Weak Algebra Bundles and Associator Varieties

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Abstract

Algebra bundles, in the strict sense, appear in many areas of geometry and physics. However, the structure of an algebra is flexible enough to vary non-trivially over a connected base, giving rise to a structure of a weak algebra bundle. We will show that the notion of a weak algebra bundle is more natural than that of a strict algebra bundle. We will give necessary and sufficient conditions for weak algebra bundles to be locally trivial. The collection of non-trivial associative algebras of a fixed dimension forms a projective variety, called associator varieties. We will show that these varieties play the role the Grassmannians play for principal $O(n)$-bundles.

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1 Introduction

Weak algebra bundles are generalizations of (strict) algebra bundles. They are monoid objects in the category of vector bundles. Algebra bundles appear more frequently in the literature. The exterior bundle and the Clifford bundle are examples. Algebra bundles are examples of weak algebra bundles. In Section 3, we look at the varieties of associative algebras of a fixed dimension, the so-called associator varieties. In Section 4, we will show that weak algebra bundles are more natural than algebra bundles by constructing the so-called classifying weak algebra bundle. In Section 5, we will give necessary and sufficient conditions for a weak algebra bundle to be locally trivial, and hence strictness. We will introduce the notion of a differential connection. Existence of a differential connection together with a technical condition guarantee local triviality.

A lot has been written for (associative) algebra bundles. See for example Chidambara-Kiranagi [2] and Kiranagi-Rajendra [7]. An almost equal amount of literature has been devoted to Lie algebra bundles. See for example Douady-Lazard [4], and the series of papers by Kiranagi et. al. [8], [9], [10], [11], [12], [13], [14], and [15]. In this article, we restrict to finite-rank weak algebra bundles. For the infinite dimensional case, many work has been done. See for example Dadarlat [3].

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2 Weak Algebra Bundles

An *algebra bundle* is a vector bundle in which the fibers are algebras rather than just vector spaces and such that the trivialization maps are algebra isomorphisms. It follows immediately the if the base space is connected then the fiber algebras are mutually isomorphic. For a weak algebra bundle, we do not require that the local trivialization maps are algebra isomorphisms. A *weak algebra bundle* over a space X is a monoid object in the category of real vector bundles $Vec(X)$ over X. More precisely, we have the following definition.

Definition 1. Let X be a topological space. A *weak algebra bundle* $A \xrightarrow{p} X$ is a vector bundle together with bundle map $A \otimes A \xrightarrow{\mu} A$ making the following diagram of bundle maps

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{id \otimes \mu} & A \otimes A \\ \Big\downarrow{\scriptstyle \mu \otimes id} & & \Big\downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array}$$

commute. If, in addition, there is a bundle map $\mathbb{1}_X \xrightarrow{\eta} A$, where $\mathbb{1}_X$ is the trivial line bundle over X, making the following diagram

$$\begin{array}{ccccc} \mathbb{1}_X \otimes A & = & A & = & A \otimes \mathbb{1}_X \\ & {\scriptstyle \eta \otimes id} \searrow & \Big\uparrow{\scriptstyle \mu} & \swarrow {\scriptstyle id \otimes \eta} & \\ & & A \otimes A & & \end{array}$$

commute, then $A \xrightarrow{p} X$ is called *unital*. The map μ is called the *bundle multiplication* or simply the multiplication while the map η is called the *bundle unit* or simply the unit.

Note that $\Gamma(X, A \otimes A) \cong \Gamma(X, A) \otimes_{C(X)} \Gamma(X, A)$ and $\Gamma(X, \mathbb{1}_X) \cong C(X)$ as $C(X)$-bimodules. The global section functor Γ induces a multiplication μ_* and a unit map η_* on $\Gamma(X, A)$ given by

$$(\sigma\tau)(x) = \sigma(x)\tau(x) \qquad \eta_*(\alpha)(x) = \eta(\alpha(x))$$

for any $\sigma, \tau \in \Gamma(X, A)$, $\alpha \in C(X)$, and $x \in X$. These maps turn $\Gamma(X, A)$ into a unital $C(X)$-algebra. Conversely, a unital $C(X)$-algebra structure on $\Gamma(X, A)$ turns $A \xrightarrow{p} X$ into a weak algebra bundle. From this equivalence, we see immediately that (strict) algebra bundles are weak algebra bundles.

3 Associator Varieties

Despite the name, weak algebra bundles are more natural than (strict) algebra bundles. In this section, we will construct a certain universal weak algebra bundle which describes *all* weak algebra bundles (strict included) of a particular rank. First, let us consider a finite-dimensional vector space A with a chosen basis $\{x_1, x_2, \ldots, x_n\}$. An associative algebra structure on A is completely determined by the structure constants α_{ij}^k, $1 \leqslant i, j, k \leqslant n$ satisfying

$$\sum_l \left(\alpha_{ij}^l \alpha_{lk}^m - \alpha_{il}^m \alpha_{jk}^l\right) = 0. \tag{1}$$

for all $1 \leqslant i, j, m, k \leqslant n$. Let χ_n be the variety defined by the n^4 equations (1), called the *rank n associator variety.* Let ξ_n be the quotient of χ_n by the equivalence relation $x \sim y$ if $A_x \cong A_y$ where A_x means the algebra structure on A corresponding to the set of structure constants $x \in \chi_n$, equipped with the quotient topology. In general, ξ_n is not a variety, just an orbivariety. For $n \in \mathbb{N}$, the *classifying weak algebra bundle* $\widehat{\mathcal{A}}$ of rank n is the weak algebra bundle $\mathcal{A}_n \xrightarrow{p} \xi_n$ such that $p^{-1}(x) = A_x$.

Let us look more closely to the associator varieties χ_n. For the purpose of this section, we let χ_n be the variety given by the system of equations 1 with the all-zero solution removed. This makes χ_n a projective variety. Before we go into the analysis of these varieties, let us give some explicit points.

Example 1. For any $1 \leqslant i, j, k \leqslant n$, let

$$\alpha_{ij}^k = \begin{cases} 1 & k = i + j \mod n \\ 0 & \text{otherwise} \end{cases}$$

The algebra A_x, where $x = (\alpha_{ij}^k)$, is the truncated polynomial algebra.

Example 2. For any $1 \leqslant i, j, k \leqslant n$, let

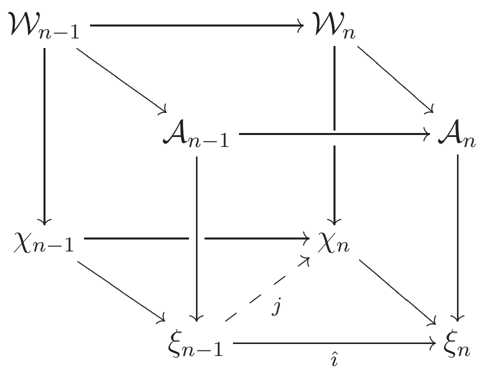
$$\alpha_{ij}^k = g(k)h(i)h(j)$$

where $g, h : \{1, 2, \ldots, n\} \longrightarrow \mathbb{C}$ are arbitrary functions. A particular example is when $g(k) = k$ and $h(j) = e^{\pi i j}$.

The associator varieties fit into a natural sequence. The variety χ_{n-1} is the intersection of χ_{n-1} with the varieties $\alpha_{in}^k = \alpha_{ni}^k = 0$, $1 \leqslant i, k \leqslant n$. This gives an inclusion of varieties $\chi_{n-1} \xrightarrow{i_{n-1}} \chi_n$. This inclusion is natural in the sense of the following proposition.

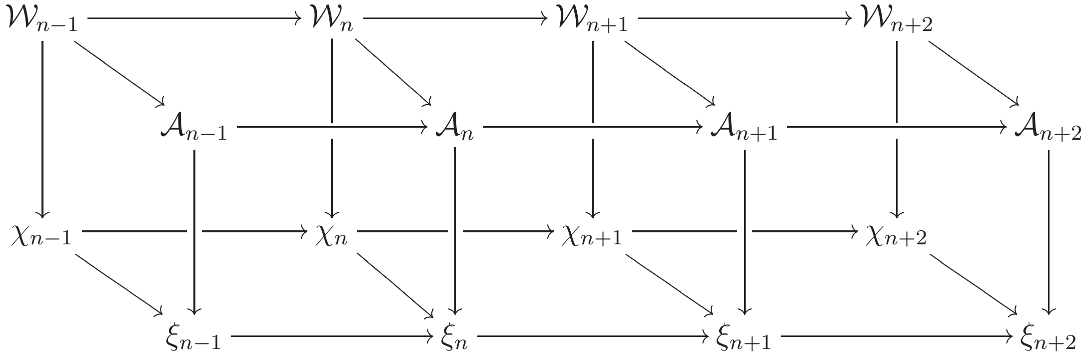
Proposition 1. *The tautological weak algebra bundle $\mathcal{W}_{n-1} \xrightarrow{r_{n-1}} \chi_{n-1}$ is the pullback of the tautological weak algebra bundle $\mathcal{W}_n \xrightarrow{r_n} \chi_n$ along the map i_{n-1} defined above.*

Proof: The proposition follows directly from the following pullback cube

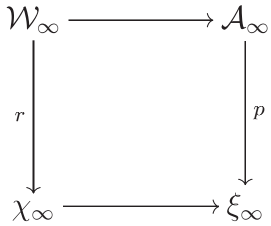


where the map j is the map induced by the universality of the quotient ξ_{n-1} and the map $\hat{\imath}$ is the composition of the j and the quotient map $\chi_n \longrightarrow \xi_n$. ■

By Proposition 1, we have several stratifications, given by the horizontal maps in the complex of spaces below.



Using maps above, we can define the spaces $\chi_\infty = \varinjlim \chi_n$. Similarly, we define ξ_∞, $\mathcal{W}_\infty$ and $\mathcal{A}_\infty$ as direct limits of the obvious sequence of spaces and maps. Then, it is immediate to check that there are weak algebra bundles $\mathcal{W}_\infty \xrightarrow{r} \chi_\infty$ and $\mathcal{A}_\infty \xrightarrow{p} \xi_\infty$. Moreover, these fit into a pullback square



where $\chi_\infty \longrightarrow \xi_\infty$ is the direct limit of the quotient maps $\chi_n \longrightarrow \xi_n$.

Let us end this section by looking at the tangent spaces of associator varieties. The following computation can be found in [19]. Consider a point $\alpha \in \chi_n$ given by $\left\{\alpha_{ij}^k | 1 \leqslant i, j, k \leqslant n\right\}$ in χ_n. Then tangent vectors $v = \left\{v_{ij}^k | 1 \leqslant i, j, k \leqslant n\right\}$ to χ_n at the point α satisfy the equation

$$\sum_l \left(\alpha_{ij}^l v_{lk}^m + \alpha_{lk}^m v_{ij}^l - \alpha_{il}^m v_{jk}^l - \alpha_{jk}^l v_{il}^m\right) = 0 \tag{2}$$

Let $x_1, \ldots, x_n$ be a basis for A giving the structure constants $\left\{\alpha_{ij}^k | 1 \leqslant i, j, k \leqslant n\right\}$. The bilinear function $f_v : A \times A \longrightarrow A$ given by $f(x_i, x_j) = \sum_k v_{ij}^k x_k$ satisfies the relation

$$xf(y,z) - f(xy,z) + f(x,yz) - f(x,y)z = 0 \tag{3}$$

for all $x, y, z \in A$. Condition 3 is called the cocycle condition. In particular, bilinear functions f satisfying 3 are called 2-cocycles. The set of all 2-cocycles in A, denoted by $Z^2(A)$ is a vector space. The map $v \mapsto f_v$ defines a linear isomorphism between the tangent space to χ_n at the point α and the space of 2-cocycles.

4 Classifying Weak Algebra Bundles

In this section, we will show that the orbivariety ξ_n is the analogue the Grassmannians play for principal $O(n)$-bundles. Let us first recall the the Grassmannian variety $G(n, \mathbb{R}^\infty)$ is the *classifying space* for $O(n)$. This in particular means that for any principal $O(n)$-bundle $Y \xrightarrow{p} X$ there is a continuous map $X \xrightarrow{f} G(n, \mathbb{R}^\infty)$ such that Y is the pull-back of $EO(n)$ along f. Here, $EO(n)$ is the total space of the *universal bundle* over $BG = G(n, \mathbb{R}^\infty)$. See [16] for more details. The following theorem jusitifies the name of the bundle $\mathcal{A}_n \xrightarrow{p} \xi_n$.

Theorem 1. *Let $\mathcal{A}_n \xrightarrow{p} \xi_n$ be the classifying weak algebra bundle of rank n. Let $\mathcal{B} \xrightarrow{q} X$ be a weak algebra bundle of rank n. Then there is a continuous map $X \xrightarrow{f} \xi_n$ such that $\mathcal{B} \cong f^*\mathcal{A}_n$ as weak algebra bundles, i.e. a map f which makes the following diagram*

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{A}_n \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & \xi_n \end{array}$$

a pullback diagram.

Proof: For a weak algebra bundle $\mathcal{B} \xrightarrow{q} X$, define the map $X \xrightarrow{f} \xi_n$ that sends $x \in X$ to $y \in \xi_n$ if $q^{-1}(x) \cong p^{-1}(y)$. Let $\{U_\alpha | \alpha \in I\}$ be an open cover of X trivializing $\mathcal{B} \xrightarrow{q} X$. Then $q^{-1}(U_\alpha) \cong U_\alpha \times B$ where B is the underlying vector space of the typical fiber of q. Let $x_1, \ldots, x_n$ be a basis for B and $\gamma_{ij}^k(x)$ be the structure functions of B_x for $x \in U_\alpha$. Then, the continuous functions $\gamma_{ij}^k : U_\alpha \longrightarrow \mathbb{R}$ satisfy the equations 1. This defines a continuous map $U_\alpha \xrightarrow{f} \xi_n$ sending $x \in U_\alpha$ to $(\gamma_{ij}^k(x) | 1 \leqslant i, j, k \leqslant n) \in \xi_n$. By Proposition 2, these functions extend globally to a continuous function $X \xrightarrow{f} \xi_n$. Finally, it is straightforward to check that, indeed, $\mathcal{B} \cong f^*\mathcal{A}_n$. ■

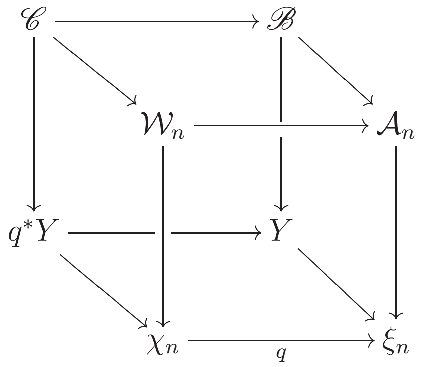
If $\mathcal{B} \xrightarrow{q} X$ is a (strict) algebra bundle the map $X \xrightarrow{f} \xi_n$ asserted by Theorem 1 is just the constant map, sending every point of X to the unique point $y \in \xi_n$ such that the typical fiber of $\mathcal{B} \xrightarrow{q} X$ is isomorphic to $p^{-1}(y)$.

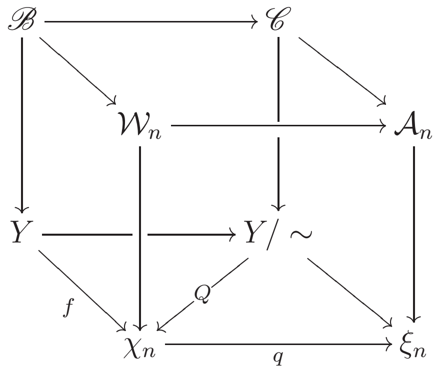
In view of Proposition 2, the non-triviality of the underlying vector bundle of a rank n weak algebra bundle $\mathcal{B} \xrightarrow{q} X$ is determined by the homotopy type of the function f asserted by Theorem 1 but not completely so. For example, a strict algebra bundle $\mathcal{B} \xrightarrow{q} X$ may have a non-trivial underlying vector bundle and yet the associated function f is constant and hence, homotopically trivial.

Let $\mathcal{W}_n \xrightarrow{r} \chi_n$ be the pullback of the rank n classifying weak algebra bundle along the quotient map $\chi_n \xrightarrow{f} \xi_n$. We call this bundle the *tautological weak algebra bundle*. The following theorem illustrates that for most purposes we can use the tautological weak algebra bundle in place of the classifying weak algebra bundle. Denote by $\mathrm{Pull}\,(\mathcal{B}, X)$ the set of all weak algebra bundles that are pullbacks of $\mathcal{B} \xrightarrow{f} X$.

Theorem 2. *There is a canonical bijection between* $\mathrm{Pull}\,(\mathcal{A}_n, \xi_n)$ *and* $\mathrm{Pull}\,(\mathcal{W}_n, \chi_n)$.

Proof: If $\mathscr{B} \longrightarrow Y$ is a pullback of the classifying weak algebra bundle, then pulling-back maps along appropriate maps as illustrated by the left cube below gives a weak algebra bundle $\mathscr{C} \longrightarrow q^*Y$ where q is the quotient map.





Conversely, let $\mathscr{B} \longrightarrow Y$ be a pullback of the tautological weak algebra bundle along a map $Y \stackrel{f}{\longrightarrow} \chi_n$. Let $\sim$ be the equivalence relation on Y defined as $y \sim y'$ if $f(y) = f(y')$. Let Q be the map induced by the universality of the quotient $Y/\sim$ and let $Y/\sim \longrightarrow \xi_n$ be the composition of Q and the quotient map q. Pulling back maps along appropriate maps according to the right cube above gives a weak algebra bundle $\mathscr{C} \longrightarrow Y/\sim$ that is a pullback of the classifying weak algebra bundle. ■

The advantage of working with $\mathcal{W}_n \stackrel{r}{\longrightarrow} \chi_n$ is the fact that χ_n is a variety and $\mathcal{W}_n \stackrel{r}{\longrightarrow} \chi_n$ is a regular vector bundle. Let us end this section by a triviality statement regarding the classifying weak algebra bundles.

Proposition 2. *As vector bundles, $\mathcal{A}_n \stackrel{p}{\longrightarrow} \xi_n$ are parallelizable for all $n \in \mathbb{N}$.*

For $i = 1, \ldots, n$, the sections $\xi_n \stackrel{\sigma_i}{\longrightarrow} \mathcal{A}_n$, $x \mapsto x_i$ gives a set of pointwise linearly independent set of n sections. This illustrates parallelizability.

5 Local Triviality of Weak Algebra Bundles

In this section, we give necessary and sufficient conditions for a weak algebra bundle to be a strict algebra bundle. For this purpose, we will specialize in the smooth case. Let X be a connected smooth manifold.

Definition 2. Let $E \twoheadrightarrow X$ be a smooth vector bundle such that the fibers are algebras whose multiplications depend on $x \in X$ continuously. A *differential connection* ∇ on E is a smooth connection such that for any vector field ν on X, we have

$$\nabla_\nu(\sigma_1\sigma_2) = \sigma_1\nabla_\nu(\sigma_2) + \nabla_\nu(\sigma_1)\sigma_2$$

for any sections $\sigma_1, \sigma_2 \in \Gamma(X, E)$. □

Surprisingly, existence of such connections is a sufficient condition for the fiber algebras to be isomorphic. For a necessary condition, one needs a stronger assumption than just having isomorphic fiber algebras. We will formalize these statements in the next two propositions.

Proposition 3. *If E has a differential connection ∇ then the fiber algebras of $E \twoheadrightarrow X$ are all isomorphic.*

Proof: Assume E has a differential connection ∇. Let $x, y \in X$ and let $\gamma : I \longrightarrow X$ be a (piecewise) smooth path in X with $\gamma(0) = x$ and $\gamma(1) = y$. Using the connection ∇, we have a parallel transport map

$$\Phi(\gamma)_x^y : E_x \longrightarrow E_y$$

which is a linear isomorphism. Thus, all we have to show is that $\Phi(\gamma)_x^y$ is multiplicative. Given $b_1, b_2 \in E_x$, there are unique smooth sections σ_1 and σ_2 of E along γ such that $\nabla_{\vec{\gamma}}\sigma_1 = \nabla_{\vec{\gamma}}\sigma_2 = 0$ and $\sigma_1(x) = b_1$ and $\sigma_2(x) = b_2$. Here, $\vec{\gamma}$ denotes the smooth tangent vector field of γ. Note that the product $\sigma_1\sigma_2$ is the unique smooth section of $E \longrightarrow\!\!\!\!\!\rightarrow X$ along γ such that $(\sigma_1\sigma_2)(x) = \sigma_1(x)\sigma_2(x) = b_1b_2$ and

$$\nabla_{\vec{\gamma}}(\sigma_1\sigma_2) = \sigma_1\nabla_{\vec{\gamma}}(\sigma_2) + \nabla_{\vec{\gamma}}(\sigma_1)\sigma_2 = 0.$$

Thus, by definition of the parallel transport map $\Phi(\gamma)_x^y$ we have

$$\Phi(\gamma)_x^y(b_1b_2) = (\sigma_1\sigma_2)(y) = \sigma_1(y)\sigma_2(y) = \Phi(\gamma)_x^y(b_1)\Phi(\gamma)_x^y(b_2)$$

which shows that $\Phi(\gamma)_x^y$ is multiplicative. ■

A strong converse of the above proposition, where the isomorphisms among fibers satisfy some coherence conditions, holds. By a coherent collection

$$\mathscr{P} = \{\Phi(\gamma)_x^y : E_x \longrightarrow E_y | \forall x, y \in X, \gamma : I \longrightarrow X \textit{ smooth}\}$$

of isomorphisms among fibers of $E \twoheadrightarrow X$, we mean a collection satisfying

(a) $\Phi(\gamma)_x^x = id$,

(b) $\Phi(\gamma)_u^y \circ \Phi(\gamma)_x^u = \Phi(\gamma)_x^y$,

(c) and Φ depends smoothly on γ, y and x.

We then have the following proposition.

Proposition 4. *A coherent collection $\mathscr{P}$ of algebra isomorphisms on fibers of $E \twoheadrightarrow X$ gives a differential connection ∇ on E.*

PROOF:

$$\nabla_\nu(\sigma) = \lim_{t\to 0} \frac{\Phi(\gamma)_{\gamma(t)}^x\sigma(\gamma(t)) - \sigma(x)}{t} = \left.\frac{d}{dt}\right|_{t=0} \Phi(\gamma)_{\gamma(t)}^x\sigma(\gamma(t))$$

for any $\sigma \in B$, $x = \gamma(0)$, and $\nu = \gamma'(0)$. That ∇ is a differential connection follows from the multiplicativity of $\Phi(\gamma)_x^y$ and the Leibniz property of $\left.\frac{d}{dt}\right|_{t=0}$. ■

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6 References

[1] Canlubo, C. R. P. *Non-commutative covering spaces and their symmetries.* PhD Thesis, University of Copenhagen (2017).

[2] Chidambara, C. and Kiranagi, B. S. *On cohomology of associative algebra bundles*, Journal of the Ramanujan Mathematical Society, 9 (1): 112, 1994.

[3] Dadarlat, M. *Continuous fields of C-algebras over finite dimensional spaces*, arXiv:math/0611405.

[4] Douady, A. and Lazard, M. *Espaces fibres en algebre de Lie et en groups*, Invent. math., Vol. 1, 133151, 1966.

[5] Dufour, J-P. and Zung, N. T. *Poisson Structures and their Normal Forms*, Progress in Mathematics 242, Birkhäuser Basel, 2005.

[6] Fell, J. M. G. *The structure of algebras of operator fields.* Acta Math. Vol. 106, No. 3-4, pp. 233-280 (1961).

[7] Kiranagi, B. S. and Rajendra, R. *Revisiting Hochschild cohomology for algebra bundles*, Journal of Algebra and its Applications, 7 (6): 685715, 2008.

[8] Kiranagi, B. S. *Lie Algebra bundles*, Bull. Sci. Math., 2e serie, 102, 57-62, 1978.

[9] Kiranagi, B. S. *Semi simple Lie algebra bundles*, Bull. Math de la Sci. Math de la R.S.de Roumaine, 27 (75), 253-257, 1983.

[10] Kiranagi, B. S. and Prema, G. *On complete reducibility of Module Bundles*, Bull. Austral. Math Soc., 28, 401-409, 1983.

[11] Kiranagi, B. S. and Prema, G. *Cohomology of Lie algebra bundles and its applications*, Ind. J. Pure and Appli. Math. 16(7), 731/735, 1985.

[12] Kiranagi, B. S. and Prema, G. *Lie algebra bundles defined by Jordan algebra bundles*, Bull. Math. Soc.Sci.Math.Rep.Soc. Roum., Noun. Ser. 33 (81), 255-264, 1989.

[13] Kiranagi, B. S. and Prema, G. *On complete reducibility of Bimodule bundles*, Bull. Math. Soc. Sci.Math. Repose; Roum, Nouv.Ser. 33 (81), 249-255, 1989.

[14] Kiranagi, B. S. and Prema, G. *A decomposition theorem of Lie algebra Bundles*, Communications in Algebra 18 (6), 1869-1877, 1990.

[15] Kiranagi, B.S., Ranjitha, K., and Prema, G. *On completely semisimple Lie algebra bundles*, Journal of Algebra and its Applications, 14 (2): 111, 2014.

[16] May, P. J. *A Concise Course in Algebraic Topology*, The University of Chicago Press, 1999.

[17] Mac Lane, S. *Categories for the Working Mathematician* Second Edition, Graduate Texts in Mathematics 5, Springer-Verlag, 1971.

[18] Radford, D. *The group of automorphisms of a semisimple Hopf algebra over a field of characteristic 0 is finite*, American Journal of Mathematics, vol. 22, pp. 331-357, 1990.

[19] Shafarevich, I. R. *Basic Algebraic Geometry 1.* 2nd rev. and expanded version, Springer Study Edition, Springer-Verlag, 1994.

[20] Shafarevich, I. R. *Basic Algebraic Geometry 2.* 2nd rev. and expanded version, Springer Study Edition, Springer-Verlag, 1994.

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