# CATEGORICAL SMOOTH COMPACTIFICATIONS AND GENERALIZED HODGE-TO-DE RHAM DEGENERATION

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Abstract. We disprove two (unpublished) conjectures of Kontsevich which state generalized versions of categorical Hodge-to-de Rham degeneration for smooth and for proper DG categories (but not smooth and proper, in which case degeneration is proved by Kaledin [\[Ka\]](#page-19-0)). In particular, we show that there exists a minimal 10-dimensional  $A_{\infty}$ -algebra over a field of characteristic zero, for which the supertrace of  $\mu_3$  on the second argument is non-zero.

As a byproduct, we obtain an example of a homotopically finitely presented DG category (over a field of characteristic zero) that does not have a smooth categorical compactification, giving a negative answer to a question of Toën. This can be interpreted as a lack of resolution of singularities in the noncommutative setup.

We also obtain an example of a proper DG category which does not admit a categorical resolution of singularities in the terminology of [\[KL\]](#page-19-1) (that is, it cannot be embedded into a smooth and proper DG category).

## **CONTENTS**



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#### 0. INTRODUCTION

<span id="page-1-0"></span>Given a smooth algebraic variety  $X$  over a field of characteristic zero, we have the Hodge-to-de Rham spectral sequence  $E_1^{p,q} = H^q(X, \Omega_X^p)$  $H_{DR}^{p+q}(X)$ . It is classically known that when X is additionally proper, this spectral sequence degenerates at  $E_1$ , that is, all differentials vanish. This follows from the classical Hodge theory for compact Kähler manifolds, and can be also proved algebraically [\[DI\]](#page-19-3).

We recall the following fundamental result of Kaledin [\[Ka\]](#page-19-0), see also [\[M\]](#page-19-4) for a different proof.

<span id="page-1-3"></span>**Theorem 0.1.** [\[Ka,](#page-19-0) Theorem 5.4] Let A be a smooth and proper DG algebra. Then the Hochschild-to-cyclic spectral sequence degenerates, so that we have an isomorphism  $HP_{\bullet}(A) = HH_{\bullet}(A)((u)).$ 

Here u denotes a variable of degree 2.

When applied to  $\text{Perf}(A) \simeq \text{Perf}(X)$  for smooth and proper variety X, Theorem [0.1](#page-1-3) gives exactly the classical Hodge-to-de Rham degeneration.

In this paper we study some generalizations of Hodge-to-de-Rham degeneration to DG categories which are not smooth and proper.

Recall that for a proper DG algebra B one has a pairing on  $HH_{\bullet}(B) \otimes HH_{\bullet}(B^{op}) \rightarrow$ k, introduced by Shklyarov [\[S\]](#page-19-5). Kontsevich [\[Ko\]](#page-19-6) proposed the following generalization of Theorem [0.1.](#page-1-3)

<span id="page-1-2"></span>**Conjecture 0.2.** Let  $B$  be a proper DG algebra. Then the composition map

(0.1) 
$$
(HH_{\bullet}(B) \otimes HC_{\bullet}(B^{op))[1] \xrightarrow{\mathrm{id} \otimes \delta} HH_{\bullet}(B) \otimes HH_{\bullet}(B^{op}) \to \mathrm{k}
$$

is zero.

Kontsevich also proposed a "dual" version of Conjecture [0.2](#page-1-2) for smooth DG algebras.

<span id="page-1-1"></span>Conjecture 0.3. Let A be a smooth DG algebra. Then the composition

$$
K_0(A\otimes A^{op})\xrightarrow{\text{ch}} (HH_\bullet(A)\otimes HH_\bullet(A^{op})_0\xrightarrow{\text{id}\otimes\delta} (HH_\bullet(A)\otimes HC_\bullet^-(A^{op}))_1
$$

vanishes on the class  $[A]$  of the diagonal bimodule.

The motivation for Conjectures [0.2](#page-1-2) and [0.3](#page-1-1) is explained in Propositions [4.1](#page-12-1) and [5.1](#page-14-1) below. Here we mention that the results of [\[KL\]](#page-19-1) imply that Conjecture [0.2](#page-1-2) holds for proper DG algebras of algebro-geometric origin: that is, for DG algebras of the form  $B = \mathbf{R} \text{End}(\mathcal{F})$ , where  $\mathcal{F} \in \text{Perf}_Z(X)$  is a perfect complex on a separated scheme X of finite type over k, supported on a *proper* closed subscheme  $Z \subset X$ . Similarly, the (weak version of) results of [\[E2\]](#page-19-7) imply that Conjecture [0.3](#page-1-1) holds for smooth DG algebras of the form  $\mathbf{R} \text{End}(\mathcal{G})$ , where  $\mathcal{G} \in D^b_{coh}(X)$  is a generator of the category  $D^b_{coh}(X)$ .

There is a closely related question formulated by B. Toen [\[To1\]](#page-19-8).

<span id="page-2-2"></span>**Question 0.4.** Is it true that any homotopically finitely presented DG category  $\beta$  is quasiequivalent to a quotient  $A/S$ , where A is smooth and proper, and  $S \subset A$  is a full subcategory?

Such a quotient presentation of  $\beta$  is called a smooth categorical compactification.

In this paper we disprove both Conjectures [0.2](#page-1-2) and [0.3.](#page-1-1) As an application, we give a negative answer to Question [0.4.](#page-2-2)

The starting point for our counterexamples is to disprove the main conjecture of [\[E1\]](#page-19-9), see Section [3.](#page-11-0) A counterexample to Conjecture [0.3](#page-1-1) is obtained in Section [4.](#page-12-0) It is deduced from the results of Section [3](#page-11-0) by some trick.

Finally, a counterexample to Conjecture [0.2](#page-1-2) is obtained in Section [5.](#page-14-0) It is deduced from our new result on nilpotent elements in the cohomology of a DG algebra (Theorem [5.3\)](#page-15-0), which is of independent interest. In particular, we obtain an example of a proper DG algebra  $B$  such that the DG category  $\text{Perf}(B)$  cannot be fully faithfully embedded into a saturated DG category. That is, it does not have a categorical resolution of singularities in the terminology of [\[KL\]](#page-19-1).

Section [5](#page-14-0) can be read independently from Sections [3](#page-11-0) and [4.](#page-12-0)

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## 1. PRELIMINARIES ON DG CATEGORIES AND  $A_{\infty}$ -ALGEBRAS

<span id="page-2-1"></span>1.1. DG categories. For the introduction on DG categories, we refer the reader to [\[Ke1\]](#page-19-10). The references for DG quotients are [\[Dr,](#page-19-11) [Ke2\]](#page-19-12). For the model structures on DG categories we refer to [\[Tab1,](#page-19-13) [Tab2\]](#page-19-14), and for a general introduction on model categories we refer to [\[Ho\]](#page-19-15). Everything will be considered over some base field k.

Mostly we will consider DG categories up to a quasi-equivalence. By a functor between DG categories we sometimes mean a quasi-functor. In some cases it is convenient for us to choose a concrete DG model or a concrete DG functor. By a commutative diagram of functors we usually mean the commutative diagram in the homotopy category  $\text{Ho}(\text{dgcat}_k)$ . Finally, we denote by  $\text{Ho}_M(\text{dgcat}_k)$  the Morita homotopy category of DG categories (with inverted Morita equivalences).

All modules are assumed to be right unless otherwise stated. For a small DG category C and a  $\mathcal C$ -module M, we denote by  $M^{\vee}$  the  $\mathcal C^{op}$ -module  $\text{Hom}_{\mathcal C}(M,\mathcal C)$ . We denote by  $M^*$ the  $\mathcal{C}^{op}$ -module  $\text{Hom}_{\mathcal{C}}(M,k)$ .

Given a small DG category C, we denote by  $D(\mathcal{C})$  its derived category of DG Cmodules. This is a compactly generated triangulated category. We denote by  $D_{\text{perf}}(\mathcal{C})$  the full triangulated subcategory of perfect  $C$ -modules. It coincides with the subcategory of compact objects.

Recall from [\[TV\]](#page-20-0) that a C-module M is pseudo-perfect if for each  $x \in \mathcal{C}$ , the complex  $M(x)$  is perfect over k (that is,  $M(x)$  has finite-dimensional total cohomology). We denote by  $D_{\text{pspe}}(\mathcal{C}) \subset D(\mathcal{C})$  the full triangulated subcategory of pseudo-perfect  $\mathcal{C}$ -modules.

For any DG category  $\mathcal{C}$ , we denote by  $[\mathcal{C}]$  its (non-graded) homotopy category, which has the same objects as C, and the morphisms are given by  $[\mathcal{C}](x, y) = H^0(\mathcal{C}(x, y))$ . We use the terminology of  $[TV,$  Definition 2.4 by calling C triangulated if the Yoneda embedding provides an equivalence  $[\mathcal{C}] \stackrel{\sim}{\to} D_{\text{perf}}(\mathcal{C})$ . In this case  $[\mathcal{C}]$  is a Karoubi complete triangulated category.

We denote by Mod<sub>C</sub> the DG category of cofibrant DG C-modules in the projective model structure (these are exactly the direct summands of semifree DG  $\mathcal{C}$ -modules). We have  $D(\mathcal{C}) \simeq [\text{Mod}_{\mathcal{C}}]$ , where  $D(\mathcal{C})$  is the derived category of DG  $\mathcal{C}$ -modules. We denote by  $\mathbf{Y}: \mathcal{C} \hookrightarrow \text{Mod}_{\mathcal{C}}$  the standard Yoneda embedding given by  $\mathbf{Y}(x) = \mathcal{C}(-,x)$ .

We write  $\text{Perf}(\mathcal{C}) \subset \text{Mod}_{\mathcal{C}}$  (resp.  $\text{PsPerf}(\mathcal{C}) \subset \text{Mod}_{\mathcal{C}}$ ) for the full DG subcategory of perfect (resp. pseudo-perfect)  $\mathcal{C}$ -modules.

For a DG functor  $\Phi:\mathcal{C}_1\to\mathcal{C}_2$  between small DG categories, we denote by  $\mathbf{L}\Phi^*:D(\mathcal{C}_1)\to\mathcal{C}_2$  $D(\mathcal{C}_2)$ , the derived extension of scalars functor. Its right adjoint functor (restriction of scalars) is denoted by  $\Phi_*: D(\mathcal{C}_2) \to D(\mathcal{C}_1)$ .

We also recall from [\[To2,](#page-20-1) Definitions 3.6] that a  $\mathcal C$ -module is called quasi-representable if it is quasi-isomorphic to a representable C-module. For two DG categories  $\mathcal{C}, \mathcal{C}'$ , a  $\mathcal{C} \otimes \mathcal{C}'$ module M is called right quasi-representable if for each object  $x \in \mathcal{C}$ , the  $\mathcal{C}'$ -module  $M(x, -)$  is quasi-representable.

We denote by  $\mathbf{R}\underline{\text{Hom}}(\mathcal{C},\mathcal{C}')\subset \text{Mod}_{\mathcal{C}^{op}\otimes\mathcal{C}'}$  the full subcategory of right quasi-representable  $\mathcal{C}^{op}\,\otimes\,\mathcal{C}'$ By  $[T<sub>0</sub>2, Theorem 6.1], this DG category (considered up to a$ quasi-equivalence) is actually the internal Hom in the homotopy category of DG categories  $\text{Ho}(\text{dgcat}_k)$  (with inverted quasi-equivalences). We have a natural quasi-functor Fun(C, C')  $\rightarrow \mathbf{R}Hom(C, C')$ , where Fun(C, C') is the naive DG category of DG functors  $\mathcal{C} \to \mathcal{C}'$ , as defined in [\[Ke1\]](#page-19-10). Moreover, if  $\mathcal{C}$  is cofibrant, this functor is essentially surjective on the homotopy categories.

A small DG category C is called smooth (resp. locally proper) if the diagonal  $C-C$ bimodule is perfect (resp. pseudo-perfect). Moreover,  $\mathcal C$  is called proper if it is locally proper and is Morita equivalent to a DG algebra (i.e. the triangulated category  $D_{\text{perf}}(\mathcal{C})$ has a classical generator).

We recall the notion of a short exact sequence of DG categories.

**Definition 1.1.** A pair of functors  $A_1 \stackrel{F_1}{\longrightarrow} A_2 \stackrel{F_3}{\longrightarrow} A_3$  is said to be a (Morita) short exact sequence of DG categories if the following conditions hold

- i) the composition  $F_2F_1$  is homotopic to zero;
- ii) the functor  $F_1$  is quasi-fully-faithful;
- iii) the induced quasi-functor  $\overline{F_2}$ :  $A_2/F_1(A_1) \rightarrow A_3$  is a Morita equivalence.

In particular, a short exact sequence of DG categories induces a long exact sequence of K-groups, where  $K_{\bullet}(\mathcal{A})$  is the Waldhausen K-theory [\[W\]](#page-20-2) of the Waldhausen category of cofibrant perfect A-modules. We will in fact need only the boundary map  $K_1(\mathcal{A}_3) \rightarrow$  $K_0(\mathcal{A}_1).$ 

<span id="page-4-0"></span>1.2.  $A_{\infty}$ -algebras and  $A_{\infty}$ -(bi)modules. All the definitions and constructions regarding DG categories which are invariant under quasi-equivalences can be translated into the world of  $A_{\infty}$ -categories. For the introduction on  $A_{\infty}$ -categories and algebras see [\[L-H,](#page-19-16) [Ke3,](#page-19-17) [KS\]](#page-19-18).

It will be sufficient for us to work with  $A_{\infty}$ -algebras (that is,  $A_{\infty}$ -categories with a single object).

In order to write down the signs in formulas it is convenient to adopt the following **Notation.** For a collection of homogeneous elements  $a_0, \ldots, a_n$  of a graded vector space A, and  $0 \leq p, q \leq n$ , we put

$$
l_p^q(a) = \begin{cases} |a_p| + \dots + |a_q| + q - p + 1 & \text{if } p \leq q; \\ |a_p| + \dots + |a_n| + |a_0| + \dots + |a_q| + n - p + q & \text{if } p > q. \end{cases}
$$

If the collection starts with  $a_1$  (and there is no  $a_0$ ) we only use  $l_p^q(a)$  for  $1 \leq p \leq q \leq n$ .

**Definition 1.2.** A non-unital  $A_{\infty}$ -structure on a graded vector space A is a sequence of multilinear operations  $\mu_n = \mu_n^A : A^{\otimes n} \to A$ , where  $\deg(\mu_n) = 2-n$ , satisfying the following relations:

$$
(1.1) \qquad \sum_{i+j+k=n+1} (-1)^{l_1^i(a)} \mu_{i+k+1}(a_1,\ldots,a_i,\mu_j(a_{i+1},\ldots,a_{i+j}),a_{i+j+1},\ldots,a_n) = 0,
$$

for  $n \ge 0$ . Here for  $1 \le p \le q \le n$  we put  $l_p^q(a) := |a_p| + \cdots + |a_q| + q - p + 1$ .

**Remark 1.3.** In our sign convention, a non-unital DG algebra B can be considered as an  $A_{\infty}$ -algebra, with  $\mu_1(a) = -d(a), \ \mu_2(a_1, a_2) = (-1)^{|a_1|} a_1 a_2, \ and \ \mu_{\geq 3} = 0.$ 

.

**Definition 1.4.** A non-unital  $A_{\infty}$ -morphism  $f : A \rightarrow B$  is given by a sequence of linear maps  $f_n: A^{\otimes n} \to B$ , where  $\deg(f_n) = 1 - n$ , satisfying the following relations:

$$
(1.2) \sum_{i_1+\dots+i_k=n} \mu_k^B(f_{i_1}(a_1,\dots,a_{i_1}),\dots,f_{i_k}(a_{i_1+\dots+i_{k-1}+1},\dots,a_n)) = \sum_{i+j+k=n} (-1)^{l_1^i(a)} f_{i+k+1}(a_1,\dots,a_i,\mu_j^A(a_{i+1},\dots,a_{i+j}),a_{i+j+1},\dots,a_n).
$$

Given an  $A_{\infty}$ -algebra A, one defines the  $A_{\infty}$ -algebra  $A^{op}$  as follows: it is equal to A as a graded vector space, and we have

$$
\mu_n^{A^{op}}(a_1,\ldots,a_n) = (-1)^{\sigma} \mu_n^A(a_n,\ldots,a_1),
$$

where  $\sigma = \sum$  $1\leq i < j \leq n$  $(|a_i|+1)(|a_j|+1).$ 

We now define the notion of an  $A_{\infty}$ -module.

**Definition 1.5.** A right  $A_{\infty}$ -module M over an  $A_{\infty}$ -algebra A is a graded vector space with a sequence of operations  $\mu_n^M : M \otimes A^{\otimes n-1} \to M$ , where  $n > 0$ ,  $\deg(\mu_n^M) = 2 - n$ , and the following relations are satisfied:

$$
(1.3) \sum_{i+j=n} \mu_{j+1}^M(\mu_{i+1}^M(m, a_1, \dots, a_i), a_{i+1}, \dots, a_n) + \sum_{i+j+k=n+1} (-1)^{|m|+l_1^i(a)} \mu_{i+k+1}(m, a_1, \dots, a_i, \mu_j(a_{i+1}, \dots, a_{i+j}), a_{i+j+1}, \dots, a_n) = 0.
$$

We also need  $A_{\infty}$ -bimodules.

**Definition 1.6.** Let A and B be non-unital  $A_{\infty}$ -algebras. An  $A_{\infty}$  A-B-bimodule M is a graded vector space with a collection of operations  $\mu_{i,j} = \mu_{i,j}^M : A^{\otimes i} \otimes M \otimes B^{\otimes j} \to M$ , where  $i, j \geq 0$ , such that for any  $n, m \geq 0$  and homogeneous  $a_1, \ldots, a_n \in A$ ,  $b_1, \ldots, b_m \in B$ ,  $m \in M$ , the following relation is satisfied:

$$
\sum_{i+j+k=n+1} (-1)^{l_1^i(a)} \mu_{i+k+1,m}^M(a_1, \ldots, \mu_j^A(a_{i+1}, \ldots, a_{i+j}), \ldots, a_n, m, b_1, \ldots, b_m)
$$
  
+ 
$$
\sum_{\substack{1 \le i \le n+1; \\0 \le j \le m}} \mu_{i-1,m-j}^M(a_1, \ldots, a_{i-1}, \mu_{n+1-i,j}^M(a_i, \ldots, a_n, m, b_1, \ldots, b_j), b_{j+1}, \ldots, b_m)
$$
  
+ 
$$
\sum_{i+j+k=m+1} (-1)^{l_1^n(a)+l_1^i(b)+|m|} \mu_{n,i+k+1}^M(a_1, \ldots, a_n, m, b_1, \ldots, \mu_j^B(b_{i+1}, \ldots, b_{i+j}), \ldots, b_m) = 0.
$$

**Remark 1.7.** 1) In our sign convention, a non-unital DG algebra  $B$  can be considered as an  $A_{\infty}$ -algebra, with  $\mu_1(b) = -d(b)$ ,  $\mu_2(b_1, b_2) = (-1)^{|b_1|}b_1b_2$ , and  $\mu_{\geq 3} = 0$ .

2) If furthermore M is a right DG B -module, then the  $A_{\infty}$  B -module structure on M is given by  $\mu_1^M(m) = d(m)$ ,  $\mu_2^M(m, a) = (-1)^{|m|+1}ma$ , and  $\mu_{\geq 3}^M = 0$ .

3) If A is another non-unital DG algebra, and M is a DG A-B -bimodule, then the  $A_{\infty}$ A-B-bimodule structure on M is given by  $\mu_{0,0}^M(m) = d(m)$ ,  $\mu_{1,0}^M(a,m) = am$ ,  $\mu_{0,1}^M(m,b) =$  $(-1)^{|m|+1}mb$ , and  $\mu_{i,j}^M = 0$  for  $i + j \geq 2$ .

We now recall the strict unitality.

**Definition 1.8.** 1) A non-unital  $A_{\infty}$ -algebra A is called strictly unital if there is a (unique) element  $1 = 1_A \in A$  such that  $\mu_1(1) = 0$ ,  $\mu_2(1, a) = a = (-1)^{|a|} \mu_2(a, 1)$  for any homogeneous element  $a \in A$ , and for  $n \geq 3$  we have  $\mu_n(a_1, \ldots, a_n) = 0$  if at least one of the arguments  $a_i$  equals 1.

2) A non-unital  $A_{\infty}$ -morphism  $f : A \to B$  between strictly unital  $A_{\infty}$ -algebras is called strictly unital if  $f_1(1_A) = 1_B$ , and for  $n \geq 2$  we have  $f_n(a_1, \ldots, a_n) = 0$  if at least one of the arguments  $a_i$  equals 1.

3) Given a strictly unital  $A_{\infty}$  -algebra A, an  $A_{\infty}$  A -module M is called strictly unital if  $\mu_2^M(m,1) = (-1)^{|m|+1}m$ , and for  $n \geq 3$  we have  $\mu_n^M(m, a_1, \ldots, a_{n-1}) = 0$  if at least one of  $a_i$  's equals 1.

4) Given strictly unital  $A_{\infty}$ -algebras A, B, an  $A_{\infty}$  A-B-bimodule is called strictly unital if  $\mu_{1,0}^M(1_A,m) = m$ ,  $\mu_{0,1}^M(m,1_B) = (-1)^{|m|+1}m$ , and for  $k+l \geq 2$  we have  $\mu_{k,l}(a_1,\ldots,a_k,m,b_1,\ldots,b_l)=0$  if at least one of  $a_i$  's equals  $1_A$  or at least one of  $b_j$  's equals  $1_B$ .

From now on, all  $A_{\infty}$ -algebras and (bi)modules will be strictly unital

Given a strictly unital  $A_{\infty}$ -algebra A, we define the DG category Mod<sup>∞</sup> -A whose objects are  $A_{\infty}$ -modules and the morphisms are defined as follows. Given  $M, N \in Mod^{\infty}$ -A, we put

$$
\operatorname{Hom}\nolimits_A^{\infty}(M,N)^{gr}:=\prod_{n\geq 0}\operatorname{Hom}\nolimits_{\mathbf k}(M\otimes A[1]^{\otimes n},N),
$$

and the differential is given by

$$
d(\varphi)_n(m, a_1, \dots, a_n) = \sum_{i=0}^n \mu_{n-i+1}^N(\varphi_i(m, a_1, \dots, a_i), a_{i+1}, \dots, a_n)
$$
  

$$
-\sum_{i=0}^n (-1)^{|\varphi|} \varphi_{n-i+1}(\mu_{i+1}^M(m, a_1, \dots, a_i), a_{i+1}, \dots, a_n)
$$
  

$$
-\sum_{1 \le i \le j \le n} (-1)^{|\varphi| + |m| + l_1^{i-1}(a)} \varphi_{n+i-j-1}(m, a_1, \dots, \mu_{j-i+1}^A(a_i, \dots, a_j), \dots, a_n).
$$

The composition is given by

$$
(\varphi\psi)_n(m,a_1,\ldots,a_n)=\sum_{i=0}^n\varphi_{n-i}(\psi_i(m,a_1,\ldots,a_i),a_{i+1},\ldots,a_n).
$$

Given a unital DG algebra B, we denote by  $P\text{sPerf}(B) \subset \text{Mod}^{\infty}$ -B the full DG subcategory formed by pseudo-perfect DG modules. We have  $[PsPerf(B)] \simeq D_{per}(B)$ .

## **Remark 1.9.** Let  $A, B$  be  $A_{\infty}$  -algebras.

1) An  $A_{\infty}$  A-B -bimodule structure on a graded vector space M is equivalent to the following data:

- the right  $A_{\infty}$  B-module structure on M;
- the  $A_{\infty}$ -morphism  $f : A \to \text{End}_{B}^{\infty}(M)$ .

Namely, given an  $A_{\infty}$ -bimodule M, the induced B-module structure is given by  $\mu_n^M$  =  $\mu_{0,n-1}^M$ , and the  $A_\infty$ -morphism is given by  $f_n(a_1,\ldots,a_n)(m,b_1,\ldots,b_l)$  =  $\mu_{n,l}(a_1, \ldots, a_n, m, b_1, \ldots, b_l).$ 

2) Also, an  $A_{\infty}$ -bimodule structure is equivalent to an

We finally define a technically useful notion of an an  $A_{\infty}$ -bimorphism of (strictly unital)  $A_{\infty}$ -algebras  $f:(A, B) \to C$ . It is given by the linear maps  $f_{r,s}: A^{\otimes r} \otimes B^{\otimes s} \to C$ , where  $r, s \geq 0$ ,  $r + s > 0$ , so that the following relations are satisfied:

$$
(1.4) \sum_{\substack{0=r_0\leq r_1\leq \cdots\leq r_k=r;\\0=s_0\leq s_1\leq \cdots\leq s_k=s}} (-1)^{\sigma} \mu_k^C(f_{r_1,s_1}(a_1,\ldots,a_{r_1};b_1,\ldots,b_{s_1}),\ldots,\n\mu_{s_1,s_2,s_3\leq \cdots\leq s_k=s}
$$
\n
$$
f_{r-r_{k-1},s-s_{k-1}}(a_{r_{k-1}+1},\ldots,a_r;b_{s_{k-1}+1},\ldots,b_s)) =
$$
\n
$$
\sum_{i+j+k=r} (-1)^{l_1^i(a)} f_{i+k+1,s}(a_1,\ldots,a_i,\mu_j(a_{i+1},\ldots,a_{i+j}),a_{i+j+1},\ldots,a_r;b_1,\ldots,b_s)+
$$
\n
$$
\sum_{i+j+k=s} (-1)^{l_1^r(a)+l_1^i(b)} f_{r,i+k+1}(a_1,\ldots,a_r;b_1,\ldots,b_i,\mu_j(b_{i+1},\ldots,b_{i+j}),b_{i+j+1},\ldots,b_s),
$$

where  $\sigma = \sum$  $1 \leq p < q \leq k$  $l_{r_{q-1}+1}^{r_q}(a)l_{s_{p-1}+1}^{s_p}(b)$ . We require that  $f_{1,0}(1_A) = 1_C = f_{0,1}(1_B)$ , and for  $k+l\geq 2$   $f_{k,l}(a_1,\ldots,a_k,b_1,\ldots,b_l)=0$  if at least one of  $a_i$ 's equals  $1_A$ , or at least one of  $b_j$ 's equals  $1_B$ .

**Remark 1.10.** One can similarly define  $A_{\infty}$  n-morphisms  $(A_1, \ldots, A_n) \rightarrow B$ , so that the category of  $A_{\infty}$ -algebras becomes a (non-symmetric) pseudo-monoidal category. In particular, the  $A_{\infty}$ -morphisms can be composed with  $A_{\infty}$ -morphisms in the natural way.

<span id="page-7-0"></span>**Remark 1.11.** If a graded vector space M is given a differential d, then an  $A_{\infty}$ -bimodule structure on M (with  $\mu_1^M = d$ ) is equivalent to an  $A_{\infty}$ -bimorphism  $f : (A, B^{op}) \rightarrow$ End<sub>k</sub> $(M)$ . Given such an  $A_{\infty}$ -bimorphism, one puts

$$
\mu_{r,s}^M(a_1,\ldots,a_r,m,b_1,\ldots,b_s):=(-1)^lf_{r,s}(a_1,\ldots,a_r,b_s,\ldots,b_1)(m),
$$

where  $l = l_1^s(b) \cdot |m| + \sum$  $1 \leq p < q \leq s$  $(|b_p| + 1)(|b_q| + 1).$  The diagonal  $A_{\infty}$  A-A-bimodule is given by A as a graded vector space, and we have

$$
\mu_{i,j}(a_1,\ldots,a_i,b,c_1,\ldots,c_j) = (-1)^{l_1^i(a)+1} \mu_{i+j+1}^A(a_1,\ldots,a_i,b,c_1,\ldots,c_j).
$$

Finally, we mention the gluing of  $A_{\infty}$ -algebras. Let M be an  $A_{\infty}$  A-B-bimodule. We denote by  $\begin{pmatrix} B & 0 \\ M & A \end{pmatrix}$ the  $A_{\infty}$ -algebra C which equals  $A \oplus B \oplus M$  as a graded vector space, so that the non-zero components of  $\mu_n^C$  are given by  $\mu_n^A$ ,  $\mu_n^B$ , and

$$
(-1)^{l_1^i(a)+1}\mu_{i,j}(a_1,\ldots,a_i,m,b_1,\ldots,b_j), \quad i+j+1=n,
$$

<span id="page-8-0"></span>where  $a_1, \ldots, a_i \in A, b_1, \ldots, b_i \in B$ .

## 2. Preliminaries on the Hochschild complex, pairings and copairings

In this section all  $A_{\infty}$ -algebras are strictly unital. For an  $A_{\infty}$ -algebra A, we put  $\overline{A} := A/k \cdot 1_A.$ 

The mixed Hochschild complex (see [\[Ke2,](#page-19-12) [KS\]](#page-19-18))  $(C_{\bullet}(A), b, B)$  of an  $A_{\infty}$ -algebra A is given as a graded vector space by

$$
C_{\bullet}(A):=\bigoplus_{n\geq 0} A\otimes (\overline{A}[1])^{\otimes n}.
$$

For convenience we write  $(a_0, \ldots, a_n)$  instead of  $a_0 \otimes \cdots \otimes a_n \in C_{\bullet}(A)$ .

The Hochschild differential is given by

$$
(2.1) \quad b(a_0, \ldots, a_n) = \sum_{0 \le i \le j \le n} (-1)^{l_0^{i-1}(a)+1} (a_0, \ldots, \mu_{j-i+1}(a_i, \ldots, a_j), \ldots, a_n) + \sum_{0 \le p < q \le n} (-1)^{l_0^{q-1}(a)l_q^n(a)+1} (\mu_{n+p+2-q}(a_q, \ldots, a_n, a_0, \ldots, a_p), a_{p+1}, \ldots, a_{q-1}).
$$

The Connes-Tsygan differential  $B$  (see [\[Co,](#page-19-19) [FT,](#page-19-20) [Ts\]](#page-20-3)) is given by

$$
B(a_0, a_1, \ldots, a_n) = \sum_{0 \leq i \leq n} (-1)^{l_0^{i-1}(a)l_i^n(a)+1} (1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1}).
$$

The Hochschild complex can be more generally defined for  $A_{\infty}$ -categories, and is Morita invariant [\[KS\]](#page-19-18). We refer to [KS] for the definition of cyclic homology  $HC_{\bullet}$ , negative cyclic homology  $HC$ <sub>•</sub> and  $HP$ •. In this paper we will in fact deal only with the first differential of the Hochschild-to-cyclic spectral sequence, which is the map  $B: HH_n(A) \to HH_{n+1}(A)$ induced by the Connes-Tsygan differential.

We recall the natural pairings and co-pairings on  $HH_{\bullet}(A)$ . Let us restrict ourselves to DG algebras for a moment. Given a DG algebra  $A$ , we have a Chern character ch :  $K_n(A) \to HH_n(A)$  (see [\[CT\]](#page-19-21); the Chern character naturally lifts to  $HC^{-}(A)$ ), but we will not need this).

In particular, given DG algebras A, B and an object  $M \in D_{\text{perf}}(A \otimes B)$ , we have a copairing

$$
ch(M) \in (HH_{\bullet}(A) \otimes HH_{\bullet}(B))_{0} \cong HH_{0}(A \otimes B).
$$

This copairing is used in the formulation of Conjecture [0.3](#page-1-1) for  $A = B^{op}$  being smooth, and  $M = A$ .

Dually [\[S\]](#page-19-5), if we have DG algebras A and B, and an object  $M \in D_{\text{pspe}}(A^{op} \otimes B^{op}),$ then we have a pairing (of degree zero)

$$
HH_{\bullet}(A) \otimes HH_{\bullet}(B) \to HH_{\bullet}(A \otimes B) \to HH_{\bullet}(\mathrm{End}_{k}(M)) \to k
$$

(the last map is an isomorphism if and only if  $M$  is not acyclic). In the formulation of Conjecture [0.2](#page-1-2) this pairing is used for  $A = B^{op}$  proper, and  $M = A$ . In this case we denote the pairing by  $\langle \cdot, \cdot \rangle$ .

We would like to obtain an explicit formula for the pairing in the  $A_{\infty}$ -setting. The reader who is not interested in (or is already familiar with) the details can skip to Corollary [2.3](#page-10-0) which is essentially all we need.

Let  $A, B, C$  be  $A_{\infty}$ -algebras. Suppose that we are given an  $A_{\infty}$ -bimorphism f:  $(A, B) \rightarrow C$ . We would like to define an explicit map of complexes

$$
f_*: C_{\bullet}(A) \otimes C_{\bullet}(B) \to C_{\bullet}(C).
$$

It is given by

<span id="page-9-0"></span>
$$
(2.2) \quad f_*((a_0, \ldots, a_n) \otimes (b_0, \ldots, b_m)) =
$$
\n
$$
\sum_{\substack{0 \le i_0 \le \cdots \le i_k \le n; \\0 \le j_0 \le \cdots \le j_k \le m; \\0 \le j_0 \le \cdots \le j_k \le m;}} (-1)^{\varepsilon(i_0, \ldots, i_k, j_1, \ldots, j_k, p, q)} (\mu_{k+p+2-q}(f_{i_{q+1}-i_q, j_q-j_{q-1}}(a_{i_{q}+1}, \ldots, a_{i_{q+1}}, b_{j_{q-1}+1}, \ldots, b_{j_q}),
$$
\n
$$
\sum_{0 \le j_0 \le \cdots \le j_k \le m; \\0 \le p < q \le k} \ldots, f_{i_{p+1}-i_{p}, j_{p}-j_{p-1}}(a_{i_{p}+1}, \ldots, a_{i_{p+1}}, b_{j_{p-1}+1}, \ldots, b_{j_p}),
$$
\n
$$
f_{i_{p+2}-i_{p+1}, j_{p+1}-j_p}(a_{i_{p+1}+1}, \ldots, a_{i_{p+2}}, b_{j_{p}+1}, \ldots, b_{j_{p+1}}), \ldots, f_{i_{q-1}+j_{q-2}}(a_{i_{q-1}+1}, \ldots, a_{i_{q}}, b_{j_{q-2}+1}, \ldots, b_{j_{q-1}})),
$$

where

$$
\varepsilon(i_0,\ldots,i_k,j_1,\ldots,j_k,p,q) = l_0^m(a) + l_{i_q+1}^n(a)l_0^{i_q}(a) + l_{j_{q-1}+1}^m(b)l_0^{j_{q-1}}(b) + 1 + \sum_{s=1}^k l_{i_{q+s}+1}^{i_{q+s+1}}(a)l_{j_{q-1}+1}^{j_{q+s-1}}(b).
$$

In this summation we mean that  $i_{s+k+1} = i_s$ ,  $j_{s+k+1} = j_s$ ,  $a_{s+n+1} = a_s$ ,  $b_{s+m+1} = b_s$ . Also, we require that for all  $s = 1, \ldots, k - 1$  we have  $(i_{s+1} - i_s) + (j_s - j_{s-1}) > 0$ , so that we don't get the (non-existing)  $f_{0,0}$  anywhere.

**Remark 2.1.** Suppose that we are in the special situation when A, B and C are DG algebras, and the  $A_{\infty}$ -bimorphism f has only two non-zero components  $f_{1,0}$  and  $f_{0,1}$ . This is equivalent to a DG algebra morphism  $A \otimes B \to C$ , which we still denote by f.

The map given by [\(2.2\)](#page-9-0) is obtained by composing the map  $C_{\bullet}(A \otimes B) \to C_{\bullet}(C)$  with the Eilenberg-Zilber map  $EZ : C_{\bullet}(A) \otimes C_{\bullet}(B) \to C_{\bullet}(A \otimes B)$ .

<span id="page-10-1"></span>**Proposition 2.2.** Let  $A_1$  and  $A_2$  be strictly unital  $A_\infty$ -algebras, and M a finite dimensional strictly unital  $A_{\infty}$   $A_1$ - $A_2$ -bimodule (we require that  $\dim \oplus_n \dim(M^n) < \infty$ ). Then the composition map

$$
\psi: HH_{\bullet}(A_1) \otimes HH_{\bullet}(A_2^{op}) \xrightarrow{\mathrm{id} \otimes B} HH_{\bullet}(A_1) \otimes HH_{\bullet}(A_2^{op}) \to HH_{\bullet}(\mathrm{End}(V)) \to \mathrm{k}
$$

is given by the following explicit formula:

$$
\psi((a_0, ..., a_n) \otimes (b_0, ..., b_m)) = \text{str}_M(m \mapsto
$$
  

$$
\mapsto (-1)^{l_0^m(b) \cdot |m|} \sum_{\substack{0 \le i \le n; \\ 0 \le j \le m}} (-1)^{\sigma_{i,j}} \mu_{n+1,m+1}(a_i, ..., a_k, ..., a_{i-1}, m, b_j, ..., b_0, b_l, ..., b_{j+1}),
$$

where

$$
\sigma_{i,j} = l_0^n(a) + l_0^{i-1}(a)l_i^n(a) + \sum_{0 \le p < q \le j} (|b_p| + 1)(|b_q| + 1) + \sum_{j+1 \le p < q \le m} (|b_p| + 1)(|b_q| + 1).
$$

*Proof.* Recall that for a finite-dimensional complex V the natural map  $HH_{\bullet}(\text{End}_{k}(V)) \to k$ (which is an isomorphism if and only if  $M$  is not acyclic) is given by the following morphism of complexes  $C_{\bullet}(\text{End}_{k}(V)) \to k$ :

$$
(a_0, \ldots, a_k) \mapsto \begin{cases} \text{str}_M(a_0) & \text{for } k = 0, |a_0| = 0; \\ 0 & \text{otherwise.} \end{cases}
$$

.

The result follows by applying the formula [\(2.2\)](#page-9-0) and Remark [1.11](#page-7-0) (and taking the strict unitality into account).  $\Box$ 

Finally, we mention one particular corollary which we need in this paper.

<span id="page-10-0"></span>**Corollary 2.3.** Let A be a finite-dimensional non-unital  $A_{\infty}$ -algebra, and  $a, b \in A$  are closed homogeneous elements such that  $|a| + |b| = 1$ . If we consider a and b as classes in  $HH_{\bullet}(A)$  and  $HH_{\bullet}(A^{op})$  respectively. Then

$$
\langle a, B(b) \rangle = (-1)^{|a|+1} \operatorname{str}_A(v \mapsto (-1)^{(|b|+1)\cdot |v|} \mu_3(a, v, b)).
$$

*Proof.* This follows immediately from Proposition [2.2.](#page-10-1)

## <span id="page-11-0"></span>3. A counterexample to the generalized degeneration conjecture

We recall the main conjecture of [\[E1\]](#page-19-9).

<span id="page-11-1"></span>**Conjecture 3.1.** [\[E1,](#page-19-9) Conjecture 1.3 for  $n = 0$ ] Let B and C be small DG categories over a field k of characteristic zero. Then the composition map

<span id="page-11-4"></span>(3.1)  $\varphi_0: K_0(\mathcal{B} \otimes \mathcal{C}) \xrightarrow{\text{ch}} (HH_\bullet(\mathcal{B}) \otimes HH_\bullet(\mathcal{C}))_0 \xrightarrow{\text{id} \otimes \delta} (HH_\bullet(\mathcal{B}) \otimes HC_\bullet^-(\mathcal{C}))_1$ 

is zero.

In this section we construct a counterexample to Conjecture [3.1.](#page-11-1) We put  $\Lambda_1 = \mathbf{k}\langle \xi \rangle / \xi^2$ , where  $|\xi| = 1$ , and (automatically)  $d\xi = 0$ . We have a quasi-equivalence Perf( $\Lambda_1$ )  $\simeq$  $\text{Perf}_{\{0\}}(\mathbb{A}_{k}^{1})$  (the free  $\Lambda_{1}$ -module of rank 1 corresponds to the skyscraper sheaf  $\mathcal{O}_{0}$ ). In particular, we have a short exact sequence

<span id="page-11-2"></span>(3.2) 
$$
0 \to \text{Perf}(\Lambda_1) \to \text{Perf}(\mathbb{A}^1) \to \text{Perf}(\mathbb{G}_m) \to 0
$$

We also denote by  $k[\varepsilon] := k[t]/t^2$  the algebra of dual numbers  $(|\varepsilon| = 0, d\varepsilon = 0)$ . Let us denote by x the coordinate on  $\mathbb{A}^1$ , and put  $T := \text{Spec}(\mathbf{k}[\varepsilon])$ . Tensoring [\(3.2\)](#page-11-2) by  $\mathbf{k}[\varepsilon]$ (and taking perfect complexes), we obtain another short exact sequence:

<span id="page-11-3"></span>(3.3) 
$$
0 \to \text{Perf}(\Lambda_1 \otimes k[\epsilon]) \to \text{Perf}(\mathbb{A}^1 \times T) \to \text{Perf}(\mathbb{G}_m \times T) \to 0.
$$

Now let us take the Cartier divisor  $D := \{x + \varepsilon = 0\} \subset \mathbb{A}^1 \times T$ . This is well-defined since  $x + \varepsilon$  is not a zero divisor in  $k[x] \otimes k[\varepsilon]$ . Moreover, we have  $D \cap (\mathbb{G}_m \times T) = \emptyset$ , since  $x + \varepsilon$  is invertible in  $k[x^{\pm 1}] \otimes k[\varepsilon]$ : we have  $(x + \varepsilon)(x^{-1} - x^{-2}\varepsilon) = 1$ . Therefore, by  $(3.3)$ , we may and will consider  $\mathcal{O}_D$  as an object of Perf( $\Lambda_1 \otimes k[\varepsilon]$ ).

**Theorem 3.2.** Conjecture [3.1](#page-11-1) does not hold for the DG algebras  $\Lambda_1$  and k[ε]. Namely, we have  $\varphi_0([\mathcal{O}_D]) \neq 0$ , where  $\varphi_0$  is defined in [\(3.1\)](#page-11-4).

*Proof.* We will prove a stronger statement:  $\psi_0([\mathcal{O}_D]) \neq 0$ , where  $\psi_0$  is the composition

$$
K_0(\Lambda_1 \otimes \mathbf{k}[\varepsilon]) \xrightarrow{\mathrm{ch}} (HH_\bullet(\Lambda_1) \otimes HH_\bullet(\mathbf{k}[\varepsilon]))_0 \xrightarrow{\mathrm{id} \otimes B} (HH_\bullet(\Lambda_1) \otimes HH_\bullet(\mathbf{k}[\varepsilon]))_1.
$$

We use the notation  $d_{dR}$  for the de Rham differential in order to avoid confusion with differentials in DG algebras.

First let us identify the Hochschild homology of  $\Lambda_1$ . Applying the long exact sequence in Hochschild homology to [\(3.2\)](#page-11-2), we see that

$$
HH_{-1}(\Lambda_1) = k[x^{\pm 1}]/k[x], \text{ and } HH_0(\Lambda_1) = k[x^{\pm 1}]d_{R}x/k[x]d_{R}x,
$$

and  $HH<sub>i</sub>(\Lambda<sub>1</sub>) = 0$  for  $i \notin \{-1, 0\}$ .

Further, for any commutative k-algebra R we have  $HH_0(R) = R$ , and  $HH_1(R) =$  $\Omega^1_{R/k}$ , (and the Connes differential  $B:HH_0(R) \to HH_1(R)$  is given by the de Rham

differential). In particular, we have  $HH_0(k[\varepsilon]) = k[\varepsilon]$ , and  $HH_1(k[\varepsilon]) = k \cdot d_{dR}\varepsilon$  (and we do not need  $HH_{\geq 2}(\mathbf{k}[\varepsilon])$  for our considerations).

**Claim.** Within the above notation, we have  $\text{ch}(\mathcal{O}_D) = \frac{d_{dR}x}{x} \otimes 1 - \frac{d_{dR}x}{x^2} \otimes \varepsilon + \frac{1}{x} \otimes d_{dR} \varepsilon$ .

*Proof.* As we already mentioned, the function  $x + \varepsilon$  is invertible on  $\mathbb{G}_m \times T$ , hence it gives an element  $\alpha \in K_1(\mathbb{G}_m \times T)$ . Moreover, the boundary map

$$
K_1(\mathbb{G}_m \times T) \to K_0(\Lambda_1 \otimes \mathbf{k}[\varepsilon])
$$

sends  $\alpha$  to  $[\mathcal{O}_D]$ . We have  $\text{ch}(\alpha) = d_{dR} \log(x + \varepsilon) \in \Omega_{\mathbb{G}_m \times T}^1 = HH_1(\mathbb{G}_m \times T)$ . Explicitly, we have

$$
d_{dR}\log(x+\varepsilon)=(x^{-1}-x^{-2}\varepsilon)d_{dR}(x+\varepsilon)=\frac{d_{dR}x}{x}-\frac{\varepsilon d_{dR}x}{x^2}+\frac{d_{dR}\varepsilon}{x}.
$$

Applying the boundary map  $HH_1(\mathbb{G}_m \times T) \to HH_0(\Lambda_1 \otimes k[\varepsilon])$ , we obtain the desired formula for ch $(\mathcal{O}_D)$ .

It follows from Claim that

$$
(\mathrm{id}\otimes B)(\mathrm{ch}([\mathcal{O}_D])) = -\frac{d_{dR}x}{x^2}\otimes d_{dR}\varepsilon \neq 0.
$$

<span id="page-12-0"></span>This proves the theorem.  $\Box$ 

4. A counterexample to Conjecture [0.3](#page-1-1)

In this section we disprove Conjecture [0.3.](#page-1-1)

<span id="page-12-1"></span>**Proposition 4.1.** Let B be a smooth DG algebra and  $F : \text{Perf}(A) \to \text{Perf}(B)$  a localization functor, where A is a smooth and proper DG algebra. Then Conjecture [0.2](#page-1-2) holds for B.

Proof. This is actually explained in [\[E1,](#page-19-9) proof of Theorem 4.6]. We explain the argument for completeness. The localization assumption implies that  $(F \otimes F^{op})^*(I_A) = I_B$ . In particular, the map  $HH_{\bullet}(A)\otimes HH_{\bullet}(A^{op})\to HH_{\bullet}(B)\otimes HH_{\bullet}(B^{op})$  takes  $\text{ch}(I_A)$  to  $\text{ch}(I_B)$ . It remains to apply the commutative diagram

$$
HH_{\bullet}(A) \otimes HH_{\bullet}(A^{op}) \xrightarrow{\text{id} \otimes \delta} HH_{\bullet}(A) \otimes HC_{\bullet}^{-}(A^{op})[-1]
$$
  

$$
\downarrow \qquad \qquad \downarrow
$$
  

$$
HH_{\bullet}(B) \otimes HH_{\bullet}(B^{op}) \xrightarrow{\text{id} \otimes \delta} HH_{\bullet}(B) \otimes HC_{\bullet}^{-}(B^{op})[1],
$$

and Theorem [0.1](#page-1-3) applied to  $A$ .

We have the following corollary, mentioned in the introduction.

**Corollary 4.2.** Let X be a separated scheme of finite type over k, and  $\mathcal{G} \in D^b_{coh}(X)$  – a generator. Then Conjecture [0.3](#page-1-1) holds for the smooth DG algebra  $A = \mathbf{R} \text{End}(\mathcal{G})$ .

*Proof.* Indeed, by  $[E2, Theorem 1.8 1]$ , there is a localization functor of the form  $D_{coh}^b(Y) \to D_{coh}^b(X)$ , where Y is a smooth projective algebraic variety over k. The result follows by Proposition [4.1.](#page-12-1) Note that here we don't even need to apply Theorem [0.1](#page-1-3) since we only use the classical Hodge-to-de Rham degeneration for  $Y$ .

**Remark 4.3.** In fact, in the formulation of Proposition [4.1](#page-12-1) we could weaken the assumption on the functor  $F$  to be a localization, requiring it only to be a homological epimorphism, which means that the functor  $D(A) \to D(B)$  a localization, see [\[E2,](#page-19-7) Section 3]. Then in the proof of Corollary [5.2](#page-15-1) we can apply the corresponding weakened version of  $E2$ , Theorem 1.8 1)] which is much easier to prove.

Clearly, Conjecture [0.3](#page-1-1) is a special case of Conjecture [3.1.](#page-11-1) On the other hand, it was proved in [\[E1\]](#page-19-9) that Conjectures [3.1](#page-11-1) and [0.3](#page-1-1) are actually equivalent (more precisely, this follows from the proof of  $[E1, Theorem 4.6]$ . However, deducing an explicit counterexample to Conjecture [0.3](#page-1-1) along the lines of [\[E1\]](#page-19-9) would require some computations, which we wish to avoid. Instead, we use some trick.

Let us take some elliptic curve E over k, with a k-rational point  $p \in E(k)$ . Choosing a local parameter  $x \in \mathcal{O}_{E,p}$ , we get an identification  $\text{Perf}(\Lambda_1) \simeq \text{Perf}_{\{p\}}(E) \subset \text{Perf}(E)$ . Let us choose some generator  $\mathcal{F} \in \text{Perf}(E)$  (e.g.  $\mathcal{F} = \mathcal{O}_E \oplus \mathcal{O}_p$ ), and put  $B_E = \mathbf{R} \text{End}(\mathcal{F})$ , so that  $\text{Perf}(B_E) \simeq \text{Perf}(E)$ . We denote by  $F : \text{Perf}(\Lambda_1) \hookrightarrow \text{Perf}(B_E)$  the resulting embedding.

Further, we denote by C the semi-free DG algebra  $k\langle t_1, t_2 \rangle$ , with  $|t_1| = 0$ ,  $|t_2| = -1$ ,  $dt_1 = 0$ , and  $dt_2 = t_1^2$ .

We take the object  $M \in \text{Perf}(\Lambda_1 \otimes C \otimes C)$  whose image in  $\text{Perf}(k[x] \otimes C \otimes C)$  is given by

$$
Cone(\mathbf{k}[x]\otimes C^{\otimes 2}\xrightarrow{x\otimes 1^{\otimes 2}+1\otimes t_1^{\otimes 2}}\mathbf{k}[x]\otimes C^{\otimes 2}).
$$

As in the previous section, we see that  $M$  is well-defined since the element

$$
x \otimes 1^{\otimes 2} + 1 \otimes t_1^{\otimes 2} \in H^0(\mathbf{k}[x^{\pm 1}] \otimes C \otimes C) = \mathbf{k}[x^{\pm 1}] \otimes \mathbf{k}[\varepsilon] \otimes \mathbf{k}[\varepsilon]
$$

is invertible.

Finally, we put  $N := (F \otimes \mathrm{id}_{C}^{\otimes 2})^{*}(M) \in \mathrm{Perf}(B_{E} \otimes C \otimes C).$ 

**Theorem 4.4.** 1) Within the above notation, the dg algebra

$$
A:=\begin{pmatrix} B_E\otimes C & 0 \\ N & C^{op} \end{pmatrix}
$$

is homotopically finitely presented (hence smooth), but it does not satisfy Conjecture [0.3.](#page-1-1)

2) The DG category  $\text{Perf}(A)$  gives a negative answer to Question [0.4.](#page-2-2)

Proof. First, by Proposition [4.1](#page-12-1) we see that 2) reduces to 1).

We now prove 1). The homotopy finiteness of A follows from  $[E2, P$ roposition 5.15 (gluing of homotopically finite DG algebras by a perfect bimodule is again homotopically finite).

The functor  $F: \text{Perf}(\Lambda_1) \to \text{Perf}(B_E) \simeq \text{Perf}(E)$  induces a map  $HH_F$  in Hochschild homology. We need the following values of  $HH_F$ . First, the morphism  $HH_F:HH_0(\Lambda_1) \rightarrow$  $HH_0(E) = H^0(\mathcal{O}_E) \oplus H^1(\omega_E) \cong k \oplus k$  is given by

$$
\frac{d_{dR}x}{x^n} \mapsto \begin{cases} (0,1) & \text{for } n = 1; \\ 0 & \text{for } n > 1. \end{cases}
$$

Further, the morphism  $HH_F: HH_{-1}(\Lambda_1) \to HH_{-1}(E) = H^1(\mathcal{O}_E)$  does not vanish on  $x^{-1}$ (because there is no rational function on E having single simple pole at  $p$ ). We denote the image  $HH_F(x^{-1})$  by  $[x^{-1}]$ .

To prove 1), it suffices to show that  $(id \otimes id \otimes B)(ch(N)) \in (HH_{\bullet}(\Lambda_1) \otimes HH_{\bullet}(C)^{\otimes 2})_1$ is non-zero. We have a natural projection  $\pi : C \to H^0(C) \cong k[\varepsilon]$ . Let us put  $\overline{N} :=$  $(id \otimes \pi^* \otimes \pi^*)(N) \in \text{Perf}(E \times T \times T)$ . Then  $\overline{N}$  is naturally isomorphic to  $\mathcal{O}_{D'}$ , where  $D' \subset E \times T \times T$  is a Cartier divisor, set-theoretically contained in  $\{p\} \times T \times T$ , and given locally by the equation  $x \otimes 1^{\otimes 2} + 1 \otimes \varepsilon^{\otimes 2} = 0$ . The computation from Section [3](#page-11-0) implies that

$$
\mathrm{ch}(\bar{N})=(0,1)\otimes 1^{\otimes 2}+[x^{-1}]\otimes d_{dR}\varepsilon\otimes \varepsilon+[x^{-1}]\otimes \varepsilon\otimes d_{dR}\varepsilon.
$$

Therefore, we obtain

$$
(\mathrm{id}\otimes \mathrm{id}\otimes B)(\mathrm{ch}(\bar{N}))=[x^{-1}]\otimes d_{dR}\varepsilon\otimes d_{dR}\varepsilon\neq 0.
$$

<span id="page-14-0"></span>By functoriality, this implies  $(id \otimes id \otimes B)(ch(N)) \neq 0$ . This proves 1).

5. A counterexample to Conjecture [0.2](#page-1-2)

In this section we disprove Conjecture [0.2.](#page-1-2)

More precisely, we will construct an example of a minimal finite-dimensional  $A_{\infty}$ -algebra B and two elements  $a, b \in B$ , such that  $|a| + |b| = 1$ , and

$$
str_B(v \mapsto (-1^{(|b|+1)|v|}) \mu_3(a, v, b)) \neq 0,
$$

thus disproving Conjecture [0.2](#page-1-2) (by Corollary [2.3\)](#page-10-0).

We first mention the following observation, which in fact motivates Conjecture [0.2.](#page-1-2)

<span id="page-14-1"></span>**Proposition 5.1.** Let B be a proper DG algebra and  $\text{Perf}(B) \hookrightarrow \text{Perf}(A)$  a quasi-fullyfaithful functor, where A is a smooth and proper DG algebra. Then Conjecture [0.2](#page-1-2) holds for B.

Proof. Indeed this follows from the commutative diagram

$$
HH_{\bullet}(B) \otimes HC_{\bullet}(B^{op})[1] \xrightarrow{\operatorname{id} \otimes \delta} HH_{\bullet}(B) \otimes HH_{\bullet}(B^{op}) \longrightarrow k;
$$
  

$$
\downarrow \qquad \qquad \downarrow
$$
  

$$
HH_{\bullet}(A) \otimes HC_{\bullet}(A^{op})[1] \xrightarrow{\operatorname{id} \otimes \delta} HH_{\bullet}(A) \otimes HH_{\bullet}(A^{op}) \longrightarrow k
$$

and Theorem [0.1](#page-1-3) applied to A.

We have the following corollary, mentioned in the introduction.

<span id="page-15-1"></span>Corollary 5.2. Let X be a separated scheme of finite type over k, and  $Z \subset X$  a closed proper subscheme. For any object  $\mathcal{F} \in \text{Perf}_Z(X)$ , Conjecture [0.2](#page-1-2) holds for the proper DG algebra  $B = \mathbf{R} \text{End}(\mathcal{F}).$ 

*Proof.* Choosing some compactification  $X \subset \overline{X}$  (which exist by Nagata's compactification theorem [\[N\]](#page-19-22)), we get  $\text{Perf}_Z(X) \simeq \text{Perf}(\bar{X})$ . Thus, we may and will assume  $X = \bar{X} = Z$ . Then the result follows by applying Proposition [5.1](#page-14-1) with [\[KL\]](#page-19-1)[Theorem 6.12]. As in the proof of Corollary [5.2,](#page-15-1) we only use here the classical Hodge-to-de Rham degeneration.  $\Box$ 

The crucial point is the following theorem, which is of independent interest.

<span id="page-15-0"></span>**Theorem 5.3.** 1) Let A be a DG algebra, and  $a \in H^0(A)$  a nilpotent element. Then the corresponding morphism  $f : k[x] \to A$  (where  $|x| = 0$ ) in Ho( $\text{dgalg}_k$ ) factors through  $k[x]/x^n$  for a sufficiently large n.

2) If moreover  $a^2 = 0$  in  $H^0(A)$ , then it suffices to take  $n = 6$ .

Before we prove Theorem [5.3,](#page-15-0) we show how it allows to construct a counterexample to Conjecture [0.2.](#page-1-2)

**Theorem 5.4.** 1) Let us denote by y the variable of degree 1. Then there exists an  $A_{\infty}$  $\text{rk}[y]/y^3$ - $\text{rk}[x]/x^6$  -bimodule structure on the 1-dimensional vector space  $V = \text{k} \cdot z$  (where  $|z|=0$ ) such that  $\mu_3^V(x,z,y)=z$ . In particular, in the glued  $A_{\infty}$ -algebra

$$
B = \begin{pmatrix} k[y]/y^3 & 0\\ V & k[x]/x^6 \end{pmatrix}
$$

we have  $str(v \mapsto \mu_3(x, v, y)) = 1$ . Therefore, by Corollary [2.3](#page-10-0) this  $A_{\infty}$ -algebra (and any quasi-isomorphic DG algebra) does not satisfy Conjecture [0.2.](#page-1-2)

2) In particular, the proper DG category  $\text{Perf}^{\infty}(B)$  does not have a categorical resolution of singularities.

*Proof.* 1) An easy computation shows that  $Ext^0_{k[y]/y^3}(k, k) = k[\varepsilon]$  (dual numbers). By Theorem [5.3](#page-15-0) 2), we have an  $A_{\infty}$ -morphism  $g: k[x]/x^6 \to \text{End}_{k[y]/y^3}^{A_{\infty}}(k)$ , such that  $\overline{g_1(x)} =$   $\varepsilon \in H^0(\text{End}_{\text{kj}/\mathcal{Y}^3}^{A_{\infty}}(k)).$  This gives the desired  $A_{\infty}$ -bimodule structure on V. The rest conclusions are clear.

2) follows from 1) and Proposition [5.1.](#page-14-1)

*Proof of Theorem [5.3,](#page-15-0) part 1).* Let us denote by  $A_f$  the k[x]-A-bimodule which equals A as an A-module, and whose  $k[x]$ -module structure comes from f. Since the algebra k[x] is smooth, we have  $A_f \in D_{\text{perf}}(k[x] \otimes A)$ . Since  $a \in H^0(A)$  is nilpotent, we have  $k[x^{\pm 1}] \otimes_{k[x]}^{\mathbf{L}} A = 0$ . We conclude that  $A_f$  is contained in the essential image of  $D_{\text{perf}}(\Lambda_1 \otimes$  $A) \hookrightarrow D_{\text{perf}}(k[x] \otimes A).$ 

Now, let us note that in Ho(dgcat<sub>k</sub>) we have  $\text{Perf}(\Lambda_1) \simeq \text{colim}_n \text{PsPerf}(k[x]/x^n)$ . It follows that we have an equivalence of triangulated categories

$$
D_{\text{perf}}(\Lambda_1 \otimes A) \simeq \operatorname{colim}_{n} D_{\text{perf}}(\text{PsPerf}(\kappa[x]/x^n) \otimes A).
$$

Therefore, there exists  $n > 0$  such that  $A_f$  is contained in the essential image of  $D_{\text{perf}}(\text{PsPerf}(k[x]/x^n) \otimes A)$ . Let us denote by  $\tilde{M} \in D_{\text{perf}}(\text{PsPerf}(k[x]/x^n) \otimes A)$  an object whose image is isomorphic to  $A_f$ . We have a natural functor

$$
\Phi: \mathrm{PsPerf}(\mathrm{k}[x]/x^n) \otimes \mathrm{Perf}(A) \to \mathbf{R\underline{Hom}}(\mathrm{k}[x]/x^n, \mathrm{Perf}(A)).
$$

By construction, the  $k[x]/x^n$ -A-bimodule  $\Phi(\tilde{M})$  is quasi-isomorphic to A as an A-module Choosing an isomorphism  $\Phi(\tilde{M})_{A} \stackrel{\sim}{\rightarrow} A$ , we obtain the following composition morphism in  $\text{Ho}(\text{dgalg}_k)$ :

$$
g: \mathbf{k}[x]/x^n \to \mathbf{R}\operatorname{End}_A(\Phi(\tilde{M})) \xrightarrow{\sim} A.
$$

By construction,  $H^0(g)(x) = a$ . Thus, g factors f through  $k[x]/x^n$ . This proves part  $\Box$ 

The proof of part 2) of Theorem [5.3](#page-15-0) requires some computations which we split into several lemmas.

First, we may replace the abstract algebra  $A$  by the concrete DG algebra  $C$  which was used in Section [4.](#page-12-0) Recall that it is freely generated by the elements  $t_1$ ,  $t_2$  with  $|t_1| = 0$ ,  $|t_2| = -1$ , and  $dt_1 = 0$ ,  $dt_2 = t_1^2$ . Indeed choosing a representative  $\tilde{a} \in A^0$  of a, and an element  $h \in A^{-1}$  such that  $dh = \tilde{a}^2$ , we obtain a morphism of DG algebras  $C \to A$ ,  $t_1 \mapsto \tilde{a}, t_2 \mapsto h$ . Thus, we may assume that  $A = C$  and  $a = \overline{t_1}$ .

It will be very useful to introduce an additional  $\mathbb{Z}$ -grading on  $C$ , which can be thought of as a  $\mathbb{G}_m$ -action. We will denote this grading by w, putting  $w(t_1) = 1$ ,  $w(t_2) = 2$ , and then extend by the rule  $w(uv) = w(u) + w(v)$ . Clearly, the differential d has degree zero with respect to w. We thus have a decomposition of C as a complex:  $C = \bigoplus C^{\bullet, n}$ .  $n \geq 0$ 

Let us define  $\hat{C} := \prod C^{\bullet, n}$ . This is also a DG algebra, and we have a map  $C \to \hat{C}$ .  $n \geq 0$ The homogeneous elements of degree  $-m$  in  $\hat{C}$  are just non-commutative power series in  $t_1, t_2$  such that in each monomial there are exactly m copies of  $t_2$ .

<span id="page-17-0"></span>**Lemma 5.5.** The cohomology algebra  $H^{\bullet}(\hat{C})$  is generated by the elements  $u_1 = \overline{t_1}$  and  $u_2 = [t_1, t_2],$  with two relations:  $u_1^2 = 0$ ,  $u_1 u_2 + u_2 u_1 = 0.$ 

*Proof.* Indeed, it is easy to see that the DG algebra  $\hat{C}$  is isomorphic to the endomorphism DG algebra  $\text{End}_{k[y]/y^3}^{A_{\infty}}(k)$ . Thus, we have an isomorphism of graded algebras  $H^{\bullet}(\hat{C}) \cong$  $\text{Ext}^{\bullet}_{k[y]/y^3}(k, k)$ . To compute this Ext-algebra, we take the semi-free resolution  $P \to k$ . The underlying graded  $k[y]/y^3$ -module is defined by

$$
P^{gr} := \bigoplus_{n=0}^{\infty} e_n \cdot k[y]/y^3,
$$

where  $|e_n| = \lfloor \frac{n}{2} \rfloor$  $\frac{n}{2}$ . The differential is given by  $d(e_0) = 0$ , and  $d(e_{2k+1}) = e_{2k}y$ ,  $d(e_{2k+2}) = 0$  $e_{2k+1}y^2$  for  $k \ge 0$ . The morphism  $P \to k$  sends  $e_0$  to 1, and  $e_n$  to 0 for  $n > 0$ . Clearly, this is a quasi-isomorphism.

We see that  $\text{Ext}^{\bullet}_{\mathbf{k}[y]/y^3}(\mathbf{k}, \mathbf{k}) \cong \text{Hom}^{\bullet}_{\mathbf{k}[y]/y^3}(P, \mathbf{k}),$  where the last complex has zero differential, and is equipped with the homogeneous basis  ${v_n}_{n\geq 0}$ , where  $|v_n| = \lfloor \frac{n}{2} \rfloor$  $\frac{n}{2}$ , and  $v_i(e_j) = \delta_{ij}$ . It is easy to see that the elements  $v_1$  and  $v_2$  correspond to the classes  $u_1, u_2 \in H^{\bullet}(\hat{C})$ , mentioned in the formulation of the lemma. Clearly, we have  $u_1^2 = 0$ . It remains to show that  $u_1u_2 = -u_2u_1$ , and  $u_1u_2^k \neq 0$  for  $k \geq 0$ . Let us choose the lifts  $\widetilde{v_n} \in \text{End}_{k[y]/y^3}(P)$  of  $v_n$ , putting

$$
\widetilde{v_{2k}}(e_n) = \begin{cases}\n(-1)^{nk}e_{n-2k} & \text{for } n \ge 2k, \\
0 & \text{otherwise;}\n\end{cases}\n\quad\n\widetilde{v_{2k+1}}(e_n) = \begin{cases}\ne_{n-2k-1} & \text{for } n \text{ odd}, n \ge 2k+1, \\
(-1)^k e_{n-2k-1}y & \text{for } n \text{ even}, n \ge 2k+2, \\
0 & \text{otherwise.}\n\end{cases}
$$

It is easy to check that  $\widetilde{v_n}$  's super-commute with the differential, and that  $\widetilde{v_1}\widetilde{v_2}+\widetilde{v_2}\widetilde{v_1} = 0$ ,<br> $\widetilde{v_1}(\widetilde{v_2})^k = (-1)^k \widetilde{v_{2k+1}}$ . This proves the lemma  $\widetilde{v}_1(\widetilde{v}_2)^k = (-1)^k \widetilde{v_{2k+1}}$ . This proves the lemma.

<span id="page-17-1"></span>**Lemma 5.6.** The natural inclusion  $C \rightarrow \hat{C}$  is a quasi-isomorphism.

*Proof.* We already know that  $\dim H^n(\hat{C}) < \infty$  for all  $n \in \mathbb{Z}$ . It remains to observe the following: for any infinite sequence of complexes of vector spaces  $\mathcal{K}_0^{\bullet}, \mathcal{K}_1^{\bullet}, \ldots$  such that dim  $H^n(\prod$  $n\geq 0$  $\mathcal{K}_n^{\bullet}$   $\leq \infty$  for all  $n \in \mathbb{Z}$ , the morphism  $\bigoplus$  $n\geq 0$  $\mathcal{K}_n^{\bullet} \to \prod$  $n\geq 0$  $\mathcal{K}_n^{\bullet}$  is a quasiisomorphism. Applying this observation to the complexes  $C^{\bullet,n}$ , we conclude the proof.  $\Box$ 

We now construct a strictly unital  $A_{\infty}$ -morphism k[x]/ $x^{6} \rightarrow C$ , using obstruction theory. First, we introduce the weight grading ( $\mathbb{G}_m$ -action) on  $k[x]/x^6$  by putting  $w(x) = 1$ .

Our  $A_{\infty}$ -morphism will be compatible with the  $\mathbb{G}_m$ -actions, and its component  $f_1$  is given by

<span id="page-18-0"></span>(5.1) 
$$
f_1(x^k) = t_1^k \text{ for } 0 \le k \le 5.
$$

Note that all the cohomology spaces  $H^n(C)$  are  $H^0(C)$ - $H^0(C)$ -bimodules, hence also over  $k[x]/x^6 - k[x]/x^6$ -bimodules (via  $f_1$ ).

Let us also note that for any  $k[x]/x^6$ -k $[x]/x^6$ -bimodule M, equipped with the compatible  $\mathbb{G}_m$ -action, the Hochschild cohomology  $HH^{\bullet}(\mathbf{k}[x]/x^6, M)$  also becomes bigraded; the second grading again corresponds to the  $G_m$ -action. For a vector space V equipped with a  $\mathbb{G}_m$ -action,  $V = \bigoplus_{n \in \mathbb{Z}} V^n$ , we denote  $V(k)$  the same space with a twisted  $\mathbb{G}_m$ -action:  $V(k)^n = V^{k+n}.$ 

<span id="page-18-1"></span>**Lemma 5.7.** We have  $HH^{2k+2}(\mathbf{k}[x]/x^6, H^{-2k}(C)) \cong \mathbf{k}[\varepsilon](6)$  for  $k \geq 0$ , and  $HH^{2k+3}(\mathbf{k}[x]/x^6, H^{-2k-1}(C)) \cong \mathbf{k}(4)$  ( $\mathbb{G}_m$ -equivariant isomorphisms).

*Proof.* We have the following  $\mathbb{G}_m$ -equivariant resolution of the diagonal bimodule:

$$
\dots \xrightarrow{d_3} k[x]/x^6 \otimes k[x]/x^6(-6) \xrightarrow{d_2} k[x]/x^6 \otimes k[x]/x^6(-1) \xrightarrow{d_1} k[x]/x^6 \otimes k[x]/x^6 \xrightarrow{m} k[x]/x^6,
$$

where  $d_{2k+1} = x \otimes 1 - 1 \otimes x$ , and  $d_{2k} = x^5 \otimes 1 + x \otimes x^4 + \ldots 1 \otimes x^5$ . Further, by Lem-mas [5.5](#page-17-0) and [5.6](#page-17-1) we know the  $\mathbb{G}_m$ -equivariant  $H^0(C)$ - $H^0(C)$ -bimodules  $H^n(C)$ . Namely,  $H^{-2k}(C) \cong k[\varepsilon](-6k)$  (twisted diagonal bimodule), and  $H^{-2k-1}(C) \cong (k[\varepsilon])_{\sigma}(-6k-3)$ – the twisted anti-diagonal bimodule. For the later, the left and right  $H^0(C)$ -actions are given respectively by  $\varepsilon \cdot 1 = \varepsilon$ ,  $1 \cdot \varepsilon = -\varepsilon$ . The result follows by an elementary computation.  $\Box$ 

We are finally able to finish the proof of the theorem.

Proof of Theorem [5.3,](#page-15-0) part 2). As we already mentioned, we will construct (or rather show the existence of) a  $\mathbb{G}_m$ -equivariant strictly unital  $A_\infty$ -morphism  $f : k[x]/x^6 \to C$ , where  $f_1$  is given by [\(5.1\)](#page-18-0). Since  $H^0(f_1)$  is a homomorphism, we can construct  $f_2$  such that the required relation is satisfied. Suppose that we have already constructed  $\mathbb{G}_m$ -equivariant  $f_1, \ldots, f_n$  (where  $n \geq 2$ ) satisfying all the relations for the  $A_{\infty}$ -morphism that involve only  $f_1, \ldots, f_n$ . We want to construct the  $(n + 1)$  components  $f_1, \ldots, f'_n, f_{n+1}$  (again, satisfying all the relevant relations) where only  $f_n$  is possibly being replaced by another map  $f'_n$ . The standard obstruction theory tells us that the obstruction to this is given by a class in  $HH^{n+1,0}(\mathbf{k}[x]/x^6, H^{1-n}(C))$  (the  $\mathbb{G}_m$ -invariant part). Applying Lemma [5.7,](#page-18-1) we see that this space vanishes. Thus, proceeding inductively we can construct the desired  $A_{\infty}$ -morphism f. This proves the theorem.

## <span id="page-19-2"></span>**REFERENCES**

<span id="page-19-22"></span><span id="page-19-21"></span><span id="page-19-20"></span><span id="page-19-19"></span><span id="page-19-18"></span><span id="page-19-17"></span><span id="page-19-16"></span><span id="page-19-15"></span><span id="page-19-14"></span><span id="page-19-13"></span><span id="page-19-12"></span><span id="page-19-11"></span><span id="page-19-10"></span><span id="page-19-9"></span><span id="page-19-8"></span><span id="page-19-7"></span><span id="page-19-6"></span><span id="page-19-5"></span><span id="page-19-4"></span><span id="page-19-3"></span><span id="page-19-1"></span><span id="page-19-0"></span>

- <span id="page-20-1"></span>[To2] B. Toën, "The homotopy theory of dg-categories and derived Morita theory", Invent. Math. 167 (2007), no. 3, 615-667.
- <span id="page-20-0"></span>[TV] B. To¨en, M. Vaqui´e, "Moduli of objects in dg-categories", Ann. Sci. Ecole Norm. Sup. (4) 40 ´ (2007), no. 3, 387-444.
- <span id="page-20-3"></span>[Ts] B. L. Tsygan, "The homology of matrix Lie algebras over rings and the Hochschild homology", Usp. Mat. Nauk 38 (2), 217-218 (1983) [Russ. Math. Surv. 38 (2), 198-199 (1983)].
- <span id="page-20-2"></span>[W] F. Waldhausen, "Algebraic K-theory of spaces," in Algebraic and Geometric Topology: Proc. Conf., New Brunswick, NJ, 1983 (Springer, Berlin, 1985), Lect. Notes Math. 1126, pp. 318-419.

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