG \mathcal{C} modules. This is a compactly generated triangulated category. We denote by $D_{\mathrm{perf}}(\mathcal{C})$ the
full triangulated subcategory of perfect \mathcal{C} -modules. It coincides with the subcategory of
compact objects.

Recall from [TV] that a \mathcal{C} -module M is pseudo-perfect if for each $x \in \mathcal{C}$, the complex M(x) is perfect over k (that is, M(x) has finite-dimensional total cohomology). We denote by $D_{\text{pspe}}(\mathcal{C}) \subset D(\mathcal{C})$ the full triangulated subcategory of pseudo-perfect \mathcal{C} -modules.

For any DG category \mathcal{C} , we denote by $[\mathcal{C}]$ its (non-graded) homotopy category, which has the same objects as \mathcal{C} , and the morphisms are given by $[\mathcal{C}](x,y) = H^0(\mathcal{C}(x,y))$. We use the terminology of [TV, Definition 2.4] by calling \mathcal{C} triangulated if the Yoneda embedding provides an equivalence $[\mathcal{C}] \xrightarrow{\sim} D_{perf}(\mathcal{C})$. In this case $[\mathcal{C}]$ is a Karoubi complete triangulated category.

We denote by $\operatorname{Mod}_{\mathcal{C}}$ the DG category of cofibrant DG \mathcal{C} -modules in the projective model structure (these are exactly the direct summands of semifree DG \mathcal{C} -modules). We have $D(\mathcal{C}) \simeq [\operatorname{Mod}_{\mathcal{C}}]$, where $D(\mathcal{C})$ is the derived category of DG \mathcal{C} -modules. We denote by $\mathbf{Y} : \mathcal{C} \hookrightarrow \operatorname{Mod}_{\mathcal{C}}$ the standard Yoneda embedding given by $\mathbf{Y}(x) = \mathcal{C}(-, x)$.

We write $\operatorname{Perf}(\mathcal{C}) \subset \operatorname{Mod}_{\mathcal{C}}$ (resp. $\operatorname{PsPerf}(\mathcal{C}) \subset \operatorname{Mod}_{\mathcal{C}}$) for the full DG subcategory of perfect (resp. pseudo-perfect) \mathcal{C} -modules.

For a DG functor $\Phi: \mathcal{C}_1 \to \mathcal{C}_2$ between small DG categories, we denote by $\mathbf{L}\Phi^*: D(\mathcal{C}_1) \to D(\mathcal{C}_2)$, the derived extension of scalars functor. Its right adjoint functor (restriction of scalars) is denoted by $\Phi_*: D(\mathcal{C}_2) \to D(\mathcal{C}_1)$.

We also recall from [To2, Definitions 3.6] that a \mathcal{C} -module is called quasi-representable if it is quasi-isomorphic to a representable \mathcal{C} -module. For two DG categories $\mathcal{C}, \mathcal{C}'$, a $\mathcal{C} \otimes \mathcal{C}'$ -module M is called right quasi-representable if for each object $x \in \mathcal{C}$, the \mathcal{C}' -module M(x, -) is quasi-representable.

We denote by $\mathbf{R}\underline{\mathrm{Hom}}(\mathcal{C},\mathcal{C}') \subset \mathrm{Mod}_{\mathcal{C}^{op}\otimes\mathcal{C}'}$ the full subcategory of right quasi-representable $\mathcal{C}^{op}\otimes\mathcal{C}'$ -modules. By [To2, Theorem 6.1], this DG category (considered up to a quasi-equivalence) is actually the internal Hom in the homotopy category of DG categories $\mathrm{Ho}(\mathrm{dgcat}_k)$ (with inverted quasi-equivalences). We have a natural quasi-functor $\mathrm{Fun}(\mathcal{C},\mathcal{C}') \to \mathbf{R}\underline{Hom}(\mathcal{C},\mathcal{C}')$, where $\mathrm{Fun}(\mathcal{C},\mathcal{C}')$ is the naive DG category of DG functors $\mathcal{C} \to \mathcal{C}'$, as defined in [Ke1]. Moreover, if \mathcal{C} is cofibrant, this functor is essentially surjective on the homotopy categories.

A small DG category \mathcal{C} is called smooth (resp. locally proper) if the diagonal $\mathcal{C}\text{-}\mathcal{C}$ -bimodule is perfect (resp. pseudo-perfect). Moreover, \mathcal{C} is called proper if it is locally proper and is Morita equivalent to a DG algebra (i.e. the triangulated category $D_{\text{perf}}(\mathcal{C})$ has a classical generator).

We recall the notion of a short exact sequence of DG categories.

Definition 1.1. A pair of functors $A_1 \xrightarrow{F_1} A_2 \xrightarrow{F_3} A_3$ is said to be a (Morita) short exact sequence of DG categories if the following conditions hold

- i) the composition F_2F_1 is homotopic to zero;
- ii) the functor F_1 is quasi-fully-faithful;
- iii) the induced quasi-functor $\overline{F_2}: A_2/F_1(A_1) \to A_3$ is a Morita equivalence.

In particular, a short exact sequence of DG categories induces a long exact sequence of K-groups, where $K_{\bullet}(\mathcal{A})$ is the Waldhausen K-theory [W] of the Waldhausen category of cofibrant perfect \mathcal{A} -modules. We will in fact need only the boundary map $K_1(\mathcal{A}_3) \to K_0(\mathcal{A}_1)$.

1.2. A_{∞} -algebras and A_{∞} -(bi)modules. All the definitions and constructions regarding DG categories which are invariant under quasi-equivalences can be translated into the world of A_{∞} -categories. For the introduction on A_{∞} -categories and algebras see [L-H, Ke3, KS].

It will be sufficient for us to work with A_{∞} -algebras (that is, A_{∞} -categories with a single object).

In order to write down the signs in formulas it is convenient to adopt the following **Notation.** For a collection of homogeneous elements a_0, \ldots, a_n of a graded vector space A, and $0 \le p, q \le n$, we put

$$l_p^q(a) = \begin{cases} |a_p| + \dots + |a_q| + q - p + 1 & \text{if } p \le q; \\ |a_p| + \dots + |a_n| + |a_0| + \dots + |a_q| + n - p + q & \text{if } p > q. \end{cases}$$

If the collection starts with a_1 (and there is no a_0) we only use $l_p^q(a)$ for $1 \le p \le q \le n$.

Definition 1.2. A non-unital A_{∞} -structure on a graded vector space A is a sequence of multilinear operations $\mu_n = \mu_n^A : A^{\otimes n} \to A$, where $\deg(\mu_n) = 2-n$, satisfying the following relations:

(1.1)
$$\sum_{i+j+k=n+1} (-1)^{l_1^i(a)} \mu_{i+k+1}(a_1, \dots, a_i, \mu_j(a_{i+1}, \dots, a_{i+j}), a_{i+j+1}, \dots, a_n) = 0,$$

for $n \ge 0$. Here for $1 \le p \le q \le n$ we put $l_p^q(a) := |a_p| + \dots + |a_q| + q - p + 1$.

Remark 1.3. In our sign convention, a non-unital DG algebra B can be considered as an A_{∞} -algebra, with $\mu_1(a) = -d(a)$, $\mu_2(a_1, a_2) = (-1)^{|a_1|} a_1 a_2$, and $\mu_{\geq 3} = 0$.

Definition 1.4. A non-unital A_{∞} -morphism $f: A \to B$ is given by a sequence of linear maps $f_n: A^{\otimes n} \to B$, where $\deg(f_n) = 1 - n$, satisfying the following relations:

(1.2)
$$\sum_{i_1+\dots+i_k=n} \mu_k^B(f_{i_1}(a_1,\dots,a_{i_1}),\dots,f_{i_k}(a_{i_1+\dots+i_{k-1}+1},\dots,a_n)) = \sum_{i_1+i_2+k=n} (-1)^{l_1^i(a)} f_{i_1+k_1}(a_1,\dots,a_i,\mu_j^A(a_{i_1+1},\dots,a_{i_2}),a_{i_1+j_2+1},\dots,a_n).$$

Given an A_{∞} -algebra A, one defines the A_{∞} -algebra A^{op} as follows: it is equal to A as a graded vector space, and we have

$$\mu_n^{A^{op}}(a_1,\ldots,a_n) = (-1)^{\sigma} \mu_n^A(a_n,\ldots,a_1),$$

where
$$\sigma = \sum_{1 \le i < j \le n} (|a_i| + 1)(|a_j| + 1).$$

We now define the notion of an A_{∞} -module.

Definition 1.5. A right A_{∞} -module M over an A_{∞} -algebra A is a graded vector space with a sequence of operations $\mu_n^M: M \otimes A^{\otimes n-1} \to M$, where n > 0, $\deg(\mu_n^M) = 2 - n$, and the following relations are satisfied:

(1.3)
$$\sum_{i+j=n} \mu_{j+1}^{M}(\mu_{i+1}^{M}(m, a_1, \dots, a_i), a_{i+1}, \dots, a_n) +$$

$$\sum_{i+j+k=n+1} (-1)^{|m|+l_1^i(a)} \mu_{i+k+1}(m, a_1, \dots, a_i, \mu_j(a_{i+1}, \dots, a_{i+j}), a_{i+j+1}, \dots, a_n) = 0.$$

We also need A_{∞} -bimodules.

Definition 1.6. Let A and B be non-unital A_{∞} -algebras. An A_{∞} A-B-bimodule M is a graded vector space with a collection of operations $\mu_{i,j} = \mu_{i,j}^M : A^{\otimes i} \otimes M \otimes B^{\otimes j} \to M$, where $i, j \geq 0$, such that for any $n, m \geq 0$ and homogeneous $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_m \in B$, $m \in M$, the following relation is satisfied:

$$\sum_{i+j+k=n+1} (-1)^{l_1^i(a)} \mu_{i+k+1,m}^M(a_1, \dots, \mu_j^A(a_{i+1}, \dots, a_{i+j}), \dots, a_n, m, b_1, \dots, b_m)$$

$$+ \sum_{\substack{1 \le i \le n+1; \\ 0 \le j \le m}} \mu_{i-1,m-j}^M(a_1, \dots, a_{i-1}, \mu_{n+1-i,j}^M(a_i, \dots, a_n, m, b_1, \dots, b_j), b_{j+1}, \dots, b_m)$$

$$+ \sum_{\substack{i+j+k=m+1}} (-1)^{l_1^n(a)+l_1^i(b)+|m|} \mu_{n,i+k+1}^M(a_1, \dots, a_n, m, b_1, \dots, \mu_j^B(b_{i+1}, \dots, b_{i+j}), \dots, b_m) = 0.$$

Remark 1.7. 1) In our sign convention, a non-unital DG algebra B can be considered as an A_{∞} -algebra, with $\mu_1(b) = -d(b)$, $\mu_2(b_1, b_2) = (-1)^{|b_1|}b_1b_2$, and $\mu_{>3} = 0$.

2) If furthermore M is a right DG B-module, then the A_{∞} B-module structure on M is given by $\mu_1^M(m) = d(m)$, $\mu_2^M(m,a) = (-1)^{|m|+1}ma$, and $\mu_{>3}^M = 0$.

3) If A is another non-unital DG algebra, and M is a DG A-B-bimodule, then the A_{∞} A-B-bimodule structure on M is given by $\mu_{0,0}^M(m) = d(m)$, $\mu_{1,0}^M(a,m) = am$, $\mu_{0,1}^M(m,b) = (-1)^{|m|+1}mb$, and $\mu_{i,j}^M = 0$ for $i+j \geq 2$.

We now recall the strict unitality.

Definition 1.8. 1) A non-unital A_{∞} -algebra A is called strictly unital if there is a (unique) element $1 = 1_A \in A$ such that $\mu_1(1) = 0$, $\mu_2(1,a) = a = (-1)^{|a|}\mu_2(a,1)$ for any homogeneous element $a \in A$, and for $n \geq 3$ we have $\mu_n(a_1, \ldots, a_n) = 0$ if at least one of the arguments a_i equals 1.

- 2) A non-unital A_{∞} -morphism $f: A \to B$ between strictly unital A_{∞} -algebras is called strictly unital if $f_1(1_A) = 1_B$, and for $n \ge 2$ we have $f_n(a_1, \ldots, a_n) = 0$ if at least one of the arguments a_i equals 1.
- 3) Given a strictly unital A_{∞} -algebra A, an A_{∞} A-module M is called strictly unital if $\mu_2^M(m,1) = (-1)^{|m|+1}m$, and for $n \geq 3$ we have $\mu_n^M(m,a_1,\ldots,a_{n-1}) = 0$ if at least one of a_i 's equals 1.
- 4) Given strictly unital A_{∞} -algebras A, B, an A_{∞} A-B-bimodule is called strictly unital if $\mu_{1,0}^M(1_A,m)=m$, $\mu_{0,1}^M(m,1_B)=(-1)^{|m|+1}m$, and for $k+l\geq 2$ we have $\mu_{k,l}(a_1,\ldots,a_k,m,b_1,\ldots,b_l)=0$ if at least one of a_i 's equals 1_A or at least one of b_j 's equals 1_B .

From now on, all A_{∞} -algebras and (bi)modules will be strictly unital

Given a strictly unital A_{∞} -algebra A, we define the DG category $\operatorname{Mod}^{\infty}$ -A whose objects are A_{∞} -modules and the morphisms are defined as follows. Given $M, N \in \operatorname{Mod}^{\infty}$ -A, we put

$$\operatorname{Hom}_A^{\infty}(M,N)^{gr} := \prod_{n>0} \operatorname{Hom}_{\mathbf{k}}(M \otimes A[1]^{\otimes n}, N),$$

and the differential is given by

$$d(\varphi)_n(m, a_1, \dots, a_n) = \sum_{i=0}^n \mu_{n-i+1}^N(\varphi_i(m, a_1, \dots, a_i), a_{i+1}, \dots, a_n)$$

$$- \sum_{i=0}^n (-1)^{|\varphi|} \varphi_{n-i+1}(\mu_{i+1}^M(m, a_1, \dots, a_i), a_{i+1}, \dots, a_n)$$

$$- \sum_{1 \le i \le j \le n} (-1)^{|\varphi| + |m| + l_1^{i-1}(a)} \varphi_{n+i-j-1}(m, a_1, \dots, \mu_{j-i+1}^A(a_i, \dots, a_j), \dots, a_n).$$

The composition is given by

$$(\varphi \psi)_n(m, a_1, \dots, a_n) = \sum_{i=0}^n \varphi_{n-i}(\psi_i(m, a_1, \dots, a_i), a_{i+1}, \dots, a_n).$$

Given a unital DG algebra B, we denote by $\operatorname{PsPerf}(B) \subset \operatorname{Mod}^{\infty} - B$ the full DG subcategory formed by pseudo-perfect DG modules. We have $[\operatorname{PsPerf}(B)] \simeq D_{\operatorname{perf}}(B)$.

Remark 1.9. Let A, B be A_{∞} -algebras.

- 1) An A_{∞} A-B-bimodule structure on a graded vector space M is equivalent to the following data:
 - the right A_{∞} B-module structure on M;
 - the A_{∞} -morphism $f: A \to \operatorname{End}_{B}^{\infty}(M)$.

Namely, given an A_{∞} -bimodule M, the induced B-module structure is given by $\mu_n^M = \mu_{0,n-1}^M$, and the A_{∞} -morphism is given by $f_n(a_1,\ldots,a_n)(m,b_1,\ldots,b_l) = \mu_{n,l}(a_1,\ldots,a_n,m,b_1,\ldots,b_l)$.

2) Also, an A_{∞} -bimodule structure is equivalent to an

We finally define a technically useful notion of an an A_{∞} -bimorphism of (strictly unital) A_{∞} -algebras $f:(A,B)\to C$. It is given by the linear maps $f_{r,s}:A^{\otimes r}\otimes B^{\otimes s}\to C$, where $r,s\geq 0,\ r+s>0$, so that the following relations are satisfied:

$$(1.4) \sum_{\substack{0=r_0 \leq r_1 \leq \cdots \leq r_k = r; \\ 0=s_0 \leq s_1 \leq \cdots \leq s_k = s}} (-1)^{\sigma} \mu_k^C(f_{r_1,s_1}(a_1, \dots, a_{r_1}; b_1, \dots, b_{s_1}), \dots, \\ f_{r-r_{k-1},s-s_{k-1}}(a_{r_{k-1}+1}, \dots, a_r; b_{s_{k-1}+1}, \dots, b_s)) = \\ \sum_{i+j+k=r} (-1)^{l_1^i(a)} f_{i+k+1,s}(a_1, \dots, a_i, \mu_j(a_{i+1}, \dots, a_{i+j}), a_{i+j+1}, \dots, a_r; b_1, \dots, b_s) + \\ \sum_{i+j+k=s} (-1)^{l_1^r(a)+l_1^i(b)} f_{r,i+k+1}(a_1, \dots, a_r; b_1, \dots, b_i, \mu_j(b_{i+1}, \dots, b_{i+j}), b_{i+j+1}, \dots, b_s),$$

where $\sigma = \sum_{1 \le p < q \le k} l_{r_{q-1}+1}^{r_q}(a) l_{s_{p-1}+1}^{s_p}(b)$. We require that $f_{1,0}(1_A) = 1_C = f_{0,1}(1_B)$, and for $k+l \ge 2$ $f_{k,l}(a_1, \ldots, a_k, b_1, \ldots, b_l) = 0$ if at least one of a_i 's equals 1_A , or at least one of b_j 's equals 1_B .

Remark 1.10. One can similarly define A_{∞} n-morphisms $(A_1, \ldots, A_n) \to B$, so that the category of A_{∞} -algebras becomes a (non-symmetric) pseudo-monoidal category. In particular, the A_{∞} -morphisms can be composed with A_{∞} -morphisms in the natural way.

Remark 1.11. If a graded vector space M is given a differential d, then an A_{∞} -bimodule structure on M (with $\mu_1^M = d$) is equivalent to an A_{∞} -bimorphism $f: (A, B^{op}) \to \operatorname{End}_k(M)$. Given such an A_{∞} -bimorphism, one puts

$$\mu_{r,s}^M(a_1, \dots, a_r, m, b_1, \dots, b_s) := (-1)^l f_{r,s}(a_1, \dots, a_r, b_s, \dots, b_1)(m),$$
where $l = l_1^s(b) \cdot |m| + \sum_{1 \le p < q \le s} (|b_p| + 1)(|b_q| + 1).$

The diagonal A_{∞} A-A-bimodule is given by A as a graded vector space, and we have

$$\mu_{i,j}(a_1,\ldots,a_i,b,c_1,\ldots,c_j) = (-1)^{l_1^i(a)+1} \mu_{i+j+1}^A(a_1,\ldots,a_i,b,c_1,\ldots,c_j).$$

Finally, we mention the gluing of A_{∞} -algebras. Let M be an A_{∞} A-B-bimodule. We denote by $\begin{pmatrix} B & 0 \\ M & A \end{pmatrix}$ the A_{∞} -algebra C which equals $A \oplus B \oplus M$ as a graded vector space, so that the non-zero components of μ_n^C are given by μ_n^A , μ_n^B , and

$$(-1)^{l_1^i(a)+1}\mu_{i,j}(a_1,\ldots,a_i,m,b_1,\ldots,b_j), \quad i+j+1=n,$$

where $a_1, \ldots, a_i \in A, b_1, \ldots, b_i \in B$.

2. Preliminaries on the Hochschild complex, pairings and copairings

In this section all A_{∞} -algebras are strictly unital. For an A_{∞} -algebra A, we put $\overline{A} := A/\mathbf{k} \cdot \mathbf{1}_A$.

The mixed Hochschild complex (see [Ke2, KS]) $(C_{\bullet}(A), b, B)$ of an A_{∞} -algebra A is given as a graded vector space by

$$C_{\bullet}(A) := \bigoplus_{n>0} A \otimes (\overline{A}[1])^{\otimes n}.$$

For convenience we write (a_0, \ldots, a_n) instead of $a_0 \otimes \cdots \otimes a_n \in C_{\bullet}(A)$.

The Hochschild differential is given by

$$(2.1) \quad b(a_0, \dots, a_n) = \sum_{0 \le i \le j \le n} (-1)^{l_0^{i-1}(a)+1}(a_0, \dots, \mu_{j-i+1}(a_i, \dots, a_j), \dots, a_n) + \sum_{0 \le p < q \le n} (-1)^{l_0^{q-1}(a)l_q^n(a)+1}(\mu_{n+p+2-q}(a_q, \dots, a_n, a_0, \dots, a_p), a_{p+1}, \dots, a_{q-1}).$$

The Connes-Tsygan differential B (see [Co, FT, Ts]) is given by

$$B(a_0, a_1, \dots, a_n) = \sum_{0 \le i \le n} (-1)^{l_0^{i-1}(a)l_i^n(a)+1} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1}).$$

The Hochschild complex can be more generally defined for A_{∞} -categories, and is Morita invariant [KS]. We refer to [KS] for the definition of cyclic homology HC_{\bullet} , negative cyclic homology HC_{\bullet}^- and HP_{\bullet} . In this paper we will in fact deal only with the first differential of the Hochschild-to-cyclic spectral sequence, which is the map $B: HH_n(A) \to HH_{n+1}(A)$ induced by the Connes-Tsygan differential.

We recall the natural pairings and co-pairings on $HH_{\bullet}(A)$. Let us restrict ourselves to DG algebras for a moment. Given a DG algebra A, we have a Chern character ch: $K_n(A) \to HH_n(A)$ (see [CT]; the Chern character naturally lifts to $HC^-(A)$), but we will not need this).

In particular, given DG algebras A, B and an object $M \in D_{perf}(A \otimes B)$, we have a copairing

$$\operatorname{ch}(M) \in (HH_{\bullet}(A) \otimes HH_{\bullet}(B))_0 \cong HH_0(A \otimes B).$$

This copairing is used in the formulation of Conjecture 0.3 for $A = B^{op}$ being smooth, and M = A.

Dually [S], if we have DG algebras A and B, and an object $M \in D_{pspe}(A^{op} \otimes B^{op})$, then we have a pairing (of degree zero)

$$HH_{\bullet}(A) \otimes HH_{\bullet}(B) \to HH_{\bullet}(A \otimes B) \to HH_{\bullet}(\operatorname{End}_{\mathbf{k}}(M)) \to \mathbf{k}$$

(the last map is an isomorphism if and only if M is not acyclic). In the formulation of Conjecture 0.2 this pairing is used for $A = B^{op}$ proper, and M = A. In this case we denote the pairing by $\langle \cdot, \cdot \rangle$.

We would like to obtain an explicit formula for the pairing in the A_{∞} -setting. The reader who is not interested in (or is already familiar with) the details can skip to Corollary 2.3 which is essentially all we need.

Let A, B, C be A_{∞} -algebras. Suppose that we are given an A_{∞} -bimorphism $f: (A, B) \to C$. We would like to define an explicit map of complexes

$$f_*: C_{\bullet}(A) \otimes C_{\bullet}(B) \to C_{\bullet}(C).$$

It is given by

$$(2.2) \quad f_*((a_0, \dots, a_n) \otimes (b_0, \dots, b_m)) = \sum_{\substack{0 \le i_0 \le \dots \le i_k \le n; \\ 0 \le j_0 \le \dots \le j_k \le m; \\ 0 \le p < q \le k}} (-1)^{\varepsilon(i_0, \dots, i_k, j_1, \dots, j_k, p, q)} (\mu_{k+p+2-q}(f_{i_{q+1}-i_q, j_q-j_{q-1}}(a_{i_q+1}, \dots, a_{i_{q+1}}, b_{j_{q-1}+1}, \dots, b_{j_q}),$$

$$\dots, f_{i_{p+1}-i_p, j_p-j_{p-1}}(a_{i_p+1}, \dots, a_{i_{p+1}}, b_{j_{p-1}+1}, \dots, b_{j_p})),$$

$$f_{i_{p+2}-i_{p+1}, j_{p+1}-j_p}(a_{i_{p+1}+1}, \dots, a_{i_{p+2}}, b_{j_p+1}, \dots, b_{j_{p+1}}), \dots,$$

$$f_{i_q-i_{q-1}, j_{q-1}-j_{q-2}}(a_{i_{q-1}+1}, \dots, a_{i_q}, b_{j_{q-2}+1}, \dots, b_{j_{q-1}})),$$

where

$$\varepsilon(i_0, \dots, i_k, j_1, \dots, j_k, p, q) = l_0^m(a) + l_{i_q+1}^n(a) l_0^{i_q}(a) + l_{j_{q-1}+1}^m(b) l_0^{j_{q-1}}(b) + 1 + \sum_{s=1}^k l_{i_{q+s}+1}^{i_{q+s}+1}(a) l_{j_{q-1}+1}^{j_{q+s}-1}(b).$$

In this summation we mean that $i_{s+k+1} = i_s$, $j_{s+k+1} = j_s$, $a_{s+n+1} = a_s$, $b_{s+m+1} = b_s$. Also, we require that for all s = 1, ..., k-1 we have $(i_{s+1} - i_s) + (j_s - j_{s-1}) > 0$, so that we don't get the (non-existing) $f_{0,0}$ anywhere.

Remark 2.1. Suppose that we are in the special situation when A, B and C are DG algebras, and the A_{∞} -bimorphism f has only two non-zero components $f_{1,0}$ and $f_{0,1}$. This is equivalent to a DG algebra morphism $A \otimes B \to C$, which we still denote by f.

The map given by (2.2) is obtained by composing the map $C_{\bullet}(A \otimes B) \to C_{\bullet}(C)$ with the Eilenberg-Zilber map $EZ : C_{\bullet}(A) \otimes C_{\bullet}(B) \to C_{\bullet}(A \otimes B)$.

Proposition 2.2. Let A_1 and A_2 be strictly unital A_{∞} -algebras, and M a finite dimensional strictly unital A_{∞} A_1 - A_2 -bimodule (we require that $\dim \oplus_n \dim(M^n) < \infty$). Then the composition map

$$\psi: HH_{\bullet}(A_1) \otimes HH_{\bullet}(A_2^{op}) \xrightarrow{\operatorname{id} \otimes B} HH_{\bullet}(A_1) \otimes HH_{\bullet}(A_2^{op}) \to HH_{\bullet}(\operatorname{End}(V)) \to \ker(A_1) \otimes HH_{\bullet}(A_2^{op})$$

is given by the following explicit formula:

$$\psi((a_0, \dots, a_n) \otimes (b_0, \dots, b_m)) = \operatorname{str}_M(m \mapsto (-1)^{l_0^m(b) \cdot |m|} \sum_{\substack{0 \le i \le n; \\ 0 \le j \le m}} (-1)^{\sigma_{i,j}} \mu_{n+1,m+1}(a_i, \dots, a_k, \dots, a_{i-1}, m, b_j, \dots, b_0, b_l, \dots, b_{j+1}),$$

where

$$\sigma_{i,j} = l_0^n(a) + l_0^{i-1}(a)l_i^n(a) + \sum_{0 \leq p < q \leq j} (|b_p| + 1)(|b_q| + 1) + \sum_{j+1 \leq p < q \leq m} (|b_p| + 1)(|b_q| + 1).$$

Proof. Recall that for a finite-dimensional complex V the natural map $HH_{\bullet}(\operatorname{End}_{\mathbf{k}}(V)) \to \mathbf{k}$ (which is an isomorphism if and only if M is not acyclic) is given by the following morphism of complexes $C_{\bullet}(\operatorname{End}_{\mathbf{k}}(V)) \to \mathbf{k}$:

$$(a_0, \dots, a_k) \mapsto \begin{cases} \operatorname{str}_M(a_0) & \text{for } k = 0, |a_0| = 0; \\ 0 & \text{otherwise.} \end{cases}$$

The result follows by applying the formula (2.2) and Remark 1.11 (and taking the strict unitality into account).

Finally, we mention one particular corollary which we need in this paper.

Corollary 2.3. Let A be a finite-dimensional non-unital A_{∞} -algebra, and $a, b \in A$ are closed homogeneous elements such that |a| + |b| = 1. If we consider a and b as classes in $HH_{\bullet}(A)$ and $HH_{\bullet}(A^{op})$ respectively. Then

$$\langle a, B(b) \rangle = (-1)^{|a|+1} \operatorname{str}_A(v \mapsto (-1)^{(|b|+1) \cdot |v|} \mu_3(a, v, b)).$$

Proof. This follows immediately from Proposition 2.2.

3. A COUNTEREXAMPLE TO THE GENERALIZED DEGENERATION CONJECTURE

We recall the main conjecture of [E1].

Conjecture 3.1. [E1, Conjecture 1.3 for n = 0] Let \mathcal{B} and \mathcal{C} be small DG categories over a field k of characteristic zero. Then the composition map

$$(3.1) \varphi_0: K_0(\mathcal{B} \otimes \mathcal{C}) \xrightarrow{\operatorname{ch}} (HH_{\bullet}(\mathcal{B}) \otimes HH_{\bullet}(\mathcal{C}))_0 \xrightarrow{\operatorname{id} \otimes \delta} (HH_{\bullet}(\mathcal{B}) \otimes HC_{\bullet}^{-}(\mathcal{C}))_1$$
is zero.

In this section we construct a counterexample to Conjecture 3.1. We put $\Lambda_1 = k\langle \xi \rangle/\xi^2$, where $|\xi| = 1$, and (automatically) $d\xi = 0$. We have a quasi-equivalence $\operatorname{Perf}(\Lambda_1) \simeq \operatorname{Perf}_{\{0\}}(\mathbb{A}^1_k)$ (the free Λ_1 -module of rank 1 corresponds to the skyscraper sheaf \mathcal{O}_0). In particular, we have a short exact sequence

$$(3.2) 0 \to \operatorname{Perf}(\Lambda_1) \to \operatorname{Perf}(\mathbb{A}^1) \to \operatorname{Perf}(\mathbb{G}_m) \to 0$$

We also denote by $k[\varepsilon] := k[t]/t^2$ the algebra of dual numbers ($|\varepsilon| = 0$, $d\varepsilon = 0$). Let us denote by x the coordinate on \mathbb{A}^1 , and put $T := \operatorname{Spec}(k[\varepsilon])$. Tensoring (3.2) by $k[\varepsilon]$ (and taking perfect complexes), we obtain another short exact sequence:

$$(3.3) 0 \to \operatorname{Perf}(\Lambda_1 \otimes \mathbf{k}[\epsilon]) \to \operatorname{Perf}(\mathbb{A}^1 \times T) \to \operatorname{Perf}(\mathbb{G}_m \times T) \to 0.$$

Now let us take the Cartier divisor $D := \{x + \varepsilon = 0\} \subset \mathbb{A}^1 \times T$. This is well-defined since $x + \varepsilon$ is not a zero divisor in $k[x] \otimes k[\varepsilon]$. Moreover, we have $D \cap (\mathbb{G}_m \times T) = \emptyset$, since $x + \varepsilon$ is invertible in $k[x^{\pm 1}] \otimes k[\varepsilon]$: we have $(x + \varepsilon)(x^{-1} - x^{-2}\varepsilon) = 1$. Therefore, by (3.3), we may and will consider \mathcal{O}_D as an object of $\operatorname{Perf}(\Lambda_1 \otimes k[\varepsilon])$.

Theorem 3.2. Conjecture 3.1 does not hold for the DG algebras Λ_1 and $k[\varepsilon]$. Namely, we have $\varphi_0([\mathcal{O}_D]) \neq 0$, where φ_0 is defined in (3.1).

Proof. We will prove a stronger statement: $\psi_0([\mathcal{O}_D]) \neq 0$, where ψ_0 is the composition

$$K_0(\Lambda_1 \otimes \mathbf{k}[\varepsilon]) \xrightarrow{\mathrm{ch}} (HH_{\bullet}(\Lambda_1) \otimes HH_{\bullet}(\mathbf{k}[\varepsilon]))_0 \xrightarrow{\mathrm{id} \otimes B} (HH_{\bullet}(\Lambda_1) \otimes HH_{\bullet}(\mathbf{k}[\varepsilon]))_1.$$

We use the notation d_{dR} for the de Rham differential in order to avoid confusion with differentials in DG algebras.

First let us identify the Hochschild homology of Λ_1 . Applying the long exact sequence in Hochschild homology to (3.2), we see that

$$HH_{-1}(\Lambda_1) = k[x^{\pm 1}]/k[x]$$
, and $HH_0(\Lambda_1) = k[x^{\pm 1}]d_{dR}x/k[x]d_{dR}x$,

and
$$HH_i(\Lambda_1) = 0$$
 for $i \notin \{-1, 0\}$.

Further, for any commutative k-algebra R we have $HH_0(R)=R$, and $HH_1(R)=\Omega^1_{R/k}$, (and the Connes differential $B:HH_0(R)\to HH_1(R)$ is given by the de Rham

differential). In particular, we have $HH_0(k[\varepsilon]) = k[\varepsilon]$, and $HH_1(k[\varepsilon]) = k \cdot d_{dR}\varepsilon$ (and we do not need $HH_{>2}(k[\varepsilon])$ for our considerations).

Claim. Within the above notation, we have $\operatorname{ch}(\mathcal{O}_D) = \frac{d_{dR}x}{x} \otimes 1 - \frac{d_{dR}x}{x^2} \otimes \varepsilon + \frac{1}{x} \otimes d_{dR}\varepsilon$.

Proof. As we already mentioned, the function $x + \varepsilon$ is invertible on $\mathbb{G}_m \times T$, hence it gives an element $\alpha \in K_1(\mathbb{G}_m \times T)$. Moreover, the boundary map

$$K_1(\mathbb{G}_m \times T) \to K_0(\Lambda_1 \otimes \mathbf{k}[\varepsilon])$$

sends α to $[\mathcal{O}_D]$. We have $\operatorname{ch}(\alpha) = d_{dR} \log(x + \varepsilon) \in \Omega^1_{\mathbb{G}_m \times T} = HH_1(\mathbb{G}_m \times T)$. Explicitly, we have

$$d_{dR}\log(x+\varepsilon) = (x^{-1} - x^{-2}\varepsilon)d_{dR}(x+\varepsilon) = \frac{d_{dR}x}{x} - \frac{\varepsilon d_{dR}x}{x^2} + \frac{d_{dR}\varepsilon}{x}.$$

Applying the boundary map $HH_1(\mathbb{G}_m \times T) \to HH_0(\Lambda_1 \otimes \mathbf{k}[\varepsilon])$, we obtain the desired formula for $\mathrm{ch}(\mathcal{O}_D)$.

It follows from Claim that

$$(\mathrm{id} \otimes B)(\mathrm{ch}([\mathcal{O}_D])) = -\frac{d_{dR}x}{x^2} \otimes d_{dR}\varepsilon \neq 0.$$

This proves the theorem.

4. A COUNTEREXAMPLE TO CONJECTURE 0.3

In this section we disprove Conjecture 0.3.

Proposition 4.1. Let B be a smooth DG algebra and $F : Perf(A) \to Perf(B)$ a localization functor, where A is a smooth and proper DG algebra. Then Conjecture 0.2 holds for B.

Proof. This is actually explained in [E1, proof of Theorem 4.6]. We explain the argument for completeness. The localization assumption implies that $(F \otimes F^{op})^*(I_A) = I_B$. In particular, the map $HH_{\bullet}(A) \otimes HH_{\bullet}(A^{op}) \to HH_{\bullet}(B) \otimes HH_{\bullet}(B^{op})$ takes $\operatorname{ch}(I_A)$ to $\operatorname{ch}(I_B)$. It remains to apply the commutative diagram

$$HH_{\bullet}(A) \otimes HH_{\bullet}(A^{op}) \xrightarrow{\operatorname{id} \otimes \delta} HH_{\bullet}(A) \otimes HC_{\bullet}^{-}(A^{op})[-1]$$

$$\downarrow \qquad \qquad \downarrow$$

$$HH_{\bullet}(B) \otimes HH_{\bullet}(B^{op}) \xrightarrow{\operatorname{id} \otimes \delta} HH_{\bullet}(B) \otimes HC_{\bullet}^{-}(B^{op})[1],$$

and Theorem 0.1 applied to A.

We have the following corollary, mentioned in the introduction.

Corollary 4.2. Let X be a separated scheme of finite type over k, and $\mathcal{G} \in D^b_{coh}(X)$ – a generator. Then Conjecture 0.3 holds for the smooth DG algebra $A = \mathbf{R} \operatorname{End}(\mathcal{G})$.

Proof. Indeed, by [E2, Theorem 1.8 1)], there is a localization functor of the form $D^b_{coh}(Y) \to D^b_{coh}(X)$, where Y is a smooth projective algebraic variety over k. The result follows by Proposition 4.1. Note that here we don't even need to apply Theorem 0.1 since we only use the classical Hodge-to-de Rham degeneration for Y.

Remark 4.3. In fact, in the formulation of Proposition 4.1 we could weaken the assumption on the functor F to be a localization, requiring it only to be a homological epimorphism, which means that the functor $D(A) \to D(B)$ a localization, see [E2, Section 3]. Then in the proof of Corollary 5.2 we can apply the corresponding weakened version of [E2, Theorem 1.8 1)] which is much easier to prove.

Clearly, Conjecture 0.3 is a special case of Conjecture 3.1. On the other hand, it was proved in [E1] that Conjectures 3.1 and 0.3 are actually equivalent (more precisely, this follows from the proof of [E1, Theorem 4.6]). However, deducing an explicit counterexample to Conjecture 0.3 along the lines of [E1] would require some computations, which we wish to avoid. Instead, we use some trick.

Let us take some elliptic curve E over k, with a k-rational point $p \in E(k)$. Choosing a local parameter $x \in \mathcal{O}_{E,p}$, we get an identification $\operatorname{Perf}(\Lambda_1) \simeq \operatorname{Perf}_{\{p\}}(E) \subset \operatorname{Perf}(E)$. Let us choose some generator $\mathcal{F} \in \operatorname{Perf}(E)$ (e.g. $\mathcal{F} = \mathcal{O}_E \oplus \mathcal{O}_p$), and put $B_E = \mathbf{R} \operatorname{End}(\mathcal{F})$, so that $\operatorname{Perf}(B_E) \simeq \operatorname{Perf}(E)$. We denote by $F : \operatorname{Perf}(\Lambda_1) \hookrightarrow \operatorname{Perf}(B_E)$ the resulting embedding.

Further, we denote by C the semi-free DG algebra $k\langle t_1, t_2 \rangle$, with $|t_1| = 0$, $|t_2| = -1$, $dt_1 = 0$, and $dt_2 = t_1^2$.

We take the object $M \in \operatorname{Perf}(\Lambda_1 \otimes C \otimes C)$ whose image in $\operatorname{Perf}(\Bbbk[x] \otimes C \otimes C)$ is given by

$$Cone(\mathbf{k}[x] \otimes C^{\otimes 2} \xrightarrow{x \otimes 1^{\otimes 2} + 1 \otimes t_1^{\otimes 2}} \mathbf{k}[x] \otimes C^{\otimes 2}).$$

As in the previous section, we see that M is well-defined since the element

$$x \otimes 1^{\otimes 2} + 1 \otimes t_1^{\otimes 2} \in H^0(\mathbf{k}[x^{\pm 1}] \otimes C \otimes C) = \mathbf{k}[x^{\pm 1}] \otimes \mathbf{k}[\varepsilon] \otimes \mathbf{k}[\varepsilon]$$

is invertible.

Finally, we put $N := (F \otimes \mathrm{id}_C^{\otimes 2})^*(M) \in \mathrm{Perf}(B_E \otimes C \otimes C)$.

Theorem 4.4. 1) Within the above notation, the dg algebra

$$A := \begin{pmatrix} B_E \otimes C & 0 \\ N & C^{op} \end{pmatrix}$$

is homotopically finitely presented (hence smooth), but it does not satisfy Conjecture 0.3.

2) The DG category Perf(A) gives a negative answer to Question 0.4.

Proof. First, by Proposition 4.1 we see that 2) reduces to 1).

We now prove 1). The homotopy finiteness of A follows from [E2, Proposition 5.15] (gluing of homotopically finite DG algebras by a perfect bimodule is again homotopically finite).

The functor $F: \operatorname{Perf}(\Lambda_1) \to \operatorname{Perf}(B_E) \simeq \operatorname{Perf}(E)$ induces a map HH_F in Hochschild homology. We need the following values of HH_F . First, the morphism $HH_F: HH_0(\Lambda_1) \to$ $HH_0(E) = H^0(\mathcal{O}_E) \oplus H^1(\omega_E) \cong \mathbb{R} \oplus \mathbb{R}$ is given by

$$\frac{d_{dR}x}{x^n} \mapsto \begin{cases} (0,1) & \text{for } n=1; \\ 0 & \text{for } n>1. \end{cases}$$

Further, the morphism $HH_F: HH_{-1}(\Lambda_1) \to HH_{-1}(E) = H^1(\mathcal{O}_E)$ does not vanish on x^{-1} (because there is no rational function on E having single simple pole at p). We denote the image $HH_F(x^{-1})$ by $[x^{-1}]$.

To prove 1), it suffices to show that $(\mathrm{id} \otimes \mathrm{id} \otimes B)(\mathrm{ch}(N)) \in (HH_{\bullet}(\Lambda_1) \otimes HH_{\bullet}(C)^{\otimes 2})_1$ is non-zero. We have a natural projection $\pi: C \to H^0(C) \cong \mathrm{k}[\varepsilon]$. Let us put $\bar{N}:=(\mathrm{id} \otimes \pi^* \otimes \pi^*)(N) \in \mathrm{Perf}(E \times T \times T)$. Then \bar{N} is naturally isomorphic to $\mathcal{O}_{D'}$, where $D' \subset E \times T \times T$ is a Cartier divisor, set-theoretically contained in $\{p\} \times T \times T$, and given locally by the equation $x \otimes 1^{\otimes 2} + 1 \otimes \varepsilon^{\otimes 2} = 0$. The computation from Section 3 implies that

$$\operatorname{ch}(\bar{N}) = (0,1) \otimes 1^{\otimes 2} + [x^{-1}] \otimes d_{dR} \varepsilon \otimes \varepsilon + [x^{-1}] \otimes \varepsilon \otimes d_{dR} \varepsilon.$$

Therefore, we obtain

$$(\mathrm{id} \otimes \mathrm{id} \otimes B)(\mathrm{ch}(\bar{N})) = [x^{-1}] \otimes d_{dR} \varepsilon \otimes d_{dR} \varepsilon \neq 0.$$

By functoriality, this implies $(id \otimes id \otimes B)(ch(N)) \neq 0$. This proves 1).

5. A COUNTEREXAMPLE TO CONJECTURE 0.2

In this section we disprove Conjecture 0.2.

More precisely, we will construct an example of a minimal finite-dimensional A_{∞} -algebra B and two elements $a, b \in B$, such that |a| + |b| = 1, and

$$\operatorname{str}_B(v \mapsto (-1^{(|b|+1)|v|})\mu_3(a,v,b)) \neq 0,$$

thus disproving Conjecture 0.2 (by Corollary 2.3).

We first mention the following observation, which in fact motivates Conjecture 0.2.

Proposition 5.1. Let B be a proper DG algebra and $Perf(B) \hookrightarrow Perf(A)$ a quasi-fully-faithful functor, where A is a smooth and proper DG algebra. Then Conjecture 0.2 holds for B.

Proof. Indeed this follows from the commutative diagram

$$HH_{\bullet}(B)\otimes HC_{\bullet}(B^{op})[1] \xrightarrow{\operatorname{id}\otimes\delta} HH_{\bullet}(B)\otimes HH_{\bullet}(B^{op}) \longrightarrow k;$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \operatorname{id} \downarrow$$

$$HH_{\bullet}(A)\otimes HC_{\bullet}(A^{op})[1] \xrightarrow{\operatorname{id}\otimes\delta} HH_{\bullet}(A)\otimes HH_{\bullet}(A^{op}) \longrightarrow k$$
and Theorem 0.1 applied to A .

We have the following corollary, mentioned in the introduction.

Corollary 5.2. Let X be a separated scheme of finite type over k, and $Z \subset X$ a closed proper subscheme. For any object $\mathcal{F} \in \operatorname{Perf}_Z(X)$, Conjecture 0.2 holds for the proper DG algebra $B = \mathbf{R} \operatorname{End}(\mathcal{F})$.

Proof. Choosing some compactification $X \subset \bar{X}$ (which exist by Nagata's compactification theorem [N]), we get $\operatorname{Perf}_Z(X) \simeq \operatorname{Perf}(\bar{X})$. Thus, we may and will assume $X = \bar{X} = Z$. Then the result follows by applying Proposition 5.1 with [KL][Theorem 6.12]. As in the proof of Corollary 5.2, we only use here the classical Hodge-to-de Rham degeneration. \square

The crucial point is the following theorem, which is of independent interest.

Theorem 5.3. 1) Let A be a DG algebra, and $a \in H^0(A)$ a nilpotent element. Then the corresponding morphism $f : k[x] \to A$ (where |x| = 0) in $Ho(dgalg_k)$ factors through $k[x]/x^n$ for a sufficiently large n.

2) If moreover $a^2 = 0$ in $H^0(A)$, then it suffices to take n = 6.

Before we prove Theorem 5.3, we show how it allows to construct a counterexample to Conjecture 0.2.

Theorem 5.4. 1) Let us denote by y the variable of degree 1. Then there exists an A_{∞} $k[y]/y^3-k[x]/x^6$ -bimodule structure on the 1-dimensional vector space $V=k\cdot z$ (where |z|=0) such that $\mu_3^V(x,z,y)=z$. In particular, in the glued A_{∞} -algebra

$$B = \begin{pmatrix} k[y]/y^3 & 0\\ V & k[x]/x^6 \end{pmatrix}$$

we have $\operatorname{str}(v \mapsto \mu_3(x, v, y)) = 1$. Therefore, by Corollary 2.3 this A_{∞} -algebra (and any quasi-isomorphic DG algebra) does not satisfy Conjecture 0.2.

2) In particular, the proper DG category $\operatorname{Perf}^{\infty}(B)$ does not have a categorical resolution of singularities.

Proof. 1) An easy computation shows that $\operatorname{Ext}_{k[y]/y^3}^0(\mathbf{k},\mathbf{k}) = k[\varepsilon]$ (dual numbers). By Theorem 5.3 2), we have an A_{∞} -morphism $g: k[x]/x^6 \to \operatorname{End}_{k[y]/y^3}^{A_{\infty}}(\mathbf{k})$, such that $\overline{g_1(x)} = \frac{1}{2} \operatorname{End}_{k[y]/y^3}^{A_{\infty}}(\mathbf{k})$

 $\varepsilon \in H^0(\operatorname{End}_{k[y]/y^3}^{A_{\infty}}(k))$. This gives the desired A_{∞} -bimodule structure on V. The rest conclusions are clear.

2) follows from 1) and Proposition 5.1.
$$\Box$$

Proof of Theorem 5.3, part 1). Let us denote by A_f the k[x]-A-bimodule which equals A as an A-module, and whose k[x]-module structure comes from f. Since the algebra k[x] is smooth, we have $A_f \in D_{\mathrm{perf}}(k[x] \otimes A)$. Since $a \in H^0(A)$ is nilpotent, we have $k[x^{\pm 1}] \otimes_{k[x]}^{\mathbf{L}} A = 0$. We conclude that A_f is contained in the essential image of $D_{\mathrm{perf}}(\Lambda_1 \otimes A) \hookrightarrow D_{\mathrm{perf}}(k[x] \otimes A)$.

Now, let us note that in $\operatorname{Ho}(\operatorname{dgcat}_k)$ we have $\operatorname{Perf}(\Lambda_1) \simeq \operatorname{colim}_n \operatorname{PsPerf}(k[x]/x^n)$. It follows that we have an equivalence of triangulated categories

$$D_{\mathrm{perf}}(\Lambda_1 \otimes A) \simeq \operatorname*{colim}_n D_{\mathrm{perf}}(\mathrm{PsPerf}(\mathtt{k}[x]/x^n) \otimes A).$$

Therefore, there exists n > 0 such that A_f is contained in the essential image of $D_{\mathrm{perf}}(\mathrm{PsPerf}(\Bbbk[x]/x^n) \otimes A)$. Let us denote by $\tilde{M} \in D_{\mathrm{perf}}(\mathrm{PsPerf}(\Bbbk[x]/x^n) \otimes A)$ an object whose image is isomorphic to A_f . We have a natural functor

$$\Phi: \operatorname{PsPerf}(\Bbbk[x]/x^n) \otimes \operatorname{Perf}(A) \to \mathbf{R}\underline{\operatorname{Hom}}(\Bbbk[x]/x^n, \operatorname{Perf}(A)).$$

By construction, the $k[x]/x^n$ -A-bimodule $\Phi(\tilde{M})$ is quasi-isomorphic to A as an A-module Choosing an isomorphism $\Phi(\tilde{M})_{|A} \xrightarrow{\sim} A$, we obtain the following composition morphism in $\operatorname{Ho}(\operatorname{dgalg}_k)$:

$$g: \mathbf{k}[x]/x^n \to \mathbf{R} \operatorname{End}_A(\Phi(\tilde{M})) \xrightarrow{\sim} A.$$

By construction, $H^0(g)(x) = a$. Thus, g factors f through $k[x]/x^n$. This proves part 1)

The proof of part 2) of Theorem 5.3 requires some computations which we split into several lemmas.

First, we may replace the abstract algebra A by the concrete DG algebra C which was used in Section 4. Recall that it is freely generated by the elements t_1 , t_2 with $|t_1| = 0$, $|t_2| = -1$, and $dt_1 = 0$, $dt_2 = t_1^2$. Indeed choosing a representative $\tilde{a} \in A^0$ of a, and an element $h \in A^{-1}$ such that $dh = \tilde{a}^2$, we obtain a morphism of DG algebras $C \to A$, $t_1 \mapsto \tilde{a}$, $t_2 \mapsto h$. Thus, we may assume that A = C and $a = \overline{t_1}$.

It will be very useful to introduce an additional \mathbb{Z} -grading on C, which can be thought of as a \mathbb{G}_m -action. We will denote this grading by w, putting $w(t_1) = 1$, $w(t_2) = 2$, and then extend by the rule w(uv) = w(u) + w(v). Clearly, the differential d has degree zero with respect to w. We thus have a decomposition of C as a complex: $C = \bigoplus C^{\bullet,n}$.

 $n \ge 0$

Let us define $\hat{C} := \prod_{n \geq 0} C^{\bullet,n}$. This is also a DG algebra, and we have a map $C \to \hat{C}$. The homogeneous elements of degree -m in \hat{C} are just non-commutative power series in t_1, t_2 such that in each monomial there are exactly m copies of t_2 .

Lemma 5.5. The cohomology algebra $H^{\bullet}(\hat{C})$ is generated by the elements $u_1 = \overline{t_1}$ and $u_2 = \overline{[t_1, t_2]}$, with two relations: $u_1^2 = 0$, $u_1u_2 + u_2u_1 = 0$.

Proof. Indeed, it is easy to see that the DG algebra \hat{C} is isomorphic to the endomorphism DG algebra $\operatorname{End}_{k[y]/y^3}^{A_\infty}(k)$. Thus, we have an isomorphism of graded algebras $H^{\bullet}(\hat{C}) \cong \operatorname{Ext}_{k[y]/y^3}^{\bullet}(k,k)$. To compute this Ext-algebra, we take the semi-free resolution $P \to k$. The underlying graded $k[y]/y^3$ -module is defined by

$$P^{gr} := \bigoplus_{n=0}^{\infty} e_n \cdot \mathbf{k}[y]/y^3,$$

where $|e_n| = \lfloor \frac{n}{2} \rfloor$. The differential is given by $d(e_0) = 0$, and $d(e_{2k+1}) = e_{2k}y$, $d(e_{2k+2}) = e_{2k+1}y^2$ for $k \geq 0$. The morphism $P \to k$ sends e_0 to 1, and e_n to 0 for n > 0. Clearly, this is a quasi-isomorphism.

We see that $\operatorname{Ext}_{\mathbf{k}[y]/y^3}^{\bullet}(\mathbf{k},\mathbf{k}) \cong \operatorname{Hom}_{\mathbf{k}[y]/y^3}^{\bullet}(P,\mathbf{k})$, where the last complex has zero differential, and is equipped with the homogeneous basis $\{v_n\}_{n\geq 0}$, where $|v_n|=\lfloor \frac{n}{2}\rfloor$, and $v_i(e_j)=\delta_{ij}$. It is easy to see that the elements v_1 and v_2 correspond to the classes $u_1,u_2\in H^{\bullet}(\hat{C})$, mentioned in the formulation of the lemma. Clearly, we have $u_1^2=0$. It remains to show that $u_1u_2=-u_2u_1$, and $u_1u_2^k\neq 0$ for $k\geq 0$. Let us choose the lifts $\widetilde{v_n}\in\operatorname{End}_{\mathbf{k}[y]/y^3}(P)$ of v_n , putting

$$\widetilde{v_{2k}}(e_n) = \begin{cases} (-1)^{nk} e_{n-2k} & \text{for } n \ge 2k, \\ 0 & \text{otherwise;} \end{cases} \qquad \widetilde{v_{2k+1}}(e_n) = \begin{cases} e_{n-2k-1} & \text{for } n \text{ odd}, n \ge 2k+1, \\ (-1)^k e_{n-2k-1}y & \text{for } n \text{ even}, n \ge 2k+2, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $\widetilde{v_n}$'s super-commute with the differential, and that $\widetilde{v_1}\widetilde{v_2} + \widetilde{v_2}\widetilde{v_1} = 0$, $\widetilde{v_1}(\widetilde{v_2})^k = (-1)^k \widetilde{v_{2k+1}}$. This proves the lemma.

Lemma 5.6. The natural inclusion $C \to \hat{C}$ is a quasi-isomorphism.

Proof. We already know that $\dim H^n(\hat{C}) < \infty$ for all $n \in \mathbb{Z}$. It remains to observe the following: for any infinite sequence of complexes of vector spaces $\mathcal{K}_0^{\bullet}, \mathcal{K}_1^{\bullet}, \ldots$ such that $\dim H^n(\prod_{n\geq 0} \mathcal{K}_n^{\bullet}) < \infty$ for all $n \in \mathbb{Z}$, the morphism $\bigoplus_{n\geq 0} \mathcal{K}_n^{\bullet} \to \prod_{n\geq 0} \mathcal{K}_n^{\bullet}$ is a quasi-isomorphism. Applying this observation to the complexes $C^{\bullet,n}$, we conclude the proof. \square

We now construct a strictly unital A_{∞} -morphism $k[x]/x^6 \to C$, using obstruction theory. First, we introduce the weight grading (\mathbb{G}_m -action) on $k[x]/x^6$ by putting w(x) = 1.

Our A_{∞} -morphism will be compatible with the \mathbb{G}_m -actions, and its component f_1 is given by

(5.1)
$$f_1(x^k) = t_1^k \text{ for } 0 \le k \le 5.$$

Note that all the cohomology spaces $H^n(C)$ are $H^0(C)$ -bimodules, hence also over $k[x]/x^6$ -k $[x]/x^6$ -bimodules (via f_1).

Let us also note that for any $k[x]/x^6$ - $k[x]/x^6$ -bimodule M, equipped with the compatible \mathbb{G}_m -action, the Hochschild cohomology $HH^{\bullet}(k[x]/x^6, M)$ also becomes bigraded; the second grading again corresponds to the G_m -action. For a vector space V equipped with a \mathbb{G}_m -action, $V = \bigoplus_{n \in \mathbb{Z}} V^n$, we denote V(k) the same space with a twisted \mathbb{G}_m -action: $V(k)^n = V^{k+n}$.

Lemma 5.7. We have $HH^{2k+2}(\mathbf{k}[x]/x^6, H^{-2k}(C)) \cong \mathbf{k}[\varepsilon](6)$ for $k \geq 0$, and $HH^{2k+3}(\mathbf{k}[x]/x^6, H^{-2k-1}(C)) \cong \mathbf{k}(4)$ (\mathbb{G}_m -equivariant isomorphisms).

Proof. We have the following \mathbb{G}_m -equivariant resolution of the diagonal bimodule:

...
$$\xrightarrow{d_3} k[x]/x^6 \otimes k[x]/x^6(-6) \xrightarrow{d_2} k[x]/x^6 \otimes k[x]/x^6(-1) \xrightarrow{d_1} k[x]/x^6 \otimes k[x]/x^6 \xrightarrow{m} k[x]/x^6$$
, where $d_{2k+1} = x \otimes 1 - 1 \otimes x$, and $d_{2k} = x^5 \otimes 1 + x \otimes x^4 + \dots 1 \otimes x^5$. Further, by Lemmas 5.5 and 5.6 we know the \mathbb{G}_m -equivariant $H^0(C)$ - $H^0(C)$ -bimodules $H^n(C)$. Namely, $H^{-2k}(C) \cong k[\varepsilon](-6k)$ (twisted diagonal bimodule), and $H^{-2k-1}(C) \cong (k[\varepsilon])_{\sigma}(-6k-3)$ – the twisted anti-diagonal bimodule. For the later, the left and right $H^0(C)$ -actions are given respectively by $\varepsilon \cdot 1 = \varepsilon$, $1 \cdot \varepsilon = -\varepsilon$. The result follows by an elementary computation.

We are finally able to finish the proof of the theorem.

Proof of Theorem 5.3, part 2). As we already mentioned, we will construct (or rather show the existence of) a \mathbb{G}_m -equivariant strictly unital A_∞ -morphism $f: \mathbf{k}[x]/x^6 \to C$, where f_1 is given by (5.1). Since $H^0(f_1)$ is a homomorphism, we can construct f_2 such that the required relation is satisfied. Suppose that we have already constructed \mathbb{G}_m -equivariant f_1, \ldots, f_n (where $n \geq 2$) satisfying all the relations for the A_∞ -morphism that involve only f_1, \ldots, f_n . We want to construct the (n+1) components $f_1, \ldots, f'_n, f_{n+1}$ (again, satisfying all the relevant relations) where only f_n is possibly being replaced by another map f'_n . The standard obstruction theory tells us that the obstruction to this is given by a class in $HH^{n+1,0}(\mathbf{k}[x]/x^6, H^{1-n}(C))$ (the \mathbb{G}_m -invariant part). Applying Lemma 5.7, we see that this space vanishes. Thus, proceeding inductively we can construct the desired A_∞ -morphism f. This proves the theorem.

References

- [CT] D.-C. Cisinski and G. Tabuada, "Non-connective K-theory via universal invariants," Compos. Math. 147, 1281-1320 (2011).
- [Co] A. Connes, "Non-commutative geometry", Academic Press, 1994.
- [DI] P. Deligne, L. Illusie, "Relévements modulo p^2 et décomposition du complexe de de Rham", Inv. Math. 89 (1987), 247-270.
- [Dr] V. Drinfeld, "DG quotients of DG categories", J. Algebra 272 (2004), no. 2, 643-691.
- [E1] A. I. Efimov, "Generalized non-commutative degeneration conjecture", Proc. Steklov Inst. Math. 290 (2015), no. 1, 1-10.
- [E2] A. I. Efimov, "Homotopy finiteness of some DG categories from algebraic geometry", arXiv:1308.0135 (preprint), to appear in JEMS.
- [FT] B. L. Feigin and B. L. Tsygan, "Cohomologies of Lie algebras of generalized Jacobi matrices", Funkts. Anal. Prilozh. 17 (2), 86-87 (1983) [Funct. Anal. Appl. 17, 153-155 (1983)].
- [Ho] M. Hovey, "Model categories", Mathematical surveys and monographs, Vol. 63, Amer. Math. Soc., Providence 1998.
- [Ka] D. Kaledin, "Spectral sequences for cyclic homology" Algebra, geometry, and physics in the 21st century, 99-129, Progr. Math., 324, Birkhuser/Springer, Cham, 2017.
- [Ke1] B. Keller, "Deriving DG categories", Ann. Sci. École Norm. Sup. (4) 27 (1994), no. 1, 63-102.
- [Ke2] B. Keller, "On the cyclic homology category of exact categories", J. Pure Appl. Algebra 136 (1) (1999) 1-56.
- [Ke3] B. Keller, "A-infinity algebras, modules and functor categories. Trends in representation theory of algebras and related topics", 67-93, Contemp. Math., 406, Amer. Math. Soc., Providence, RI, 2006.
- [Ko] M. Kontsevich, private communication, 2012.
- [KS] M. Kontsevich, Y. Soibelman, "Notes on A-infinity algebras, A-infinity categories and noncommutative geometry, I". In: Homological Mirror Symmetry. Lecture Notes in Physics, vol 757. Springer, Berlin, Heidelberg.
- [KL] A. Kuznetsov, V. Lunts, "Categorical resolutions of irrational singularities", International Mathematics Research Notices, Volume 2015, Issue 13, 1 January 2015, Pages 4536-4625.
- [L-H] K. Lefèvre-Hasegawa, "Sur les A_{∞} -catégories", Ph.D. thesis, Université Paris 7, U.F.R. de Mathématiques, http://arXiv.org/abs/math.CT/0310337, 2003.
- [M] A. Mathew, "Kaledin's degeneration theorem and topological Hochschild homology", arXiv:1710.09045 (preprint).
- [N] M. Nagata, A generalization of the imbedding problem of an abstract variety in a complete variety. J. Math. Kyoto Univ. 3 1963 89-102.
- [S] D. Shklyarov, "Hirzebruch-Riemann-Roch-type formula for DG algebras", Proceedings of the London Mathematical Society, Volume 106, Issue 1, 1 January 2013, Pages 1-32.
- [Tab1] G. Tabuada, "Théorie homotopique des DG-catégories", Thése de L'Université Paris Diderot Paris 7.
- [Tab2] G. Tabuada, "Une structure de catégorie de modéles de Quillen sur la catégorie des dg-catégories",
 C. R. Acad. Sci. Paris Ser. I Math. 340 (1) (2005), 15-19.
- [To1] B. Toën, private communication, 2012.

- [To2] B. Toën, "The homotopy theory of dg-categories and derived Morita theory", Invent. Math. 167 (2007), no. 3, 615-667.
- [TV] B. Toën, M. Vaquié, "Moduli of objects in dg-categories", Ann. Sci. École Norm. Sup. (4) 40 (2007), no. 3, 387-444.
- [Ts] B. L. Tsygan, "The homology of matrix Lie algebras over rings and the Hochschild homology", Usp. Mat. Nauk 38 (2), 217-218 (1983) [Russ. Math. Surv. 38 (2), 198-199 (1983)].
- [W] F. Waldhausen, "Algebraic K-theory of spaces," in Algebraic and Geometric Topology: Proc. Conf., New Brunswick, NJ, 1983 (Springer, Berlin, 1985), Lect. Notes Math. 1126, pp. 318-419.

Steklov Mathematical Institute of RAS, Gubkin str. 8, GSP-1, Moscow 119991, Russia, National Research University Higher School of Economics, Russian Federation *E-mail address*: efimov@mccme.ru