

GAUSS SUMS OF SOME MATRIX GROUPS OVER $\mathbb{Z}/n\mathbb{Z}$

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ABSTRACT. In this paper, we will explicitly calculate Gauss sums for the general linear groups and the special linear groups over \mathbb{Z}_n , where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $n > 0$ is an integer. For r being a positive integer, the formulae of Gauss sums for $\mathrm{GL}_r(\mathbb{Z}_n)$ can be expressed in terms of classical Gauss sums over \mathbb{Z}_n , while the formulae of Gauss sums for $\mathrm{SL}_r(\mathbb{Z}_n)$ can be expressed in terms of hyper-Kloosterman sums over \mathbb{Z}_n . As an application, we count the number of $r \times r$ invertible matrices over \mathbb{Z}_n with given trace by using the formulae of Gauss sums for $\mathrm{GL}_r(\mathbb{Z}_n)$ and the orthogonality of Ramanujan sums.

1. INTRODUCTION

Classically, there are two kinds of Gauss sums: one is defined over \mathbb{F}_q , i.e., the finite field with q elements, and the other one is defined over \mathbb{Z}_n , i.e., the residue class ring of \mathbb{Z} modulo $n\mathbb{Z}$. For the case of $q = n = p$ being a prime, these two kinds of Gauss sums coincide, and are defined by

$$(1) \quad G(\mathbb{F}_p, \chi, \lambda) = \sum_{a \in \mathbb{F}_p^*} \chi(a)\lambda(a),$$

where χ is a multiplicative character of \mathbb{F}_p^* , i.e., the group of units of \mathbb{F}_p and λ is an additive character of \mathbb{F}_p . It is well known that Gauss initially used $G(\mathbb{F}_p, \chi, \lambda)$ with χ being a quadratic character, to prove the quadratic reciprocity law. For the general case, these two kinds of Gauss sums behave differently and need to be treated separately (see p.30 of [1]).

The Gauss sums are ubiquitous in number theory. For example, they naturally occur in the Stickelberger's theorem and the functional equations of Dirichlet L -functions. Therefore, the Gauss sums attract many researcher's interests and have many generalizations.

The Gauss sums for classical groups over a finite field have been extensively studied by D.S.Kim et al. in a series of papers [7, 8, 9, 10, 11, 12, 13, 14]. In [8], Kim got the explicit formulae of Gauss sums for the general linear group $\mathrm{GL}_r(\mathbb{F}_q)$ and the special linear group $\mathrm{SL}_r(\mathbb{F}_q)$ by using the Bruhat decomposition. Indeed, the Gauss sums for $\mathrm{GL}_r(\mathbb{F}_q)$ and $\mathrm{SL}_r(\mathbb{F}_q)$ are defined by

$$(2) \quad G(\mathrm{GL}_r(\mathbb{F}_q), \chi, \lambda) = \sum_{X \in \mathrm{GL}_r(\mathbb{F}_q)} \chi(\det X)\lambda(\mathrm{tr} X),$$

$$(3) \quad G(\mathrm{SL}_r(\mathbb{F}_q), \lambda) = \sum_{X \in \mathrm{SL}_r(\mathbb{F}_q)} \lambda(\mathrm{tr} X),$$

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where λ is an additive character of \mathbb{F}_q and χ is a multiplicative character of \mathbb{F}_q^* . In [8], Kim proved that for nontrivial λ ,

$$(4) \quad G(\mathrm{GL}_r(\mathbb{F}_q), \chi, \lambda) = q^{r(r-1)/2} G(\mathbb{F}_q, \chi, \lambda)^r,$$

$$(5) \quad G(\mathrm{SL}_r(\mathbb{F}_q), \lambda) = q^{r(r-1)/2} K_r(\mathbb{F}_q, \lambda),$$

where $G(\mathbb{F}_q, \chi, \lambda)$ is the classical Gauss sum over \mathbb{F}_q , defined similarly as (1), and

$$(6) \quad K_r(\mathbb{F}_q, \lambda) = \sum_{\substack{x_1 x_2 \cdots x_r = 1 \\ x_1, x_2, \dots, x_r \in \mathbb{F}_q}} \lambda(x_1 + x_2 + \cdots + x_r),$$

is the hyper-Kloosterman sum over \mathbb{F}_q .

As Kim remarked, the formulae (4) and (5) already appeared in the work of Eichler [3] and Lamprecht [17] (see the introduction of [8]). Fulman [5] also got the formulae (4) by using the cycle index techniques.

In [18], Li and Hu studied the Gauss sums for $\mathrm{GL}_r(\mathbb{F}_q)$ and $\mathrm{SL}_r(\mathbb{F}_q)$ in a more general setting and got the explicit formulae for

$$(7) \quad G(\mathrm{GL}_r(\mathbb{F}_q), U, \chi, \lambda) = \sum_{X \in \mathrm{GL}_r(\mathbb{F}_q)} \chi(\det X) \lambda(U \cdot X),$$

and

$$(8) \quad G(\mathrm{SL}_r(\mathbb{F}_q), U, \lambda) = \sum_{X \in \mathrm{SL}_r(\mathbb{F}_q)} \lambda(U \cdot X),$$

by a different method, where $U \in \mathrm{M}_r(\mathbb{F}_q)$ is an arbitrary $r \times r$ matrix over \mathbb{F}_q and $U \cdot X = \mathrm{tr} U^t X$ is the inner product of U and X . Note that for a fixed non-trivial additive character λ , as U varies over $\mathrm{M}_r(\mathbb{F}_q)$, $\lambda(U \cdot \quad)$ runs over all additive characters of $\mathrm{M}_r(\mathbb{F}_q)$. As an application, they [18] calculated N_β , the number of matrices in $\mathrm{GL}_r(\mathbb{F}_q)$ with trace β , where $\beta \in \mathbb{F}_q$. Explicitly, we have

$$(9) \quad N_\beta = \begin{cases} q^{n(n-1)/2-1} \cdot \left(\prod_{i=1}^n (q^i - 1) - (-1)^n \right) & \text{if } \beta \neq 0 \\ q^{n(n-1)/2-1} \cdot \left(\prod_{i=1}^n (q^i - 1) + (-1)^n (q - 1) \right) & \text{if } \beta = 0 \end{cases}$$

Gauss sums (and more generally exponential sums) for matrix groups can be used to study the uniform distribution property for matrix groups (e.g., see [4], [6], [21] and [27]). Also they can be applied in coding theory (see [15] and [22, 23]).

The aim of this paper is to generalize formulae (4), (5), (9) to general linear groups and special linear group over \mathbb{Z}_n .

From now on, we always assume that χ is a multiplicative character of \mathbb{Z}_n^* , i.e., the unit group of \mathbb{Z}_n , and λ is an additive character of \mathbb{Z}_n . Explicitly, λ can be uniquely written as

$$\lambda(x) = \exp\left(\frac{2\pi\sqrt{-1}ax}{n}\right), \text{ where } x \in \mathbb{Z}_n, \text{ for some } a \in \mathbb{Z}_n.$$

The Gauss sums for the general linear group $\mathrm{GL}_r(\mathbb{Z}_n)$ and the special linear group $\mathrm{SL}_r(\mathbb{Z}_n)$ are defined as

$$(10) \quad G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda) = \sum_{X \in \mathrm{GL}_r(\mathbb{Z}_n)} \chi(\det X) \lambda(\mathrm{tr} X),$$

$$(11) \quad G(\mathrm{SL}_r(\mathbb{Z}_n), \lambda) = \sum_{X \in \mathrm{SL}_r(\mathbb{Z}_n)} \lambda(\mathrm{tr} X),$$

Note that for $r = 1$, equation (10) is just the definition of classical Gauss sums for \mathbb{Z}_n , i.e.

$$(12) \quad G(\mathbb{Z}_n, \chi, \lambda) = \sum_{X \in \mathbb{Z}_n^*} \chi(x)\lambda(x),$$

In this paper, we will calculate the Gauss sums $G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda)$ and $G(\mathrm{SL}_r(\mathbb{Z}_n), \lambda)$ explicitly (see Theorems 3.4, 3.5 and 3.8 below). Note that the expression of $G(\mathrm{SL}_r(\mathbb{Z}_n), \lambda)$ involves the hyper-Kloosterman sums, which are previously studied by a lot of researchers (e.g. see [19], [24] [28], [30] and [31, 32, 33]).

The key ingredient in calculation of (10) and (11) is averaging such sums over Borel subgroup, i.e. the subgroups consisting of upper triangular matrices. This method is similar to that of [18] and is originally inspired by [4].

Note that, Maeda [20] explicitly determine the following kind of Gauss sums for $\mathrm{GL}_2(\mathbb{Z}_{p^m})$:

$$(13) \quad G(\mathrm{GL}_2(\mathbb{Z}_{p^m}), \Psi, \lambda) = \sum_{X \in \mathrm{GL}_2(\mathbb{Z}_{p^m})} \Psi(X)\lambda(\mathrm{tr} X),$$

where Ψ is any irreducible character of group representations of $\mathrm{GL}_2(\mathbb{Z}_{p^m})$, p is an odd prime and $m \geq 2$ is an integer (The case of $m = 1$ is contained in the results of [16] by Kondo, previously). In the abstract of [20], Maeda remarked that ‘‘While there are several studies of the Gauss sums on finite algebraic groups defined over a finite field, this paper seems to be the first one which determines the Gauss sums on a matrix group over a finite ring.’’

Finally, as an application, we count the number of matrices in $\mathrm{GL}_r(\mathbb{Z}_n)$ with trace β , where $\beta \in \mathbb{Z}_n$ (see Theorem 4.1 and Theorem 4.3). The calculation relies on the orthogonality of Ramanujan sums (e.g. see [29]), which can be viewed as a special kind of Gauss sums $G(\mathbb{Z}_n, \chi, \lambda)$ with trivial multiplicative character χ .

2. PRELIMINARIES ON GAUSS SUMS OVER \mathbb{Z}_n

Write the additive character

$$(14) \quad \lambda(x) = \exp\left(\frac{2\pi\sqrt{-1}ax}{n}\right), \quad \text{with } d = \gcd(a, n),$$

for some $a \in \mathbb{Z}$, where $x \in \mathbb{Z}_n$ and $\gcd(\cdot, \cdot)$ represents the greatest common divisor. Let f be the conductor of the multiplicative character χ , i.e. f is the smallest positive integer such that χ factors through \mathbb{Z}_f^* .

First, we collect some results on Gauss sums over \mathbb{Z}_n from Cohen’s book (see p.31-33 and p.94-95 of [1]).

$$(15) \quad G(\mathbb{Z}_n, \chi, \lambda) = \begin{cases} 0 & \text{if } f \nmid n/d \\ \frac{\varphi(n)}{\varphi(n/d)} G(\mathbb{Z}_{n/d}, \chi', \lambda) & \text{if } f \mid n/d \end{cases},$$

where χ' is the character of $\mathbb{Z}_{n/d}^*$ induced by χ , i.e.

$$(16) \quad \chi'(c \bmod n/d) = \chi(c \bmod n), \quad \text{for } \gcd(c, n) = 1,$$

and φ is the Euler’s totient function.

In equation (15), the case of $f \nmid n/d$ is just Proposition 2.1.40 of [1]. For $f \mid n/d$, both λ and χ can be defined modulo n/d . We have

$$(17) \quad \sum_{x \in \mathbb{Z}_n^*} \chi(x)\lambda(x) = \frac{\varphi(n)}{\varphi(n/d)} \sum_{y \in \mathbb{Z}_{n/d}^*} \chi'(y)\lambda(y)$$

as $\varphi(n)/\varphi(n/d)$ many $x \in \mathbb{Z}_n^*$ maps to one $y \in \mathbb{Z}_{n/d}^*$. This yields the desired identity.

From equation (15), we only need to consider the case $d = \gcd(a, n) = 1$. Under this condition,

$$(18) \quad G(\mathbb{Z}_n, \chi, \lambda) = \overline{\chi(a)} \sum_{x \in \mathbb{Z}_n^*} \chi(x) \exp\left(\frac{2\pi\sqrt{-1}x}{n}\right),$$

(see Proposition 2.1.39 of [1]).

Now, for $a = 1$,

$$(19) \quad G(\mathbb{Z}_n, \chi, \lambda) = \mu\left(\frac{n}{f}\right)\chi_f\left(\frac{n}{f}\right)G(\mathbb{Z}_f, \chi_f, \lambda_f),$$

where χ_f is the character of \mathbb{Z}_f^* such that

$$\chi_f(c \bmod f) = \chi_f(c \bmod n) \quad \text{for } \gcd(c, n) = 1;$$

$$\lambda_f(x) = \exp\left(\frac{2\pi ix}{f}\right) \quad \text{for } x \in \mathbb{Z}_f;$$

and μ is the Möbius function (see exercise 12 on page 95 of [1]).

It is also known that

$$(20) \quad |G(\mathbb{Z}_f, \chi_f, \lambda_f)| = f^{\frac{1}{2}}$$

(see Proposition 2.1.45 of [1]).

3. MAIN RESULTS

Theorem 3.1. *Assume the order of λ is n , i.e. $d = \gcd(a, n) = 1$ in equation (14). Then*

$$(21) \quad G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda) = G(\mathbb{Z}_n, \chi, \lambda)^r n^{\frac{r(r-1)}{2}}.$$

Proof. Let $B_r(\mathbb{Z}_n)$ be the Borel subgroup of $\mathrm{GL}_r(\mathbb{Z}_n)$, i.e., the group of upper triangular invertible matrices over \mathbb{Z}_n . Averaging equation (10) over $B_r(\mathbb{Z}_n)$ and changing the order of summation, we get

$$(22) \quad \begin{aligned} & \sum_{X \in \mathrm{GL}_r(\mathbb{Z}_n)} \chi(\det X)\lambda(\mathrm{tr} X) \\ &= \frac{1}{\varphi(n)^r n^{\frac{r(r-1)}{2}}} \sum_{B \in B_r(\mathbb{Z}_n)} \sum_{X \in \mathrm{GL}_r(\mathbb{Z}_n)} \chi(\det BX)\lambda(\mathrm{tr} BX) \\ &= \frac{1}{\varphi(n)^r n^{\frac{r(r-1)}{2}}} \sum_{X \in \mathrm{GL}_r(\mathbb{Z}_n)} \chi(\det X) \sum_{B \in B_r(\mathbb{Z}_n)} \chi(\det B)\lambda(\mathrm{tr} BX). \end{aligned}$$

Writing

$$(23) \quad B = (b_{ij}) \in B_r(\mathbb{Z}_n), X = (x_{ij}) \in \mathrm{GL}_r(\mathbb{Z}_n)$$

and using the multiplicative property of characters, we get

$$(24) \quad \begin{aligned} & \frac{1}{\varphi(n)^r n^{\frac{r(r-1)}{2}}} \sum_{X \in \mathrm{GL}_r(\mathbb{Z}_n)} \chi(\det X) \sum_{(b_{ij}) \in \mathrm{B}_r(\mathbb{Z}_n)} \prod_{i=1}^r \chi(b_{ii}) \lambda(b_{ii} x_{ii}) \prod_{i < j} \lambda(b_{ij} x_{ji}) \\ &= \frac{1}{\varphi(n)^r n^{\frac{r(r-1)}{2}}} \sum_{X \in \mathrm{GL}_r(\mathbb{Z}_n)} \chi(\det X) \prod_{i=1}^r \sum_{b_{ii} \in \mathbb{Z}_n^*} \chi(b_{ii}) \lambda(b_{ii} x_{ii}) \prod_{i < j} \sum_{b_{ij} \in \mathbb{Z}_n} \lambda(b_{ij} x_{ji}) \end{aligned}$$

By the assumption $\gcd(a, n) = 1$, we know that $\lambda(x_{ji} \cdot)$ is a nontrivial character of \mathbb{Z}_n if $x_{ji} \neq 0$ in \mathbb{Z}_n , where $1 \leq i < j \leq r$. Therefore, we have

$$(25) \quad \sum_{b_{ij} \in \mathbb{Z}_n} \lambda(b_{ij} x_{ji}) = \begin{cases} n & \text{if } x_{ji} = 0 \\ 0 & \text{otherwise} \end{cases},$$

by the orthogonality of characters, where $1 \leq i < j \leq r$.

From (25), the summation item for $X \in \mathrm{GL}_r(\mathbb{Z}_n)$ in (24) is nonzero only when $X \in \mathrm{B}_r(\mathbb{Z}_n)$ happens. Then, substituting (25) into (24), we obtain

$$(26) \quad \begin{aligned} & G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda) \\ &= \frac{1}{\varphi(n)^r n^{\frac{r(r-1)}{2}}} \sum_{X \in \mathrm{B}_r(\mathbb{Z}_n)} \chi(\det X) \prod_{i=1}^r \sum_{b_{ii} \in \mathbb{Z}_n^*} \chi(b_{ii}) \lambda(b_{ii} x_{ii}) \cdot n^{\frac{r(r-1)}{2}} \\ &= \frac{1}{\varphi(n)^r} \sum_{X \in \mathrm{B}_r(\mathbb{Z}_n)} \prod_{i=1}^r \sum_{b_{ii} \in \mathbb{Z}_n^*} \chi(b_{ii} x_{ii}) \lambda(b_{ii} x_{ii}) \\ &= G(\mathbb{Z}_n, \chi, \lambda)^r n^{\frac{r(r-1)}{2}} \end{aligned}$$

□

To treat the general case $d = \gcd(a, n) \neq 1$, we need the following lemmas. Let $n = p_1^{m_1} \cdots p_s^{m_s}$ be the prime factorization of n . Then by the Chinese Remainder Theorem, it can be seen that

$$(27) \quad \mathrm{GL}_r(\mathbb{Z}_n) \simeq \mathrm{GL}_r(\mathbb{Z}_{p_1^{m_1}}) \times \mathrm{GL}_r(\mathbb{Z}_{p_2^{m_2}}) \times \cdots \times \mathrm{GL}_r(\mathbb{Z}_{p_s^{m_s}}).$$

Lemma 3.2. *Assume p is a prime and m is a positive integer. Then the cardinality of $\mathrm{GL}_r(\mathbb{Z}_{p^m})$ is*

$$(28) \quad |\mathrm{GL}_r(\mathbb{Z}_{p^m})| = p^{mr^2} \prod_{i=1}^r (1 - p^{-i}).$$

Proof. See Proposition 11.15 of [25] on page 182. □

Lemma 3.3. *The cardinality of $\mathrm{GL}_r(\mathbb{Z}_n)$ is*

$$(29) \quad |\mathrm{GL}_r(\mathbb{Z}_n)| = n^{r^2} \prod_{p|n} \prod_{i=1}^r (1 - p^{-i}).$$

Proof. Combine (27) and (28) together. □

Theorem 3.4. *Let the order of λ be n/d , i.e. $d = \gcd(a, n)$ in (14). Let f be the conductor of χ . Then, for $f|n/d$, we have*

$$(30) \quad G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda) = n^{\frac{r(r-1)}{2}} d^{\frac{r(r+1)}{2}} \prod_{\substack{p|n \\ p \nmid n/d}} \prod_{i=1}^r (1-p^{-i}) G(\mathbb{Z}_{n/d}, \chi', \lambda)^r$$

where χ' is the character of $\mathbb{Z}_{n/d}^*$ satisfying (16). Otherwise, for $f \nmid n/d$, we have $G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda) = 0$.

Proof. Similarly as in the proof of Theorem 3.1, we have $G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda)$ equals to the right hand side of (24), i.e.

$$(31) \quad \frac{1}{\varphi(n)^r n^{\frac{r(r-1)}{2}}} \sum_{X \in \mathrm{GL}_r(\mathbb{Z}_n)} \chi(\det X) \prod_{i=1}^r \sum_{b_{ii} \in \mathbb{Z}_n^*} \chi(b_{ii}) \lambda(b_{ii} x_{ii}) \prod_{i < j} \sum_{b_{ij} \in \mathbb{Z}_n} \lambda(b_{ij} x_{ji})$$

If $f \nmid n/d$, then by equation (15), the Gauss sums

$$(32) \quad \sum_{b_{ii} \in \mathbb{Z}_n^*} \chi(b_{ii}) \lambda(b_{ii} x_{ii}) = 0,$$

for $1 \leq i \leq r$. Combining (31) and (32), we get $G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda) = 0$, for $f \nmid n/d$.

Now assume $f|n/d$. Both χ and λ are defined modulo n/d . Consider the natural homomorphism,

$$(33) \quad \mathrm{GL}_r(\mathbb{Z}_n) \xrightarrow{\pi} \mathrm{GL}_r(\mathbb{Z}_{n/d}) : X \mapsto X \pmod{n/d}, \text{ for } X \in \mathrm{GL}_r(\mathbb{Z}_n)$$

which is clearly surjective and whose kernel has the cardinality:

$$(34) \quad \begin{aligned} |\ker \pi| &= |\mathrm{GL}_r(\mathbb{Z}_n)| / |\mathrm{GL}_r(\mathbb{Z}_{n/d})| \\ &= d^{r^2} \prod_{\substack{p|n \\ p \nmid n/d}} \prod_{i=1}^r (1-p^{-i}) \end{aligned}$$

by Lemma 3.3.

Clearly, if $\pi(X) = \pi(X')$, i.e. $X \equiv X' \pmod{n/d}$, then

$$\det X \equiv \det X' \pmod{n/d}, \text{ and } \mathrm{tr} X \equiv \mathrm{tr} X' \pmod{n/d}.$$

Therefore,

$$(35) \quad \chi(\det X) \lambda(\mathrm{tr} X) = \chi(\det X') \lambda(\mathrm{tr} X') = \chi'(\det \pi(X)) \lambda(\mathrm{tr} \pi(X)).$$

Combining (10), (34) and (35), we get

$$(36) \quad G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda) = d^{r^2} \prod_{\substack{p|n \\ p \nmid n/d}} \prod_{i=1}^r (1-p^{-i}) G(\mathrm{GL}_r(\mathbb{Z}_{n/d}), \chi', \lambda).$$

Substituting (21) into (36), we get

$$(37) \quad G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda) = n^{\frac{r(r-1)}{2}} d^{\frac{r(r+1)}{2}} \prod_{\substack{p|n \\ p \nmid n/d}} \prod_{i=1}^r (1-p^{-i}) G(\mathbb{Z}_{n/d}, \chi', \lambda)^r$$

which concludes the proof. \square

Combining equation (15) and Theorem 3.4, we get the following uniform result.

Theorem 3.5. *Let the order of λ be n/d , i.e. $d = \gcd(a, n)$ in (14). Then*

$$(38) \quad G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda) = n^{\frac{r(r-1)}{2}} d^{\frac{r(r-1)}{2}} \prod_{\substack{p|n \\ p \nmid n/d}} \prod_{i=1}^r (1-p^{-i}) / (1-p^{-1}) G(\mathbb{Z}_n, \chi, \lambda)^r.$$

Proof. Let f be the conductor of χ . If $f \nmid n/d$, then the both sides of (38) are zero by Theorem 3.4. Otherwise, by (15), we have

$$(39) \quad G(\mathbb{Z}_n, \chi, \lambda) = d \prod_{\substack{p|n \\ p \nmid n/d}} (1 - \frac{1}{p}) G(\mathbb{Z}_{n/d}, \chi', \lambda)$$

Substituting (39) into (30), we get the desired result. \square

Corollary 3.6. *Let f be the conductor of χ and n/d be the order of λ . Then*

$$(40) \quad |G(\mathrm{GL}_r(\mathbb{Z}_n), x, \lambda)| = n^{\frac{r(r-1)}{2}} d^{\frac{r(r+1)}{2}} f^{\frac{r}{2}} \prod_{\substack{p|n \\ p \nmid n/d}} \prod_{i=1}^r (1-p^{-i})$$

if $f|n/d$; $n/(df)$ is squarefree and coprime to f . Otherwise $G(\mathrm{GL}_r(\mathbb{Z}_n), \chi, \lambda) = 0$.

Proof. Combining (30), (18), (19) and (20), we get the desired result. \square

Next, we consider the case of special linear groups.

Theorem 3.7. *Assume the order of λ is n , i.e. $d = \gcd(a, n) = 1$ in (14). Then*

$$(41) \quad G(\mathrm{SL}_r(\mathbb{Z}_n), \lambda) = n^{\frac{r(r-1)}{2}} K_r(\mathbb{Z}_n, \lambda),$$

where

$$(42) \quad K_r(\mathbb{Z}_n, \lambda) = \sum_{\substack{x_1 \cdots x_r = 1 \\ x_1, \dots, x_r \in \mathbb{Z}_n}} \lambda(x_1 + x_2 + \cdots + x_r)$$

is the hyper-Kloosterman sum over \mathbb{Z}_n .

Proof. Let $\tilde{\mathrm{B}}_r(\mathbb{Z}_n)$ be the Borel subgroup of $\mathrm{SL}_r(\mathbb{Z}_n)$, i.e., the group of upper triangular matrices with determinate 1 over \mathbb{Z}_n . Averaging equation (11) over $\tilde{\mathrm{B}}_r(\mathbb{Z}_n)$ and changing the order of summation, we get

$$(43) \quad \sum_{X \in \mathrm{SL}_r(\mathbb{Z}_n)} \lambda(\mathrm{tr} X) = \frac{1}{\varphi(n)^{r-1} n^{\frac{r(r-1)}{2}}} \sum_{X \in \mathrm{SL}_r(\mathbb{Z}_n)} \sum_{B \in \tilde{\mathrm{B}}_r(\mathbb{Z}_n)} \lambda(\mathrm{tr} BX)$$

Substituting $X = (x_{ij})$ and $B = (b_{ij})$ into (43) and using the multiplicative property of λ , we obtain

$$(44) \quad \begin{aligned} & \frac{1}{\varphi(n)^{r-1} n^{\frac{r(r-1)}{2}}} \sum_{X \in \mathrm{SL}_r(\mathbb{Z}_n)} \sum_{B \in \tilde{\mathrm{B}}_r(\mathbb{Z}_n)} \prod_{i \leq j} \lambda(b_{ij} x_{ji}) \\ &= \frac{1}{\varphi(n)^{r-1} n^{\frac{r(r-1)}{2}}} \sum_{X \in \mathrm{SL}_r(\mathbb{Z}_n)} \sum_{b_{11} \cdots b_{rr} = 1} \lambda\left(\sum_{i=1}^r b_{ii} x_{ii}\right) \cdot \prod_{i < j} \sum_{b_{ij} \in \mathbb{Z}_n} \lambda(b_{ij} x_{ji}) \end{aligned}$$

By the assumption $d = \gcd(a, n) = 1$, we have

$$(45) \quad \sum_{b_{ij} \in \mathbb{Z}_n} \lambda(b_{ij} x_{ji}) = \begin{cases} n, & x_{ji} = 0, \\ 0, & x_{ji} \neq 0. \end{cases}$$

Therefore, the summation item for $X \in \mathrm{SL}_r(\mathbb{Z}_n)$ in (44) is nonzero only if $X \in \widetilde{\mathrm{B}}_r(\mathbb{Z}_n)$. Substituting (45) into (44), we get

$$(46) \quad G(\mathrm{SL}_r(\mathbb{Z}_n), \lambda) = \frac{1}{\varphi(n)^{r-1}} \sum_{X \in \widetilde{\mathrm{B}}_r(\mathbb{Z}_n)} \sum_{b_{11} \cdots b_{rr} = 1} \lambda(b_{11} x_{11} + \cdots + b_{rr} x_{rr}).$$

As $x_{11} x_{22} \cdots x_{rr} = 1$, we get the desired identity (41). \square

Theorem 3.8. *Let n/d be the order of λ . Then*

$$(47) \quad G(\mathrm{SL}_r(\mathbb{Z}_n), \lambda) = n^{r(r-1)/2} d^{r(r+1)/2-1} \prod_{\substack{p|n \\ p \nmid n/d}} \prod_{i=2}^r (1 - p^{-i}) \cdot K_r(\mathbb{Z}_{n/d}, \lambda)$$

where $K_r(\mathbb{Z}_{n/d}, \lambda)$ is the hyper-Kloosterman sum over $\mathbb{Z}_{n/d}$ defined in (42).

Proof. Consider the natural homomorphism

$$(48) \quad \mathrm{SL}_r(\mathbb{Z}_n) \xrightarrow{\pi} \mathrm{SL}_r(\mathbb{Z}_{n/d}), \quad X \mapsto X \bmod n/d, \quad \text{for } X \in \mathrm{SL}_r(\mathbb{Z}_n)$$

By a result of Shimura (see the proof of Lemma 1.38 of [26] on page 21), we know that π is surjective.

Clearly

$$(49) \quad |\mathrm{SL}_r(\mathbb{Z}_n)| = |\mathrm{GL}_r(\mathbb{Z}_n)| / |\mathbb{Z}_n^*|.$$

Therefore, by Lemma 2, we have

$$(50) \quad |\mathrm{SL}_r(\mathbb{Z}_n)| = n^{r^2-1} \prod_{p|n} \prod_{i=2}^r (1 - p^{-i}).$$

Combining the subjectivity of π and (50), we get

$$(51) \quad |\ker \pi| = d^{r^2-1} \prod_{\substack{p|n \\ p \nmid n/d}} \prod_{i=2}^r (1 - p^{-i})$$

Note that for $X, X' \in \mathrm{SL}_r(\mathbb{Z}_n)$, if $\pi(X) = \pi(X')$, i.e. $X \equiv X' \pmod{n/d}$, then $\mathrm{tr} X \equiv \mathrm{tr} X' \pmod{n/d}$, which implies

$$\lambda(\mathrm{tr} X) = \lambda(\mathrm{tr} X') = \lambda(\mathrm{tr} \pi(X)).$$

Therefore,

$$(52) \quad G(\mathrm{SL}_r(\mathbb{Z}_n), \lambda) = |\ker \pi| \cdot G(\mathrm{SL}_r(\mathbb{Z}_{n/d}), \lambda).$$

Substituting (41) and (51) into (52), we get the desired identity. \square

Remark. In [28] and [24], some bounds of hyper-Kloosterman sums $K_r(\mathbb{Z}_n, \lambda)$ are given. Using their results, we can derive bounds of Gauss sums $G(\mathrm{SL}_r(\mathbb{Z}_n), \lambda)$ by Theorem 3.8.

For convinience of readers, here we rewrite their results by our notations.

Let $n = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$ be the prime factorization of n . Denote $p_i^{m_i}$ by q_i , for $1 \leq i \leq s$. Let $w_1, w_2, \dots, w_s \in \mathbb{Z}$ satisfy

$$(53) \quad w_1 \frac{n}{q_1} + w_2 \frac{n}{q_2} + \cdots + w_s \frac{n}{q_s} = 1$$

and denote $w_i n / q_i$ by e_i , where $1 \leq i \leq s$. By Theorem 1 of [28], we have

$$(54) \quad K_r(\mathbb{Z}_n, \lambda) = K_r(\mathbb{Z}_{q_1}, \lambda_{e_1}) \cdots K_r(\mathbb{Z}_{q_s}, \lambda_{e_s})$$

where λ_{e_i} is a character of \mathbb{Z}_{q_i} such that

$$(55) \quad \lambda_{e_i}(x) = \lambda(e_i x), \quad \forall x \in \mathbb{Z}_{q_i}$$

where $1 \leq i \leq s$.

Now assume the order of λ is n , i.e., $d = \gcd(a, n) = 1$

Then by Theorem 6 of [28], we have

$$(56) \quad |K_r(\mathbb{Z}_n, \lambda)| \leq n^{\frac{r-1}{2}} d_r(n)$$

where $d_r(n)$ denotes the number of representations of n as a product of r factors. Explicitly,

$$(57) \quad d_r(n) = \prod_{p^m \parallel n} \binom{m+r-1}{m}$$

Next, we will state the bounds of [24].

By (54), to bound $K_r(\mathbb{Z}_n, \lambda)$, it suffices to consider the case $n = p^m$ being a prime power. We still assume the order of λ is n , which is sufficient for our purpose.

Let $h = v_p(r)$ be the exact power of p dividing r and $\lfloor x \rfloor$ be the floor function, i.e. the greatest integer $\leq x$. From Example 1.17 of [24], we know that

$$|K_r(\mathbb{Z}_{p^m}, \lambda)| \leq \begin{cases} rp^{\frac{(r-1)}{2}}, & m = 1; \\ rp^{\frac{m(r-1)}{2}}, & h = 0; \\ p^{v_p(2)-h/2} rp^{\frac{m(r-1)}{2}}, & h > 0 \text{ and } m \geq 3h + 2 + 4v_p(2); \\ p^{v_p(2)+\min\{h, \lfloor m/2 \rfloor - 1 - v_p(2)\}} \gcd(r, p-1) p^{\frac{m(r-1)}{2}}, & \text{otherwise.} \end{cases}$$

In the above inequality, the case of $m = 1$ is originally due to Deligne [2].

4. APPLICATIONS

For $\beta \in \mathbb{Z}_n$, let

$$(58) \quad N_\beta = |\{x \in \text{GL}_r(\mathbb{Z}_n) \mid \text{tr } X = \beta\}|$$

In this section, we will calcute N_β by applying Theorem 3.5.

Note that for $c \in \mathbb{Z}_n^*$, $N_{c\beta} = N_\beta$ since multiplication by c is a bijection of $\text{GL}_r(\mathbb{Z}_n)$. Therefore,

$$(59) \quad N_\beta = N_{\gcd(\beta, n)}.$$

and only N_l for $l|n$ needs to be computed.

For our purpose, we also need some knowledge of Ramanujan sums (see p.1-2 of [29]). By definition, the Ramanujan sum $C_n(k)$ is the sum of k -th powers of n -th primitive roots of unity ($k \in \mathbb{Z}$), i.e.

$$(60) \quad C_n(k) = \sum_{j \in \mathbb{Z}_n^*} \exp\left(\frac{jk2\pi\sqrt{-1}}{n}\right).$$

Clearly, Ramanujan sums are Gauss sums with trivial multiplicative characters $\chi = 1$, i.e.

$$(61) \quad C_n(k) = G(\mathbb{Z}_n, 1, \lambda_k),$$

where λ_k is the character of \mathbb{Z}_n such that

$$(62) \quad \lambda_k(x) = \exp\left(\frac{kx2\pi\sqrt{-1}}{n}\right), \quad \forall x \in \mathbb{Z}_n$$

It can also be expressed as

$$(63) \quad C_n(k) = \sum_{d|\gcd(k,n)} d\mu(n/d).$$

The Ramanujan sums enjoy the following orthogonal properties:

$$(64) \quad \frac{1}{\text{lcm}(l,n)} \sum_{k=1}^{\text{lcm}(l,n)} C_l(k)C_n(k) = \begin{cases} \varphi(n), & \text{if } l = n, \\ 0, & \text{otherwise.} \end{cases}$$

where lcm represents the least common multiple.

Theorem 4.1. *For $\beta \in \mathbb{Z}_n$, let $l = \gcd(\beta, n)$. Then,*

$$(65) \quad N_\beta = \frac{1}{\varphi(n/l)} n^{r(r-1)/2-1} \sum_{d|n} C_{n/l}(d) C_n^r(d) \varphi(n/d) \\ \times d^{r(r-1)/2} \prod_{\substack{p|n \\ p \nmid n/d}} \prod_{i=1}^r (1-p^{-i}) / (1-p^{-1})$$

Proof. By definition, $G(\text{GL}_r(\mathbb{Z}_n), 1, \lambda_k)$ equals to

$$(66) \quad \sum_{X \in \text{GL}_r(\mathbb{Z}_n)} \lambda_k(\text{tr } X) \\ = \sum_{l|n} \sum_{\gcd(\text{tr } X, n)=l} \lambda_k(\text{tr } X) \\ = \sum_{l|n} N_l \times \sum_{\substack{j=1 \\ \gcd(j, n/l)=1}}^{n/l} \exp\left(\frac{kjl2\pi\sqrt{-1}}{n}\right).$$

The last equality is by using (59), replacing $\text{tr } X = jl$, and using (62). Substituting (60) into (66) with n by n/l , we get

$$(67) \quad \sum_{l|n} N_l C_{n/l}(k) = G(\text{GL}_r(\mathbb{Z}_n), 1, \lambda_k).$$

Substituting Theorem 3.5 into (67) and using (61), we get

$$(68) \quad \sum_{l|n} N_l C_{n/l}(k) = n^{\frac{r(r-1)}{2}} C_n^r(k) A_n(k)$$

where

$$(69) \quad A_n(k) = d^{\frac{r(r-1)}{2}} \prod_{\substack{p|n \\ p \nmid n/d}} \prod_{i=1}^r (1-p^{-i}) / (1-p^{-1}) \text{ with } d = \gcd(k, n).$$

Letting k be $1, 2, \dots, n$ in equation (68), we get a system of linear equations:

$$(70) \quad \begin{pmatrix} C_n(1) & \cdots & C_{n/l}(1) & \cdots & C_1(1) \\ C_n(2) & \cdots & C_{n/l}(2) & \cdots & C_1(2) \\ \vdots & & \vdots & & \vdots \\ C_n(k) & \cdots & C_{n/l}(k) & \cdots & C_1(k) \\ \vdots & & \vdots & & \vdots \\ C_n(n) & \cdots & C_{n/l}(n) & \cdots & C_1(n) \end{pmatrix} \begin{pmatrix} N_1 \\ \vdots \\ N_l \\ \vdots \\ N_n \end{pmatrix} = n^{\frac{r(r-1)}{2}} \begin{pmatrix} C_n^r(1)A_n(1) \\ C_n^r(2)A_n(2) \\ \vdots \\ C_n^r(k)A_n(k) \\ \vdots \\ C_n^r(n)A_n(n) \end{pmatrix}$$

By (64), the columns of coefficient matrix in (70) are orthogonal. Taking the inner product, we get

$$(71) \quad l \cdot n/l \cdot \varphi(n/l) \cdot N_l = n^{\frac{r(r-1)}{2}} \sum_{k=1}^n C_{n/l}(k) C_n^r(k) A_n(k)$$

by (64) again. Since the summation item in the right hand side of (71) only depends on the $d = \gcd(k, n)$, we get

$$(72) \quad N_l = \frac{1}{\varphi(n/l)} n^{\frac{r(r-1)}{2}-1} \sum_{d|n} C_{n/l}(d) C_n^r(d) A_n(d) \varphi(n/d)$$

Substituting (69) into (72), we get the desired result. \square

The Theorem 4.1 looks complicated for general n . So we calculate the special case of n being a prime power. The result seems to be much simpler.

Theorem 4.2. *Let $n = p^m$ be a prime power with $m \geq 1$. For $\beta \in \mathbb{Z}_n$, let $p^t = \gcd(\beta, p^m)$ with $0 \leq t \leq m$. Then*

$$(73) \quad N_\beta = \begin{cases} p^{m(r^2-1)} \left(\prod_{i=1}^r (1-p^{-i}) - (-1)^r p^{-r(r+1)/2} \right), & \text{if } t = 0, \\ p^{m(r^2-1)} \left(\prod_{i=1}^r (1-p^{-i}) + (-1)^r p^{-r(r+1)/2} (p-1) \right), & \text{otherwise.} \end{cases}$$

Proof. By Theorem 4.1, we have

$$(74) \quad N_\beta = \frac{1}{\varphi(p^{m-t})} p^{m(r(r-1)/2-1)} \sum_{u=0}^m C_{p^{m-t}}(p^u) C_{p^m}^r(p^u) \varphi(p^{m-u}) p^{ur(r-1)/2} \\ \times \prod_{p|p^{m-u}} \prod_{i=1}^r (1-p^{-i}) / (1-p^{-1})$$

From (63), we have

$$(75) \quad C_{p^m}(p^u) = \begin{cases} \varphi(p^m), & u \geq m, \\ -p^{m-1}, & u = m-1, \\ 0, & 0 \leq u < m-1. \end{cases}$$

Therefore only for $u = m, m-1$, the corresponding summation items in (74) are nonzero, which are

$$(76) \quad \varphi(p^{m-t})\varphi(p^m)^r p^{mr(r-1)/2} \prod_{i=1}^r (1-p^{-i})/(1-p^{-1})$$

and

$$(77) \quad C_{p^{m-t}}(p^{m-1})(-p^{m-1})^r \varphi(p) p^{(m-1)r(r-1)/2}$$

respectively. By (75), we have

$$(78) \quad C_{p^{m-t}}(p^{m-1}) = \begin{cases} -p^{m-1}, & \text{if } t = 0, \\ \varphi(p^{m-t}), & \text{if } 0 < t \leq m. \end{cases}$$

Substituting (78) into (77), then putting (76) and (77) into (74), we get the desired result. \square

Remark. In Theorem 4.2, it is interesting to note that N_β has only two values depending $p|\beta$ or not. This is clearly true in the case of $m = 1$. In fact, Theorem 3.1 of [18] gives the exact value of N_β for $m = 1$, e.g.

$$(79) \quad N_0 = p^{r^2-1} \left(\prod_{i=1}^r (1-p^{-i}) + (-1)^r p^{-r(r+1)/2} (p-1) \right)$$

For general m , this has the following explanations. Consider the natural homomorphism:

$$(80) \quad \mathrm{GL}_r(\mathbb{Z}_{p^m}) \xrightarrow{\pi} \mathrm{GL}_r(\mathbb{F}_p), \quad X \mapsto X \bmod p,$$

where $X \in \mathrm{GL}_r(\mathbb{Z}_{p^m})$. Denote $Y = \pi(X)$. Then clearly,

$$(81) \quad \mathrm{tr} Y \equiv \mathrm{tr} X \pmod{p}, \quad \det Y \equiv \det X \pmod{p}.$$

Fixed $Y \in \mathrm{GL}_r(\mathbb{F}_p)$, let X_0 be any matrix in $\mathrm{GL}_r(\mathbb{Z}_{p^m})$ such that $X_0 \equiv Y \pmod{p}$. Then

$$(82) \quad \pi^{-1}(Y) = \{X_0 + Z \mid Z \in p\mathrm{M}_r(\mathbb{Z}_{p^m})\}$$

since $\det(X_0 + Z) \equiv \det Y \not\equiv 0 \pmod{p}$, where $\mathrm{M}_r(\mathbb{Z}_{p^m})$ is the set of $r \times r$ square matrices over \mathbb{Z}_{p^m} .

Now assume $p|\beta$. If $\mathrm{tr} X = \beta$, then $\mathrm{tr} \pi(X) = 0$ in \mathbb{F}_p .

Therefore

$$(83) \quad \{X \in \mathrm{GL}_r(\mathbb{Z}_{p^m}) \mid \mathrm{tr} X = \beta\} = \bigcup_{\mathrm{tr} Y=0} \{X \in \pi^{-1}(Y) \mid \mathrm{tr} X = \beta\}.$$

Fixed Y , by (82), we have

$$(84) \quad \{X \in \pi^{-1}(Y) \mid \mathrm{tr} X = \beta\} = \{X_0 + Z \mid \mathrm{tr} Z = \beta - \mathrm{tr} X_0, Z \in p\mathrm{M}_r(\mathbb{Z}_{p^m})\}.$$

This implies that

$$(85) \quad |\{X \in \pi^{-1}(Y) \mid \mathrm{tr} X = \beta\}| = p^{(m-1)(r^2-1)},$$

which does not depend on β .

Combining (79), (83) and (85), we have, for $p|\beta$,

$$(86) \quad N_\beta = p^{m(r^2-1)} \left(\prod_{i=1}^r (1 - p^{-i}) + (-1)^r p^{-r(r+1)/2} (p-1) \right).$$

Clearly, for $p \nmid \beta$, $N_\beta = N_1$ does not depend on β , which can be easily computed by combining Lemma 3.2 and equation (86).

Summing up, this actually gives another proof of Theorem 4.2 based on Theorem 3.1 of [18].

From the Chinese Remainder Theorem,

$$\mathrm{GL}_r(\mathbb{Z}_n) \simeq \mathrm{GL}_r(\mathbb{Z}_{p_1^{m_1}}) \times \mathrm{GL}_r(\mathbb{Z}_{p_2^{m_2}}) \times \cdots \times \mathrm{GL}_r(\mathbb{Z}_{p_s^{m_s}})$$

where $n = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$ is the prime factorization of n . Therefore

$$(87) \quad |\{X \in \mathrm{GL}_r(\mathbb{Z}_n) | \mathrm{tr} X = \beta\}| = \prod_{i=1}^s |\{X \in \mathrm{GL}_r(\mathbb{Z}_{p_i^{m_i}}) | \mathrm{tr} X = \beta_i\}|$$

where $\beta_i \equiv \beta \pmod{p_i^{m_i}}$.

Combining Theorem 4.2 and (87), we get a product formula of N_β .

Theorem 4.3. *For $\beta \in \mathbb{Z}_n$, let $l = \gcd(n, \beta)$. Then*

$$(88) \quad N_\beta = n^{r^2-1} \prod_{p|l} \left(\prod_{i=1}^r (1 - p^{-i}) + (-1)^r p^{-r(r+1)/2} (p-1) \right) \\ \times \prod_{\substack{p|n \\ p \nmid l}} \left(\prod_{i=1}^r (1 - p^{-i}) - (-1)^r p^{-r(r+1)/2} \right)$$

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