Expansions of the Riemann Zeta function in the critical strip

B. Candelpergher [1]

Introduction

We introduce the functions defined for $t \in]0, +\infty[$ by

$$\Psi_m(t) = \sqrt{2}\ (\frac{t^2-1}{t^2+1})^m \frac{1}{\sqrt{1+t^2}}$$

where $m \geq 0$ is an integer. Their Mellin transform are

$$\mathcal{M}(\Psi_m)(s) = \int_0^{+\infty} t^{s-1}\Psi_m(t)dt = \frac{1}{\sqrt{2\pi}}\Gamma(\frac{s}{2})\Gamma(\frac{1-s}{2})Q_m(s)$$

where Q_m are polynomials in $\mathbb{R}[X]$ with their roots on the line $Re(s) = 1/2$.
We use these functions Ψ_m to get the expansion

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2} - 1 - \frac{1}{t} = \sum_{m\geq 0} \alpha_{2m}\Psi_{2m}(t) \quad \text{for } t \in]0, +\infty[$$

with

$$\alpha_{2m} = \frac{2^{-4m}}{(2m)!}\Big(\sum_{n\in\mathbb{Z}} H_{4m}(\sqrt{2\pi}\ n)e^{-\pi n^2} - 2\frac{(4m)!}{(2m)!}\Big)$$

where H_n are the Hermite polynomials.
In the strip $0 < Re(s) < 1$ the well-known classical result

$$\int_0^{+\infty} t^{s-1}(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2} - 1 - \frac{1}{t})dt = \Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s)$$

allows us to conjecture the following expansion of Zeta for $0 < Re(s) < 1$

$$\zeta(s) = \frac{1}{\sqrt{2\pi}}\ \pi^{\frac{s}{2}}\ \Gamma(\frac{1-s}{2})\sum_{m\geq 0}\alpha_{2m}\ Q_{2m}(s)$$

[1] Univ. de Nice, Lab. J.A.Dieudonné , email: candel@unice.fr

1 Functions related to the quantum harmonic oscillator

1.1 Hermite functions

For every integer $m \geq 0$ let us consider the Hermite function

$$\Phi_m(x) = H_m(\sqrt{2\pi}x)e^{-\pi x^2}$$

where $H_m \in \mathbb{R}[X]$ are the Hermite polynomials defined by the generating function

$$e^{-t^2+2xt} = \sum_{m\geq 0} \frac{H_m(x)}{m!}t^m$$

or directly by $H_m(x) = (-1)^m e^{x^2}\partial^m e^{-x^2}$.

The Hermite functions $\Phi_m \in L^2(\mathbb{R})$ are known to form an orthogonal system of eigenfunctions of the quantum harmonic oscillator

$$2\pi(x^2 - \frac{1}{4\pi^2}\partial^2)\Phi_m = (2m+1)\Phi_m \quad \text{with} \quad \int_{\mathbb{R}} (\Phi_m(x))^2 dx = \frac{1}{\sqrt{2}}2^m m!$$

The function Φ_m has same parity as m. We are interested with the even functions Φ_{2m}, we have (cf. [4])

$$\int_{\mathbb{R}} e^{-2i\pi x\xi}\Phi_{2m}(x)dx = (-1)^m\Phi_{2m}(\xi)$$

Thus for $\xi = 0$

$$\int_{\mathbb{R}} \Phi_{2m}(x)dx = (-1)^m\Phi_{2m}(0) = \frac{(2m)!}{m!}$$

The function Φ_{2m} is bounded (cf. [4]) by

$$B1) \qquad |\Phi_{2m}(x)| \leq K2^m\sqrt{(2m)!} \quad \text{with } K = 1.086435 \quad \text{for } x \in \mathbb{R}$$

The function Φ_{2m} is (cf. [5]) oscillating in the interval

$$I_m = [-\frac{1}{\sqrt{\pi}}\sqrt{2m+1}, \frac{1}{\sqrt{\pi}}\sqrt{2m+1}]$$

and exponentially decreasing when $x \notin I_m$, more precisely (cf. [4]) we have

$$(B2) \qquad |\Phi_{2m}(x)| \leq \frac{(2m)!}{m!}e^{2x\sqrt{2\pi m}}e^{-\pi x^2} \text{ for } x > 0$$

In the series expansions of the following sections we use the normalized sums

$$S_{2m} = \frac{2^{-2m}}{m!} \sum_{n \in \mathbb{Z}} \Phi_{2m}(n)$$

Lemma 0
For $m \to +\infty$ we have

$$\frac{2^{-2m}}{m!} \sum_{|n| \geq 2\sqrt{2m}} |\Phi_{2m}(n)| = O(e^{-2\pi\sqrt{2m}})$$

and

$$\frac{2^{-2m}}{m!} \sum_{n \in \mathbb{Z}} |\Phi_{2m}(n)| = O(m^{1/4})$$

Proof
We have for $m \geq 1$

$$2x\sqrt{2\pi m} - \pi x^2 \leq -\pi x \quad \text{for} \quad x \geq 2\sqrt{2m}$$

thus using inequality (B2) we get

$$|\frac{2^{-2m}}{m!} \Phi_{2m}(x)| \leq \frac{2^{-2m}(2m)!}{(m!)^2} e^{-\pi|x|} \text{ for } |x| \geq 2\sqrt{2m}$$

Thus by summation for $|n| \geq 2\sqrt{2m}$ and with the Stirling formula we get

$$\frac{2^{-2m}}{m!} \sum_{|n| \geq 2\sqrt{2m}} |\Phi_{2m}(n)| \leq \frac{2^{-2m+1}(2m)!}{(m!)^2} \frac{e^{-\pi(2\sqrt{2m})}}{1 - e^{-\pi}} = O(e^{-\pi 2\sqrt{2m}})$$

From inequality (B1) we deduce that

$$\begin{aligned} \frac{2^{-2m}}{m!} \sum_{n \in \mathbb{Z}} |\Phi_{2m}(n)| &= \sum_{|n| < 2\sqrt{2m}} \frac{2^{-2m}}{m!} |\Phi_{2m}(n)| + \frac{2^{-2m}}{m!} \sum_{|n| \geq 2\sqrt{2m}} |\Phi_{2m}(n)| \\ &\leq K\sqrt{2m} \frac{2^{-m+1}\sqrt{(2m)!}}{m!} + O(e^{-\pi 2\sqrt{2m}}) \end{aligned}$$

thus by Stirling formula we get $\frac{2^{-2m}}{m!} \sum_{n \in \mathbb{Z}} |\Phi_{2m}(n)| = O(m^{1/4})$.
□

1.2 The functions Ψ_m

The function $x \mapsto e^{-2\pi a^2 x^2}$, $Re(a^2) > -\frac{1}{2}$, expands (cf. [6] p.71-75) as the following series of Hermite polynomials, for $x \in \mathbb{R}$ we have

$$e^{-2\pi a^2 x^2} = \frac{1}{\sqrt{1+a^2}} \sum_{m \geq 0} \frac{(-1)^m a^{2m}}{2^{2m}(1+a^2)^m m!} H_{2m}(\sqrt{2\pi}\ x)$$

Multiplying by $e^{-\pi x^2}$ we get, with $t^2 = 1 + 2a^2$

$$e^{-\pi x^2 t^2} = \sum_{m \geq 0} (-1)^m \frac{1}{2^{2m}} \Phi_{2m}(x) \frac{\Psi_m(t)}{m!} \text{ for } Re(t^2) > 0 \tag{1}$$

where we define for $t \in S = \{re^{i\theta} |\ r > 0, -\frac{\pi}{4} < \theta < \frac{\pi}{4}\}$ the function

$$\Psi_m(t) = \sqrt{2}\ (\frac{t^2-1}{t^2+1})^m \frac{1}{\sqrt{1+t^2}}$$

Lemma 1
The functions Ψ_m are related to the Hermite functions by

$$\frac{\Psi_m}{m!} = (\frac{2\sqrt{2}}{x} e^{-\pi/x^2})\ * \frac{\Phi_{2m}}{(2m)!}$$

where $*$ is the multiplicative convolution of functions defined on $]0, +\infty[$

$$(f * g)(t) = \int_0^{+\infty} f(\frac{t}{x}) g(x) \frac{1}{x} dx$$

Proof
With the classical relation

$$\int_{-\infty}^{+\infty} e^{-ax^2} e^{bx} dx = \sqrt{\frac{\pi}{a}}\ e^{\frac{b^2}{4a}} \quad \text{where } a > 0,\ b \in \mathbb{C}$$

we get

$$e^{z^2 \frac{t^2-1}{t^2+1}} \frac{t}{\sqrt{1+t^2}} = \int_{-\infty}^{+\infty} e^{-\pi x^2 \frac{t^2+1}{t^2}} e^{-z^2 + 2\sqrt{2\pi} x z} dx$$

and using the power series expansion

$$e^{-z^2+2\sqrt{2\pi}xz} = \sum_{m\geq 0} \frac{z^m}{m!} H_m(\sqrt{2\pi}x)$$

we get by identification

$$\frac{1}{m!}\sqrt{2}\ (\frac{t^2-1}{t^2+1})^m \frac{t}{\sqrt{1+t^2}} = 2\sqrt{2}\int_0^{+\infty} e^{-\pi x^2 \frac{t^2+1}{t^2}} \frac{1}{(2m)!} H_{2m}(\sqrt{2\pi}x)dx$$

This gives

$$\frac{1}{m!}\Psi_m(t) = 2\sqrt{2}\int_0^{+\infty} e^{-\pi x^2/t^2}\frac{1}{t}\frac{\Phi_{2m}(x)}{(2m)!}dx$$

and we see that this last integral is the multiplicative convolution

$$(f*g)(t) = \int_0^{+\infty} f(\frac{t}{x})g(x)\frac{1}{x}dx$$

with $f(x) = 2\sqrt{2}\ e^{-\pi/x^2}\frac{1}{x}$ and $g(x) = \frac{\Phi_{2m}(x)}{(2m)!}$.
□

2 Series expansions

Let $z \mapsto \sqrt{z}$ the principal determination of the square root, the holomorphic function

$$u \mapsto t = \sqrt{\frac{1+u}{1-u}}$$

maps the open unit disk $D(0,1) = \{z \in \mathbb{C} |\ \ |z| < 1\}$ onto the sector

$$S = \{re^{i\theta} |\ r > 0, -\frac{\pi}{4} < \theta < \frac{\pi}{4}\}$$

For any function f holomorphic in S let us define the function

$$Tf(u) = \frac{1}{\sqrt{1-u}} f\Big(\sqrt{\frac{1+u}{1-u}}\Big)$$

which is holomorphic in the open disk $D(0,1)$.
For every integer $m \geq 0$ we verify immediately that we have

$$T\Psi_m(u) = u^m$$

For a function f defined on S the expansion

$$f(t) = \sum_{m\geq 0} a_m \frac{\Psi_m(t)}{m!}$$

follows the Taylor expansion of Tf

$$Tf(u) = \sum_{m\geq 0} \frac{a_m}{m!}\ u^m$$

Remark. Note that $\Psi_m(t) = (-1)^m \frac{1}{t}\Psi_m(\frac{1}{t})$ for all $t \in S$. For a function f on S the relation

$$f(t) = \frac{1}{t} f(\frac{1}{t})$$

is equivalent to the parity of the function Tf

$$Tf(u) = Tf(-u)$$

in this case the expansion of f is of the form

$$f(t) = \sum_{m\geq 0} a_{2m} \frac{\Psi_{2m}(t)}{(2m)!}$$

Example
For the function $f = \frac{1}{1+t}$, $t \in S$, we have

$$Tf(u) = \frac{\sqrt{1+u} - \sqrt{1-u}}{2u}$$

This gives for $t \in S$

$$\frac{1}{1+t} = \frac{1}{2} \sum_{m\geq 0} \frac{(4m)!}{2^{4m}(2m+1)!} \frac{\Psi_{2m}(t)}{(2m)!} \tag{2}$$

2.1 Expansion of the theta function

The theta function defined for $t \in S$ by

$$G(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2}$$

is holomorphic in S and we have for $u \in D(0,1)$

$$TG(u) = \frac{1}{\sqrt{1-u}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2 \frac{1+u}{1-u}}$$

Let

$$TG(u) = \sum_{m\geq 0} g_m \frac{1}{m!} u^m$$

be the power series expansion of the holomorphic function TG in the open disk $D(0,1)$. The Jacobi identity (cf.[3])

$$\frac{1}{t} G(\frac{1}{t}) = G(t)$$

gives the parity of TG and we get $TG(u) = \sum_{n\geq 0} g_{2m} \frac{1}{(2m)!} u^{2m}$.

Thus we have for $t \in S$

$$G(t) = \sum_{m\geq 0} g_{2m} \frac{\Psi_{2m}(t)}{(2m)!}$$

Lemma 2

We have for $t \in S = \{re^{i\theta} |\ r > 0, -\frac{\pi}{4} < \theta < \frac{\pi}{4}\}$

$$G(t) = \sum_{m\geq 0} S_{4m} \Psi_{2m}(t) \quad \text{where} \quad S_{4m} = \frac{2^{-4m}}{(2m)!} \sum_{n\in\mathbb{Z}} \Phi_{4m}(n)$$

Proof

Take the relation (1) with $x = n \in \mathbb{Z}$, by summation we get

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2} = \sum_{n\in\mathbb{Z}} \sum_{m\geq 0} \frac{(-1)^m}{m!} 2^{-2m} \Phi_{2m}(n) \Psi_m(t)$$

$$= \sum_{m\geq 0} \frac{(-1)^m}{m!} 2^{-2m} \Psi_m(t) \sum_{n\in\mathbb{Z}} \Phi_{2m}(n)$$

To justify the interchange of summations $\sum_{n\in\mathbb{Z}}\sum_{m\geq 0} = \sum_{m\geq 0}\sum_{n\in\mathbb{Z}}$ we observe that

$$|\frac{t^2-1}{t^2+1}| < 1 \text{ for } t \in S$$

and by Lemma 0 we have $\frac{2^{-2m}}{m!}\sum_{n\in\mathbb{Z}}|\Phi_{2m}(n)| = O(m^{1/4})$ thus for $t \in S$

$$\sum_{m\geq 0}\frac{2^{-2m}}{m!}\sum_{n\in\mathbb{Z}}|\Phi_{2m}(n)||\Psi_m(t)| < +\infty$$

This gives

$$G(t) = \sum_{m\geq 0}(-1)^m S_{2m}\Psi_m(t)$$

Since we have seen that the function TG is even we deduce that in this last sum, only the constants S_{4m} are non zero.

□

Remark

We have also (cf. Appendix) for the constants S_{4m} another expression

$$S_{4m} = \frac{(\frac{\pi}{2})^{2m}}{(2m)!}\frac{1}{S_0\sqrt{2}}\sum_{(k,l)\in\mathbb{Z}^2}(-1)^{kl}e^{-\pi\frac{1}{2}(k^2+l^2)}(k+il)^{4m}$$

Theorem

For $t \in S = \{re^{i\theta}|\ r > 0, -\frac{\pi}{4} < \theta < \frac{\pi}{4}\}$ we have

$$G(t) - 1 - \frac{1}{t} = \sum_{m\geq 0}\alpha_{2m}\Psi_{2m}(t) \text{ with } \alpha_{2m} = S_{4m} - \frac{2^{-4m+1}(4m)!}{(2m)!(2m)!}$$

Proof

Using Lemma 2, to get the expansion of $G(t) - 1 - \frac{1}{t}$ in terms of $\Psi_m(t)$ it is now sufficient to expand $1 + \frac{1}{t}$. For $f(t) = 1 + \frac{1}{t}$ one has

$$Tf(u) = \frac{1}{\sqrt{1+u}} + \frac{1}{\sqrt{1-u}}$$

and we obtain for $t \in S$

$$1 + \frac{1}{t} = 2\sum_{m\geq 0}\frac{(4m)!}{2^{4m}(2m)!}\frac{\Psi_{2m}(t)}{(2m)!}$$

□

Remark

We see that

$$\alpha_{2m} = \frac{2^{-4m}}{(2m)!}\Big(\sum_{n\neq 0}\Phi_{4m}(n) - [\Phi_{4m}(0) + \int_{\mathbb{R}}\Phi_{4m}(x)dx]\Big)$$

This is easily explained if we look at the general Müntz formula (cf. [7]):

let F be an even continuously differentiable function such that F and F' are $O(x^{-a})$, $(a > 1)$ when $x \to \infty$, then for $0 < Re(s) < 1$ we have

$$2\ \zeta(s)\mathcal{M}F(s) = \mathcal{M}\Big(G(t) - [F(0) + \int_{\mathbb{R}} F(xt)dx]\Big)(s)$$

with $G(t) = \sum_{n\in\mathbb{Z}} F(nt)$, this is our case with $F(x) = e^{-\pi x^2}$.

If there exist functions a sequence of functions φ_m and ψ_m such that we have an expansion

$$F(xt) = \sum_{m\geq 0}\varphi_m(x)\psi_m(t)$$

then, at least formally, we get

$$G(t) - [F(0) + \int_{\mathbb{R}} F(xt)dx] = \sum_{m\geq 0}\Big(\sum_{n\in\mathbb{Z}}\varphi_m(n) - [\varphi_m(0) + \int_{\mathbb{R}}\varphi_m(x)dx]\Big)\psi_m(t)$$

in our case $\varphi_m = \frac{2^{-4m}}{(2m)!}\Phi_{4m}$ and $\psi_m(t) = \Psi_{2m}(t)$.

3 Mellin transforms

3.1 The polynomials Q_m

For $Re(s) > 0$, the Mellin transforms of the Hermite functions Φ_{2m} are

$$\int_0^{+\infty}\frac{\Phi_{2m}(x)}{(2m)!}x^{s-1}dx = \frac{1}{2}\pi^{-s/2}\Gamma(s/2)\frac{Q_m(s)}{m!}$$

where Q_m are polynomials in $\mathbb{R}[X]$. This is simply a consequence of the relation

$$\int_0^{+\infty} e^{-\pi x^2}x^{s+2k-1}dx = \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\ \pi^{-k}\ \frac{s}{2}(\frac{s}{2}+1)...(\frac{s}{2}+k-1)$$

We get $Q_0(s) = 1$, $Q_1(s) = 2s - 1$, $Q_2(s) = \frac{4}{3}s^2 - \frac{4}{3}s + 1, \ldots$.

More generally we have

$$Q_m(s) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{(m-k)!} \frac{2^{2k}}{(2k)!} s(s+2)\ldots(s+2(k-1))$$

and (cf. [4]) an expression of Q_m in terms of the hypergeometric function

$$Q_m(s) = (-1)^m \; {}_2F_1(-m, s/2; 1/2; 2)$$

The roots of Q_m are on the line $Re(s) = 1/2$ (cf. [1], [2]). This can be proved (cf. [1]) by observing that the orthogonality relation of the Hermite functions Φ_{2m} implies the orthogonality of the family of polynomials

$$t \mapsto Q_m(\frac{1}{2} + it)$$

with respect to the Borel measure $|\Gamma(\frac{1}{4} + i\frac{t}{2})|^2 dt$ on $\mathbb{R}$.

More explicitly, using the Parseval's formula for Mellin transform

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \big(\mathcal{M}(f)\overline{\mathcal{M}(g)}\big)(\frac{1}{2} + it)dt = \int_0^{+\infty} (f\overline{g})(x)dx$$

we get

$$\frac{1}{4\pi\sqrt{\pi}} \int_{\mathbb{R}} |\Gamma(\frac{1}{4} + i\frac{t}{2})|^2 \; (\frac{Q_{m_1}}{m_1!} \frac{\overline{Q_{m_2}}}{m_2!})(\frac{1}{2} + it)dt = \int_{\mathbb{R}} (\frac{\Phi_{2m_1}}{(2m_1)!} \frac{\Phi_{2m_2}}{(2m_2)!})(x)dx$$

3.2 Mellin transform of Ψ_m

Lemma 3

For $0 < Re(s) < 1$ we have

$$\int_0^{+\infty} t^{s-1} \Psi_m(t)dt = \frac{1}{\sqrt{2\pi}} \Gamma(\frac{s}{2})\Gamma(\frac{1-s}{2})Q_m(s)$$

Proof

By Mellin transform of the relation of Lemma 1, we get

$$\int_0^{+\infty} t^{s-1} \frac{1}{m!} \Psi_m(t)dt = \Big(\int_0^{+\infty} 2\sqrt{2}e^{-\pi/u^2} \frac{1}{u} \, u^{s-1}du\Big)\Big(\int_0^{+\infty} \frac{1}{(2m)!} \Phi_{2m}(x)x^{s-1}dx\Big)$$

that is

$$\int_0^{+\infty} t^{s-1}\frac{1}{m!}\Psi_m(t)dt = \sqrt{2}\pi^{\frac{s-1}{2}}\Gamma(\frac{1-s}{2})\int_0^{+\infty}\frac{1}{(2m)!}\Phi_{2m}(x)x^{s-1}dx$$

□

Remark
Using

$$\frac{1}{t}\Psi_m(\frac{1}{t}) = (-1)^m\Psi_m(t)$$

we get with the change of variable $t \mapsto \frac{1}{t}$

$$\int_0^{+\infty} t^{s-1}\frac{1}{m!}\Psi_m(t)dt = (-1)^m\int_0^{+\infty} t^{-s}\frac{1}{m!}\Psi_m(t)dt$$

for $0 < Re(s) < 1$.
By the preceding lemma this gives

$$Q_m(1-s) = (-1)^m Q_m(s)$$

As a consequence of this relation we see that for $s = \frac{1}{2} + it$ the polynomials $t \mapsto Q_{2m}(\frac{1}{2} + it)$ are in $\mathbb{R}[X]$.

3.3 Expansion of Mellin transforms in terms of the polynomials Q_m

If we have for a function f holomorphic in S an expansion

$$f(t) = \sum_{m\geq 0} a_{2m}\frac{\Psi_{2m}(t)}{(2m)!}$$

and if we can evaluate the Mellin transform of f for $0 < Re(s) < 1$ by integration of the terms of the series :

$$\int_0^{+\infty}\Big(\sum_{m\geq 0} a_{2m}\frac{\Psi_{2m}(t)}{(2m)!}\Big)t^{s-1}dt = \sum_{m\geq 0}\frac{a_{2m}}{(2m)!}\int_0^{+\infty} t^{s-1}\Psi_{2m}(t)dt$$

then we get for $0 < Re(s) < 1$

$$\int_0^{+\infty} f(t)t^{s-1}dt = \frac{1}{\sqrt{2\pi}}\Gamma(\frac{s}{2})\Gamma(\frac{1-s}{2})\sum_{m\geq 0}\frac{a_{2m}}{(2m)!}Q_{2m}(s)$$

A simple condition to justify this calculation is

$$\sum_{m\geq 0}\frac{|a_{2m}|}{(2m)!}<+\infty$$

Since in this case we have for $0<Re(s)<1$

$$\int_0^{+\infty}\sum_{m\geq 0}|t^{s-1}a_{2m}\frac{\Psi_{2m}(t)}{(2m)!}|dt\leq\sum_{m\geq 0}\frac{|a_{2m}|}{(2m)!}\int_0^{+\infty}t^{Re(s)-1}\frac{\sqrt{2}}{\sqrt{1+t^2}}dt<+\infty$$

Example

We have by relation (2)

$$\frac{1}{1+t}=\frac{1}{2}\sum_{m\geq 0}a_{2m}\frac{\Psi_{2m}(t)}{(2m)!}\ \text{with}\ \ a_{2m}=\frac{(4m)!}{2^{4m}(2m+1)!}$$

By Stirling formula we have $\frac{a_{2m}}{(2m)!}=O(m^{-\frac{3}{2}})$ thus $\sum_{m\geq 0}\frac{|a_{2m}|}{(2m)!}<+\infty$.
And for $0<Re(s)<1$ we get

$$\frac{\pi}{\sin(\pi s)}=\frac{1}{2\sqrt{2\pi}}\Gamma(\frac{s}{2})\Gamma(\frac{1-s}{2})\sum_{m\geq 0}\frac{(4m)!}{(2m)!(2m+1)!}2^{-4m}Q_{2m}(s)$$

3.4 A conjecture for an expansion of Zeta in the critical strip

For $0<Re(s)<1$ it is known (cf. [3]) that the Mellin transform of the function $t\mapsto G(t)-1-\frac{1}{t}$ is

$$\int_0^{+\infty}t^{s-1}(G(t)-1-\frac{1}{t})dt=\Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s)$$

We have seen in 2.1 that

$$G(t)-1-\frac{1}{t}=\sum_{m\geq 0}\alpha_{2m}\Psi_{2m}(t)\ \ \text{with}\ \ \alpha_{2m}=S_{4m}-\frac{2^{-4m+1}(4m)!}{(2m)!(2m)!}$$

If we proceed by integration of the terms of the preceding series we get

$$\Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s)=\frac{1}{\sqrt{2\pi}}\Gamma(\frac{s}{2})\Gamma(\frac{1-s}{2})\sum_{m\geq 0}\alpha_{2m}Q_{2m}(s)$$

Unfortunately it seems that in this case $\sum_{m\geq 0}|\alpha_{2m}| = +\infty$ and the justification of the preceding section does not work.

Conjecture

For $0 < Re(s) < 1$ the evaluation of the Mellin transform of $G(t) - 1 - \frac{1}{t}$ by integration of the terms of the preceding series is valid and we get

$$\zeta(s) = \frac{1}{\sqrt{2\pi}}\pi^{\frac{s}{2}}\Gamma(\frac{1-s}{2})\sum_{m\geq 0}\alpha_{2m}\ Q_{2m}(s)$$

$$\text{with } \alpha_{2m} = S_{4m} - \frac{2^{-4m+1}(4m)!}{(2m)!(2m)!}$$

As we have seen the polynomials

$$Q_{2m}(s) = \ {}_2F_1(-2m, s/2; 1/2; 2)$$

are related to Mellin transforms of the Hermite functions Φ_{4m} and they have their roots on the line $Re(s) = 1/2$.

Remarks

1) For the Riemann-Hardy function (cf. [3]) defined for $t \in \mathbb{R}$ by

$$Z(t) = \pi^{\frac{-it}{2}}\frac{\Gamma(\frac{1}{4}+i\frac{t}{2})}{|\Gamma(\frac{1}{4}+i\frac{t}{2})|}\zeta(\frac{1}{2}+it)$$

the preceding conjecture gives

$$Z(t) = \frac{1}{\sqrt{2\pi}}\sum_{m\geq 0}\alpha_{2m}f_{2m}(t)$$

where the functions

$$f_{2m}(t) = \pi^{\frac{1}{4}}|\Gamma(\frac{1}{4}+i\frac{t}{2})|Q_{2m}(\frac{1}{2}+it)$$

are orthogonal in $L^2(\mathbb{R})$.

2) Other expressions of $\zeta(s)$ in the critical strip are obtained by the use of the Müntz formula (cf. [7]): for a continuously differentiable function F on $[0,+\infty[$ such that F and F' are $O(x^{-a})$, $(a>1)$ when $x\to\infty$, we have for $0<Re(s)<1$

$$\zeta(s)\mathcal{M}F(s)=\mathcal{M}\Big(\sum_{n\geq 1}F(nt)-\frac{1}{t}\int_0^{+\infty}F(x)dx\Big)(s)$$

We now show that, with our preceding method, we can obtain a simple expansion of Zeta in the critical strip by applying this formula to the function $F(x)=e^{-2\pi x}$.

For $0<Re(s)<1$ we have

$$(2\pi)^{-s}\Gamma(s)\zeta(s)=\mathcal{M}(\sum_{n\geq 1}e^{-2\pi nt}-\frac{1}{2\pi t})=\int_0^{+\infty}f(t)t^{s-1}dt$$

where

$$f(t)=\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}$$

It is possible to get an expansion of $f(t)$ using the Laguerre functions

$$\varphi_m(x)=e^{-x}L_m(2x)$$

defined by the generating function $\frac{1}{1-u}e^{-x\frac{1+u}{1-u}}=\sum_{m\geq 0}e^{-x}L_m(2x)u^m$.

These functions are orthogonal in $L^2(]0,+\infty[)$ and if J_0 is the Bessel function of order 0 then (cf.[6])

$$\int_0^{+\infty}J_0(2\sqrt{\xi x})\varphi_m(x)dx=(-1)^m\varphi_m(\xi)$$

For $t>0$ we set

$$\psi_m(t)=(\frac{t-1}{t+1})^m\frac{2}{1+t}$$

Using the generating function of the φ_m we get

$$e^{-2\pi xt}=\sum_{m\geq 0}\varphi_m(2\pi x)\psi_m(t) \tag{3}$$

Summing (3) for $x = n \geq 1$ we have formally for $Re(t) > 0$

$$\frac{1}{e^{2\pi t} - 1} = \sum_{m \geq 0} s_m \psi_m(t) \text{ with } s_m = \sum_{n \geq 1} \varphi_m(2\pi n)$$

Since $\frac{1}{2\pi t} = \frac{1}{2\pi} \sum_{m \geq 0} (-1)^m \psi_m(t)$ we get for $Re(t) > 0$

$$f(t) = \sum_{m \geq 0} \big(s_m - (-1)^m \frac{1}{2\pi}\big) \psi_m(t) \tag{4}$$

The functions ψ_m and φ_m are related by the multiplicative convolution

$$\psi_m = 2(-1)^m (e^{-\frac{1}{x}} \frac{1}{x}) * \varphi_m \tag{5}$$

The Mellin transform of φ_m is (cf. [4]) for $Re(s) > 0$

$$\int_0^{+\infty} \varphi_m(x) x^{s-1} dx = \Gamma(s) q_m(s)$$

where q_m is the polynomial $q_m(s) = {}_2F_1(-m, s; 1; 2)$.

By the orthogonality relation of the φ_m we deduce that the polynomials $t \mapsto q_m(\frac{1}{2} + it)$ are orthogonal with respect to the Borel measure $|\Gamma(\frac{1}{2} + i\frac{t}{2})|^2 dt$ on $]0, +\infty[$. Thus q_m has his roots on the line $Re(s) = \frac{1}{2}$.

By (5) for $0 < Re(s) < 1$ we have the Mellin transform of ψ_m

$$\mathcal{M}(\psi_m)(s) = 2\Gamma(s)\Gamma(1-s)(-1)^m q_m(s)$$

By Mellin transform of (4) we get formally

$$\zeta(s) = 2(2\pi)^s \Gamma(1-s) \sum_{m \geq 0} \Big((-1)^m s_m - \frac{1}{2\pi}\Big) q_m(s)$$

A more simple expansion can be obtained using the Mellin transform of the function $g(t) = \frac{1}{t} f(\frac{1}{t})$.

Using the Poisson formula we have

$$g(t) = \frac{1}{t} \sum_{n \geq 1} e^{-2\pi \frac{n}{t}} - \frac{1}{2\pi} = \frac{1}{\pi} \sum_{n \geq 1} \frac{1}{1 + n^2 t^2} - \frac{1}{2t}$$

By Müntz formula we get

$$\mathcal{M}(g)(s) = \mathcal{M}\Big(\frac{1}{\pi}\sum_{n\geq 1}\frac{1}{1+n^2t^2} - \frac{1}{2t}\Big)(s) = \zeta(s)\frac{1}{2\pi}\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})$$

We now apply our preceding method, to the function $F(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.
We verify immediately that

$$\frac{1}{\pi}\frac{1}{1+x^2t^2} = \frac{1}{2\pi}\sum_{m\geq 0}(-1)^m\psi_m(x^2)\psi_m(t^2) \tag{6}$$

We have for $t > 0$

$$\frac{1}{\pi}\sum_{n\geq 1}\frac{1}{1+n^2t^2} = \frac{1}{2\pi}\sum_{m\geq 0}(-1)^m\sigma_m\psi_m(t^2) \quad \text{where} \quad \sigma_m = \sum_{n\geq 1}\psi_m(n^2)$$

With $u = \frac{t^2-1}{t^2+1}$ we have $\frac{1}{t} = \frac{2}{1+t^2}(1-u^2)^{-1/2}$, thus we get

$$\frac{1}{t} = \sum_{m\geq 0}c_m\psi_m(t) \text{ with } c_{2n} = \frac{(2n)!}{2^{2n}(n!)^2} \text{ and } c_{2n+1} = 0$$

Finally we have the expansion

$$g(t) = \frac{1}{2\pi}\sum_{m\geq 0}(-1)^m(\sigma_m - \pi c_m)\psi_m(t^2) \tag{7}$$

By Mellin transform of (7) we get formally

$$\zeta(s) = \sum_{m\geq 0}\Big(\sigma_m - \pi c_m\Big)q_m(\frac{s}{2})$$

Note that, unlike the preceding expansions related to Hermite and Laguerre functions, in this expansion the sequence

$$m \mapsto \sigma_m - \pi c_m = \sum_{n\geq 1}(\frac{n^2-1}{n^2+1})^m\frac{2}{1+n^2} - \pi c_m$$

has a very regular oscillation with amplitude near $\frac{\sqrt{\pi}}{\sqrt{2}}\frac{1}{\sqrt{m}}$, but the polynomials $s \mapsto q_m(\frac{s}{2})$ have their roots on the line $Re(s) = 1$.

References

[1] D.Bump, K.K.Choi, P.Kurlberg, J.Vaaler. A local Riemann hypothesis. Math. Zeitschrift 233. (2000).

[2] D.Bump , E.K.-S.Ng. On the Riemann Zeta Function. Math. Zeitschrift 192. (1986).

[3] H.M.Edwards. Riemann Zeta function. Dover. (1974).

[4] I.S.Gradshteyn, I.M.Ryzhik. Tables of Integrals, Series and Products. Academic Press, Inc. (1994).

[5] E.Hille. A Class of Reciprocal Functions. Annals of Maths. Second Series. Vol. 27 N4 (1926)

[6] N.N.Lebedev. Revised and translated by R.A.Silverman. Special functions and their applications. Dover (1972).

[7] E.C. Titchmarsh. D.R. Heat-Brown. The theory of the Riemann Zeta function. Clarendon 1986.

Acknowledgments.

My warmest thanks go to F. Rouvière, M. Miniconi and J.F. Burnol for their interest and helpful comments.

4 Appendix. Another expression for the constants S_{4m}

Using Poisson summation formula we deduce that for $\varphi \in \mathcal{S}(\mathbb{R})$

$$\sum_{(k,l)\in\mathbb{Z}^2} \int_{\mathbb{R}} e^{-2i\pi xk} e^{-\pi(x-l)^2} \varphi(x)dx = \sum_{l\in\mathbb{Z}} e^{-\pi l^2} \sum_{k\in\mathbb{Z}} \varphi(k) = S_0 \sum_{k\in\mathbb{Z}} \varphi(k)$$

Taking $u \in \mathbb{C}$ and $\varphi(x) = e^{-2\pi xu} e^{-\pi x^2}$, we have

$$\int_{\mathbb{R}} e^{-2i\pi xk} e^{-\pi(x-l)^2} \varphi(x)dx = \frac{1}{\sqrt{2}} (-1)^{kl} e^{-\pi\frac{1}{2}(k^2+l^2)} e^{i\pi u(k+il)} e^{\pi u^2/2}$$

This gives for $u \in \mathbb{C}$ the relation

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2+2\pi nu-\pi u^2/2} = \frac{1}{S_0\sqrt{2}} \sum_{(k,l)\in\mathbb{Z}^2} (-1)^{kl} e^{-\pi\frac{1}{2}(k^2+l^2)} e^{i\pi u(k+il)} \tag{8}$$

Let us now define for every integer $m \geq 0$

$$T_m = \sum_{(k,l)\in\mathbb{Z}^2} (-1)^{kl} e^{-\pi\frac{1}{2}(k^2+l^2)}(k+il)^m$$

We have clearly $T_{2m+1} = 0$ since

$$(-1)^{-k(-l)} e^{-\pi\frac{1}{2}((-k)^2+(-l)^2)}(-k-il)^{2m+1} = -[(-1)^{kl} e^{-\pi\frac{1}{2}(k^2+l^2)}(k+il)^{2m+1}]$$

thus $T_{4m+1} = T_{4m+3} = 0$ and also $T_{4m+2} = 0$ because

$$(-1)^{(-k)l} e^{-\pi\frac{1}{2}(l^2+(-k)^2)}(-k+il)^{4m+2} = -[(-1)^{kl} e^{-\pi\frac{1}{2}(k^2+l^2)}(l+ik)^{4m+2}]$$

Thus only the constants T_{4m} are non zero, and by derivation with respect to u of the holomorphic function defined by the right side of (6) we have

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2+2\pi nu-\pi u^2/2} = \frac{1}{S_0\sqrt{2}} \sum_{m\geq 0} \pi^{4m} T_{4m} \frac{u^{4m}}{(4m)!}$$

Now using the generating function of Hermite polynomials we have

$$e^{-\pi n^2+2\pi nu-\pi u^2/2} = \sum_{m\geq 0} (\frac{\pi}{2})^{\frac{m}{2}} \Phi_m(n) \frac{u^m}{m!}$$

By summation with $n \in \mathbb{Z}$ of this relation we deduce that for $|u| < 1$

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2+2\pi nu-\pi u^2/2} = \sum_{m\geq 0} (\frac{\pi}{2})^{\frac{m}{2}} \Big(\sum_{n\in\mathbb{Z}} \Phi_m(n)\Big) \frac{u^m}{m!}$$

(the interchange of $\sum_{n\in\mathbb{Z}}$ and $\sum_{m\geq 0}$ is easily justified using Lemma 0).
Thus we have for $|u| < 1$

$$\sum_{m\geq 0} (\frac{\pi}{2})^{\frac{m}{2}} \Big(\sum_{n\in\mathbb{Z}} \Phi_m(n)\Big) \frac{u^m}{m!} = \frac{1}{S_0\sqrt{2}} \sum_{m\geq 0} \pi^{4m} T_{4m} \frac{u^{4m}}{(4m)!}$$

and by identification we get

$$\sum_{n\in\mathbb{Z}} \Phi_{4m+2}(n) = 0 \quad \text{and} \quad \sum_{n\in\mathbb{Z}} \Phi_{4m}(n) = \frac{(2\pi)^{2m}}{S_0\sqrt{2}} T_{4m}$$