

On Unfoldings of Some Integrals of Automorphic Functions on General Linear Groups

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Abstract

We use results about Fourier coefficients appearing in [T] (and some more obtained here), to obtain information for certain among the integrals of the form

$$I = \int_{GL_n(\mathfrak{k})Z_n(\mathbb{A}_{\mathfrak{k}})\backslash GL_n(\mathbb{A}_{\mathfrak{k}})} \varphi(g)\phi(g)F(E)(\tilde{j}(g))dg$$

where: $\mathbb{A}_{\mathfrak{k}}$ is the adèle ring of a number field \mathfrak{k} ; φ is a $GL_n(\mathbb{A}_{\mathfrak{k}})$ -cuspidal automorphic form; ϕ is a $GL_n(\mathbb{A}_{\mathfrak{k}})$ -automorphic function (even the trivial for some results); E is a $GL_N(\mathbb{A}_{\mathfrak{k}})$ -automorphic form for a multiple N of n ; $F(E)$ is a Fourier coefficient of E for certain choices of additive functions F in a set \mathcal{B}_N which we defined in [T]; \tilde{j} is a diagonal embedding of GL_n in GL_N ; of course $\tilde{j}(GL_n) \in \text{Stab}_{GL_N}(\mathbb{F})$; and Z_n is the center of GL_n .

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1 Integrals of automorphic functions

Results from [T] and simple refinements of them that we use are given in the appendix. Unless otherwise specified, we retain the meaning of the notations and definitions in [T]; the main exception is in relation to the mirabolic group in the definition below. In some occasions we recall or reference notations from [T], but we soon and frequently use them without mention (even in their first occurrence here); in particular, notations about \mathcal{A} -trees are freely used in the present paper: in many proofs and in the appendix.

Relations of the present paper to the literature are given in Remark 1.30 and Definition 1.21.

We choose an integer $n \geq 2$. Whenever N appears, it is assumed to be a positive multiple of n . By “AF” we mean additive function. For F being an “AF” we denote by D_F its domain. Let H be an algebraic group containing D_F as an algebraic subgroup. For $\gamma \in H$ we define the AF γF given by $\gamma F(u) = F(\gamma^{-1}u\gamma)$ for all $u \in D_{\gamma F} := \gamma D_F \gamma^{-1}$. We also define $\text{Stab}_H(F) = \{\gamma \in H : \gamma F = F\}$.

Definition 1.1 ($P_n^{\text{mir}}, M_n^{\text{mir}}, U_n^{\text{mir}}$). We denote by P_n^{mir} the GL_n -parabolic subgroup with Levi isomorphic to $GL_{n-1} \times GL_1$ so that GL_1 appears in the lower right corner. The unipotent radical (resp. the Levi) of P_n^{mir} is denoted by U_n^{mir} (resp. M_n^{mir}). △

Some standard definitions and conventions 1.2. Unless otherwise specified, for a product of general linear groups: the standard choice is assumed for root data (including parabolic and Levi subgroups); root data, rows, columns, and entries, are with respect to the biggest such group that is mentioned. The previous sentence also holds for 2a1d-groups, which we recall in Definition 2.2, but to be clear, it does not affect statements for diagonally embedded general linear groups. The Weyl group of GL_n is denoted by W_n and its elements are identified with permutation matrices.

In case not otherwise specified, an index variable is assumed to take all the values for which the statement containing it has a precise meaning¹.

¹For example, due to this rule, in (ii) in the first definition below, we do not need to mention that i takes as values: $1, \dots, t$. However, there are some occasions in which we mention (for emphasis) the values obtained by this rule.

We denote by U_n the set of upper triangular unipotent matrices of GL_n , and by $U_{n,(i,j)}$ or $U_{(i,j)}$, the root group of GL_n which is nontrivial on the (i,j) (matrix) entry.

The transpose of a matrix A is denoted by A^t .

We identify M_N^{mir} with $GL_{N-1} \times GL_1$ as needed. For example (by recalling the concept “ $\mathcal{O}_F(F)$ ” from 2.6) as a result of this identification the following statement is **wrong**: $\mathcal{O}_F(F) = \emptyset$ for all F with $D_F \subseteq M_n^{\text{mir}}$. \triangle

The AFs F as in the abstract that we consider, belong to a set $\mathcal{R}_{n,N}$ defined below which easily² turns out to be a subset of \mathcal{B}_N (Part (i) in Lemma 1.15). The meaning of \mathcal{B}_N is recalled in Definition 2.5, where we also define extensions of \mathcal{B}_N which we need (for Lemma 1.15 and the proof of Proposition 1.22). Except for the last statement for the integrals I in the abstract (which is Proposition 1.27) we restrict to cases in which one among the GL_N -parabolic subgroups from which E is induced is P_N^{mir} . We first encounter integrals I as in the abstract, in Proposition 1.9, where we only unfold until I is expressed as a (finite) sum of integrals over a factorizable domain of Fourier coefficients. The AFs corresponding to these Fourier coefficients are first studied in Lemma 1.15 after Part (i), where they are proved to belong to certain among the previously mentioned extensions of \mathcal{B}_N . Then in Proposition 1.22 we return to the integrals I , and the information in the appendix together with Lemma 1.15 are essential. Finally (the already mentioned) Proposition 1.27 only addresses cases of I in which F is the trivial AF; also, the proof of Proposition 1.27 uses the appendix to a lesser extent than Proposition 1.22, and by restricting the proofs of Propositions 1.22 and 1.27 to the intersection of these two propositions, the proofs obtained have differences (see Remark 1.29).

As it happens in many cases³ in the Rankin-Selberg method, in Propositions 1.22 and 1.27, a dimension equation formulated by D. Ginzburg—after we extend the set of integrals on which it has been formulated—(see Definition 1.21) is satisfied, or is not satisfied but in ways involving a modest extra effort at most.

Definition 1.3 ($\mathcal{R}_{n,N}, P_i, Q_i$). Consider an AF F , and subgroups P_i, Q_i, V_i of GL_N for $1 \leq i \leq t$ (for a positive integer t), and also Q_{t+1} , all of them defined over \mathfrak{k} (which is a number field fixed throughout the paper), as follows:

- (i) $D_F = V_1 \dots V_t$.
- (ii) V_i is the maximal abelian unipotent normal algebraic subgroup of P_i which: normalizes the unipotent radical U_{Q_i} of Q_i , and $V_i U_{Q_i}$ is the unipotent radical of P_i (hence, if $P_i = Q_i$, then V_i is the unipotent radical of P_i).
- (iii) $F|_{V_i}$ is in the open orbit of the action by conjugation of P_i on the AFs with domain V_i .
- (iv) P_1 is a maximal GL_N -parabolic subgroup and $Q_1 = P_1$. Let Q_{i+1} be defined so that we have the semidirect product of algebraic subgroups: $\text{Stab}_{P_i}(F|_{V_i}) = V_i \rtimes Q_{i+1}$. We require that: Q_i is equal to a general linear group (that is, any isomorphic copy of GL_x (for some x)) or equal to a parabolic subgroup of a general linear group; Q_{t+1} is a general linear group; P_i is equal to Q_i or equal to a maximal parabolic subgroup of Q_i ; and P_i is not a general linear group. To avoid confusion, the usual assumption about varying indices (mentioned in 1.2) gives here (in (iv)) that i takes the values $1, \dots, t$.
- (v) Here we just fix a convenient choice among conjugate ones. We require that: $D_F \subseteq U_N$, all the V_i are generated by 2a1d-groups (which we recall in Definition 2.2), any two V_i and V_j have no nontrivial entries in common, Q_{t+1} is equal to $\tilde{j}(GL_n)$ where $\tilde{j}: GL_n \rightarrow GL_N$ is the embedding given by

$$\tilde{j}(g) = \begin{pmatrix} g & & \\ & \ddots & \\ & & g \end{pmatrix}, \quad (1)$$

and $J_{F|_{V_i}}$ (recalled in Definition 2.3) is a conjugate of a Jordan matrix by an element in W_N .

²Easy in the sense that it is an easy refinement of a well known argument, that is, we proceed as in the Whittaker expansion of $GL_N(\mathbb{A}_{\mathfrak{k}})$ -cusp forms, except that we deal with blocks and the process can become increasingly diagonal.

³Cases different from these include many of the often called “new way integrals”.

We denote by $\mathcal{R}_{n,N}$ the set of such AFs F . Notations depending on F without mentioning F (e.g. P_i and Q_i) are assumed to be defined each time with respect to the way F is chosen in $\mathcal{R}_{n,N}$. \triangle

One can directly obtain a more explicit description of $\mathcal{R}_{n,N}$; for example, the unipotent radical of P_i is either abelian or two step nilpotent; we mention more such information in Observation 1.12, but only use it in the proof of Lemma 1.15 (and there it is used freely).

Even though each Q_i is reductive if and only if it is a general linear group, we frequently use phrases such as “ Q_i is reductive”.

Example 1.4. Two examples of $F \in \mathcal{R}_{n,N}$ (with only two “diagonality levels”⁴), appear in the two pictures in Remark 1.17. Part of a choice of data describing the choice of F described in the left (resp. right) picture is: P_1 with Levi isomorphic to $GL_{12} \times GL_2$, $t = 4$, $P_i = Q_i$ except for $i = 2$ (resp. the Levi of P_1 is isomorphic to $GL_{14} \times GL_2$, $t = 5$, and $P_i = Q_i$ except for $i = 3$). Of course in the left choice of F there is a second way to choose the data, that is the Levi of P_1 is isomorphic to $GL_6 \times GL_8$, $t = 4$ (then we always have $P_i = Q_i$).

Lemma 1.5 (and definition of w_o). *Let $F \in \mathcal{R}_{n,N}$. The double cosets of $P_N^{\text{mir}}(\mathfrak{k}) \backslash GL_N(\mathfrak{k}) / (\tilde{j}(GL_N)D_F)(\mathfrak{k})$ are all represented by elements in W_N . We denote by w_o the minimal length element in W_N representing the open double coset.*

Proof. Let P be the GL_N -parabolic subgroup with unipotent radical D_F . For any embedding of the form $h \rightarrow w \begin{pmatrix} h & & \\ & \ddots & \\ & & h \end{pmatrix} w^{-1}$ for all $h \in GL_m$ for some m dividing N , all blocks except one are contained in P_N^{mir} . Hence

$$P_N^{\text{mir}}(\mathfrak{k}) \backslash GL_N(\mathfrak{k}) / P(\mathfrak{k}) = P_N^{\text{mir}}(\mathfrak{k}) \backslash GL_N(\mathfrak{k}) / (\tilde{j}(GL_N)D_F^{\text{root}})(\mathfrak{k})$$

and

$$P_N^{\text{mir}}(\mathfrak{k}) \backslash GL_N(\mathfrak{k}) / (\tilde{j}(GL_N)(V_1 \dots V_{i+1})^{\text{root}} V_{i+2} \dots V_t)(\mathfrak{k}) = P_N^{\text{mir}}(\mathfrak{k}) \backslash GL_N(\mathfrak{k}) / (\tilde{j}(GL_N)(V_1 \dots V_i)^{\text{root}} V_{i+1} \dots V_t)(\mathfrak{k})$$

where for any 2ald-group L we denote by L^{root} or $(L)^{\text{root}}$ the smallest group containing L which is generated by root groups. From these equalities and the Bruhat decomposition we are done. \square

Definition 1.6 ($F_{\emptyset,H}$). For an algebraic group H we define $F_{\emptyset,H}$ to be the AF with domain the trivial subgroup of H . \triangle

Definition 1.7. Let F be an AF and $a : G \rightarrow H$ be an algebraic homomorphism of algebraic groups G, H , such that D_F and G are algebraic subgroups of a (common to both) algebraic group. We call $a(F)$, if it exists, the AF with domain $a(D_F \cap G)$ satisfying

$$a(F)(a(u)) = F(u) \quad \forall u \in D_F \cap G. \quad \triangle$$

Definition 1.8 (\prod, \times). In addition to using \prod and \times for direct products of groups we also use them as follows: Given group homomorphisms $j_i : H \rightarrow H_i$ for $1 \leq i \leq x$ we denote the homomorphism of H on $\prod_i H_i$ mapping each $h \in H$ to $(j_1(h), \dots, j_x(h))$ by $\prod_i j_i$ or by $j_1 \times \dots \times j_x$. For Z_1, \dots, Z_x being AFs with domain a unipotent algebraic subgroup of H , in case $D_{Z_1} \dots D_{Z_x}$ is a direct product (of the D_{Z_i}), two other (and preferred) names for $Z_1 \circ \dots \circ Z_x$ are $\prod_i Z_i$ and $Z_1 \times \dots \times Z_x$. \triangle

Proposition 1.9 (and definitions of $W_n, \tilde{j}_{p,w}, p$, and $Y(\mathbb{A}_{\mathfrak{k}})$). *Let $F \in \mathcal{R}_{n,N}$. Let I, φ, ϕ , and E be as in the abstract, and further assume that E admits an absolutely convergent Eisenstein series expansion $E(g) = \sum_{\gamma \in P_N^{\text{mir}}(\mathfrak{k}) \backslash GL_N(\mathfrak{k})} f(\gamma g)$. Then*

$$I = \sum_w \int_{Y_w(\mathbb{A}_{\mathfrak{k}})} (-W_n)(\varphi)(g) (\text{id} \times \tilde{j}_{p,w})(W_n) \circ (F_{\emptyset,n} \times p(wF))(\phi f)(g, wh\tilde{j}(g)) dh dg \quad (2)$$

⁴That is, $x(t) = 2$, where $x(\dots)$ is defined Definition 1.11.

where: W_n is an AF with domain U_n which is nontrivial on all simple root groups; p is the projection⁵ of P_N^{mir} onto its Levi; w varies over the elements in W_N which are the smallest choice of representatives (one) for each double coset in the lemma above for which $p(wF)$ is defined (by Definition 1.7) (equivalent to “ $p(wF)$ is defined” is that the restriction of wF on the intersection of D_{wF} with the unipotent radical of P_N^{mir} is trivial); $\tilde{j}_{p,w}$ is the function given by $\tilde{j}_{p,w}(g) = p(w\tilde{j}(g)w^{-1})$ (for all possible g); $Y_w(\mathbb{A}_{\mathfrak{k}}) := (Z_n\tilde{j}(U_n)D_F \cap w^{-1}P_N^{\text{mir}}w)(\mathbb{A}_{\mathfrak{k}}) \setminus (\tilde{j}(GL_n)D_F)(\mathbb{A}_{\mathfrak{k}})$, and id is the identity function on GL_n .

Proof. As in familiar special cases of I , we unfold it by first using the Eisenstein series expansion of E over P_N^{mir} and then by using the Fourier expansion of φ over $U_n(\mathfrak{k}) \setminus U_n(\mathbb{A}_{\mathfrak{k}})$, except that the Fourier expansion is applied to every term obtained by Lemma 1.5 (except the terms for which $p(wF)$ is not defined). \square

As we mentioned earlier, Definition 2.5 ($\mathcal{B}_H, \mathcal{B}_H(k), \dots$) is needed for Lemma 1.15 below. As for the definitions below preceding the Lemma, (they are of course also needed but) deferring to read them to various extents, is likely efficient; after the proof of Lemma 1.15, only $A(F)$ is used among them (in Proposition 1.22).

Definition 1.10 (C -root group, \cap^{ent}). Let T_N be the maximal standard torus of GL_N . For any subgroup H of GL_N , let $N_{GL_N}(H)$ be the normalizer of H in GL_N (that is $\{\gamma \in GL_N : \gamma H \gamma^{-1} = H\}$). Let C be a subgroup of GL_N , which is generated by 2a1d-groups and elements in T_N ; then for a 2a1d-group L contained in C for which $T_N \cap N_{GL_N}(C) \subseteq N_{GL_N}(L)$, we say that L is a C -root group. Also, for C being generated by 2a1d-groups, and L' being a subgroup of GL_N generated by root groups, we define $C \cap^{\text{ent}} L'$ to be the subgroup of C generated by the C -root groups which have a nontrivial entry in common with L' . \triangle

Definition 1.11 ($\text{row}(L), \text{col}(L), y_i, i_1, i_2, \dots, s(i), x(i)$). For L being a 2a1d-group let $\text{row}(L)$ (resp. $\text{col}(L)$) be the set of r such that L is nontrivial in the r -th row (resp. r -th column).

Let y_i be the number of elements (which are disjoint sets) in the set

$$\{\text{row}(L) : L \text{ is a } V_i\text{-root group and } F(L) \neq \{0\}\}$$

(of course replacing “ $\text{row}(L)$ ” with “ $\text{col}(L)$ ” does not affect y_i).

We define $i_1 = 1$, and for $r > 1$, we define i_r to be the smallest number for which $y_{i_r} < y_{i_{r-1}}$. We define $s(i)$ to be the biggest number for which $i_{s(i)} \leq i$.

We define $x(1) := 1$, and

$$x(i+1) := \begin{cases} x(i) & \text{if } x'(i+1) = x'(i) \\ x(i) + 1 & \text{if } x'(i+1) > x'(i) \end{cases}$$

where $x'(i)$ (which is not mentioned again) is the number of entries on which a V_i -root group is nontrivial. \triangle

Observation 1.12. Let $F \in \mathcal{R}_{n,N}$ and r be chosen so that i_r and i_{r+1} are defined. Then the collection of numbers $y_{i_r}, y_{i_{r+1}}, \dots, y_{i_{r+1}}$ expressed in the same order takes the form

$$\underbrace{y_{i_r}, \dots, y_{i_r}}_{a_1 \text{ times}} \underbrace{z_1, \dots, z_1}_{b_1 \text{ times}} \underbrace{y_{i_r}, \dots, y_{i_r}}_{a_2 \text{ times}} \underbrace{z_2, \dots, z_2}_{b_2 \text{ times}} \dots \underbrace{y_{i_r}, \dots, y_{i_r}}_{a_k \text{ times}} \underbrace{z_k, \dots, z_k}_{b_k \text{ times}} \underbrace{y_{i_r}, \dots, y_{i_r}}_{a_{k+1} \text{ times}} \quad (3)$$

where: in the cases not otherwise specified, any of the numbers appearing in (3) (e.g. k) can even be zero; $z_j > y_{i_r}$; if $i_r < i < i_{r+1}$, we have

$$x(i+1) > x(i) \iff (y_i = z_j \text{ for some } j \text{ and } y_{i+1} \neq y_i);$$

if Q_{i_r} is not reductive we have $a_1 = 1, b_1 = 0$, and $x(i_r+1) > x(i_r)$; if $b_j = 0$ then $j = 1$; if $b_j > 0$ we have

$$z_j = \begin{cases} a_{j+1}y_{i_r} + b_{j+1}z_{j+1} & \text{if } 1 \leq j < k \\ a_{k+1}y_{i_r} + y_{i_{r+1}} & \text{if } j = k \end{cases}.$$

⁵That is, p is trivial on the unipotent radical of P_N^{mir} and restricts to the identity function on the Levi.

Recall with the superscript t we denote transpose.

Definition 1.13 ($A(F)$). We define $A(F)$ to be the set of w as in (2) **except** the ones for which α or β or γ hold.

α There is an i such that; $V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w$ is nontrivial, V_i is nontrivial in more rows than columns; $i_{s(i)} < i < i_{s(i)+1}$ and $\alpha.1$ or $\alpha.2$ below holds.

$\alpha.1$ $x(i) = x(i+1)$. $F(V_{i+1} \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w) \neq \{0\}$ (note that for $i+1 < i_{s(i)+1}$ this is implied from the part of the sentence preceding the last “;”). There is a number i' such that $i_{s(i)} \leq i' < i$, $y_{i'} = y_{i_{s(i)}}$ and $x(i') < x(i)$; choose the biggest such i' ; then for the $V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w$ -root group, say L , on which F is nontrivial we have

$$\text{col}(L) \cap \left(\bigcup_{L'} \text{row}(L') \right) = \emptyset$$

where L' varies over the $V_{i'}$ -root groups satisfying $F(L') \neq \{0\}$.

$\alpha.2$ $Q_{i_{s(i)}}$ is reductive and $x(i_{s(i)} + 1) = x(i_{s(i)+1})$.

β There is an s such that: V_{i_s} intersects more columns than rows; $V_{i_s} \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w$ is trivial; and $\beta.1$ or $\beta.2$ below holds.

$\beta.1$ $x(i_{s+1}) > x(i_s) + 1$; for every i for which $i_s < i < i_{s+1}$ and $y_{i-1} < y_i$, the group $V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w$ is trivial;

$\beta.2$ Q_{i_s} is reductive.

γ There is an s such that $V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w$ is trivial for all $i_s \leq i \leq i_{s+1}$, and Q_{i_s} is reductive.

\triangle

Remark 1.14. Of course $w_o \in A(F)$. For an $F \in \mathcal{R}_{n,N}$ for which D_F is not generated by root groups, we have $A(F) = \{w_o\}$ if and only if: for each r , at least one among Q_{i_r} , and $Q_{i_{r+1}}$ is reductive; and if $Q_{i_{r+1}}$ is not reductive then V_{i_r} is nontrivial in more rows than in columns and $y_i = y_{i_r}$ for all $i_r \leq i < i_{r+1}$.

\triangle

Lemma 1.15. Let $F \in \mathcal{R}_{n,N}$. Consider any w as in (2). Then:

(i) $F \in \mathcal{B}_N$;

(ii) $p(wF) \in \mathcal{B}_{M_N^{\text{mir}}}(\dim(D_{p(wF)}) - \dim(D_{p(w_oF)}))$;

(iii) For $w \notin A(F)$ we have $p(wF) \in \mathcal{B}_{M_N^{\text{mir}}}(\dim(D_{p(wF)}) - \dim(D_{p(w_oF)}) - 1)$;

(iv) There is an $(F_{\emptyset,n} \times p(wF) \rightarrow (\text{id} \times \tilde{j}_{p,w})(W_n) \circ (F_{\emptyset,n} \times p(wF)), \mathcal{B}_{GL_n \times M_N^{\text{mir}}})$ -path (note for example this together with (ii) implies that $(\text{id} \times \tilde{j}_{p,w_o})(W_n) \circ (F_{\emptyset,n} \times p(w_oF)) \in \mathcal{B}_{GL_n \times M_N^{\text{mir}}}$).

Proof. Consider the $(F_{\emptyset,N} \rightarrow F)$ -path Ξ with labels of vertices given at increasing depth: $F_{\emptyset,N}$, $F|_{V_1}$, $F|_{V_1V_2}$, ..., F . For each $1 \leq i \leq t$ that is possible, let H_i^1 (resp. H_i^2) be an algebraic subgroup of P_i which:

- acts by conjugation freely and transitively on an open subset of the set of terms of the e -step of quasi(Ξ) with input $AF|_{V_1 \dots V_{i-1}}$, and this open subset contains $F|_{V_1 \dots V_i}$.
- every matrix in it is trivial on all nondiagonal entries (r, c) with r, c belonging in

$\{x : F|_{V_i} \text{ is nontrivial on a } V_i\text{-root group, such that this group is nontrivial on the } x\text{-th row (resp. column)}\}$.

(4)

We see that for each $1 \leq i \leq t$ at least one between H_i^1 and H_i^2 is defined, and hence Ξ is a \mathcal{B}_N -path, and hence we obtain (i).

Unless more restrictively specified, let w be any element as in (2).

Given an edge of an \mathcal{A} -tree, we identify it with the \mathcal{A} -path with only two vertices for which this is the (unique) edge it has. Also if such a path is a $\mathcal{B}_H(k)$ -path (for some choice of H and k), we also call it a $\mathcal{B}_H(k)$ -edge.

Let X be a 2ald-group and X' be a V_i -root group. Assume that there are numbers l_1, \dots, l_k such that: X is nontrivial on the l_1 -th row (resp. column); for each $1 < r \leq k$, there is a j with $y_j = y_i$, and a V_j -root group, say X_r , which is nontrivial on the (l_r, l_{r-1}) entry (resp. (l_{r-1}, l_r) entry) and such that $F(L_r) \neq \{0\}$; and $X_k = X'$. Then we say that X is linked to X' .

Let Ξ_w be the \mathcal{A} -path obtained by replacing in Ξ each label $F|_{V_1 \dots V_i}$ with the label $p(w(F|_{V_1 \dots V_i}))$. Consider the i -th edge of Ξ_w ; then notice that one among $wH_i^1w^{-1}$ and $wH_i^2w^{-1}$, contains a subgroup the action of which implies that this edge is a $\mathcal{B}_{M_N^{\text{mir}}}(k_i)$ -edge where

- (a) $k_i = 0$, if one among (a.1), (a.2), and (a.3) below hold (of course (a.1.1) (resp. (a.1.2), (a.3.1)) is considered part of (a.1) (resp. (a.1), (a.3)):

(a.1) $V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}}w$ and $V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w$ are both trivial; and (a.1.1) or (a.1.2) holds.

(a.1.1) (resp. (a.1.2)) In case $i = i_{s(i)}$, Q_i is not reductive, and V_i intersects nontrivially more rows (resp. columns) than columns (resp. rows), then F is trivial on all V_i -root groups (resp. nontrivial on a V_i -root group) linked to a: $w^{-1}V_jw$ -root group contained in $w^{-1}U_N^{\text{mir}^t}w$ or in $w^{-1}U_N^{\text{mir}}w$ for a j satisfying $i_{s(i)-1} \leq j < i_{s(i)}$.

(a.2) $V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w$ is nontrivial, and either V_i intersects nontrivially at least as many columns as rows or $F(V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w) = \{0\}$;

(a.3) $F(V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}}w) \neq \{0\}$; and (a.3.1): in case $V_{i_{s(i)}}$ nontrivially intersects more columns than rows we also require that a $V_{i_{s(i)}}$ -root group on which F is nontrivial is linked to a $w^{-1}V_iw$ -root group contained in $w^{-1}U_N^{\text{mir}}w$.

- (b) $k_i = y_i$, if: $V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}}w$ is nontrivial and $F(V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}}w) = \{0\}$; or the negation of (a.1.2) holds.

- (c) $k_i = y_i - y_{i_{s(i)}}$, if: $F(V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}}w) \neq \{0\}$ and the negation of (a.3.1) holds.

- (d) $k_i = \sum_{i < j < i_{s(i)+1} \text{ and } x(i) \leq x(j)} y_j$ if: either (all three) (d.1) (d.2) and (d.3) hold, or the negation of (a.1.1) holds.

$k_i = \sum_{i < j \text{ and } x(i) = x(j)} y_j$ if: (d.1) (d.2) and the negation of (d.3) hold.

(d.1) V_i nontrivially intersects more rows than columns.

(d.2) $F(V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w) \neq \{0\}$.

(d.3) A $V_{i_{s(i)}}$ -root group on which F is nontrivial is linked to a $V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w$ -root group.

Hence (since Ξ_w is a $\mathcal{B}_{M_N^{\text{mir}}}(\sum_i k_i)$ -path) we obtain (ii), and the restriction of (iii) to the w with the added condition: α does not hold (" α " appears in the definition of $A(F)$). Hence consider a choice of w for which α holds. Then we may have $\sum_i k_i = \dim(D_{p(wF)}) - \dim(D_{p(w_oF)})$, and hence we will describe a modification Ξ'_w of Ξ_w which will turn out to be an $(F_{\emptyset, M_N^{\text{mir}}} \rightarrow p(wF), \mathcal{B}_{M_N^{\text{mir}}}(\dim(D_{p(wF)}) - \dim(D_{p(w_oF)}) - 1))$ -path. Let L_i be the $V_i \cap^{\text{ent}} w^{-1}U_N^{\text{mir}^t}w$ -root group such that $F(L_i) \neq \{0\}$. We define Ξ'_w to be obtained from Ξ_w so that:

- If $\alpha.1$ holds, we replace the vertex with label $p(w(F|_{V_1 \dots V_i}))$ with the e-edge with input and output AFs respectively being $p(w(F|_{V_1 \dots V_{i-1}V_i^1}))$ and $p(w(F|_{V_1 \dots V_{i-1}V_i^2V_{i+1})))$, where V_i^1 (resp. V_i^2) is generated by all the V_i -root groups except the ones, say L , for which $\text{col}(L) = \text{col}(L_i)$ (resp. (again) $\text{col}(L) = \text{col}(L_i)$ and

$$\text{row}(L) \in \{\text{row}(L') : L' \text{ is a } V_i\text{-root group with } F(L') \neq \{0\}\}.$$

- Consider now the case that $\alpha.1$ does not hold. Hence $F(V_{i+1} \cap^{\text{ent}} w^{-1}U_N^{\text{mir}}w) = \{0\}$, $i = i_{s(i)+1} - 1$, $Q_{i_{s(i)}}$ is reductive, $x(i_{s(i)}) = x(i_{s(i)+1})$ and y_j is the same for all $i_{s(i)} \leq j \leq i$. Hence there is an alternative way we can choose V_j for $i_{s(i)} \leq j \leq i_{s(i)+1}$ (and still obtain the same F), which we denote by V'_j . Then replace in Ξ_w the \mathcal{A} -subpath with labels $p(w(F|_{V_1 \dots V_{i_{s(i)+1}}}), \dots, p(w(F|_{V_1 \dots V_{i_{s(i)+1}}}))$, with the \mathcal{A} -path with labels

$$p(w(F|_{V_1 \dots V_{i_{s(i)}} V'_{i_{s(i)+1}}}), \dots, p(w(F|_{V_1 \dots V_{i_{s(i)}} V'_{i_{s(i)+1}} \dots V'_{i_{s(i)+1}}})) \quad (5)$$

We see that:

- If $\alpha.1$ holds, then in the three (successive) edges in Ξ'_w not encountered in Ξ_w we have:
 - the first (that is the one with input AF being $p(wF|_{V_1 \dots V_{i-1}})$) and last one is a $\mathcal{B}_{M_N^{\text{mir}}}$ -edge;
 - the middle one is a $\mathcal{B}_{M_N^{\text{mir}}}((\sum_j y_j) - 1)$ -edge, where in case $y_i = y_{i_{s(i)}}$ (resp. $y_i > y_{i_{s(i)}}$) j varies over the values satisfying $i < j \leq i_{s(i)+1}$ (resp. $i < j$ and $y_i = y_j$).
 - In case F is trivial in all $V_{i_{s(i)}}$ -root groups (resp. nontrivial in a $V_{i_{s(i)}}$ -root group) linked to L_i , the last edge is a $\mathcal{B}_{M_N^{\text{mir}}}$ -edge (resp. $\mathcal{B}_{M_N^{\text{mir}}}(y_i - y_{i_{s(i)}})$ -edge).
- If $\alpha.1$ does not hold, then the path with vertices as in (5) is a $\mathcal{B}_{M_N^{\text{mir}}}(y_{i_{s(i)+1}})$ -path (because its first edge is a $\mathcal{B}_{M_N^{\text{mir}}}(y_{i_{s(i)+1}})$ -edge and the next ones are $\mathcal{B}_{M_N^{\text{mir}}}$ -edges).

Hence Ξ'_w is as claimed (in the first sentence it was mentioned) and (iii) is (fully) obtained.

Since $\text{Stab}_{GL_n \times M_N^{\text{mir}}}(p(wF))$ contains an appropriate copy of the mirabolic subgroup of GL_n , by applying $\text{id} \times \tilde{j}_{p,w}$ and then $\circ(F_{\emptyset,n} \times p(wF))$ to an appropriate $(F_{\emptyset,n} \rightarrow W_n, \mathcal{B}_n)$ -path, we obtain (iv). \square

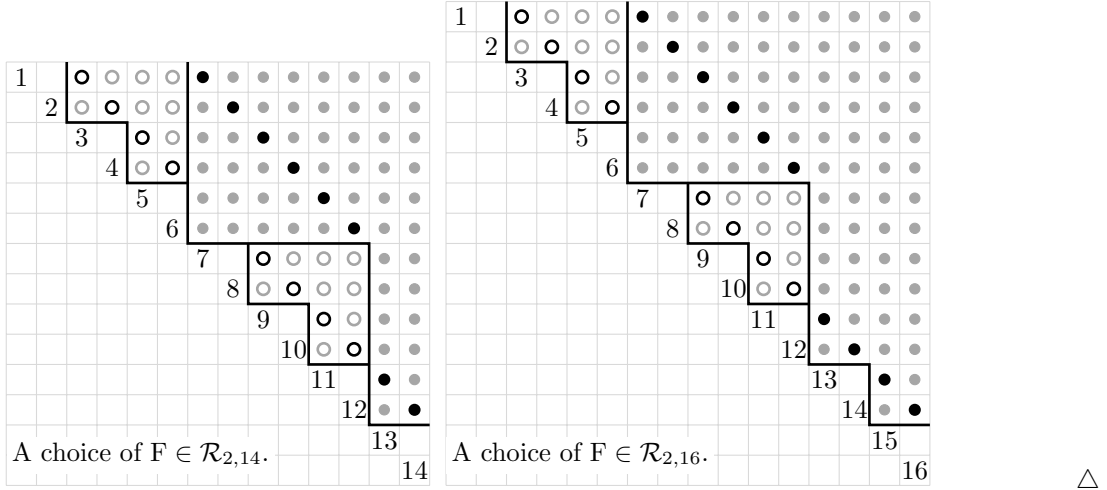
Remark 1.16. In the proof of Proposition 1.22 we obtain bounds on the dimensions of orbits by using Main corollary 2.7 and the Lemma above. More precicely the uses of the Lemma are * together with (iv) for * =(i), (ii), (iii). Even though not used, it may worth mentioning that the bounds obtained in the same way by only using (ii) from the Lemma above—that is, $\frac{\dim(a)}{2} \geq \dim(D_{p(w_o F)})$ for every $a \in \mathcal{O}(p(wF))$ —can also be obtained as follows.

Among the $2a1d$ -groups which are contained in $\text{Stab}_{GL_N}(F)$ and are nontrivial in the entry (n, i) , let X_i be the one which is nontrivial in as few entries as possible. Let F' be an AF which: has domain $D_{F'} = (\prod_{1 \leq i \leq n-1} X_i)D_F$, is nontrivial in $\prod_{1 \leq i \leq n-1} X_i$, and $F'|_{D_F} = F$. By (i) in the Lemma above we easily obtain $F' \in \mathcal{B}_N$. For a $J \in \mathbf{X}(p(wF))$ choose a $J' \in \mathbf{X}(wF') \cap \text{Lie}(P_N^{\text{mir}})^{\dagger}$ which projects (through the differential of p) to J . Hence (by 7.1.1 in [CM]) $\dim(\mathcal{O}(J)) \geq \dim(\mathcal{O}(J')) - N + 1$, and by also using Main corollary 2.7 for $[F \leftarrow F']$ we are done. \triangle

Remark 1.17. The dimension bound obtained from (i) in the Lemma above and Main corollary 2.7 is optimal. That is there is an $a \in \mathcal{O}(F)$ (resp. $a \in \mathcal{O}(p(w_o F))$) satisfying $\frac{\dim(a)}{2} = \dim(D_F)$ (resp. $\frac{\dim(a)}{2} = \dim(D_{p(w_o F)})$); we obtain such an orbit as in the proof of Lemma 3 in [T2].

In the picture on the left (resp. right) below, we describe a choice of F for which the dimension bound obtained as in the previous paragraph by replacing (i) with (ii) (resp. (ii) and (iv)) is optimal. The picture rules are the same as in [T] (see Definition 8.2.3 and the paragraph slightly below this definition finishing with “ \triangle fixing a picture”).

More precisely, choose $a = [3, 3, 4, 3]$ (resp. $a = [4, 2, 5, 4]$), w so that for V being a positive root group in the forth row, the root group wVw^{-1} is negative; then one can check that $a \in \mathcal{O}(p(wF))$ (resp. $a \in \mathcal{O}(p(w(\tilde{j}(W_2) \circ F)))$) and $\frac{\dim(a)}{2} = \dim(D_{p(w_o F)})$ (resp. $\frac{\dim(a)}{2} = \dim(D_{p(w_o(\tilde{j}(W_2) \circ F))})$). The order of the numbers inside the brackets—which does not affect the meaning of a —is chosen in the way one “encounters them” by following an appropriate initial \mathcal{A} -subpath of $\Xi^{\text{st}}(F)$ with its output AF in $\mathcal{C}_{13}[a]$ (resp. $\mathcal{C}_{15}[a]$).



In the rest of the present paper we concentrate on the I “belonging” in a set \mathcal{S} defined below, for which the choices of ϕ are products of certain Fourier coefficients of automorphic forms. The scare quotes on the word belonging are because \mathcal{S} is a set of integral expressions (precisely described as tuples and in a way specific to the needs of the present paper).

Definition 1.18 (\mathcal{S}). We denote by \mathcal{S} the set of tuples of the form

$$I := (F_1, \dots, F_{k-1}, F, \pi_{\text{cusp}}, \pi_1, \dots, \pi_{k-1}, \pi)$$

where: $F \in \mathcal{R}_{n,N}$, π_{cusp} is a $GL_n(\mathbb{A}_{\mathfrak{t}})$ -cuspidal automorphic representation, π is a $GL_N(\mathbb{A}_{\mathfrak{t}})$ -automorphic representation, k is a positive integer; and for $1 \leq i \leq k-1$ we define positive integers N_i , AFs $F_i \in \mathcal{B}_{N_i}$, embeddings $\tilde{j}_i : GL_n \rightarrow GL_{N_i}$ for which $\tilde{j}_i(GL_n) \in \text{Stab}_{GL_{N_i}}(F_i)$, and $GL_{N_i}(\mathbb{A}_{\mathfrak{t}})$ -automorphic representations π_i such that the vector space $\{\text{restriction of } F(\varphi_i) \text{ on } \tilde{j}_i(GL_n(\mathbb{A}_{\mathfrak{t}})) : \varphi_i \in \pi_i\}$ is neither trivial nor one dimensional.

For the tuple I in \mathcal{S} we obtain the functional $I : \pi_{\text{cusp}} \times \prod_{i \leq k-1} \pi_i \times \pi \rightarrow \mathbb{C}$, which maps each tuple $(\varphi, \varphi_1, \dots, \varphi_{k-1}, E)$ in its domain to the integral

$$I(\varphi, \varphi_1, \dots, \varphi_{k-1}, E) = \int_{GL_n(\mathfrak{t})Z_n(\mathbb{A}_{\mathfrak{t}}) \backslash GL_n(\mathbb{A}_{\mathfrak{t}})} \varphi(g) \left(\prod_{1 \leq i \leq k-1} F_i(\varphi_i)(\tilde{j}_i(g)) \right) F(E)(\tilde{j}(g)) dg. \quad (6)$$

Whenever we have a text of the form “ $I = *$ ” for $*$ being any text, we mean “ $I(\varphi, \varphi_1, \dots, \varphi_{k-1}, E) = *$ for all $\varphi, \varphi_1, \dots, \varphi_{k-1}, E$ in the domain of I ”.

To be clear, whenever an $I \in \mathcal{S}$ appears, all notations within the present definition (e.g. I) are adopted for this choice of I . △

Definition 1.19 ($\mathcal{N}_H, \mathcal{O}(X)$, blocks). Let $H = \prod_{1 \leq i \leq k} GL_{n_i}$ (for some positive integers k, n_1, \dots, n_k). Then we refer to each GL_{n_i} as a block. We denote by \mathcal{N}_H the set of nilpotent orbits of the action by conjugation of H on $\text{Lie}(H)$. We mostly write \mathcal{N}_n instead of \mathcal{N}_{GL_n} and we identify its elements with positive integer partitions $n = n_1 + \dots$ which we denote by $[n_1, \dots]$. For $X \in \text{Lie}(H)$ we denote by $\mathcal{O}(X)$ the orbit in \mathcal{N}_H that contains X .

As usual the order we use for \mathcal{N}_H is: given two different $a, b \in \mathcal{N}_H$ with $a = [a_{1,1}, a_{1,2}, \dots] \times [a_{2,1}, \dots] \times \dots$ and $b = [b_{1,1}, b_{1,2}, \dots] \times [b_{2,1}, \dots] \times \dots$, we say a is bigger from b if and only if $\sum_{1 \leq j \leq k} a_{i,j} \geq \sum_{1 \leq j \leq k} b_{i,j}$ (for every possible choice of i, k).

For an $a \in \mathcal{N}_H$ consider the (again) form $a = a_1 \times \dots \times a_k$, for $a_i \in \mathcal{N}_{n_i}$; then if a_i is trivial (resp. nontrivial) we say that a is trivial (resp. nontrivial) on the i -th block. Usually we prefer to call leftmost (resp. rightmost) block the first (resp. last) block. △

Definition 1.20 ($\dim(\pi)$). Let π be a $GL_N(\mathbb{A}_{\mathfrak{t}})$ -automorphic representation. Let $\mathcal{O}(\pi)$ be the set consisting of the maximal orbits of

$$\{\mathcal{O}(J_F) : F \in \mathcal{C}_{\mathfrak{t},N} \text{ and } F(\pi) \neq 0\}$$

where the definition of the matrix J_F (the set \mathcal{C}_N) is recalled (and extended) in Definition 2.3 (resp. 2.4). The equivalence of this definition of $\mathcal{O}(\pi)$ with the one found for example in [G4] follows from the proof of Corollary 9.3.5 in [T].

To my knowledge, the proof that $\mathcal{O}(\pi)$ is a singleton has not appeared for all choices of π ; hence we fix throughout the paper a choice, say a_π , in $\mathcal{O}(\pi)$, and we define $\dim(\pi) := \frac{\dim(a_\pi)}{2}$. \triangle

Definition 1.21 (Dimension equation of D. Ginzburg, $d(\mathbf{I})$). D. Ginzburg has formulated a dimension equation which is satisfied by many familiar Rankin-Selberg integral expressions, and is significantly related to them being factorizable. Without making any change to the equation, we extend the set of integral expressions for which D. Ginzburg formulated⁶ it, so that it contains the elements in \mathcal{S} when they are viewed as integral expressions in the way “suggested” in Definition 1.18. Main corollary 2.7 justifies to some extent not making any change. Let $\mathbf{I} \in \mathcal{S}$. We define $d(\mathbf{I})$ to be the difference of the two sides of this equation, and hence D. Ginzburg’s equation reads: $d(\mathbf{I}) = 0$. Since cuspidal representations are generic we have:

$$d(\mathbf{I}) = \left(\sum_i \dim(\pi_i) \right) + \dim(\pi) - \frac{n(n+1)}{2} + 1 - \left(\sum_i \dim(D_{F_i}) \right) - \dim(D_F). \quad (7)$$

\triangle

Proposition 1.22. *Consider an $\mathbf{I} \in \mathcal{S}$ such that the elements in π admit an absolutely convergent Eisenstein series expansion over P_N^{mir} . Let A_0 be the set consisting of the w as in 2 for which $\mathcal{O}((\prod \tilde{j}_i \times \tilde{j}_{p,w})(W_n) \circ (\prod_i F_i \times p(wF)))$ contains an orbit with dimension $2\dim(D_{p(w \circ F)}) + 2\sum_i(\dim(D_{F_i})) + n(n-1)$ (of course by Lemma 1.15 and Main corollary 2.7, the dimension couldn’t be any smaller, and $A_0 \subseteq A(F)$).*

i If $d(\mathbf{I}) = 0$ and $k = 1$, then

$$I = \sum_{w \in A_0} \int_{Y_w(\mathbb{A}_\mathfrak{k})} (-W_n)(\varphi)(g)p(w(\tilde{j}(W_n) \circ F))(f)(wh\tilde{j}(g))dhdg.$$

ii If $d(\mathbf{I}) = 0$, $k = 2$, and $(\tilde{j}_1(W_n) \circ F_1)(\pi_1) = 0$, then

$$I = \sum_{w \in A_0} \int_{Y_w(\mathbb{A}_\mathfrak{k})} (-W_n)(\varphi)(g)(\tilde{j}_1(W_n^{\text{deg}} \circ F_1)(\varphi_1)(g)p(w(\tilde{j}(W_n^{\text{deg}}) \circ F))(f)(wh\tilde{j}(g))dhdg,$$

where $W_{n,1}^{\text{deg}}$ (resp. W_n^{deg}) is an AF with domain U_n which is nontrivial exactly on the root groups $U_{n,(1,2)}, U_{n,(3,4)}, \dots$ (resp. $U_{n,(2,3)}, U_{n,(4,5)}, \dots$).

iii If $d(\mathbf{I}) = 0$, $k = 2$, and $(\tilde{j}_1(W_n) \circ F_1)(\pi_1) \neq 0$, then

$$I = \sum_{w \in A_0} \int_{Y_w(\mathbb{A}_\mathfrak{k})} (-W_n)(\varphi)(g)(\tilde{j}_1(W_n) \circ F_1)(\varphi_1)(g)p(w(F))(f)(wh\tilde{j}(g))dhdg.$$

iv If $d(\mathbf{I}) < 0$ or $k > 2$, then $I = 0$.

Remark 1.23. Some readers may prefer to only consider this proposition in the special case $N_1 = \dots = N_{k-1} = n$ (and hence $F_1 = \dots = F_{k-1} = F_{\emptyset,n}$) due to the following expectation:

Expectation 1.24. *Consider a \mathfrak{k} -AFF $\mathcal{B} \in \mathcal{B}_N$, and a $GL_N(\mathbb{A}_\mathfrak{k})$ -automorphic representation π . We assume that a \mathfrak{k} -copy of GL_n in GL_N is contained in $\text{Stab}_{GL_N}(F)$, and that $0 \leq \dim(F) - \dim(\pi) \leq \frac{n(n-1)}{2}$. We identify GL_n with this copy. Then there is a $GL_n(\mathbb{A}_\mathfrak{k})$ -automorphic representation π' such that*

$$\dim(\pi') = \dim(\pi) - \dim(D_F)$$

and for $F' \in \mathcal{C}_{\mathfrak{k},n}[\mathcal{O}(\pi')]$ we have $\{(F' \circ F)(\varphi) : \varphi \in \pi\} = \{F'(\varphi') : \varphi' \in \pi'\}$.

⁶By “formulate” I only mean to precisely state the equation (it does not have to be satisfied). The integral expressions for which D. Ginzburg formulated the equation include the ones in [G2] in (2).

△

Proof of Proposition 1.22. To be clear, all nilpotent orbits that are mentioned in the present proof are in $\mathcal{N}_{GL_{N_1} \times \dots \times GL_{N_{k-1}} \times M_N^{\text{mir}}}$. By Lemma 1.15 and Main corollary 2.7 we have:

- I all the terms of I in (2) except possibly the ones corresponding to $w \in A_0$ vanish;
- II if $d(I) < 0$, the terms corresponding to $w \in A_0$ also vanish;
- III For the special case of the proposition in which $d(I) = 0$ and $k > 2$, we are left with proving the following claim:

Claim. For $w \in A_0$, fix an orbit in

$$\mathcal{O}_{\text{sd}} \left(\left(\left(\prod_i \tilde{j}_i \right) \times \tilde{j}_{p,w} \right) (W_n) \circ \left(\left(\prod_i F_i \right) \times p(wF) \right) \right). \quad (8)$$

Then there are at most two blocks on which this orbit is nontrivial; and if they are two, one of them is the rightmost one.

- IV if $d(I) = 0$ and $k = 2$, it is sufficient for $w \in A_0$ to find an $(j_1 \times \tilde{j}_{p,w})(W_n) \circ (F_1 \times p(wF)) \rightarrow (\tilde{j}_1(W_{n,1}^{\text{deg}}) \circ F_1) \times p(w(\tilde{j}(W_n^{\text{deg}}) \circ F))$ -path—it is trivial to find one (only use e-steps)—and prove that

$$\mathcal{O}_{\text{sd}} ((\tilde{j}_1 \times \tilde{j}_{p,w})(W_n) \circ (F_1 \times p(wF))) \cap \mathcal{O} \subseteq \mathcal{O}((\tilde{j}_1(W_{n,1}^{\text{deg}}) \circ F_1) \times p(w(\tilde{j}(W_n^{\text{deg}}) \circ F))) \quad (9)$$

where \mathcal{O} consists of the orbits in $\mathcal{N}_{GL_{N_1} \times M_N^{\text{mir}}}$ which are nontrivial in both blocks.

We are left with proving: the Claim, and (9) (with the conditions in IV). We make plenty of use of “resp.” and any two uses in different sentences are independent. In case F is nontrivial (resp. trivial) let $H := \prod_{1 \leq i \leq k-1} GL_n \times P_n^{\text{mir}}$ (resp. $H := \prod_{1 \leq i \leq k-1} GL_n \times GL_{n-1}$). Let \tilde{j}_{hor} be the embedding of U_n in H given by $\tilde{j}_{\text{hor}}(u) = (u, \dots, u, u_k)$ where in case F is nontrivial we have $u_k := u$ and in case F is trivial we define every entry, say (i, j) , of u_k to be the $(i + 1, j + i)$ entry of u . In case F is nontrivial or $k > 2$ (resp. F is trivial and $k = 2$), we define:

- R_1, R_2, \dots to respectively be $U_{n,(1,n)}, U_{n,(1,n-1)}, \dots, U_{n,(1,2)}$ (resp. $U_{n,(2,n)}, U_{n,(2,n-1)}, \dots, U_{n,(2,3)}$);
- \tilde{j}'_{hor} and \tilde{j}''_{hor} to be the embeddings of U_n in H such that they are both equal to \tilde{j}_{hor} when we restrict to any root group except for R_1, R_2, \dots , and (for each i): $\tilde{j}'_{\text{hor}}(R_i)$ differs from $\tilde{j}_{\text{hor}}(R_i)$ only on the leftmost block of H and in the leftmost block is trivial; $\tilde{j}''_{\text{hor}}(R_i)$ is trivial on all the blocks except the leftmost block, and in the leftmost block is equal to R_i .

We define by $\tilde{j}'_{\text{hor}}(W_n)^+$ (resp. $\tilde{j}''_{\text{hor}}(W_n)^+$) to be the trivial extension of $\tilde{j}'_{\text{hor}}(W_n)$ (resp. $\tilde{j}''_{\text{hor}}(W_n)$) with domain $\tilde{j}'_{\text{hor}}(U_n)\tilde{j}''_{\text{hor}}(U_n)$. Consider the $(\tilde{j}_{\text{hor}}(W_n), e)$ -quasipath (resp. $(\tilde{j}'_{\text{hor}}(W_n), e)$ -quasipath; $(\tilde{j}''_{\text{hor}}(W_n), e)$ -quasipath) over $\tilde{j}_{\text{hor}}(R_1), \tilde{j}_{\text{hor}}(R_2), \dots$ (resp. $\tilde{j}'_{\text{hor}}(R_1), \tilde{j}'_{\text{hor}}(R_2), \dots; \tilde{j}'_{\text{hor}}(R_1), \tilde{j}'_{\text{hor}}(R_2), \dots$). Notice that in any e-step in any of these three quasipaths, all the terms except the ones that are constant with respect to a 2a1d-group are conjugate by an element in H . Hence we obtain:

$$\tilde{j}_{\text{hor}}(W_n) \xrightarrow{f, H} \{\tilde{j}'_{\text{hor}}(W_n)^+, \tilde{j}''_{\text{hor}}(W_n)^+\} \quad \tilde{j}'_{\text{hor}}(W_n) \xrightarrow{f, H} \tilde{j}'_{\text{hor}}(W_n)^+ \quad \tilde{j}''_{\text{hor}}(W_n) \xrightarrow{f, H} \tilde{j}''_{\text{hor}}(W_n)^+. \quad (10)$$

Let \mathcal{X}' be the set consisting of the AFs in \mathcal{A}_H which are nontrivial on the leftmost block. In case $k = 2$ (resp. $k > 2$) let \mathcal{X}'' be the set consisting of the AFs in \mathcal{A}_H which are nontrivial on the rightmost block (nontrivial on all the blocks except possibly the leftmost and the rightmost). By using Proposition 7.1 in [T] and Theorem 2.1 we obtain:

$$\text{if F is nontrivial or } k > 2 \text{ then } \tilde{j}'_{\text{hor}}(W_n) \xrightarrow{f, \mathcal{X}'^H} \emptyset; \quad (11)$$

$$\text{if F is trivial or } k > 2 \text{ then } \tilde{j}''_{\text{hor}}(W_n) \xrightarrow{f, \mathcal{X}''^H} \emptyset. \quad (12)$$

Make a choice of \mathcal{A} -trees which justifies⁷ the equations (10) (11) (12); and for each one of them, say Ξ , consider the tree⁸ $((\prod_i \tilde{j}_i) \times \tilde{j}_{p,w})(\Xi) \circ (F_1 \times \dots \times F_{k-1} \times p(wF))$. Then by Exchange corollary 2.9 (and proving⁹ containments in $\mathcal{B}_{GL_{N_1} \times \dots \times GL_{N_{i-1}} \times M_N^{\text{mir}}}(\dim(D_{p(wF)}) - \dim(D_{p(w_oF)}))$) we directly obtain the Claim, and we also obtain (9) as follows. If $k = 2$ and F is trivial, by successive uses of (10) and (12) for n replaced by $n, n-1, \dots$, (also when n is replaced by $n-1, n-3, \dots$ we interchange the right with the left), we obtain

$$\mathcal{O}_{\text{sd}}((\tilde{j}_1 \times \tilde{j}_{p,w})(W_n) \circ (F_1 \times p(wF))) \cap \mathcal{O} = \mathcal{O}_{\text{sd}}\left(\left(\tilde{j}_1(W_{n,1}^{\text{deg|gen}}) \circ F_1\right) \times p(w(\tilde{j}(W_n^{\text{deg|gen}}) \circ F))\right) \quad (13)$$

where $W_{n,1}^{\text{deg|gen}}$ (resp. $W_n^{\text{deg|gen}}$) is the restriction of $W_{n,1}^{\text{deg}}$ (resp. W_n^{deg}) on the rows on which it is nontrivial. In case $k = 2$ and F is nontrivial, we again obtain (13), by starting with a use of (10) and (11) and then proceeding as in the previous sentence. From the last two sentences, from $(\tilde{j}_1(W_{n,1}^{\text{deg|gen}}) \circ F_1) \times p(w(W_n^{\text{deg|gen}} \circ F)) \xrightarrow{f} (\tilde{j}_1(W_{n,1}^{\text{deg}}) \circ F_1) \times p(w(W_n^{\text{deg}} \circ F))$ (which follows trivially), and since $\mathcal{O}_{\text{sd}}(X) \subseteq \mathcal{O}_f(X)$ for any X (Main corollary 2.7), we obtain (9). \square

Remark 1.25. The arguments in the proof above are similar to arguments found inside the proofs of Theorems 8.3.11, 8.3.12 and 8.3.18 in [T]. For example—in relation to (10)—see Property 2 inside the proofs of these theorems¹⁰. Also, in the special case that F is trivial, the proposition is implied by Theorems 8.3.11 and 8.3.12 in [T] as follows.

Let $N_{\text{hor}} := (k-1)n + n - 1$, and $j : H \rightarrow GL_{N_{\text{hor}}}$ be the embedding given by

$$(g_1, \dots, g_k) \rightarrow \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{pmatrix}.$$

We adopt Definition 8.3.9 in [T] for $l = k-1$ and $N = N_{\text{hor}}$; therefore $\tilde{j}_{F_{n,k,k-1}}$ is a restriction of $j \cdot \tilde{j}_{\text{hor}}$. Recall from the appendix the information about $\mathbf{X}(\dots)$ given in Definition 2.3 and in Proposition-Definition 2.6. Notice that for every $J \in \mathbf{X}(\tilde{j}_{\text{hor}}(W_n))$, we can find (and we fix) a $J_{N_{\text{hor}}} \in \mathbf{X}(F_{n,k,k-1})$ which belongs to the Lie algebra of the $GL_{N_{\text{hor}}}$ -parabolic with Levi $j(H)$, and with projection to $\text{Lie}(j(H))$ being equal to $j(J)$. We have

$$\dim(D_{\tilde{j}_{\text{hor}}(W_n)}) + \dim(U_P) = \dim(D_{F_{n,k,k-1}}) \leq \frac{\dim(\mathcal{O}(J_{N_{\text{hor}}}))}{2} \leq \frac{\dim(\mathcal{O}(J))}{2} + \dim(U_P)$$

where the first (resp. second) inequality follows for example from A in Main corollary 2.7 (resp. 7.1.1 in [CM]). Therefore if $\dim(D_{\tilde{j}_{\text{hor}}(W_n)}) = \frac{\dim(\mathcal{O}(J))}{2}$ the inequalities above become equalities and hence:

- By Main corollary 2.7 we have $\mathcal{O}(J_{N_{\text{hor}}}) \in \mathcal{O}_f(F_{n,k,k-1})$; the set $\mathcal{O}_f(F_{n,k,k-1})$ is calculated in Theorems 8.3.11 and 8.3.12 in [T].
- By using 7.1.1 in [CM] we obtain that $\mathcal{O}(J_{N_{\text{hor}}})$ is induced from $\mathcal{O}(J)$.

By using these theorems from [T] and 7.2.5 in [CM] we see how and which among the orbits in $\mathcal{O}_f(F_{n,k,k-1})$ are induced from $\text{Lie}(H)$ and hence we are done. \triangle

Proposition 1.26. *Let $H := GL_{n_1} \times \dots \times GL_{n_k}$ and $n := \max\{n_1, \dots, n_k\}$. Let (for $1 \leq i \leq k$) let j_i be an embedding of GL_{n_i} on GL_n given by $j_i(g) = \begin{pmatrix} I_{x_i} & & \\ & g & \\ & & I_{n-x_i-n_i} \end{pmatrix}$ for each $g \in GL_{n_i}$*

⁷That is, as in Definition 2.8.

⁸Here (in contrast to Definition 1.8) $(\prod_i \tilde{j}_i) \times \tilde{j}_{p,w}$ is defined in H and maps each (h_1, \dots, h_k) to $(j_1(h_1), \dots, j_{k-1}(h_{k-1}), \tilde{j}_{p,w}(h_k))$.

⁹Use (ii) in Lemma 1.15 and then continue with an \mathcal{A} -path obtained by applying $\circ((\prod_i F_i) \times p(wF))$ to an embedding of an appropriate $(F_{\emptyset,n} \rightarrow W_n, \mathcal{B}_n)$ -path.

¹⁰Note that the proof of Property 2.1 becomes simpler (and more similar) by replacing the ue-step there with a c-step.

(and some choice of x_i), and let $\tilde{j}_{\text{hor}} := j_1^{-1} \times \dots \times j_k^{-1}$. Let $\mathbf{F} \in \mathcal{A}_{\mathfrak{F}}(U_n)$, $[b_1, \dots] := \mathcal{O}(J_{\mathbf{F}})$, and $a := [a_{1,1}, \dots] \times \dots \times [a_{1,k}, \dots] \in \mathcal{N}_H$ be such that there is an $(\tilde{j}_{\text{hor}}(\mathbf{F}) \rightarrow \mathcal{C}_H[a])$ -path. Then

$$\sum_i (b_i - 1) \leq \sum_{i,j} (a_{i,j} - 1).$$

Proof. Since $\sum_{i,j} (a_{i,j} - 1)$ admits the smallest value for $a \in \mathcal{O}(\tilde{j}_{\text{hor}}(\mathbf{F}))$ (and since this set does not depend on the choice of \mathcal{A} -tree defining it), it is sufficient to fix any $(\tilde{j}_{\text{hor}}(\mathbf{F}) \rightarrow \mathcal{C}_H)$ -tree and replace in the statement of the proposition the information “there is an $(\tilde{j}_{\text{hor}}(\mathbf{F}) \rightarrow \mathcal{C}_H[a])$ -path” with the information “an output AF of the \mathcal{A} -tree we just fixed belongs to $\mathcal{C}_H[a]$ ”.

The proof of this modification of the proposition for a certain choice of the fixed \mathcal{A} -tree follows inductively on k from the claim below and from Proposition 7.1 (Part 1) in [T]. This claim in turn easily follows from the proof of Theorem 2.1.

Claim. Recall the meaning of $\Xi^{\text{st}}(\dots)$, which appears in the appendix in the proof of Theorem 2.1; here we identify $P_{n_1}^{\text{mir,as},0}$ with the subgroup $P_{n_1}^{\text{mir,as},0} \times 1 \times \dots \times 1$ of H . Let \mathbf{I} be an initial path of $\Xi^{\text{st}}(\tilde{j}_{\text{hor}}(\mathbf{F}))$ with output vertex an output vertex of $\Xi^{\text{st}}(\tilde{j}_{\text{hor}}(\mathbf{F}))$, and let \mathbf{Z} be the label of this vertex. Let w_1, \dots, w_y be the elements in W_n such that in the i -th use of Case 3 in the proof of Theorem 2.1 when we follow the path \mathbf{I} , we have $w = w_i$ (we include the cases in which w is trivial). Consider the e -steps of $\text{quasi}(\mathbf{I})$ which occur as in Case 2 and so that their label on the vertex they share with \mathbf{I} is not their constant term. Let c'_1, \dots, c'_y be the numbers such that the i -th such e -step of $\text{quasi}(\mathbf{I})$ lies in the c'_i -th column (of $n_1 \times n_1$ matrices). Let c_1, \dots, c_y be the numbers such that $(\prod_{1 \leq j < i} w_j) j_1^{-1}(c_i) (\prod_{1 \leq j < i} w_j)^{-1} = c'_i$ where here c_i (resp. c'_i) is identified with the torus, say T , in \mathcal{D}_n (resp. \mathcal{D}_{n_1}) for which $\text{Set}(T) = \{c_i\}$ (resp. $\text{Set}(T) = \{c'_i\}$). Then $\mathbf{Z} = \mathbf{Z}_1 \times (\prod_{2 \leq i \leq k} j_i^{-1})(\mathbf{Z}_2)$ where $\mathbf{Z}_1 \in \mathcal{A}(U_{n_1})$,

$$D_{\mathbf{Z}_2} \supseteq \prod_{i < j \notin \{c_1, \dots, c_y\}} U_{(i,j)} \quad (14)$$

and \mathbf{F} and \mathbf{Z}_2 have the same restriction on the group on the right hand side of (14). \square

Proposition 1.27. Consider any $\mathbf{I} \in \mathcal{S}$ with: $d(\mathbf{I}) \leq 0$; $\mathbf{F}_1, \dots, \mathbf{F}_{k-1}, \mathbf{F}$ being trivial; $k > 2$; and the elements of any among π_1, \dots, π_{k-1} , and π , admitting absolutely convergent Eisenstein series expansion over discrete data¹¹. Then $I = 0$.

Proof. It is known that:

Nilpotent orbit dimension formula 1.28. The dimension of an orbit $[a_1, \dots, a_z] \in \mathcal{N}_n$ where $a_1 \geq \dots \geq a_z$, is equal to two times the number of nondiagonal upper triangular entries (of $n \times n$ matrices) except for the entries in the $\sum_{1 \leq i \leq j} a_i$ -th row for $j = 1, \dots, z$.

Since $d(\mathbf{I}) \leq 0$, by the classification of automorphic forms, Corollary 9.3.5 in [T], and Nilpotent orbit dimension formula 1.28, we obtain a trivial representation of a general linear group, say GL_t , which appears in the discrete data of one among π_1, \dots, π_{k-1} , and w.l.o.g we assume this is the case for π . By Proposition 1.22 we are also assuming w.l.o.g that GL_1 does not appear in the inducing data of any among $\pi_1, \dots, \pi_{k-1}, \pi$. Hence, by replacing E with an appropriate automorphic form admitting it as a residue (and by the way the dimension of a nilpotent orbit changes after applying induction), we are reduced to proving:

Claim. Let \mathbf{I} be as in the statement of the proposition except that: we replace “ $d(\mathbf{I}) \leq 0$ ” with “ $d(\mathbf{I}) \leq n - 3$ ”, we assume that π is induced from P_n^{mir} in exactly one way, that the representation on the GL_{n-1} -copy in the inducing data of this induction is nontrivial, that the elements in π admit absolutely convergent Eisenstein series expansion over P_n^{mir} , and none among π_1, \dots, π_{k-1} are induced from P_n^{mir} . Then $I = 0$.

By unfolding I as usual (Eisenstein series expansion of E over P_n^{mir} , and Fourier expansion of φ over $U_n(\mathfrak{k}) \setminus U_n(\mathbb{A}_{\mathfrak{k}})$) and then using Proposition 1.26, we see the Claim is obtained from:

¹¹To be clear, we count the representations in the discrete spectrum as such representations. Also, in [T] we denoted the set of such representations by $\text{Aut}_{\mathfrak{k}, n, >}$.

Claim'. Let a_1, \dots, a_{k-1} be orbits in \mathcal{N}_n , and a_k be an orbit in \mathcal{N}_{n-1} , and assume that none of them is the trivial or the minimal (that is, $[1, 1, \dots]$ or $[2, 1, 1, \dots]$). Let $[a_{i,1}, a_{i,2}, \dots] := a_i$. Assume that $\sum_{i,j} (a_{i,j} - 1) \geq n - 1$. Then

$$\sum_i \frac{\dim(a_i)}{2} > \frac{n(n-1)}{2} + n - 3. \quad (15)$$

(of course, by the usual convention, i varies over $1, \dots, k$.) By using the Nilpotent orbit dimension formula 1.28 we easily obtain any of the claims below (at least some of which are of course familiar); among them, we only prove the last one, the other are even easier. Then Claim' is obtained as follows:

- (i). Nilpotent orbit dimension formula 1.28 and Claim 1 reduces it to the case, which we call Claim'', in which: $a_{i,1} \geq 3$ for at most one i , and if such an i exist, we have $a_{i,j} \geq 2$ for all j .
- (ii). Then by Claim 2 we are reduced to the modification of Claim'', which we call Claim''', in which, in case the i as in (i) exists: we have $i = k$, we replace " $\sum_{i,j} (a_{i,j} - 1) \geq n - 1$ " with " $\sum_{i,j} (a_{i,j} - 1) \geq n - 2$ ", and we subtract from the right hand side of (15) the number $\min\{a_{k,1}, a_{k,2}, \dots\} - 1$.
- (iii). Then by using Claim 3 we reduce Claim''' to the special case which we call Claim 4.

Claim 1 (follows from [CM], Theorem 6.2.5). *If for two $a, b \in \mathcal{N}_n$ we have $a > b$, then $\dim(a) > \dim(b)$. In particular, if a is obtained from b by changing two terms $r \geq r'$ of the later one with the terms $r + 1, r' - 1$, then $\dim(a) \geq \dim(b)$.*

Claim 2. *Let $[a_1, \dots, a_y] \in \mathcal{N}_n$ be such $a_1 \geq \dots \geq a_y \geq 2$ and $a_1 > 2$. Then*

$$\dim([a_1, \dots, a_y]) + \dim([2^l, 1^{n-1-l}]) \geq \dim([2^l, 1^{n-l}]) + \dim([a_1, \dots, a_{y-1}, a_y - 1]) + 2(a_y - 1).$$

Claim 3. *Consider positive integers $l_1 > l_2$ which are smaller from $\frac{n}{2}$. Then*

$$\dim([2^{l_2}, 1^{n-l_2}]) + \dim([2^{l_1}, 1^{n-l_1}]) \geq \dim([2^{l_2-1}, 1^{n-l_2+1}]) + \dim([2^{l_1+1}, 1^{n-l_1-1}]).$$

Claim 4. Let $a = [a_1, \dots, a_y] \in \mathcal{N}_{n-1}$ and $n \leq l \leq 2$ be such that:

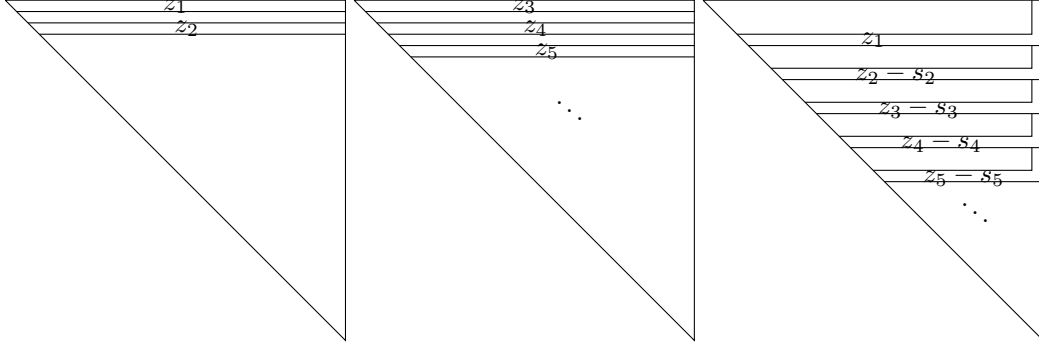
$$2 + l + \sum_i (a_i - 1) \geq n - 2; \quad (16)$$

$a_1 \geq \dots \geq a_y$; if $a_1 \geq 3$ then $a_y \geq 2$; and if $a_1 = 2$ then the left hand side of (16) is bigger or equal to $n - 1$. If the left hand side of (16) is bigger or equal to $n - 1$ we define $x := 0$; otherwise we define $x := a_y - 1$. We have:

$$\frac{\dim([2^2, 1^{n-4}])}{2} + \frac{\dim([2^l, 1^{n-l}])}{2} + \frac{\dim(a)}{2} > \frac{n(n-1)}{2} - x + n - 3. \quad (17)$$

Proof of Claim 4. We are easily reduced to assuming that the left hand side of (16) is equal to $n - 1$ or $n - 2$ (and hence l is respectively equal to $y - 2$ or $y - 3$). Then, due to Nilpotent orbit dimension formula 1.28, the left hand side of (17) is equal to $\frac{n(n-1)}{2} - x' + s_2 + \dots + s_{l+2}$ where: $s_2 = a_1 - 2$, $s_3 := a_1 + a_2$, for $2 < i \leq l + 2$ we have $s_i := a_1 + \dots + a_{i-1} - 2i + 6$, and if $x = 0$ (resp. $x = a_y - 1$) we define $x' := 0$ (resp. $x' := a_y$). Indeed: consider the three triangles below, each of them containing the strictly upper triangular entries of $n \times n$ matrices (which are $\frac{n(n-1)}{2}$ in total), and the trapeziums on the third triangle with horizontal bases have heights (that is number of rows nontrivially intersecting them) $a_1 - 1, \dots, a_y - 1$ starting from the uppermost one and moving downwards; for $1 \leq i \leq l + 2$ let z_i be the number of entries in the trapezium within which z_i lies in one of the first two triangles; we then notice that each

$z_i - s_i$ is also the number of entries of the area enclosed by the polygon containing this number (this time in the third triangle); then by using Nilpotent orbit dimension formula 1.28 we obtain $\frac{\dim([2^2, 1^{n-4}])}{2} = z_1 + z_2$, $\frac{\dim([2^l, 1^{n-l}])}{2} = \sum_{3 \leq i \leq l+2} z_i$, and $\frac{\dim(a)}{2}$ is equal to the number of entries of the trapeziums (with horizontal bases) on the third triangle. Hence we finish the proof by checking¹² that $s_2 + \dots + s_{l+2} > n - 2$.



□Claim 4 □

Remark 1.29 (Avoiding the use of Main corollary 2.7.). In the beginning of the proof above, we made use of Proposition 1.22, in the proof of which in turn we used Main corollary 2.7. We can avoid using Proposition 1.22 by instead proving the modification of Claim' in which: the orbits a_1, \dots, a_k are allowed to be equal to the minimal one, a_k can even be the trivial one, and we remove $n - 3$ in the right hand side of (15). This modification of Claim' is obtained for example by repeating (i), (ii), and (iii), except that in (iii) we are reduced to a statement easier than Claim 4. \triangle

Remark 1.30 (Relations of the present paper with the literature). In the present paper we used slight refinements of results on Fourier coefficients appearing in [T] (and we explain there how this paper in turn relates to the literature).

As for the unfolding of integrals until we are left with Fourier coefficients (that is, (2)), except for the use of Lemma 1.5, we proceeded as in the most familiar $GL_n \times GL_n$ Rankin-Selberg integrals¹³. Proposition 1.22 in the special case $k = 2$, $N_1 = N = n$ (and hence $F_1 = F = F_{\emptyset, n}$) is found in [G1] (in Section 5). Proposition 1.22 in the special case that $k = 1$ and D_F is generated by root groups appears in [T] (in 8.4).

Next we discuss the results about vanishing of integrals. Conjecture 1 in [G2] has as special case: a special case of Proposition 1.22 in which F_1, \dots, F_{k-1}, F are among certain AFs, say Z , satisfying $\mathcal{O}(Z) = \{\mathcal{O}(J_Z)\}$. Proposition 1.27 is the restriction of Conjecture 1 in [G3] to representations with elements admitting absolutely convergent Eisenstein series expansion over discrete data. Since a proof of many cases of Conjecture 2 in [G3] appear in [G3], and since I see no mention of my name, I point out that at least 4 months before the submission of [G3] (on the arXiv), the author of [G3] received emails from me which included most of the work¹⁴ for obtaining any of the three following proofs of Conjecture 2:

Proof 1. Use: Main corollary 2.7 (and Exchange corollary 2.9), (10), (11), and (12). □

Proof 2. Use: Main corollary 2.7, the claim inside the proof of Proposition 1.26 only for I being an e-path, Claim 1 and Claim 3 in the proof of Proposition 1.27, and Nilpotent orbit dimension formula 1.28. □

Proof 3. Change Proof 2 by: Removing the use of Main corollary 2.7, and using (this time more of the essence of) Proposition 1.26. □

He received these emails while I was a postdoc in Tel Aviv University and he was the person over there to whom I was presenting progress in my research. In particular, in 14 May 2016 I emailed him a document which contained: Theorem 3.1 of [T] (both statement and proof); most of the

¹²After checking a few cases we are reduced to assuming: $a_1 \geq 3$ and $y \geq 6$. Then $s_2 + \dots + s_{l+2} \geq a_1 - 2 + (a_1 + a_2)(y - 3) \geq +1 + (a_1 + a_2)\frac{y}{2} \geq 1 + (a_1 + \dots + a_y) \geq n$.

¹³To be clear, the present paper contains no information about which L -functions appear.

¹⁴in the situation that one has available the part of the literature preceding [G3] (and hence preceding [T]).

work of section 6 in [T]. I mention Theorem 3.1 of [T] because it overlaps with the proof in [G3] and because it leads to Proof 3 above, which is much shorter. I mention Section 6 in [T] because together with Theorem 3.1 it gives even shorter proofs (Proofs 1 and 2 above). Note that Proof 3 above remains much shorter from the proof in [G3] even after uncovering as needed the proof of Theorem 3.1 in [T]. In the proof in [G3] the representations that are addressed are not addressed simultaneously. \triangle

2 Appendix: general results on \mathcal{A} -trees

Below we recall and consider simple extensions of results and definitions from [T]. The differences in the proofs are minor. We present the proof of the first Theorem below in a self contained way, and we mostly skip the other proofs (which are even more identical to the ones in [T]).

Recall from [T] that for a \mathfrak{k} -group V , the set of AFs with domain V is denoted by $\mathcal{A}(V)$ (and hence the subset of \mathfrak{k} -AFs with $\mathcal{A}_{\mathfrak{k}}(V)$). Also, for an algebraic group H , the set of AFs with domain contained in H is denoted by \mathcal{A}_H .

Theorem 2.1 (It extends 3.1 in [T]). *Let $P_n^{\text{mir,as},0} := j(SL_{n-1}) \prod_{1 < j \leq n} U_{(1,j)}$ where j is the lower right corner embedding of SL_{n-1} in GL_n (“as” stands for “associate”). Let F be an AF over \mathfrak{k} such that there is a (linear) \mathfrak{k} -group containing both D_F and $P_n^{\text{mir,as},0}$ as \mathfrak{k} -subgroups. We assume that D_F normalizes $P_n^{\text{mir,as},0}$ and that there is a \mathfrak{k} -homomorphism $f : P_n^{\text{mir,as},0} D_F \rightarrow P_n^{\text{mir,as},0}$ such that its restriction to $P_n^{\text{mir,as},0}$ is the identity morphism and $f(D_F) \subseteq U_n$. Then there is an*

$$(F \rightarrow \mathcal{A}_{\mathfrak{k}}(U_n) D_F, P_n^{\text{mir,as},0} D_F, \mathfrak{k}) \text{-tree.}$$

Proof. We freely use notations from [T]. Let \mathcal{X}_n be the set of AFs F which are required to satisfy everything in the statement of the theorem except for the last sentence. For each $F \in \mathcal{X}_n$ we inductively define an $(F \rightarrow \mathcal{A}_{\mathfrak{k}}(U_n) D_F, P_n^{\text{mir,as},0} D_F, \mathfrak{k})$ -tree which we call $\Xi^{\text{st}}(F)$. The induction is on n and downwards on $\dim(D_F \cap U_n)$. More precisely: if $D_F \supseteq U_n$ we define $\Xi^{\text{st}}(F)$ to be the trivial F -tree (that is, $\Xi^{\text{st}}(F)$ has only one vertex); and otherwise we define $\Xi^{\text{st}}(F)$, assuming that $\Xi^{\text{st}}(F')$ is defined for all n' and $F' \in \mathcal{X}_{n'}$ satisfying one among 1) and 2) below

$$1) \ n' < n \quad 2) \ n' = n \text{ and } \dim(D_{F'} \cap U_n) > \dim(D_F \cap U_n).$$

Let j be the lower right corner embedding of GL_{n-1} in GL_n . Assume that an AF $F^{\text{ih}} \in \mathcal{X}_n$ satisfies

$$F^{\text{ih}} = j(F') \circ Y \tag{18}$$

where

- Y is an AF with domain $D_Y = \prod_{1 < j \leq n} U_{(1,j)}$, which is trivial on all root groups contained in D_Y except possibly $U_{(1,2)}$ (therefore $j(P_{n-1}^{\text{mir,as},0}(\mathfrak{k})) \in \text{Stab}_{P_n^{\text{mir,as},0}}(Y)$);
- $F' \in \mathcal{X}_{n-1}$, and $j(F')$ is defined by extending j so that its domain contains $D_{F'}$ and the third sentence in the statement of the theorem holds after replacing D_F with $j(D_{F'})$.

Then we define $\Xi^{\text{st}}(F^{\text{ih}}) := j(\Xi^{\text{st}}(F')) \circ Y$.

We will define an $(F, P_n^{\text{mir,as},0} D_F, \mathfrak{k})$ -tree ξ , for each output AF $F^{\text{ih}'}$ of which: we have $f(D_{F^{\text{ih}'}}) \subseteq U_n$, and either satisfies $\dim(D_{F^{\text{ih}'}}) > \dim(D_F)$ or is chosen as in (18) in the place of F^{ih} . After that, the proof ends by defining

$$\Xi^{\text{st}}(F) := \Xi^{\text{st}}(F^{\text{ih}'}) \vee_{F^{\text{ih}'}} \xi$$

where $F^{\text{ih}'}$ varies over all output AFs of ξ .

In the cases that ξ will turn out to contain an e-step, say ξ_e , this is its last \mathcal{A} -step, because the dimension of the intersection of U_n with the domain of the terms of ξ_e (that is, the output AFs of ξ_e) is bigger from $\dim(D_F \cap U_n)$.

Consider the smallest number $m \geq 2$ for which the set $A_m := \prod_{m \leq i \leq n} U_{(1,i)}$ is contained in D_F . To be clear, by $\prod_{n+1 \leq i \leq n} U_{(1,i)}$ we mean the trivial subgroup of GL_n .

Case 1: $F(A_m) = \{0\}$ and $2 < m$. In this case ξ is chosen to be the (F, e) -step over $U_{(1, m-1)}$.
1

Case 2: $m = 2$ and $F(A_2) = \{0\}$. In this case (18) is true for $[F^{\text{ih}} \leftarrow F]$, and hence we choose ξ to be the trivial F -tree. □Case 2

Case 3: $F(A_m) \neq \{0\}$. We start with some definitions and observations, and return to defining ξ in the next paragraph. Consider integers $1 < l \leq n$ and $1 \leq k < l$, and an algebraic subgroup D of $U_n D_F$. We define $L_{(k, l)}$ to be the biggest algebraic subgroup of $U_n D_F$ such that $f(L_{(k, l)})$ is generated by all the root groups in U_n except the $U_{(i, l)}$ with $k < i < l$. For example $L_{(l-1, l)} = U_n D_F$. For $k > 1$, the group $L_{(k-1, l)}$ is a normal subgroup of $L_{(k, l)}$, and hence $D \cap L_{(k-1, l)}$ is a normal subgroup of $D \cap L_{k, l}$. Since unipotent algebraic groups are connected, the identity embedding

$$j_{D, k, l} : D \cap L_{(k-1, l)} \setminus D \cap L_{(k, l)} \rightarrow L_{(k-1, l)} \setminus L_{(k, l)}, \quad (19)$$

is either trivial or an isomorphism.

Back to defining ξ , let l_1 be the biggest number such that $F(U_{(1, l_1)}) \neq \{0\}$. The \mathcal{A} -tree ξ will either finish with an e -step in at most $l_1 - 2$ \mathcal{A} -steps or it will consist of $l_1 - 1$ \mathcal{A} -steps each one being a ue -step or a c -step.

Let $F^0 := F$. Assume that for a number i satisfying $1 \leq i \leq l_1 - 2$, the first $i - 1$ \mathcal{A} -steps of ξ have been defined, that none of them was an e -step, and that the output AF of the $i - 1$ -th one has been given the name F^{i-1} . To avoid confusion, if $i = 1$, this means that no \mathcal{A} -steps have been defined. The i -th \mathcal{A} -step in ξ —and in case it is not an e -step, its output AF F^i —are defined as follows:

1. If $j_{D_{F^{i-1}}, l_1-i, l_1}$ is trivial and $U_{(1, l_1-i)} \not\subseteq D_{F^{i-1}}$, the i -th \mathcal{A} -step in ξ is the (F^{i-1}, e) -step over $U_{(1, l_1-i)}$. Hence this is the last \mathcal{A} -step in ξ .
2. If $j_{D_{F^{i-1}}, l_1-i, l_1}$ is trivial and $U_{(1, l_1-i)} \subseteq D_{F^{i-1}}$, there is an element $u \in U_{(l_1-i, l_1)}(\mathfrak{k})$, such that $uF^{i-1}(U_{(1, l_1-i)}) = \{0\}$. We choose the i -th \mathcal{A} -step in ξ to be the $(F^{i-1} \rightarrow uF^{i-1}, c)$ -step and we define $F^i := uF^{i-1}$.
3. Finally assume that $j_{D_{F^{i-1}}, l_1-i, l_1}$ is an isomorphism. Let F^i be the AF defined by

$$D_{F^i} = U_{(1, l_1-i)}(D_{F^{i-1}} \cap L_{(l_1-i-1, l_1)}), \quad F^i(U_{(1, l_1-i)}) = \{0\}, \quad \text{and} \\ F^i(u) = F^{i-1}(u) \quad \forall u \in D_{F^{i-1}} \cap L_{(l_1-i-1, l_1)}. \quad (20)$$

The i -th \mathcal{A} -step in ξ is the $(F^{i-1} \rightarrow F^i, ue)$ -step.

Assuming that no e -step was encountered, we have

$$\prod_{1 < j \leq n} U_{(1, j)} \subseteq D_{F^{l_1-2}} \subseteq L_{(1, l_1)}, \quad (21)$$

and the last \mathcal{A} -step of ξ is the c -step obtained from the minimal length element $w \in W_n$ such that (18) is correct for $[F^{\text{ih}} \leftarrow wF^{l_1-2}]$. □Case 3 □

Unless otherwise specified, **in the rest of the appendix**,

$$H := \prod_{1 \leq i \leq k} GL_{n_i}$$

(for some positive integers k, n_1, \dots, n_k).

Definition 2.2 (2a1d-groups). Let S be a finite set of root groups in H such that for any roots α, β for which U_α and U_β belong in S we have: neither $\alpha + \beta$ nor $\alpha - \beta$ is a root, and $\alpha \neq -\beta$. Any at most one dimensional algebraic subgroup of $\prod_{V \in S} V$ is called a 2a1d-group.

We say that a 2a1d-group, say L , is nontrivial on an entry, if and only if this entry is nondiagonal and a matrix in L is nontrivial on this entry. △

Definition 2.3 ($\mathbf{X}(\mathbb{F}), J_{\mathbb{F}}$). Let $\mathbb{F} \in \mathcal{A}_H$. We define

$$\mathbf{X}(\mathbb{F}) := \{(J_1, \dots, J_k) \in \text{Lie}(H) : \sum_{1 \leq i \leq k} \text{tr}(J_i u_i) = \mathbb{F}(u_1, \dots, u_k) \quad \forall (u_1, \dots, u_k) \in D_{\mathbb{F}}\}$$

(where tr denotes the trace). Further assume that $D_{\mathbb{F}}$ is generated by root groups; then we denote by $J_{\mathbb{F}}$ the unique element in $\mathbf{X}(\mathbb{F}) \cap D_{\mathbb{F}}^t$ (where $D_{\mathbb{F}}^t := \{(J_1, \dots, J_k) : (J_1^t, \dots, J_k^t) \in D_{\mathbb{F}}\}$, and t denotes transpose). \triangle

Definition 2.4 ($\mathcal{C}_H, \mathcal{C}_H[a]$). We define

$$\mathcal{C}_H := \{\mathbb{F} \in \mathcal{A}_H : D_{\mathbb{F}} \text{ is a unipotent radical of an } H\text{-parabolic subgroup}\},$$

and for an $a \in \mathcal{N}_H$, we also define $\mathcal{C}_H[a] = \{\mathbb{F} \in \mathcal{C}_H : \mathcal{O}(J_{\mathbb{F}}) = a\}$.

We frequently write \mathcal{C}_n (resp. $\mathcal{C}_n[a, \dots]$) in place of \mathcal{C}_{GL_n} (resp. $\mathcal{C}_{GL_n}[a, \dots]$). \triangle

Definition 2.5 ($\mathcal{B}_H(k), \mathcal{B}_H(k)$ -paths). For two AFs $\mathbb{F}', \mathbb{F} \in \mathcal{A}_H$, consider an $(\mathbb{F}' \rightarrow \mathbb{F}, H)$ -path called Ξ . For every e-step ξ of $\text{quasi}(\Xi)$, say the i -th one, consider an algebraic subgroup of H which acts by conjugation freely and transitively on a subset, say S_i , of the variety consisting of the terms of this e-step, so that the vertex of Ξ which is also an output vertex of ξ , is labeled with an AF in S_i (here we do not identify any different vertices).

For every nonnegative integer k for which we can choose these actions so that

$$k \geq \dim(D_{\mathbb{F}}) - \dim(D_{\mathbb{F}'}) - \sum_i \dim(S_i)$$

we say that Ξ is an $(\mathbb{F}' \rightarrow \mathbb{F}, \mathcal{B}_H(k))$ -path or (just) a $\mathcal{B}_H(k)$ -path. We denote by $\mathcal{B}_H(k)$ the set of AFs \mathbb{F} admitting an $(\mathbb{F}_{\emptyset, H} \rightarrow \mathbb{F}, \mathcal{B}_H(k))$ -path. We frequently write \mathcal{B}_H instead of $\mathcal{B}_H(0)$, and (similarly to other notations) \mathcal{B}_n instead of \mathcal{B}_{GL_n} . \triangle

Proposition-Definition 2.6 ($\mathcal{O}(\mathbb{F}), \text{mult}(a, \mathbb{F}), \mathcal{O}_{\mathbb{F}}(\mathbb{F})$). Let $\mathbb{F} \in \mathcal{A}_H$ and Ξ be an $(\mathbb{F} \rightarrow \mathcal{C}_H, H)$ -tree (known to exist by iterative uses of Theorem 2.1). We define $\mathcal{O}(\mathbb{F})$ to be the set consisting of the minimal elements of

$$\{a \in \mathcal{N}_H : \text{An output AF of } \Xi \text{ belongs to } \mathcal{C}_H[a]\}.$$

Let $a \in \mathcal{O}(\mathbb{F})$. If the output vertices of Ξ having label in $\mathcal{C}_H[a]$ are finitely many, we denote by $\text{mult}(a, \mathbb{F})$ the number of such vertices. If these vertices are infinitely many we write $\text{mult}(a, \mathbb{F}) = \infty$. It turns out that $\mathcal{O}(\mathbb{F})$ is also the set of minimal elements in $\{a \in \mathcal{N}_H : \mathbf{X}(\mathbb{F}) \cap a \neq \emptyset\}$; hence we obtain that $\mathcal{O}(\mathbb{F})$ does not depend on the choice of Ξ , and with a similar argument $\text{mult}(a, \mathbb{F})$ is also independent from Ξ (see 6.15 in [T]). Finally we define

$$\mathcal{O}_{\mathbb{F}}(\mathbb{F}) := \{a \in \mathcal{O}(\mathbb{F}) : \text{mult}(a, \mathbb{F}) < \infty\}. \quad \triangle$$

Main corollary 2.7 (It extends 6.17 in [T]; here we also define $\mathcal{O}_{\text{sd}}(\mathbb{F})$). Consider $\mathbb{F} \in \mathcal{A}_H$ and a nonnegative integer k such that there is a $(\mathbb{F}_{\emptyset, H} \rightarrow \mathbb{F}, \in \mathcal{B}_H(k))$ -path. Consider an orbit $a \in \mathcal{N}_H$, such that there is an $(\mathbb{F} \rightarrow \mathcal{C}_H, H)$ -tree with an output AF belonging in $\mathcal{C}_H[a]$ (equivalently $\mathbf{X}(\mathbb{F}) \cap a \neq \emptyset$). Then:

A. $\frac{\dim(a)}{2} \geq \dim(D_{\mathbb{F}}) - k$;

B. If

$$\frac{\dim(a)}{2} = \dim(D_{\mathbb{F}}) - k, \quad (22)$$

then: $\mathbb{F} \in \mathcal{B}_H(k')$ if and only if $k' \geq k$; we define $\mathcal{O}_{\text{sd}}(\mathbb{F}) = \{b \in \mathcal{O}(\mathbb{F}) : \dim(b) = \dim(D_{\mathbb{F}}) - k\}$. If for no data a, k as in the first two sentences of the present corollary formula (22) holds, we define $\mathcal{O}_{\text{sd}}(\mathbb{F}) := \emptyset$.

C. $\mathcal{O}_{\text{sd}}(\mathbb{F}) \subseteq \{b \in \mathcal{O}(\mathbb{F}) : \text{mult}(b, \mathbb{F}) = 1\}$.

D. Assume that $F \in \mathcal{B}_H$. Then

$$\left\{ b \in \mathcal{O}(F) : \dim(D_F) = \frac{\dim(b)}{2} \right\} = \mathcal{O}_{\text{sd}}(F) = \mathcal{O}_f(F) = \{b \in \mathcal{O}(F) : \text{mult}(b, F) = 1\}$$

(of course the first equality follows from the definition of $\mathcal{O}_{\text{sd}}(F)$).

Proof. The differences with the proof of Main corollary 6.17 in [T] are small, the two that matter the most are:

- As already mentioned in the definition above, the existence of an $(F \rightarrow \mathcal{C}_H, H)$ -tree is obtained by iterative uses of Theorem 2.1 (instead of a single use of Theorem 3.1 in [T]).
- To obtain C, we again prove that $\mathbf{X}(F) \cap b$ is a connected variety for $b \in \mathcal{O}_{\text{sd}}(F)$; in fact, $\mathbf{X}(F) \cap b$ is irreducible (again), and we obtain this irreducibility inductively by again using that $\mathbf{X}(F'') \cap b$ is an open nonempty subset of $\mathbf{X}(F') \cap b$, where F' and F'' are any two successive labels in a path Ξ as in Definition 2.5 (by making the smallest choice of k); but in contrast to the case $k = 0$, to obtain this openness we use the information $b \in \mathcal{O}_{\text{sd}}(F)$ (instead of only using that b is a subvariety of $\text{Lie}(H)$ nontrivially intersecting $\mathbf{X}(F)$).

□

Definition 2.8 ($\overset{f, \mathcal{X}}{\rightarrow} H$). Let (more generally than in the rest of the appendix) H be an algebraic subgroup of a product of general linear groups. Let $F \in \mathcal{A}_H$, and $\mathcal{X} \subseteq \mathcal{A}_H$. Consider an (F, H) -tree Ξ . Let \mathcal{V} be a set consisting of output vertices of Ξ , and for each $u \in \mathcal{V}$ let F_u be the label of u . For every AF Z appearing as the label of an output vertex of Ξ which is not a vertex in \mathcal{V} , we assume there are infinitely many output vertices of Ξ with label equal to Z . We then write

$$F \overset{f, \mathcal{X}}{\rightarrow} \{F_u : u \in \mathcal{V}\} \cap \mathcal{X}. \quad (23)$$

To be clear, the only notation introduced in (23) is “ $\overset{f, \mathcal{X}}{\rightarrow} H$ ” (the right hand side is a set intersection as usual). In case H is a direct product of general linear groups (resp. $\mathcal{X} = \mathcal{A}_H$), we remove H (resp. \mathcal{X}) from this notation. \triangle

Exchange corollary 2.9 (It extends 8.3.5 in [T]). *Let k, k_i be nonnegative integers, $F \in \mathcal{B}_n(k)$, $F_i \in \mathcal{B}_n(k_i)$, and $F'_i \in \mathcal{A}_n$, where i varies over the elements of a set, say X . Let $d := \dim(D_F) - k$. Assume that*

1. there is an $a \in \mathcal{O}(F)$ with $\frac{\dim(a)}{2} = d$
2. $\dim(D_{F_i}) - k_i = d$,
3. $F \overset{f}{\rightarrow} \{F'_i : i \in X\}$,
4. $F_i \overset{f}{\rightarrow} F'_i$.

Then

$$\mathcal{O}_{\text{sd}}(F) = \bigcup_i \{a \in \mathcal{O}(F_i) : \dim(a) = d\},$$

also for each i the set $\{a \in \mathcal{O}(F_i) : \dim(a) = d\}$ is equal to $\mathcal{O}_{\text{sd}}(F_i)$ or to \emptyset .

Proof. It follows directly from Main corollary 2.7. □

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