ON THE COHOMOLOGY OF CONGRUENCE SUBGROUPS OF GL_3 OVER THE EISENSTEIN INTEGERS

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ABSTRACT. Let F be the imaginary quadratic field of discriminant -3 and $\mathcal{O}_F$ its ring of integers. Let Γ be the arithmetic group $\mathrm{GL}_3(\mathcal{O}_F)$, and for any ideal $\mathfrak{n} \subset \mathcal{O}_F$ let $\Gamma_0(\mathfrak{n})$ be the congruence subgroup of level $\mathfrak{n}$ consisting of matrices with bottom row $(0, 0, *)$ mod $\mathfrak{n}$. In this paper we compute the cohomology spaces $H^{\nu-1}(\Gamma_0(\mathfrak{n}); \mathbb{C})$ as a Hecke module for various levels $\mathfrak{n}$, where ν is the virtual cohomological dimension of Γ. This represents the first attempt at such computations for GL_3 over an imaginary quadratic field, and complements work of Grunewald–Helling–Mennicke [22] and Cremona [11], who computed the cohomology of GL_2 over imaginary quadratic fields. In our results we observe a variety of phenomena, including cohomology classes that apparently correspond to nonselfdual cuspforms on GL_3 / F.

1. Introduction

1.1. Let F be a CM field with ring of integers $\mathcal{O}_F$. Let $\mathbf{G}$ be the Weil restriction $\mathbf{G} = \mathrm{R}_{F/\mathbb{Q}} \mathrm{GL}_n$, and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. Then according to a theorem of Franke [18], the complex group cohomology $H^*(\Gamma; \mathbb{C})$ provides a geometric incarnation of certain automorphic forms; this should be thought of as analogous to the classical Eichler–Shimura isomorphism, which computes the (parabolic) cohomology of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ in terms of holomorphic modular forms. Moreover, thanks to recent work of Harris–Lan–Taylor–Thorne [28] and Scholze [32], we know that these automorphic forms correspond to certain Galois representations, in the following sense. Suppose $\xi \in H^*(\Gamma; \mathbb{C})$ is a Hecke eigenclass, let p be a rational prime, and choose an isomorphism $\overline{\mathbb{Q}}_p \to \mathbb{C}$. Then there is a continuous semisimple Galois representation $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that the Hecke eigenvalues of ξ at almost all primes $\mathfrak{p} \in \mathcal{O}_F$ can be computed in terms of the (inverse) characteristic polynomial of $\rho(\mathrm{Frob}_\mathfrak{p})$, where $\mathrm{Frob}_\mathfrak{p}$ is the Frobenius at $\mathfrak{p}$. We refer to §3 for more details.

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1.2. Hence the complex cohomology of Γ provides a concrete method to compute certain automorphic forms and to explicitly investigate certain Galois representations. The goal of this paper is to carry out this investigation in the case of $\Gamma \subset \mathrm{GL}_3(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers in the imaginary quadratic field F of discriminant -3. In particular, for any ideal $\mathfrak{n} \subset \mathcal{O}_F$ let $\Gamma_0(\mathfrak{n}) \subset \mathrm{GL}_3(\mathcal{O}_F)$ be the subgroup of matrices with bottom row congruent to $(0, 0, *) \bmod \mathfrak{n}$. The top (potentially) nonvanishing cohomological degree of Γ is $\nu = 6$, and it is known that the cohomological degrees where the cuspidal automorphic forms can appear are $3, 4, 5$, with the contributions to any degree essentially equivalent to each other. Then our main results are the following:

- We compute the cohomology spaces just below the cohomological dimension, namely $H^5(\Gamma_0(\mathfrak{n}); \mathbb{C})$, for levels $\mathfrak{n}$ of norm $\mathrm{N}(\mathfrak{n}) \leq 919$.
- For most of the levels $\mathfrak{n}$ above, we compute the action of the Hecke operators on the cohomology for a range of primes $\mathfrak{p}$.
- Finally, for most of the eigenclasses we computed, we determine their motivic nature.

By the last point, we mean that in most cases we were able to identify the source of Galois representation that is apparently attached to the eigenclass. For example, we find that some eigenclasses correspond to elliptic curves over F via Eisenstein cohomology, and that some are attached to Hecke characters over F. The most interesting classes we find are apparently attached to nonselfdual automorphic forms over F. These are the first concretely constructed nonselfdual classes over an imaginary quadratic field, and indeed over any number field other than $\mathbb{Q}$ (classes for congruence subgroups of $\mathrm{GL}_3 / \mathbb{Q}$ were first constructed by Ash–Grayson–Green [2] and later by van Geemen–van der Kallen–Top–Verberkmoes [35]).

1.3. To place this work in context, we recall prior work. Similar computations for GL_2 over imaginary quadratic fields were done by Grunewald–Helling–Mennicke [22], Cremona [11], Cremona–Whitley [13], Bygott [10], and Lingham [30]. In particular these authors not only computed cohomology, but also compute the Hecke action and investigated the motivic nature of the cohomology. The first cohomology computations for arithmetic groups in GL_3 over an imaginary quadratic field were done by Staffeldt [33], who computed the cohomology of GL_3 of the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$. Later computations were done by two of us (PG and DY) with Dutour Sikirić–Gangl–Hanke–Schürmann [15] for GL_3 over a variety of imaginary quadratic fields and for GL_4 over the Gaussian and Eisenstein integers. However, in these higher $\mathbb{Q}$-rank papers no Hecke operators were computed. Indeed, it is the Hecke computations, together with the explicit nonselfdual eigenclasses, that represent the most novel aspects of the current paper.

1.4. Here is a guide to the paper. In §2 we outline the techniques used to carry out the cohomology computation and give an explicit description of the cell complex. In §3, we describe the techniques used to compute the Hecke action on the cohomology. Finally, in §4 we summarize the results of the computation.

2. Cohomology computation

2.1. Let $G = \mathbf{G}(\mathbb{R}) = \mathrm{GL}_3(\mathbb{C})$ be the group of real points of $\mathbf{G}$, and let $K = \mathrm{U}(3) \subset G$ be a maximal compact subgroup. Let $\Gamma \subset \mathrm{GL}_3(\mathcal{O}_F)$ be a finite-index subgroup. The group Γ acts on the symmetric space $X = G/K$ by left multiplication. Since X is contractible, we have $H^*(\Gamma; \mathbb{C}) \simeq H^*(\Gamma\backslash X; \mathbb{C})$.

Thus one can compute the complex group cohomology of Γ by computing cohomology of the locally symmetric space $\Gamma\backslash X$. This is the approach we follow, as in the works [2, 35]. More precisely, we a generalization of Voronoi's theory of perfect quadratic forms allows one to construct an $\mathrm{GL}_3(\mathcal{O}_F)$-equivariant polyhedral subdivision of X. Such a subdivision was first constructed by two of us with Dutour Sikirić–Gangl–Hanke–Schürmann [15]; we recall the setup and results below and refer to [15] for more details. Using this polyhedral subdivision we form a chain complex that computes $H^*(\Gamma\backslash X; \mathbb{C})$.

We remark that although the Hecke operators act on cohomology, they do not act directly on this chain complex. Computing the Hecke action requires different techniques, which we describe in §3.

2.2. Let V be the $\mathbb{R}$-vector space of 3×3 Hermitian matrices over $\mathbb{C}$. Inside V is the cone C of positive-definite Hermitian matrices. The cone C is preserved by homotheties $z \mapsto \lambda z$, $\lambda \in \mathbb{R}_{>0}$, and the quotient $\pi(C) = C/\mathbb{R}_{>0}$ can be identified with the symmetric space X. Under this identification, the left action of $\gamma \in \mathrm{GL}_3(\mathcal{O}_F)$ on X becomes $x \mapsto \gamma x \gamma^*$ on C, where the star denotes Hermitian conjugate.

Let $\Lambda = \mathcal{O}_F^3$, thought of as column vectors, and let $\Lambda' = \Lambda \smallsetminus \{0\}$. We have a map $q \colon \Lambda \to V$ given by $q(v) = vv^*$. Note that $q(v) = q(\varepsilon v)$ for any unit $\varepsilon \in \mathcal{O}_F^\times$. As v ranges over Λ', the points $q(v)$ range over various rank one Hermitian forms in the closure $\bar{C} \subset V$. The convex hull Π of these images is called the *Voronoi polyhedron.*

The group $\mathrm{GL}_3(\mathcal{O}_F)$ acts on Λ by left-multiplication and acts on Π. The cones on the faces of Π determine a rational polyhedral fan Σ in the closed cone $\bar{C}$. Let $\tilde{\Sigma} \subset \Sigma$ be the subset of cones that meet C nontrivially. Then the images $\pi(\sigma)$ of the cones $\sigma \in \tilde{\Sigma}$ form polyhedral cells in $\pi(C) \simeq X$. We call this subdivision of X the *Voronoi decomposition.*

2.3. It is clear that the group $\mathrm{GL}_3(\mathcal{O}_F)$ acts cellularly on the Voronoi decomposition. Moreover, one can prove that there are only finitely many cells modulo $\mathrm{GL}_3(\mathcal{O}_F)$. Let $\Sigma_k^*(\Gamma)$ denote a set of representatives modulo Γ of the k-dimensional Voronoi cells. One can use the Σ_k^* to construct a chain complex $(V_*(\Gamma), d)$ computing $H^*(\Gamma\backslash X; \mathbb{C})$, although some care must be taken since the boundary of a cell in $\tilde{\Sigma}$ may contain faces lying in $\bar{C} \smallsetminus C$ (such cells are said to lie *at infinity*), and since one must consider the action of the stabilizer subgroups. Details can be found in [15, §3], which in turn relies on [16]. Here we content ourselves with the following statement (cf. [15, Theorem 3.7]):

2.4. Theorem. *Let $\Gamma \subset \mathrm{GL}_3(\mathcal{O}_F)$ be a finite index subgroup, and let $V_k(\Gamma)$ be free abelian group on the Voronoi cells $\Sigma_k^*(\Gamma)$ mod Γ. Then there is a differential $d\colon \Sigma_k^*(\Gamma) \to \Sigma_{k-1}^*(\Gamma)$ such that the homology of the resulting complex $(V_* \otimes \mathbb{C}, d)$ is isomorphic to the cohomology $H^*(\Gamma\backslash X; \mathbb{C})$. More precisely, we have an isomorphism*

$$H_k(V_* \otimes \mathbb{C}) \longrightarrow H^{8-k}(\Gamma\backslash X; \mathbb{C}).$$

2.5. We finish this section by giving a complete description of the sets Σ_k^* for the case of interest, $F = \mathbb{Q}(\sqrt{-3})$. We encode cells by giving the lists of vectors in Λ that give the vertices of the corresponding faces of Π. Such vectors are called *minimal vectors*, because they correspond to the vectors on which certain positive-definite Hermitian forms attain their minima.

Let $\omega = \frac{1+\sqrt{-3}}{2} \in \mathcal{O}_F$. Let A denote the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ -\omega-1 & -\omega-1 & -\omega & -1 & -\omega & -1 & 1 & 1 & 0 & -\omega & -1 & 0 & 1 \\ 1 & \omega & 0 & 0 & \omega & 1 & -1 & 0 & 1 & \omega^2 & \omega & 0 & -\omega \end{bmatrix}.$$

We describe Σ^* in terms of the columns of A. In what follows, the list of integers $[m_1, \dots, m_k]$ denotes the cell in X that is the image of the cone $\sigma \in \tilde{\Sigma}$ generated by the points $q(v)$, where v is taken from the columns of A indexed by $m_1, \dots, m_k$.

The top dimensional cells in Σ^* have dimension that of $\dim X = 8$. There are two equivalence classes 8-dimensional cells, with representatives

$$a_1 = [1,2,3,4,5,6,7,8,9] \quad \text{and} \quad a_2 = [2,3,4,5,6,7,8,9,10,11,12,13].$$

Recall that the f-vector of a d-dimensional polytope σ is the vector $(f_{-1}, f_0, f_1, \dots, f_d)$, where f_k is the number of k-dimensional faces of σ and $f_{-1} := 1$. The f-vector of a_1 is

$$f(a_1) = (1, 9, 36, 84, 126, 126, 84, 36, 9, 1).$$

Since a_1 has nine vertices in an 8-dimensional space, we see that a_1 is a simplex, and this is indeed compatible with its f-vector. As such, the faces of a_1 are easily described as all of the subsets of $\{1, 2, \dots, 9\}$. The cone a_2 is not a simplex, but it is a simplicial polytope since all of its 81 facets are simplices. The f-vector of a_2 is

$$f(a_2) = (1, 12, 66, 216, 459, 648, 594, 324, 81, 1).$$

The 7-dimensional cells in Σ^* fall into two equivalence classes under the action of Γ, namely

$$b_1 = [1,2,3,4,5,6,7,8] \quad \text{and} \quad b_2 = [4,6,7,8,9,11,12,13];$$

these are both simplexes (this implies all lower-dimensional cells are also simplexes). All nine of the facets of a_1 are of type b_1. Of the 81 facets of a_2, nine are of type b_1 and 72 are of type b_2.

The 6-dimensional cells in Σ^* fall into three equivalence classes

$$c_1 = [1,2,3,4,5,6,7], \quad c_2 = [4,6,8,9,11,12,13], \quad \text{and} \quad c_3 = [4,6,7,8,9,11,13].$$

The eight facets of b_1 are type c_1. There are six facets of b_2 of type c_2 and one each of types c_1 and c_3.

The 5-dimensional cells of Σ^* fall into four equivalence classes

$$d_1 = [1,2,3,4,5,6], \qquad d_2 = [1,2,3,4,5,7],$$
$$d_3 = [4,6,8,9,11,12], \qquad d_4 = [4,8,9,11,12,13].$$

The cell c_1 has one facet of type d_1 and six facets of type d_2. The cell c_2 has three facets of type d_2, three facets of type d_3, and one facet of type d_4. The cell c_3 has one facet of type d_1 and three facets of type d_3.

The 4-dimensional cells of Σ^* fall into three equivalence classes

$$e_1 = [1,2,3,4,5], \quad e_2 = [1,2,3,4,7], \quad \text{and} \quad e_3 = [4,6,8,9,11].$$

All six of the facets of d_1 are type e_1. The cell d_2 has three facets of type e_1 and e_2 each. The cell d_3 has one facet of type e_1, three facets of type e_2, and two facets of type e_3. All six of the facets of d_4 are type e_2.

The 3-dimensional cells of Σ^* fall into two equivalence classes

$$f_1 = [1,2,3,4] \quad \text{and} \quad f_2 = [1,2,4,5].$$

The cell e_1 has three facets of type f_1 and two facets of type f_2. The cell e_2 has one facet of type e_1 and four facets of type f_2. The cell e_3 has one facet at infinity and four facets of type f_2.

The 2-dimensional cells of Σ^* fall into a single equivalence class

$$g_1 = [1,2,3].$$

All of the facets of f_1 are type g_1. The cell f_2 has one facet at infinity and three facets of type g_1. The boundary of g_1 consists of three 1-cells, all of which lie at infinity.

Since our goal is to compute $H^5(\Gamma;\mathbb{C})$, we are primarily interested in the Voronoi complex in degrees $2,3,4$.

3. Hecke operators

3.1. In this section we describe the techniques used to compute the Hecke action on the cohomology. Our basic technique is that of [23], adapted to the setting of congruence subgroups $\Gamma \subset \mathrm{GL}_3(\mathcal{O}_F)$. This technique has already been applied in several other settings: GL_2 over a totally complex quartic field [24, 29], GL_2 over a real quadratic field [25], GL_2 over a non real cubic field [14, 26], and GL_4 over $\mathbb{Q}$ [3–6]. The common feature all these cases have is that the cohomology group of interest is $H^{\nu-1}(\Gamma;\mathbb{C})$, where ν is the virtual cohomological dimension of the relevant arithmetic group. Indeed, for GL_n over a number field, the cohomological degrees where the cuspidal automorphic forms can contribute to cohomology (the *cuspidal range*) depends only on n and the signature of the field. The above cases, together with the current paper, cover all cases where the top of the cuspidal range coincides with $\nu - 1$.

Before presenting details of our method, we remark that the current case (GL_3 over an imaginary quadratic field F) is much more like GL_4 over $\mathbb{Q}$ than the GL_2 examples cited above. This is because the field F has no units of infinite order, which implies that certain complicated constructions in [24–26] are not necessary.

3.2. We begin by describing the construction of the Hecke operators. Fix a level $\mathfrak{n}$, let $\Gamma = \Gamma_0(\mathfrak{n})$, and let $\mathfrak{p}$ be a prime ideal not dividing $\mathfrak{n}$. Let π be a generator for $\mathfrak{p}$ ($\mathfrak{p}$ is always principal since F has class number 1). Consider the two diagonal matrices $g_1 = \mathrm{diag}(1, 1, \pi)$ and $g_2 = \mathrm{diag}(1, \pi, \pi)$. The matrices g_i each determine correspondences on $\Gamma\backslash X$ as follows. Let $\Gamma' = g^{-1}\Gamma g \cap \Gamma$, where g either equals g_1 or g_2. The group Γ' has finite index in both Γ and $g^{-1}\Gamma g$. We have a diagram $C(g)$

$$\begin{array}{ccc} & \Gamma'\backslash X & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\ \Gamma\backslash X & & \Gamma\backslash X \end{array}$$

called a *Hecke correspondence*. The map s is induced by the inclusion $\Gamma' \subset \Gamma$, while t is induced by the inclusion $\Gamma' \subset g^{-1}\Gamma g$ followed by the diffeomorphism $g^{-1}\Gamma g\backslash X \to \Gamma\backslash X$ given by left multiplication by g. Specifically,

$$s(\Gamma' x) = \Gamma x, \quad t(\Gamma' x) = \Gamma g x, \quad x \in X.$$

The maps s and t are finite-to-one, since the indices $[\Gamma' : \Gamma]$ and $[\Gamma' : g^{-1}\Gamma g]$ are finite. This implies that we obtain maps on cohomology

$$s^*\colon H^*(\Gamma\backslash X) \longrightarrow H^*(\Gamma'\backslash X), \quad t_*\colon H^*(\Gamma'\backslash X) \longrightarrow H^*(\Gamma\backslash X).$$

Here the map s^* is the usual induced map on cohomology, while the "wrong-way" map t_* is given by summing a class over the finite fibers of t. These maps can be composed to give a map

$$T_g := t_* s^*\colon H^*(\Gamma\backslash X;\mathbb{C}) \longrightarrow H^*(\Gamma\backslash X;\mathbb{C}),$$

which is called the *Hecke operator* associated to g. There is an obvious notion of isomorphism of Hecke correspondences. One can show that up to isomorphism, the correspondence $C(g)$ and thus the Hecke operator T_g depend only on the double coset $\Gamma g \Gamma$.

3.3. Thus for each prime $\mathfrak{p}$ prime to $\mathfrak{n}$, we have defined two operators on cohomology which we denote $T(\mathfrak{p}, k)$, $k = 1, 2$. To compute them, we need to understand how the double cosets $\Gamma g_i \Gamma$ break up into left cosets. More precisely, for $g \in \{g_1, g_2\}$, the double coset $\Gamma g \Gamma$ can be written as a disjoint union of left cosets,

$$\Gamma g \Gamma = \coprod_{h \in \Omega} \Gamma h$$

for a certain finite subset Ω of 3×3 matrices with entries in $\mathcal{O}_F$. We now compute Ω explicitly for the two operators. Let $R \subset \mathcal{O}_F$ denote a set of representatives for the residue field $\mathcal{O}_F/\mathfrak{p}$. For the operator $T(\mathfrak{p}, 1)$, the set Ω can be taken to be

$$\Omega = \left\{ \begin{bmatrix} \pi & & \\ & 1 & \\ & & 1 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} 1 & & \alpha \\ & 1 & \beta \\ & & \pi \end{bmatrix} : \alpha, \beta \in R \right\} \cup \left\{ \begin{bmatrix} 1 & \alpha & \\ & \pi & \\ & & 1 \end{bmatrix} : \alpha \in R \right\}.$$

For $T(\mathfrak{p}, 2)$, the set Ω can be taken to be

$$\Omega = \left\{ \begin{bmatrix} \pi & & \\ & \pi & \\ & & 1 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} 1 & \alpha & \beta \\ & \pi & \\ & & \pi \end{bmatrix} : \alpha, \beta \in R \right\} \cup \left\{ \begin{bmatrix} \pi & & \\ & 1 & \alpha \\ & & \pi \end{bmatrix} : \alpha \in R \right\}.$$

If f is an eigenclass with $T(\mathfrak{p}, k) = a(\mathfrak{p}, k) f$, then we define the GL_3 *Hecke polynomial*

$$h_{\mathfrak{p}}(t) = h(f, \mathfrak{p}) = 1 - a(\mathfrak{p}, 1)t + a(\mathfrak{p}, 2)\,\mathrm{N}(\mathfrak{p})t^2 - \mathrm{N}(\mathfrak{p})^3 t^3. \tag{3.1}$$

If we set $t = \mathrm{N}(\mathfrak{p})^{-s}$, where s is a complex variable, then the resulting Dirichlet polynomial is the inverse of the local factor at $\mathfrak{p}$ of the L-function attached to f. It is also this polynomial that, for almost all primes $\mathfrak{p}$, matches the inverse characteristic polynomial of $\rho(\mathrm{Frob}_{\mathfrak{p}})$, where ρ is the Galois representation attached to the eigenclass.

3.4. As mentioned before, the Hecke operators do not act on the Voronoi cells, and thus do not directly act on the Voronoi complex. To remedy this, we define another complex with Hecke action, the *sharbly complex*, that also computes the cohomology of Γ. This complex was originally introduced by Ash [1].

Recall that for any $x \in \Lambda'$, we have constructed a point $q(x) \in \bar{C}$ (see §2.2). Write $x \sim y$ if $q(x)$ and $q(y)$ determine the same point in $\bar{X}$; this happens if and only if the rays generated by $q(x)$ and $q(y)$ coincide, i.e., $\mathbb{R}_{\geq 0} q(x) = \mathbb{R}_{\geq 0} q(y)$. Let $\mathcal{S}_k$, $k \geq 0$, be the Γ-module A_k/C_k, where A_k is the set of formal $\mathbb{C}$-linear sums of symbols $\mathbf{u} = [x_1, \ldots, x_{k+3}]$, where each x_i is in Λ', and C_k is the submodule generated by

(1) $[x_{\sigma(1)}, \ldots, x_{\sigma(k+3)}] - \mathrm{sgn}(\sigma)[x_1, \ldots, x_{k+3}]$,
(2) $[x, x_2, \ldots, x_{k+3}] - [y, x_2, \ldots, x_{k+3}]$ if $x \sim y$, and
(3) $\mathbf{u}$ if $x_1, \ldots, x_{k+3}$ are contained in a hyperplane (we say $\mathbf{u}$ is *degenerate*).

We define a boundary map $\partial\colon \mathcal{S}_{k+1} \to \mathcal{S}_k$ by

$$\partial[x_1, \ldots, x_{k+3}] = \sum_{i=1}^{k+3} (-1)^i [x_1, \ldots, \hat{x}_i, \ldots, x_{k+3}], \tag{3.2}$$

where $\hat{x}_i$ means omit x_i. The resulting complex $(\mathcal{S}_*, \partial)$ is called the *sharbly complex*.

3.5. The group Γ acts on $\mathcal{S}_*$ via the obvious left action of Γ on Λ, and one can show that the homology of the complex of coinvariants $(\mathcal{S}_*)_\Gamma$ is isomorphic to the cohomology $H^*(\Gamma; \mathbb{C})$. More precisely, we have an isomorphism

$$H_i((\mathcal{S}_*)_\Gamma) \to H^{\nu-i}(\Gamma; \mathbb{C}),$$

where $\nu = 6$ is the virtual cohomological dimension of Γ; for details see [1, §1.4]. The Hecke operators also act on cohomology through this action. More precisely, let $\xi \in \mathcal{S}_*$ be a sharbly chain that is a cycle in $(\mathcal{S}_*)_\Gamma$, and write $\xi = \sum n(\mathbf{u})\mathbf{u}$. Then if Ω is the set of coset representatives described in §3.3 for the Hecke operator T, then the chain

$$T(\xi) = \sum n(\mathbf{u}) \sum_{g \in \Omega} g \cdot \mathbf{u} \tag{3.3}$$

is a well-defined cycle mod Γ.

3.6. Recall that all Voronoi cells of dimension less than or equal to 4 are simplices (§2.5). Thus in these dimensions an oriented Voronoi k-cell corresponds to an ordered list of $k+1$ vectors in Λ. This means one can naturally identify subspaces of $\mathcal{S}_{(k+1)-3}$ for $k = 2, 3, 4$ corresponding to the Voronoi cells of these dimensions. This identification is compatible with the boundary maps and the Γ-action, so we can think of $V_*(\Gamma)$ as being a subcomplex of $(\mathcal{S}_*)_\Gamma$ in these degrees. The main task in computing the Hecke action on $H^5(\Gamma; \mathbb{C})$, our cohomology group of interest, is then the following: given any cycle $\xi \in (\mathcal{S}_1)_\Gamma$ representing a cohomology class in $H^5(\Gamma; \mathbb{C})$, find a cycle $\xi' \in (\mathcal{S}_1)_\Gamma$ supported on $V_3(\Gamma)$ equivalent to it.

3.7. We now describe how to accomplish this. Given a point $q(x)$ with $x \in \Lambda$, let $R(x)$ be the point $y \in \Lambda'$ such that $y \sim x$ and $q(y)$ is closest to the origin along the ray $\mathbb{R}_{\geq 0} q(x)$. We call $R(x)$ the *spanning point* of x; for any sharbly $\mathbf{u} = [x_1, \ldots, x_n]$ we can speak of its set of spanning points. Any sharbly $\mathbf{u} = [x_1, \ldots, x_n]$ determines a closed cone $\sigma(\mathbf{u})$ in $\bar{C}$. We call a sharbly $\mathbf{u} = [x_1, \ldots, x_n]$ *reduced* if its spanning points are a subset of the spanning points of some fixed Voronoi cone; similarly we call a sharbly cycle reduced if each sharbly in its support is reduced. Note that since Voronoi cells are not simplicial in general, if $\mathbf{u}$ is reduced it does not mean that $\sigma(\mathbf{u})$ is the face of some Voronoi cell, and thus a reduced sharbly cycle need not come from a Voronoi cycle. However, it is clear that there are only finitely many reduced sharbly cycles modulo Γ. Moreover, it is not difficult to write any reduced sharbly cycle in terms of Voronoi cycles directly, so our main challenge is to rewrite $T(\xi)$ in terms of reduced sharbly cycles.

Computing directly if a sharbly is reduced is very expensive. Hence we introduce another measure on cycles, called *size*. For any 0-sharbly $\mathbf{u} = [x_1, x_2, x_3]$, define the size of $\mathbf{u}$ to be $\mathrm{Size}(\mathbf{u}) = |\mathrm{N}(\det(R(x_1), R(x_2), R(x_3))|$. We extend this to general sharblies by first defining the size of $\mathbf{u} = [x_1, \ldots, x_n]$ to be the maximum of the size of the sub 0-sharblies $[x_i, x_j, x_k]$, and then defining the size $\mathrm{Size}(\xi)$ of a chain $\xi = \sum n(\mathbf{u})\mathbf{u}$ to be the maximum size occuring in its support. Clearly the smallest size any chain can have is 1. We remark that any chain of size 1 is reduced, but the converse is not true: there are reduced sharbly cycles that have size > 1. Nevertheless, experimentally we found that every cohomology class we considered can be represented by a sharbly cycle of size 1. Thus size gives a convenient measure of success in our reduction algorithm.

3.8. We now turn to the reduction algorithm itself. The overall structure proceeds as described in [23, 25], and we refer to there for more details. Let ξ be a nonreduced 1-sharbly chain. For each 1-sharbly $\mathbf{u}$ in the support of ξ, we Γ-equivariantly choose a collection of *reducing points* for the nonreduced maximal faces of $\mathbf{u}$.[1] These points, together with the original spanning points of the $\mathbf{u}$, are assembled into a new 1-sharbly chain ξ' homologous mod Γ to ξ as described in [23, Algorithm 4.13] and as recalled below. Experimentally, the new chain ξ' always satisfies $\mathrm{Size}(\xi') < \mathrm{Size}(\xi)$, although we cannot prove this. The reduction process is repeated until we obtain a 1-sharbly chain of size 1.

We now describe how individual 1-sharblies $\mathbf{u}$ are modified during the algorithm. Each $\mathbf{u}$ is treated according to the number of its nonreduced faces (cf. Figure 1). Suppose $\mathbf{u} = [x_1, x_2, x_3, x_4]$, and for any $i = 1, \ldots, 4$ let f_i denote the 0-sharbly face $[\ldots, \hat{x}_i, \ldots]$.

Suppose first that $\mathbf{u}$ has four nonreduced faces. Then for each face f_i, we choose a reducing point w_i. The reduction step replaces $\mathbf{u}$ with a sum of eleven new 1-sharblies

$$\begin{aligned}\mathbf{u} \longmapsto &- [x_1, x_4, w_2, w_3] + [x_1, w_2, w_3, w_4] + [x_1, x_3, w_2, w_4] + [x_2, x_4, w_1, w_3] \\ &- [x_2, w_1, w_3, w_4] - [x_2, x_3, w_1, w_4] - [x_1, x_2, w_3, w_4] - [x_3, x_4, w_1, w_2] \\ &- [x_4, w_1, w_2, w_3] - [w_1, w_2, w_3, w_4] + [x_3, w_1, w_2, w_4].\end{aligned} \tag{3.4}$$

If $\mathbf{u}$ has three nonreduced faces, arrange that the face f_4 is reduced, and choose reducing points w_1, w_2, and w_3. Then the reduction step replaces $\mathbf{u}$ with a sum of eight new 1-sharblies

$$\begin{aligned}\mathbf{u} \longmapsto &[x_1, x_2, x_3, w_1] - [x_1, x_3, w_1, w_2] + [x_1, w_1, w_2, w_3] + [x_1, x_2, w_1, w_3] \\ &- [x_3, x_4, w_1, w_2] - [x_4, w_1, w_2, w_3] + [x_2, x_4, w_1, w_3] - [x_1, x_4, w_2, w_3].\end{aligned} \tag{3.5}$$

[1] In [23], such points were called *candidates*.

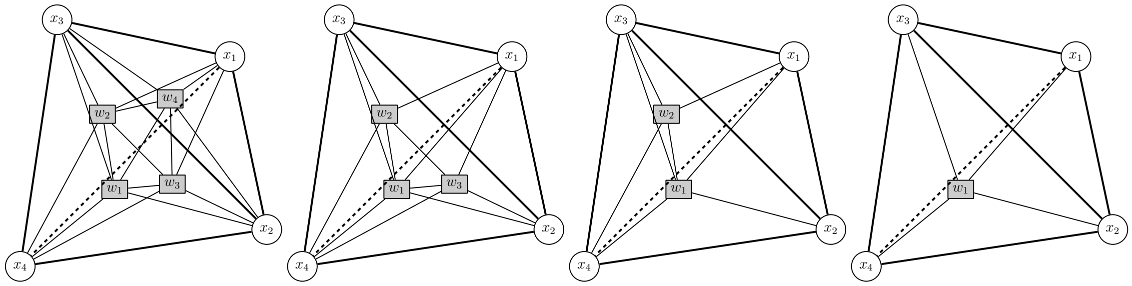

FIGURE 1. From left to right, we have from 4 to 1 nonreduced faces, as described in (3.4), (3.5), (3.6), and (3.7).

If $\mathbf{u}$ has two nonreduced faces, arrange that f_1 and f_2 are not reduced, and choose reducing points w_1 and w_2. Then the reduction step replaces $\mathbf{u}$ with a sum of five new 1-sharblies

$$\begin{aligned}\mathbf{u} \longmapsto &- [x_1, x_2, x_4, w_1] - [x_1, x_3, w_1, w_2] + [x_1, x_2, x_3, w_1] + [x_1, x_4, w_1, w_2] \\ &- [x_3, x_4, w_1, w_2].\end{aligned} \tag{3.6}$$

Finally, if $\mathbf{u}$ has one nonreduced face, arrange that f_1 is not reduced, and choose reducing point w_1. Then the reduction step replaces $\mathbf{u}$ with a sum of three new 1-sharblies

$$\mathbf{u} \longmapsto [x_1, x_2, x_3, w_1] - [x_1, x_2, x_4, w_1] + [x_1, x_3, x_4, w_1]. \tag{3.7}$$

4. COMPUTATIONAL RESULTS

4.1. Overview of the data. We computed $H^5(\Gamma_0(\mathfrak{n}); \mathbb{C})$ for ideals $\mathfrak{n} \subset \mathcal{O}_F$ of norm less than or equal to 919. If a level $\mathfrak{n}$ satisfies $\mathfrak{n} \neq \bar{\mathfrak{n}}$, then we only computed the cohomology for one of $\mathfrak{n}, \bar{\mathfrak{n}}$. There are 515 ideals of norm less than or equal to 919 which yield 303 up to conjugation. Of these, 108 levels had nontrivial cohomology. For each of these levels, we give relevant data in Table 1. We give the dimension d_3 of $H^5(\Gamma_0(\mathfrak{n}); \mathbb{C})$, the dimension of the new subspace d_3^{new} as described in §4.8. We give the dimension c_2^{new} of the new cuspidal subspace of Bianchi forms as described in §4.3. We also give the number g^{pr} of primitive Hecke Grössencharacters as described in §4.5. The last column Δ is the dimension of the space that remains.

The ideals are indexed using an HNF-label, which encodes their Hermite normal forms with respect to a certain basis of $\mathcal{O}_F$. For any ideal $\mathfrak{n} \subset \mathcal{O}_F$, there are rational integers a, c, d such that $\{a, d\omega + c\}$ is a $\mathbb{Z}$-basis of $\mathfrak{n}$, with $0 \leq c < a$; $d > 0$; $ad = \mathrm{N}(\mathfrak{n})$; $d \mid a$; and $d \mid c$. The HNF-label is the uniquely determined triple $[N, c, d]$.

Table 1: Data for nontrivial cohomology groups $H^5(\Gamma_0(\mathfrak{n});\mathbb{C})$ for $\mathrm{N}(\mathfrak{n}) \leq 919$. We give the dimension d_3 of $H^5(\Gamma_0(\mathfrak{n});\mathbb{C})$, the dimension of the new subspace d_3^{new} as described in §4.8, the dimension c_2^{new} of the new cuspidal subspace of Bianchi forms as described in §4.3, and the number g^{pr} of primitive Hecke Grössencharacters as described in §4.5. The last column Δ is the dimension of the space that remains, and these forms are dicsussed in §4.13.

HNF($\mathfrak{n}$)	HNF($\mathfrak{n}'$)	d_3	d_3^{new}	g^{pr}	c_2^{new}	Δ
[49,18,1]	[49,30,1]	2	2	1	0	0
[73,8,1]	[73,64,1]	2	2	0	1	0
[75,5,5]	[75,5,5]	2	2	0	1	0
[81,0,9]	[81,0,9]	2	2	1	0	0
[121,0,11]	[121,0,11]	2	2	0	1	0
[124,10,2]	[124,50,2]	2	2	0	1	0
[144,0,12]	[144,0,12]	2	2	1	0	0
[147,7,7]	[147,7,7]	2	2	0	1	0
[147,67,1]	[147,79,1]	6	0	0	0	0
[169,22,1]	[169,146,1]	4	4	2	0	0
[171,21,3]	[171,33,3]	2	2	0	1	0
[192,8,8]	[192,8,8]	2	2	0	1	0
[196,0,14]	[196,0,14]	2	2	0	1	0
[196,36,2]	[196,60,2]	6	0	0	0	0
[219,64,1]	[219,154,1]	6	0	0	0	0
[225,0,15]	[225,0,15]	6	0	0	0	0
[228,14,2]	[228,98,2]	2	2	0	1	0
[241,15,1]	[241,225,1]	2	2	0	1	0
[243,9,9]	[243,9,9]	6	0	0	0	0
[256,0,16]	[256,0,16]	4	4	2	0	0
[273,16,1]	[273,256,1]	2	2	0	1	0
[273,100,1]	[273,172,1]	2	2	0	1	0
[283,44,1]	[283,238,1]	2	2	0	1	0
[289,0,17]	[289,0,17]	2	2	0	1	0
[292,16,2]	[292,128,2]	6	0	0	0	0
[300,10,10]	[300,10,10]	8	2	0	1	0
[324,0,18]	[324,0,18]	8	2	0	1	0

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Table 1 – continued from previous page

HNF($\mathfrak{n}$)	HNF($\mathfrak{n}'$)	d_3	d_3^{new}	g^{pr}	c_2^{new}	Δ
[343,14,7]	[343,28,7]	8	2	0	1	0
[343,18,1]	[343,324,1]	6	0	0	0	0
[361,0,19]	[361,0,19]	2	2	0	1	0
[361,68,1]	[361,292,1]	6	6	3	0	0
[363,11,11]	[363,11,11]	8	2	0	1	0
[372,50,2]	[372,134,2]	6	0	0	0	0
[379,51,1]	[379,327,1]	2	2	0	1	0
[399,163,1]	[399,235,1]	2	2	0	1	0
[400,0,20]	[400,0,20]	2	2	0	1	0
[412,92,2]	[412,112,2]	2	2	0	1	0
[417,181,1]	[417,235,1]	2	2	0	1	0
[432,12,12]	[432,12,12]	6	0	0	0	0
[441,0,21]	[441,0,21]	8	2	0	1	0
[441,54,3]	[441,90,3]	14	2	1	0	0
[475,35,5]	[475,55,5]	2	2	0	1	0
[481,211,1]	[481,269,1]	2	2	0	1	0
[484,0,22]	[484,0,22]	6	0	0	0	0
[496,20,4]	[496,100,4]	6	0	0	0	0
[507,13,13]	[507,13,13]	6	6	0	3	0
[507,22,1]	[507,484,1]	12	0	0	0	0
[511,81,1]	[511,429,1]	6	0	0	0	0
[511,137,1]	[511,373,1]	6	0	0	0	0
[513,21,3]	[513,147,3]	8	2	0	1	0
[523,60,1]	[523,462,1]	2	2	0	1	0
[525,20,5]	[525,80,5]	6	0	0	0	0
[529,0,23]	[529,0,23]	4	4	0	2	0
[532,60,2]	[532,204,2]	4	4	0	2	0
[553,102,1]	[553,450,1]	2	2	0	1	0
[567,18,9]	[567,36,9]	6	0	0	0	0
[576,0,24]	[576,0,24]	12	0	0	0	0
[577,213,1]	[577,363,1]	2	2	0	1	0
[579,277,1]	[579,301,1]	4	4	0	2	0
[588,14,14]	[588,14,14]	14	2	0	1	0
[588,134,2]	[588,158,2]	18	0	0	0	0
[603,87,3]	[603,111,3]	2	2	0	1	0
[613,65,1]	[613,547,1]	2	2	0	1	0

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Table 1 – continued from previous page

HNF($\mathfrak{n}$)	HNF($\mathfrak{n}'$)	d_3	d_3^{new}	g^{pr}	c_2^{new}	Δ
[625,0,25]	[625,0,25]	8	8	4	0	0
[637,165,1]	[637,471,1]	6	0	0	0	0
[637,263,1]	[637,373,1]	6	0	0	0	0
[651,25,1]	[651,625,1]	2	2	0	1	0
[651,67,1]	[651,583,1]	2	2	0	1	0
[657,24,3]	[657,192,3]	12	0	0	0	0
[673,255,1]	[673,417,1]	2	2	0	1	0
[675,15,15]	[675,15,15]	12	0	0	0	0
[676,0,26]	[676,0,26]	4	4	0	2	0
[676,44,2]	[676,292,2]	12	0	0	0	0
[679,158,1]	[679,520,1]	2	2	0	1	0
[679,326,1]	[679,352,1]	2	2	0	1	0
[684,42,6]	[684,66,6]	12	0	0	0	0
[700,20,10]	[700,40,10]	2	2	0	1	0
[703,26,1]	[703,676,1]	2	2	0	1	0
[721,46,1]	[721,674,1]	2	2	0	1	0
[723,256,1]	[723,466,1]	8	2	0	1	0
[729,0,27]	[729,0,27]	19	7	2	1	1
[739,320,1]	[739,418,1]	2	2	0	0	2
[741,334,1]	[741,406,1]	2	2	0	1	0
[756,24,6]	[756,96,6]	2	2	0	1	0
[757,27,1]	[757,729,1]	2	2	0	1	0
[768,16,16]	[768,16,16]	20	2	0	1	0
[784,0,28]	[784,0,28]	6	0	0	0	0
[784,72,4]	[784,120,4]	16	4	2	0	0
[793,230,1]	[793,562,1]	2	2	0	1	0
[804,74,2]	[804,326,2]	2	2	0	1	0
[819,27,3]	[819,243,3]	6	0	0	0	0
[819,48,3]	[819,222,3]	6	0	0	0	0
[832,24,8]	[832,72,8]	4	4	0	2	0
[837,75,3]	[837,201,3]	4	4	0	1	2
[841,0,29]	[841,0,29]	4	4	0	2	0
[847,22,11]	[847,44,11]	6	0	0	0	0
[849,238,1]	[849,610,1]	6	0	0	0	0
[853,220,1]	[853,632,1]	4	4	0	1	2
[867,17,17]	[867,17,17]	14	8	0	3	2

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Table 1 – continued from previous page

HNF($\mathfrak{n}$)	HNF($\mathfrak{n}'$)	d_3	d_3^{new}	g^{pr}	c_2^{new}	Δ
[868,50,2]	[868,382,2]	6	0	0	0	0
[868,134,2]	[868,298,2]	12	6	0	3	0
[871,230,1]	[871,640,1]	2	2	0	1	0
[876,128,2]	[876,308,2]	20	2	0	1	0
[900,0,30]	[900,0,30]	26	2	0	1	0
[903,79,1]	[903,823,1]	2	2	0	1	0
[903,436,1]	[903,466,1]	2	2	0	1	0
[912,28,4]	[912,196,4]	6	0	0	0	0
[919,52,1]	[919,866,1]	4	4	0	2	0

4.2. Eisenstein cohomology and interior cohomology. Before presenting interpretations of our data, we recall some background about Eisenstein cohomology [27]. Recall that $X = \mathrm{GL}_3(\mathbb{C})/\,\mathrm{U}(3)$ is the global symmetric space. Let X^{BS} be the partial compactification constructed by Borel and Serre [7]. The quotient $Y := \Gamma_0(\mathfrak{n})\backslash X$ is an orbifold, and the quotient $Y^{\mathrm{BS}} := \Gamma_0(\mathfrak{n})\backslash X^{\mathrm{BS}}$ is a compact orbifold with corners. We have

$$H^*(\Gamma_0(\mathfrak{n};\mathbb{C}) \simeq H^*(Y;\mathbb{C}) \simeq H^*(Y^{\mathrm{BS}};\mathbb{C}).$$

Let $\partial Y^{\mathrm{BS}} = Y^{\mathrm{BS}} \smallsetminus Y$. The Hecke operators act on the cohomology of the boundary $H^*(\partial Y^{\mathrm{BS}};\mathbb{C})$, and the inclusion of the boundary $\iota\colon \partial Y^{\mathrm{BS}} \to Y^{\mathrm{BS}}$ induces a map on cohomology $\iota^*\colon H^*(Y^{\mathrm{BS}};\mathbb{C}) \to H^*(\partial Y^{\mathrm{BS}};\mathbb{C})$ compatible with the Hecke action. The kernel $H^*_!(Y^{\mathrm{BS}};\mathbb{C})$ of ι^* is called the *interior cohomology*; it equals the image of the cohomology with compact supports. The goal of Eisenstein cohomology is to use Eisenstein series and cohomology classes on the boundary to construct a Hecke-equivariant section $s\colon H^*(\partial Y^{\mathrm{BS}};\mathbb{C}) \to H^*(Y^{\mathrm{BS}};\mathbb{C})$ mapping onto a complement $H^*_{\mathrm{Eis}}(Y^{\mathrm{BS}};\mathbb{C})$ of the interior cohomology in the full cohomology. We call classes in the image of s *Eisenstein classes.*

4.3. Eisenstein cohomology I: classes coming from cuspforms on GL_2/F. Some cohomology classes that we found correspond to weight 2 Bianchi modular cuspforms. Such forms are the analogue of classical holomorphic weight 2 cuspforms for the group $\mathrm{GL}_2(\mathcal{O}_F)$ [17]. The contribution of these forms is apparently via Eisenstein cohomology [27] coming from the two types of maximal parabolic subgroups of $\mathrm{GL}_3(F)$. This phenomenon also occurs in the top degree cohomology of subgroups of $\mathrm{SL}_3(\mathbb{Z})$ [2].

More precisely, let f be a Bianchi cuspidal newform of level $\mathfrak{n}$. Then f gives rise to 2 eigenclasses ϕ and ϕ' in $H^5(\Gamma_0(\mathfrak{n});\mathbb{C})$ with associated Hecke polynomials

$$\begin{aligned} h(\phi,\mathfrak{p}) &= (1-\mathrm{N}(\mathfrak{p})^2 t)(1-a(\mathfrak{p},f)t+\mathrm{N}(\mathfrak{p})t^2) \\ h(\phi',\mathfrak{p}) &= (1-t)(1-\mathrm{N}(\mathfrak{p})a(\mathfrak{p},f)t+\mathrm{N}(\mathfrak{p})^3t^2). \end{aligned} \tag{4.1}$$

In particular, let $a(\mathfrak{p},i)$, $i=1,2$ be the Hecke eigenvalues for ϕ, and let $a'(\mathfrak{p},i)$, $i=1,2$ be the Hecke eigenvalues for ϕ'. Comparing to (3.1), we get

$$\begin{aligned} a(\mathfrak{p},1) &= a'(\mathfrak{p},2) = \mathrm{N}(\mathfrak{p})^2 + a(\mathfrak{p},f), \\ a(\mathfrak{p},2) &= a'(\mathfrak{p},1) = 1 + \mathrm{N}(\mathfrak{p})a(\mathfrak{p},f). \end{aligned} \tag{4.2}$$

4.4. Example. Let $\mathfrak{p}_{73} = (-9\omega+1)$ be a prime ideal above 73. The space of Bianchi cusp forms of level $\mathfrak{p}_{73}$ is 1-dimensional and new, corresponding the isogeny class of the elliptic curve[2] `73.1-a3` with Weierstrass equation

$$y^2 + (\omega+1)xy + y = x^3 + x^2.$$

This class contributes to the cohomology for GL_3 over F in two ways with Hecke polynomials as given in (4.1). The cohomology $H^5(\Gamma_0(\mathfrak{p}_{73});\mathbb{C})$ is 2-dimensional, so the Bianchi cuspform accounts for all of the cohomology at level $\mathfrak{p}_{73}$.

4.5. Eisenstein cohomology II: Hecke Grössencharacters. Some of the cohomology that arises can be explained by Hecke Grössencharacters of weight 1. We expect that these classes are also Eisenstein cohomology classes, again attached to the two types of maximal parabolic subgroups in GL_3/F. We briefly recall some of the details about Grössencharacters from [36], specialized to our field F.

An ∞-type $\mathbf{T}$ is a pair of integers $[n_1, n_2]$ indexed by the embeddings of F in $\mathbb{C}$. The evaluation of $\mathbf{T}$ on an element $\alpha \in F$ is given by $\mathbf{T}(\alpha) = \alpha_1^{n_1}\alpha_2^{n_2}$. For example, the ∞-type $[1,1]$ gives the norm. The *weight* of $\mathbf{T}$ is the sum n_1+n_2. For a given modulus $\mathfrak{m}$, the ∞-type $\mathbf{T}$ is *coherent* if $\mathbf{T}(\alpha)=1$ for all units $\alpha \equiv 1 \bmod \mathfrak{m}$. Given a modulus $\mathfrak{m}$ and a coherent ∞-type $\mathbf{T}$, define $\mathbf{G}((\alpha)) = \mathbf{T}(\alpha)$ for all $\alpha \equiv 1 \bmod \mathfrak{n}$. This defines a Hecke Grössencharacter $\mathbf{G}$ up to a finite quotient that is exactly the Hecke character group for $\mathfrak{m}$, the dual of the ray class group for modulus $\mathfrak{m}$. (what is this?) In particular, a Hecke character χ and an ∞-type $\mathbf{T}$ determine a Hecke Grössencharacter, which we denote $\mathbf{G}_{\chi,\mathbf{T}}$.

We observe primitive weight 1 Hecke Grössencharacters contributing to the cohomology as follows. Let H denote the Hecke character group of modulus $\mathfrak{m}$, and let D denote the Dirichlet character group of modulus $\mathfrak{m}$. Suppose $\#H = \varphi(\#D)/2$ and $\#D \equiv 0 \pmod 6$. Let ζ be a primitive $\#D$th root of unity. The values of the Grössencharacters are in a subfield K of $\mathbb{Q}(\zeta)$ and give rise to Hecke polynomials with coefficients in a real subfield L of K. Note that K is the cyclotomic field that is the codomain for the Dirichlet characters, so K is smaller than $\mathbb{Q}(\zeta)$ in some cases. Let

[2] This and other similar labels refer to the *L-functions and modular forms database* [34].

$\chi \in H$ be a character with primitive Grössencharacter $\mathbf{G}_{\chi,[1,0]}$. Then χ contributes to the cohomology in two ways. Specifically, χ gives rise to 2 eigenclasses ϕ, ϕ' of level $\mathfrak{n}$ with corresponding Hecke polynomials that have coefficients in L and factor over K as

$$\begin{aligned} h(\phi, \mathfrak{p}) &= (1 - \mathrm{N}(\mathfrak{p})^2 t)(1 - \mathbf{G}_{\chi,[1,0]}(\mathfrak{p})t)(1 - \mathbf{G}_{\chi^{-1},[0,1]}(\mathfrak{p})t), \\ h(\phi', \mathfrak{p}) &= (1 - t)(1 - \mathrm{N}(\mathfrak{p})\mathbf{G}_{\chi,[1,0]}(\mathfrak{p})t)(1 - \mathrm{N}(\mathfrak{p})\mathbf{G}_{\chi^{-1},[0,1]}(\mathfrak{p})t). \end{aligned} \tag{4.3}$$

Note $\mathbf{G}_{\chi^{-1},[0,1]}(\mathfrak{p})$ is the complex conjugate of $\mathbf{G}_{\chi,[1,0]}(\mathfrak{p})$, so

$$\mathbf{G}_{\chi,[1,0]}(\mathfrak{p})\mathbf{G}_{\chi^{-1},[0,1]}(\mathfrak{p}) = \mathrm{N}(\mathfrak{p}).$$

Let $a(\mathfrak{p}, i)$, $i = 1, 2$ be the Hecke eigenvalues for ϕ, and let $a'(\mathfrak{p}, i)$, $i = 1, 2$ be the Hecke eigenvalues for ϕ'. Comparing to (3.1), we find

$$\begin{aligned} a(\mathfrak{p}, 1) = a'(\mathfrak{p}, 2) &= \mathrm{N}(\mathfrak{p})^2 + \mathbf{G}_{\chi,[1,0]}(\mathfrak{p}) + \mathbf{G}_{\chi^{-1},[0,1]}(\mathfrak{p}) \\ a(\mathfrak{p}, 2) = a'(\mathfrak{p}, 1) &= 1 + \mathrm{N}(\mathfrak{p})(\mathbf{G}_{\chi,[1,0]}(\mathfrak{p}) + \mathbf{G}_{\chi^{-1},[0,1]}(\mathfrak{p})). \end{aligned} \tag{4.4}$$

As in §4.3, we expect the classes (4.3) to be Eisenstein cohomology classes attached to the maximal parabolic subgroups.

We note that sometimes these classes correspond to elliptic curves E over F with CM by an order in F. In this case $\mathbf{G}_{\chi,[1,0]}(\mathfrak{p}) + \mathbf{G}_{\chi^{-1},[0,1]}(\mathfrak{p})$ is in $\mathbb{Q}$ for all prime ideals $\mathfrak{p}$. The elliptic curve E is a twist of a curve E' defined over $\mathbb{Q}$. The base-change from $\mathbb{Q}$ to F of the classical holomorphic cuspform corresponding to E' is not a cuspidal Bianchi modular form. In fact, we have $L(E, s) = L(f, s)$, where f is an Eisenstein series. Note in particular that such Bianchi modular forms apparently contribute to the Eisenstein cohomology as in §4.3, although they are not cuspidal.

4.6. Example. Let $\mathfrak{p}_7$ be a prime ideal above 7, and let $\mathfrak{n} = \mathfrak{p}_7^2$. The cohomology $H^5(\Gamma_0(\mathfrak{n}); \mathbb{C})$ is 2-dimensional and can be explained by a Hecke Grössencharacter giving rise to a elliptic curve over F with CM by an order in F.

The Hecke character group of modulus $\mathfrak{p}_7$ is trivial.[3] Let e denote the trivial character. Then e is not primitive, but the corresponding Grössencharacter $\mathbf{G}_{e,[1,0]}$ is primitive. The Dirichlet character group of modulus $\mathfrak{p}_7$ is cyclic of order 6, so we are in the situation described above with ζ a 6th root of unity. Thus $K = \mathbb{Q}(\zeta) = F$ and $L = \mathbb{Q}$. In particular, the Grössencharacter gives rise to a pair of Hecke polynomials that have rational coefficients and split over F as given in (4.3). Because the Hecke polynomials have $\mathbb{Q}$ coefficients, there should be an elliptic curve over F of conductor $\mathfrak{n}$ with CM by an order in F. A quick search in LMFDB [34] yields the curve `49.3-CMa1` with Weierstrass equation

$$y^2 + (\omega + 1)y = x^3 + (-\omega - 1)x^2 + \omega x - \omega.$$

[3]We used M. Watkins's implementation of Hecke Grössencharacters in `Magma` [9] in these computations.

Let $\mathfrak{p}_7'$ denote the other prime ideal above 7. Then the cohomology at level $\mathfrak{p}_7'^2$ is also 2-dimensional and corresponds to the isogeny class of the CM elliptic curve `49.1-CMa1`. with Weierstrass equation

$$y^2 + \omega y = x^3 + (\omega + 1)x^2 + \omega x.$$

4.7. Example. We now give an example that does not correspond to an elliptic curve. Let $\mathfrak{p}_{13}$ be a prime ideal above 13. The Hecke character group is cyclic of order 2, which we denote $\{e, \chi\}$. The corresponding Grössencharacters $\mathbf{G}_{e,[1,0]}$ and $\mathbf{G}_{\chi,[1,0]}$ are both primitive. The Dirichlet character group of modulus $\mathfrak{p}_{13}$ is cyclic of order 12, so we are in the situation described above with ζ a 12th root of unity. We have Hecke polynomials with coefficients in the real quadratic field $L = \mathbb{Q}(\sqrt{3})$ that split over the degree 4 cyclotomic field $K = \mathbb{Q}(\zeta)$. The cohomology at level $\mathfrak{n} = \mathfrak{p}_{13}^2$ is 4-dimensional, so this accounts for all the cohomology. Specifically, the Hecke operators $T(\mathfrak{p}, 1)$ and $T(\mathfrak{p}, 2)$ diagonalize over $\mathbb{Q}(\sqrt{3})$. The resulting Hecke polynomials have coefficients in $\mathbb{Q}(\sqrt{3})$ and factor over $\mathbb{Q}(\zeta)$ as given in (4.3).

4.8. Old cohomology classes. We observe "old" phenomena in cohomology consistent with what happens for $\mathrm{GL}_3/\mathbb{Q}$, according to van Geeman, et. al. [35], who cite Reeder [31]. In particular, suppose f is a newform for GL_3 over F at level $\mathfrak{m}$. Then we observe f contributing to the cohomology at level $\mathfrak{n}$ with multiplicity 3 if $\mathfrak{n}/\mathfrak{m}$ is prime, with multiplicity 6 if $\mathfrak{n}/\mathfrak{m}$ is a square of a prime, and with multiplicity 9 if $\mathfrak{n}/\mathfrak{m}$ is a product of distinct primes. We call the subspace complementary to the old space *new*.

4.9. Example. Let $\mathfrak{p}_3$ be the prime above 3, and let $\mathfrak{p}_7$ be a prime above 7. Let $\mathfrak{n}$ be the norm 147 ideal $\mathfrak{n} = \mathfrak{p}_3\mathfrak{p}_7^2$. The cohomology at level $\mathfrak{n}$ is 6-dimensional. The cohomology at level $\mathfrak{p}_7^2$ is new and 2-dimensional (see Example 4.6). It contributes to $H^5(\Gamma_0(\mathfrak{n}); \mathbb{C})$ with multiplicity 3 since $\mathfrak{n}/\mathfrak{p}_7^2$ is prime, so all of the cohomology at level $\mathfrak{n}$ is old.

4.10. Example. Let $\mathfrak{n}$ be the norm 576 ideal $\mathfrak{n} = (24) = \mathfrak{p}_3^2\mathfrak{q}_2^3$, where $\mathfrak{p}_3$ is the prime above 3 and $\mathfrak{q}_2$ is the prime above 2. The cohomology at level $(12) = \mathfrak{p}_3^2\mathfrak{q}_2^2$ is 2-dimensional and new, corresponding to the isogeny class of the CM elliptic curve `144.1-CMa1` with Weierstrass equation $y^2 = x^3 + 1$. The cohomology at level $\mathfrak{p}_3\mathfrak{q}_2^3$ is 2-dimensional and new, corresponding to isogeny class of the elliptic curve `192.1-a1` with Weierstrass equation

$$y^2 = x^3 + \omega x^2 + (11\omega - 6)x + 11\omega - 1.$$

Each of these spaces contribute with multiplicity 3 to the cohomology at level $\mathfrak{n}$ since $\mathfrak{n}/(\mathfrak{p}_3^2\mathfrak{q}_2^2)$ and $\mathfrak{n}/(\mathfrak{p}_3\mathfrak{q}_2^3)$ are both prime. The cohomology at level $\mathfrak{n}$ is 12-dimensional, so oldforms account for all of the cohomology at level $\mathfrak{n}$.

4.11. Example. Let $\mathfrak{n}$ be the norm 588 ideal $\mathfrak{n} = \mathfrak{p}_3\mathfrak{q}_2\mathfrak{p}_7^2$, where $\mathfrak{p}_3$ is the prime above 3, $\mathfrak{q}_2$ is the prime above 2, and $\mathfrak{p}_7$ is a prime above 7. The cohomology at level

$\mathfrak{p}_7^2$ is new and 2-dimensional (see Example 4.6). It contributes with multiplicity 9 to the cohomology at level $\mathfrak{n}$ since $\mathfrak{n}/\mathfrak{p}_7^2$ is the product of two distinct primes. The cohomology at level $\mathfrak{n}$ is 18-dimensional, so oldforms account for all of the cohomology at level $\mathfrak{n}$.

4.12. Example. Let $\mathfrak{n}$ be the norm 657 ideal $\mathfrak{n} = \mathfrak{p}_3^2\mathfrak{p}_{73}$, where $\mathfrak{p}_3$ is the prime above 3 and $\mathfrak{p}_{73}$ is a prime above 73. The cohomology at level $\mathfrak{p}_{73}$ is 2-dimensional (see example Example 4.4). It contributes with multiplicity 6 to the cohomology at level $\mathfrak{n}$, since $\mathfrak{n}/\mathfrak{p}_{73}$ is the square of a prime ideal. The cohomology at level $\mathfrak{n}$ is 12-dimensional, so oldforms account for all of the cohomology at level $\mathfrak{n}$.

4.13. The remaining cases concern levels where, after removing the classes corresponding to Eisenstein cohomology and old classes as above, we still had some eigenclasses left over. We believe these classes are in the interior cohomology. There were two different phenomena we observed:

- For one level $\mathfrak{n} = \mathfrak{p}_3^6$, where $\mathfrak{p}_3$ is the prime over 3, we found an excess 1-dimensional space. We believe this class is the symmetric square lift of a weight 2 Bianchi modular cuspform. Some details can be found in §4.14.
- For four remaining levels, we found cohomology classes that appear to coming from nonselfdual automorphic representations. We give details in §4.15 below.

4.14. Interior cohomology I: A symmetric square lift. Let $\mathfrak{n} = \mathfrak{p}_3^6$ be the ideal of norm 729 generated by 27. The cohomology $H^5(\Gamma_0(\mathfrak{n}); \mathbb{C})$ is 19-dimensional. This is the only cohomology group in the range of computations with odd dimension.

First we describe the old classes in the cohomology. The cohomology at level $\mathfrak{p}_3^4$ is new and 2-dimensional, coming from the isogeny class of `81.1-CMa1` with Weierstrass equation

$$y^2 + y = x^3,$$

as described in §4.5. The cohomology at level $\mathfrak{p}_3^5$ is 6-dimensional, so it is all old, accounted for by the cohomology at level $\mathfrak{p}_3^4$. Thus there is a 12-dimensional old subspace in the cohomology at level $\mathfrak{n}$, as described in §4.8.

Next we consider Eisenstein classes. The space of Bianchi new cuspforms of level $\mathfrak{n}$ is 1-dimensional, and corresponds to the isogeny class of the elliptic curve `729.1-a1` with Weierstrass equation

$$y^2 + xy + \omega y = x^3 - x^2 - 2\omega x + \omega. \tag{4.5}$$

This contributes a 2-dimensional subspace to the GL_3 over F cohomology at level $\mathfrak{n}$, as described in §4.3.

The Hecke character group at level $\mathfrak{p}_3^3$ is cyclic of order 3. Let $\chi \in H$ denote a generator. The Dirichlet character group is isomorphic to $C_3 \times C_6$, so we are in the situation described in §4.5 with $K = F$ and $L = \mathbb{Q}$. The Grössencharacters $\mathbf{G}_{\chi,[1,0]}$ and $\mathbf{G}_{\chi^2,[1,0]}$ are primitive. They give rise to a 4-dimensional subspace of the

TABLE 2. Hecke polynomials for the symmetric square eigenclass at the norm 729 level $\mathfrak{p}_3^6$, described in §4.14.

HNF($\mathfrak{p}$)	$h(\phi, \mathfrak{p})$
$[4, 0, 2]$	$(-4t + 1)(16t^2 + 7t + 1)$
$[7, 2, 1]$	$(-7t + 1)(49t^2 + 10t + 1)$
$[7, 4, 1]$	$(-7t + 1)(49t^2 + 10t + 1)$
$[13, 3, 1]$	$(-13t + 1)(169t^2 + 25t + 1)$
$[13, 9, 1]$	$(-13t + 1)(169t^2 + 25t + 1)$

cohomology at level $\mathfrak{n}$ corresponding to the two isogeny classes of the elliptic curves over F of conductor $\mathfrak{n}$ with CM in $\mathcal{O}_F$. A search in LMFDB [34] gives `729.1-CMa1` with Weierstrass equation

$$y^2 + \omega y = x^3,$$

and `729.1-CMb1` with Weierstrass equation

$$y^2 + (\omega + 1)y = x^3 - \omega.$$

After accounting for the above eigenclasses, we find a 1-dimensional subspace left. Let ϕ be the Hecke eigenclass spanning this complement. Some Hecke polynomials for ϕ are given in Table 2.

We claim that ϕ appears to be the symmetric square lift of the class corresponding to a Bianchi newform `729.1-a` of level $\mathfrak{n}$, in fact of the same form attached to the elliptic curve (4.5) above. Indeed, let f be this form, and for any prime $\mathfrak{p} \nmid \mathfrak{n}$ let $a_\mathfrak{p}$ be the eigenvalue of the GL_2-Hecke operator $T_\mathfrak{p}$. Let $\alpha_\mathfrak{p}$, $\beta_\mathfrak{p}$ be the complex roots of the Hecke polynomial $1 - a_\mathfrak{p} t + \mathrm{N}(\mathfrak{p})t^2$. Then one expects the Hecke polynomial of the symmetric square $\mathrm{Sym}^2 f$ to have the form

$$(1-\alpha_\mathfrak{p}^2 t)(1-\alpha_\mathfrak{p}\beta_\mathfrak{p} t)(1-\beta_\mathfrak{p}^2 t) = 1-(a_\mathfrak{p}^2-\mathrm{N}(\mathfrak{p}))T+(a_\mathfrak{p}^2-\mathrm{N}(\mathfrak{p}))\,\mathrm{N}(\mathfrak{p})t^2-\mathrm{N}(\mathfrak{p})^3t^3. \tag{4.6}$$

The form f has eigenvalues $a_2 = -1$, $a_{\mathfrak{p}_7} = a_{\mathfrak{p}_7'} = 2$, and $a_{\mathfrak{p}_{13}} = a_{\mathfrak{p}_{13}'} = -1$, which after insertion into (4.6) recovers Table 2.[4]

[4]We remark that there is more to be said about this Bianchi modular form f. It is itself a base change of a weight two newform g (`81.2.1.a`) on $\Gamma_0(81) \subset \mathrm{SL}_2(\mathbb{Z})$ with coefficients in the quadratic field $\mathbb{Q}(\eta)$ of discriminant 12, where $\eta^2 = 3$. The eigenvalue of T_p, $p \nmid 3$ on g away is rational (respectively in $\mathbb{Z} \cdot \eta$) exactly when p is inert (respectively splits) in our imaginary quadratic field F. Thus g corresponds to an abelian surface with extra twist as in [12].

4.15. Interior cohomology II: Nonselfdual classes. We come at last to the most interesting classes in the paper, those that apparently correspond to nonselfdual automorphic representations. For a recent reference on automorphic forms and representations, we refer to [19–21]. For the connections between automorphic forms and cohomology of arithmetic groups, we refer to [8].

Let E be a number field, let $\mathbb{A}_E$ the adeles of E, and let π be a cuspidal automorphic representation for $\mathrm{GL}_n(\mathbb{A}_E)$. Then π is called *selfdual* if it is isomorphic to its contragredient. We say a cuspform f is selfdual if its corresponding automorphic representation is selfdual, and that a cohomology class is selfdual if its corresponding automorphic form is. We can similarly define *nonselfdual* for these objects.

We return to GL_3 over our field F. Let $\mathfrak{p} \nmid 3$ be a prime ideal and suppose that ϕ is a Hecke eigenclass with Hecke polynomial

$$h(\phi,\mathfrak{p}) = 1 - a(\mathfrak{p},1)t + a(\mathfrak{p},2)\,\mathrm{N}(\mathfrak{p})t^2 - \mathrm{N}(\mathfrak{p})^3t^3 \tag{4.7}$$

at $\mathfrak{p}$. If ϕ is not an Eisenstein class, then at least one Hecke polynomial must be irreducible. One also knows that the eigenvalues $a(\mathfrak{p},1)$ and $a(\mathfrak{p},2)$ are complex conjugates:

$$a(\mathfrak{p},1) = \overline{a(\mathfrak{p},2)}. \tag{4.8}$$

Moreover, if ϕ is in addition a nonselfdual cuspidal class with Hecke polynomial (4.7), then the field determined by the Hecke eigenvalues should be nonreal, and there must be another nonselfdual class ϕ' complementing ϕ and with Hecke polynomial at $\mathfrak{p}$ given by

$$h(\phi',\mathfrak{p}) = 1 - a(\mathfrak{p},2)t + a(\mathfrak{p},1)\,\mathrm{N}(\mathfrak{p})t^2 - \mathrm{N}(\mathfrak{p})^3t^3. \tag{4.9}$$

To summarize, when detecting nonselfdual cuspidal cohomology classes, one looks for the following criteria:

- One wants to see (at least one) irreducible Hecke polynomial.[5]
- The Hecke eigenvalues should determine a nonreal field.
- The classes should come in pairs, with the pattern of eigenvalues given by (4.7)–(4.9).

We observed classes with these properties at four levels having norms 739, 837, 853, and 867. The classes at level norm 867 appear to be base changes of classes on $\Gamma_0(153) \subset \mathrm{SL}_3(\mathbb{Z})$ originally found in [35]. We give details in Examples 4.16–4.19 below.

4.16. Example. Let $\mathfrak{p}_{739}$ be the prime ideal above 739 generated by $7\omega - 30$. The cohomology at level $\mathfrak{p}_{739}$ is 2-dimensional. Since the level is prime, there are no oldforms or forms coming from Hecke Grössencharacters. The space of Bianchi cuspforms is

[5]Strictly speaking, this does not imply that the cohomology class is cuspidal, only that it appears in the *interior* cohomology of Γ (see [2, 27] for the definition). For $\mathrm{GL}_3/\mathbb{Q}$ being interior implies cuspidal. For our purposes, we will abuse notation and ignore this distinction.

trivial at level $\mathfrak{n}$. Thus the cohomology $H^5(\Gamma_0(\mathfrak{p}_{739});\mathbb{C})$ is different from the types explained above.

There are 2 eigenclasses ϕ and ϕ' with corresponding Hecke polynomials that have coefficients in the imaginary quadratic field $\mathbb{Q}(\sqrt{-123})$. The Hecke polynomials for ϕ and ϕ' are given in Table 3.

Let $a(\mathfrak{p}, i)$, $i = 1, 2$ be the Hecke eigenvalues for ϕ, and let $a'(\mathfrak{p}, i)$, $i = 1, 2$ be the Hecke eigenvalues for ϕ'. Then we observe that

$$a(\mathfrak{p}, 1) = a'(\mathfrak{p}, 2) = \overline{a'(\mathfrak{p}, 1)} = \overline{a(\mathfrak{p}, 2)} \tag{4.10}$$

in the range of the computation.

4.17. Example. Let $\mathfrak{n}$ be the ideal of norm $837 = 3^3 \cdot 31$ generated by $21\omega + 12$. The cohomology at level $\mathfrak{n}$ is 4-dimensional.

The space of Bianchi cuspforms is 1-dimensional and new at level $\mathfrak{n}$, coming from the isogeny class of the elliptic curve `837.1-a1` with Weierstrass equation

$$y^2 + xy + \omega y = x^3 + \omega x^2 - x.$$

This contributes a 2-dimensional subspace to $H^5(\Gamma_0(\mathfrak{n});\mathbb{C})$, as described in §4.3.

Let ϕ and ϕ' be Hecke eigenclasses that generate the remaining 2-dimensional subspace of the cohomology. The corresponding Hecke polynomials have coefficients in F and are given in Table 4.

Let $a(\mathfrak{p}, i)$, $i = 1, 2$ be the Hecke eigenvalues for ϕ, and let $a'(\mathfrak{p}, i)$, $i = 1, 2$ be the Hecke eigenvalues for ϕ'. Then we observe that as before,

$$a(\mathfrak{p}, 1) = a'(\mathfrak{p}, 2) = \overline{a'(\mathfrak{p}, 1)} = \overline{a(\mathfrak{p}, 2)} \tag{4.11}$$

in the range of the computation.

4.18. Example. Let $\mathfrak{p}_{853}$ be the prime ideal above 853 generated by $27\omega - 31$. The cohomology $H^5(\Gamma_0(\mathfrak{p}_{853});\mathbb{C})$ is 4-dimensional. Since the level is prime, there are no oldforms or forms coming from Hecke Grössencharacters. The space of Bianchi cuspforms is 1-dimensional and new at level $\mathfrak{p}_{853}$, coming from the the isogeny class of elliptic curve `853.1-a1` over F with Weierstrass equation

$$y^2 + xy + (\omega + 1)y = x^3 + (\omega - 1)x^2 + (-\omega - 1)x - \omega,$$

as described in §4.3.

Let ϕ and ϕ' be Hecke eigenclasses that generate the remaining 2-dimensional subspace of the cohomology. The corresponding Hecke polynomials that have coefficients in the imaginary quadratic field $\mathbb{Q}(\sqrt{-31})$ and are given in Table 5.

Let $a(\mathfrak{p}, i)$, $i = 1, 2$ be the Hecke eigenvalues for ϕ, and let $a'(\mathfrak{p}, i)$, $i = 1, 2$ be the Hecke eigenvalues for ϕ'. Then as in Example 4.16, we observe that

$$a(\mathfrak{p}, 1) = a'(\mathfrak{p}, 2) = \overline{a'(\mathfrak{p}, 1)} = \overline{a(\mathfrak{p}, 2)} \tag{4.12}$$

in the range of the computation.

TABLE 3. Hecke polynomials for the nonselfdual eigenclasses at level $\mathfrak{p}_{739}$ described in Example 4.16. Here, α generates $\mathbb{Q}(\sqrt{-123})$ and satisfies the polynomial $x^2 - x + 31$.

HNF($\mathfrak{p}$)	$h(\phi_1, \mathfrak{p})$ and $h(\phi_2, \mathfrak{p})$
$[3, 1, 1]$	$-27t^3 + (-3\alpha - 9)t^2 + (-\alpha + 4)t + 1$ $-27t^3 + (3\alpha - 12)t^2 + (\alpha + 3)t + 1$
$[4, 0, 2]$	$-64t^3 + (4\alpha - 12)t^2 + (\alpha + 2)t + 1$ $-64t^3 + (-4\alpha - 8)t^2 + (-\alpha + 3)t + 1$
$[7, 2, 1]$	$-343t^3 + (-7\alpha - 14)t^2 + (-\alpha + 3)t + 1$ $-343t^3 + (7\alpha - 21)t^2 + (\alpha + 2)t + 1$
$[7, 4, 1]$	$-343t^3 + (7\alpha - 28)t^2 + (\alpha + 3)t + 1$ $-343t^3 + (-7\alpha - 21)t^2 + (-\alpha + 4)t + 1$
$[13, 3, 1]$	$-2197t^3 + (13\alpha + 78)t^2 + (\alpha - 7)t + 1$ $-2197t^3 + (-13\alpha + 91)t^2 + (-\alpha - 6)t + 1$
$[13, 9, 1]$	$-2197t^3 + (-13\alpha - 78)t^2 + (-\alpha + 7)t + 1$ $-2197t^3 + (13\alpha - 91)t^2 + (\alpha + 6)t + 1$
$[19, 7, 1]$	$-6859t^3 + (57\alpha - 171)t^2 + (3\alpha + 6)t + 1$ $-6859t^3 + (-57\alpha - 114)t^2 + (-3\alpha + 9)t + 1$
$[19, 11, 1]$	$-6859t^3 + (-57\alpha - 152)t^2 + (-3\alpha + 11)t + 1$ $-6859t^3 + (57\alpha - 209)t^2 + (3\alpha + 8)t + 1$
$[25, 0, 5]$	$-15625t^3 + (-25\alpha - 125)t^2 + (-\alpha + 6)t + 1$ $-15625t^3 + (25\alpha - 150)t^2 + (\alpha + 5)t + 1$
$[31, 5, 1]$	$-29791t^3 + (-31\alpha + 1302)t^2 + (-\alpha - 41)t + 1$ $-29791t^3 + (31\alpha + 1271)t^2 + (\alpha - 42)t + 1$
$[31, 25, 1]$	$-29791t^3 + (-155\alpha - 744)t^2 + (-5\alpha + 29)t + 1$ $-29791t^3 + (155\alpha - 899)t^2 + (5\alpha + 24)t + 1$

4.19. Example. Let $\mathfrak{n}$ be the norm 867 ideal $\mathfrak{n} = \mathfrak{p}_3\mathfrak{q}_{17}$, where $\mathfrak{p}_3$ is the prime above 13 and $\mathfrak{q}_{17}$ is the prime above 17. The cohomology $H^5(\Gamma_0(\mathfrak{n}); \mathbb{C})$ is 14-dimensional. There is a 6-dimensional contribution of old forms, as described in §4.8, coming from level the 2-dimensional cohomology at level $\mathfrak{q}_{17}$, so the new subspace of $H^5(\Gamma_0(\mathfrak{n}); \mathbb{C})$ is 8-dimensional. The space of Bianchi cuspforms at level $\mathfrak{n}$ is 5-dimensional. The new subspace is 3-dimensional, including a rational cuspidal newform corresponding

TABLE 4. Hecke polynomials for the nonselfdual eigenclasses at the norm 837 level generated by $21\omega + 12$ described in Example 4.17.

$\mathrm{HNF}(\mathfrak{p})$	$h(\phi_1, \mathfrak{p})$ and $h(\phi_2, \mathfrak{p})$
$[4, 0, 2]$	$-64t^3 + (-24\omega + 4)t^2 + (-6\omega + 5)t + 1$ $-64t^3 + (24\omega - 20)t^2 + (6\omega - 1)t + 1$
$[7, 2, 1]$	$(-7t + 1)(49t^2 + 10t + 1)$ $(-7t + 1)(49t^2 + 10t + 1)$
$[7, 4, 1]$	$(7t + 1)(-49t^2 + (12\omega - 6)t + 1)$ $(7t + 1)(-49t^2 + (-12\omega + 6)t + 1)$
$[13, 9, 1]$	$(-13t + 1)(169t^2 + 4t + 1)$ $(-13t + 1)(169t^2 + 4t + 1)$

to the isogeny class of 867.1-a1 with Weierstrass equation

$$y^2 + y = x^3 + x^2 - 59x - 196.$$

The new subspace of Bianchi forms contributes a 6-dimensional subspace in $H^5(\Gamma_0(\mathfrak{n}); \mathbb{C})$ as described in §4.3. Let ϕ and ϕ' be the Hecke eigenclasses that generate the remaining 2-dimensional subspace of the cohomology. The corresponding Hecke polynomials have coefficients in the imaginary quadratic field $\mathbb{Q}(\sqrt{-2})$ and are given in Table 6.

Let $a(\mathfrak{p}, i)$, $i = 1, 2$ be the Hecke eigenvalues for ϕ, and let $a'(\mathfrak{p}, i)$, $i = 1, 2$ be the Hecke eigenvalues for ϕ'. Then as in Example 4.16, we observe that

$$a(\mathfrak{p}, 1) = a'(\mathfrak{p}, 2) = \overline{a'(\mathfrak{p}, 1)} = \overline{a(\mathfrak{p}, 2)} \tag{4.13}$$

in the range of the computation.

Since our classes meet the criteria in §4.15, we expect them to be nonselfdual cuspidal classes. This appears to be the case, and in fact they appear to be base changes from $\mathbb{Q}$ to F of cohomology classes on $\Gamma_0(153) \subset \mathrm{SL}_3(\mathbb{Z})$ appearing in [35]. Indeed, if so one expects the L-functions to satisfy

$$L(g, s) = L(f, s)L(f \otimes \chi, s),$$

where g (respectively f) is the base change on GL_3/F (resp., the original automorphic form on $\mathrm{GL}_3/\mathbb{Q}$) and χ is the quadratic character corresponding to the extension $F/\mathbb{Q}$. In particular, one expects the following identification of Hecke eigenvalues away from the primes 3 and 17:

$$a(\mathfrak{p}, 1) = a(p, 1) \quad \text{if } \mathfrak{p} \mid p \text{ and } p \text{ splits,} \tag{4.14}$$

$$a(\mathfrak{p}, 1) = a(p, 1)^2 - 2pa(p, 2) \quad \text{if } \mathfrak{p} = (p) \text{ and } p \text{ is inert.} \tag{4.15}$$

TABLE 5. Hecke polynomials for the nonselfdual eigenclasses at level $\mathfrak{p}_{853}$ described in Example 4.18. Here, β generates $\mathbb{Q}(\sqrt{-31})$ and satisfies the polynomial $x^2 - x + 8$.

HNF($\mathfrak{p}$)	$h(\phi_1, \mathfrak{p})$ and $h(\phi_2, \mathfrak{p})$
$[3, 1, 1]$	$(-3t + 1)(9t^2 + 5t + 1)$ $(-3t + 1)(9t^2 + 5t + 1)$
$[4, 0, 2]$	$(4t + 1)(-16t^2 + (-2\beta + 1)t + 1)$ $(4t + 1)(-16t^2 + (2\beta - 1)t + 1)$
$[7, 2, 1]$	$-343t^3 + (28\beta - 56)t^2 + (4\beta + 4)t + 1$ $-343t^3 + (-28\beta - 28)t^2 + (-4\beta + 8)t + 1$
$[7, 4, 1]$	$(-7t + 1)(49t^2 + 10t + 1)$ $(-7t + 1)(49t^2 + 10t + 1)$
$[13, 3, 1]$	$-2197t^3 + (104\beta - 195)t^2 + (8\beta + 7)t + 1$ $-2197t^3 + (-104\beta - 91)t^2 + (-8\beta + 15)t + 1$
$[13, 9, 1]$	$-2197t^3 + (-52\beta + 26)t^2 + (-4\beta + 2)t + 1$ $-2197t^3 + (52\beta - 26)t^2 + (4\beta - 2)t + 1$
$[19, 7, 1]$	$-6859t^3 - 152\beta t^2 + (-8\beta + 8)t + 1$ $-6859t^3 + (152\beta - 152)t^2 + 8\beta t + 1$
$[19, 11, 1]$	$(-19t + 1)(361t^2 + 23t + 1)$ $(-19t + 1)(361t^2 + 23t + 1)$

Here we have written $a(\mathfrak{p}, k)$ for the eigenvalues of the class on $\mathrm{GL}_3(\mathcal{O}_F)$ and $a(p, k)$ for the eigenvalues of the class on $\mathrm{SL}_3(\mathbb{Z})$. From [35], the relevant Hecke eigenvalues $a(p, 1)$ are

$$a(2, 1) = 1, \quad a(7, 1) = -6\gamma - 3, \quad a(13, 1) = -9\gamma - 12,$$

which agrees with (4.14)–(4.15) since 2 is inert and 7, 13 both split in F.

We remark that the classes at level norms 739, 837, and 853 *cannot* be base changes from $\mathbb{Q}$, since they have different eigenvalues at some conjugate pairs of primes of $\mathcal{O}_F$.

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TABLE 6. Hecke polynomials for nonselfdual eigenclasses at level $\mathfrak{p}_3\mathfrak{q}_{17}$ of norm 867 described in Example 4.19. Here, $\gamma = \sqrt{-2}$.

HNF($\mathfrak{p}$)	$h(\phi_1, \mathfrak{p})$ and $h(\phi_2, \mathfrak{p})$
$[4, 0, 2]$	$-64t^3 - 12t^2 + 3t + 1$
	$-64t^3 - 12t^2 + 3t + 1$
$[7, 2, 1]$	$-343t^3 + (42\gamma - 21)t^2 + (6\gamma + 3)t + 1$
	$-343t^3 + (-42\gamma - 21)t^2 + (-6\gamma + 3)t + 1$
$[7, 4, 1]$	$-343t^3 + (42\gamma - 21)t^2 + (6\gamma + 3)t + 1$
	$-343t^3 + (-42\gamma - 21)t^2 + (-6\gamma + 3)t + 1$
$[13, 9, 1]$	$-2197t^3 + (-156\gamma - 117)t^2 + (-12\gamma + 9)t + 1$
	$-2197t^3 + (156\gamma - 117)t^2 + (12\gamma + 9)t + 1$

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