SU(3) Knot Solitons: Hopfions in the F_2 Skyrme-Faddeev-Niemi model

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We discuss the existence of knot solitons (Hopfions) in a Skryme-Faddeev-Niemi-type model on the target space $SU(3)/U(1)^2$, which can be viewed as an effective theory of both the SU(3) Yang-Mills theory and the SU(3) anti-ferromagnetic Heisenberg model. We derive the knot solitons with two different types of ansatz: the first is a trivial embedding configuration of SU(2) into SU(3), and the second is a non-embedding configuration that can be generated through the Bäcklund transformation. The resulting Euler-Lagrange equations for both ansatz reduce exactly to those of the CP^1 Skyrme-Faddeev-Niemi model. We also examine some quantum aspects of the solutions using the collective coordinate zero-mode quantization method.

I. INTRODUCTION

It is of great importance to consider SU(3) generalizations of the O(3) nonlinear σ -model, because they may possibly play crucial roles in relevant limits of fundamental theories — for example, the low-energy limit of the SU(3) pure Yang-Mills theory and the continuum limit of the SU(3) Heisenberg models. The main achievement of the present paper is that we have successfully constructed novel soliton solutions, called "Hopfions," on the flag manifold $F_2 = SU(3)/U(1)^2$. Hopfions are topological solitons with knotted structures characterized by a Hopf invariant. Such knotted structures appear in various branches of physics: quantum chromodynamics (QCD) [1–3], Bose-Einstein condensates [4, 5], superconductors [6], liquid crystals, [7] and so on.

A typical theory that includes Hopfions is the Skyrme-Faddeev-Niemi (SFN) model [2, 8], which is an O(3) nonlinear σ -model in (3+1)-dimensional Minkowski space-time. The scalar field theory for this model is defined by the Lagrangian density

$$\mathcal{L} = M^2 \partial_{\mu} \vec{n} \cdot \partial^{\mu} \vec{n} - \frac{1}{2e^2} \left(\partial_{\mu} \vec{n} \times \partial_{\nu} \vec{n} \right)^2 \tag{1}$$

where M has the dimension of mass, e is a dimensionless coupling constant, and \vec{n} is a three-component vector of unit length; i.e., $\vec{n} \cdot \vec{n} = 1$. The second term on the right-hand side in (1), the Skyrme term, was introduced by Faddeev [8] in order for the theory to satisfy Derrick's criteria for the existence of stable soliton solutions. Solutions of toroidal shape, which have the lower Hopf numbers H=1 or 2, were first found under an axially symmetric ansatz by Gladikowski and Hellmund [9], and by Faddeev and Niemi [1]. Hopfions with higher charge — including twisted tori, linked loops, and knots — were subsequently constructed by means of full 3D energy minimization [10–14].

By means of the Cho-Faddeev-Niemi-Shabanov decomposition, Faddeev and Niemi showed in detail that the SFN model (1) can be derived as an effective theory that describes the confinement phase of the SU(2) pure Yang-Mills theory [2]. From this point of view, Hopfions are considered as natural candidates for glueballs that can be interpreted as closed fluxtubes. This model is sometimes referred to as the $\mathbb{C}P^1$ SFN

model, because it is based on a formula for the Lagrangian that can be described in terms of a complex scalar field via the stereographic projection $S^2 \to CP^1$; i.e.,

$$\vec{n} = \frac{1}{\Delta} (u + u^*, -i(u - u^*), |u|^2 - 1)$$
 (2)

where u is a complex scalar field and $\Delta=1+|u|^2$. For a finite-energy configuration, the field \vec{n} has to approach a constant vector at spatial infinity. This makes the points at infinity identical, and the space \mathbb{R}^3 is compactified to S^3 . The field \vec{n} defines a mapping $S^3 \to S^2$, and the field configurations are characterized by an integer, called a Hopf invariant, that corresponds to an element of $\pi_3(S^2)=\mathbb{Z}$. Since this invariant is nonlocal, an integral form of the invariant cannot be written in terms of \vec{n} or u; in order to define it we need introduce the complex vector $\vec{Z}=(Z_0,Z_1)^T$, with $|\vec{Z}|^2=1$, which satisfies $u\equiv \mathcal{Z}_1/\mathcal{Z}_0$. Then, the Hopf invariant can be defined as

$$H_{CP^1} = \frac{1}{4\pi^2} \int \mathcal{A} \wedge d\mathcal{A}, \quad \mathcal{A} = i\vec{\mathcal{Z}}^{\dagger} d\vec{\mathcal{Z}}.$$
 (3)

In this paper, we construct Hopfions in a generalization of the SFN model for the case of SU(3), the gauge group of QCD. For SU(N+1), where $N \geq 2$, there are several possibilities for the field decomposition associated with dynamical symmetry-breaking patterns. Most studies of field decomposition are based on the following two options: the maximal case $SU(N+1) \rightarrow U(1)^N$ [3] and the minimal case $SU(3) \rightarrow U(2)$ [15, 16]. Depending upon the options chosen, SFN-type models have been proposed on both the relevant target spaces, $F_N = SU(N+1)/U(1)^N$ and $CP^N = SU(N+1)/U(N)$, in [3] and [17], respectively. Note that $CP^1 = F_1 = SU(2)/U(1)$ is equivalent to S^2 , the target space of the standard SFN model. However, note also that the \mathbb{CP}^N $(N \geq 2)$ model cannot possess knot solitons as a static stable solution in three-dimensional space, because the corresponding homotopy group is trivial; i.e., $\pi_3(CP^N) = 0$ for $N \geq 2$. In 2(N+1)-dimensional space-time, the existence of Hopfions associated with $\pi_{2N+1}\left(CP^{N}\right)=\mathbb{Z}$ is discussed in [18]. In addition, if N is odd, then 3D, time-dependent, non-topological solitons — called Q-balls and Q-shells — are obtained in a CP^N model with a V-shaped potential [19].

Contrary to the case of CP^2 , the third homotopy group of the flag manifold is nontrivial; i.e., $\pi_3(F_2) = \mathbb{Z}$. Thus, we expect Hopfions to exist in the F_2 SFN model, which is composed of the F_2 nonlinear σ -model with quadratic terms in the derivatives. The main purpose of the present paper is to confirm the existence of the F_2 Hopfions and understand their detailed structures. It has recently been found

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that the 2-dimensional F_2 nonlinear σ -model possesses vortexlike solutions (2D instantons), both of the embedding type [20] and of the genuine (non-embedding) type [21, 22]. The Hopfions considered in this paper are probably the vortices with knot structures. In [21], Bykov introduced the so-called Kalb-Ramond field and found that the model is integrable for specific coefficients of the field [23, 24]. Though the Kalb-Ramond field appears naturally for some continuum limits of the SU(3) antiferromagnetic spin chain, for the moment we do not consider this field. The reason is that, if one derives F_2 nonlinear σ -models from other fundamental theories, it is unclear whether the field can appear naturally. The genuine solutions are constructed by using the CP^2 Din-Zakrzewski tower generated by the Bäcklund transformation [25] which implies that the solution is a composite of solitons and antisolitons.

The F_2 nonlinear σ -model has been derived from the SU(3) antiferromagnetic Heisenberg model as an effective model on a square lattice [26], a triangular lattice [27], and a 1D chain [28]. It has also been proposed to describe some phenomena of a quantum spin-nematic system [27] and of a color superconductor in high-density quark matter [29]. The present work can thus be applied to several areas of condensed matter physics as well as to QCD.

This paper is organized as follows. In Sec.II we introduce the model and some important quantities, together with a particularly nice parametrization that makes the computations transparent. In Sec.III, we derive the formal Euler-Lagrange equation, which we solve with some ansatz. We give a brief analysis of some quantum aspects of the solutions in Sec.IV, and we conclude with Sec.V.

II. THE MODEL

A. The static energy, topological charge and torsion

The fundamental degrees of freedom of F_2 nonlinear σ -models are given by su(3)-valued fields, called "color-direction fields" in the context of QCD. They are defined by

$$\mathfrak{n}_a = U h_a U^{\dagger}, \qquad a = 1, 2 \tag{4}$$

where U is an element of SU(3), and the matrices h_a are the Cartan generators in su(3). The F_2 SFN model is defined by the following Lagrangian density in (3+1)-dimensional Minkowski space-time [3]:

$$\mathcal{L} = \sum_{a=1}^{2} \left\{ M^{2} \left\langle \partial_{\mu} \mathfrak{n}_{a}, \partial^{\mu} \mathfrak{n}_{a} \right\rangle - \frac{1}{e^{2}} F^{a}_{\mu\nu} F^{a\mu\nu} \right\}$$
 (5)

where the angle brackets denote the inner product on su(3); i.e. $\langle A, B \rangle = \text{Tr}\left(A^{\dagger}B\right)$ for $A, B \in su(3)$. The second-rank tensors are defined as

$$F_{\mu\nu}^{a} = -\frac{i}{2} \sum_{b=1}^{2} \langle \mathfrak{n}_{a}, [\partial_{\mu} \mathfrak{n}_{b}, \partial_{\nu} \mathfrak{n}_{b}] \rangle$$
 (6)

and the 2-forms $F^a = \frac{1}{2} F^a_{\mu\nu} dx^\mu \wedge dx^\nu$ are called the Kirillov-Kostant (KK) symplectic forms. The Lagrangian (5) is invariant under the left global SU(3) transformation $U \to gU, g \in SU(3)$, and the local $U(1)^2$ transformation $U \to Uk, k \in$

 $U(1)^2$. From these symmetries, one can understand that the target space of this model is the coset space $SU(3)/U(1)^2$, which is equivalent to the flag manifold F_2 .

The static energy functional associated with (5) is given by

$$E = \int d^3x \sum_{a=1}^{2} \left\{ \langle \partial_i \mathfrak{n}_a, \partial_i \mathfrak{n}_a \rangle + F_{ij}^a F_{ij}^a \right\}. \tag{7}$$

where for simplicity we use the length unit $(Me)^{-1}$ and the energy unit 4M/e. Since the energy consists of both quadratic and quartic terms, three dimensional particle-like configurations evidently evade Derrick's no-go theorem.

We reformulate the energy functional (7) into a more tractable form that is given solely in terms of the off-diagonal components of the Maurer-Cartan form $U^{\dagger}\partial_{\mu}U$. We decompose the Maurer-Cartan form in terms of the SU(3) Cartan-Weyl basis as

$$U^{\dagger} \partial_{\mu} U = i A^a_{\mu} h_a + i J^p_{\mu} e_p \tag{8}$$

where we use a basis of the form

$$\begin{split} h_1 &= \frac{1}{\sqrt{2}} \lambda_3, \quad h_2 = \frac{1}{\sqrt{2}} \lambda_8, \\ e_{\pm 1} &= \frac{1}{2} \left(\lambda_1 \pm i \lambda_2 \right), \ e_{\pm 2} = \frac{1}{2} \left(\lambda_4 \mp i \lambda_5 \right), \ e_{\pm 3} = \frac{1}{2} \left(\lambda_6 \pm i \lambda_7 \right). \end{split}$$

Since the basis set is orthonormal, the currents can be written as

$$A^a_\mu = -i \left\langle h_a, U^\dagger \partial_\mu U \right\rangle, \quad J^p_\mu = -i \left\langle e_p, U^\dagger \partial_\mu U \right\rangle.$$
 (9)

Note that A^a_{μ} are real and $J^{-p}_{\mu} = (J^p_{\mu})^*$. Under the gauge transformation $U \to Uk$, with $k = \exp(i\theta^a h_a)$, A^a_{μ} transforms as a gauge field and J^p_{μ} as a charged particle; i.e.,

$$A^a_\mu \to A^a_\mu + \partial_\mu \theta^a, \qquad J^p_\mu \to J^p_\mu e^{-i\theta^a \alpha^p_a}$$
 (10)

where α_a^p is the a-th component of the root vector corresponding to e_p . Now the root vectors are given by

$$\alpha^1 = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}, \ \alpha^2 = \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \ \alpha^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$
 (11)

with
$$\alpha^{-p} = -\alpha^p$$
 for $p = 1, 2, 3$.

For the nonlinear σ -model, the quadratic term in (5), can be written using only the off-diagonal components of J^p_μ . In addition, one can write the KK forms as $F^a = \mathrm{d}A^a = -i\sum_p \alpha^p_a J^p \wedge J^{-p}$ where $A^a = A^a_\mu \ \mathrm{d}x^\mu$, and $J^p = J^p_\mu \ \mathrm{d}x^\mu$. Thus the static energy can be written as

$$E = \int d^3x \sum_{q=1}^{3} \left[J_i^q J_i^{-q} - \frac{1}{4} \left(J_{[i}^q J_{j]}^{-q} - J_{[i}^{q+1} J_{j]}^{-(q+1)} \right)^2 \right]$$
(12)

where q is a mod 3 number; i.e. $q \equiv q+3 \pmod{3}$. Note that $J_{[i}^p J_{j]}^{-p} \equiv J_i^p J_j^{-p} - J_j^p J_i^{-p}$ is purely imaginary, and therefore the energy functional is positive definite. It is worth noting that, similar to the CP^1 case [30], the energy functional (12) can be interpreted as a gauge-fixing functional for a nonlinear maximal Abelian gauge, without making the Abelian subgroup components fixed.

To ensure the finiteness of the energy functional the fields \mathfrak{n}_a must approach constant matrices at spatial infinity, so that the space \mathbb{R}^3 is topologically compactified to S^3 , and the fields \mathfrak{n}_a define the map $S^3 \to F_2 = SU(3)/U(1)^2$. Consequently,

the finite energy configurations can be characterized by elements of the homotopy group $\pi_3\left(SU(3)/U(1)^2\right)=\mathbb{Z}$. The corresponding topological charge, the Hopf invariant, is given by

$$H_{F_2} = \frac{1}{8\pi^2} \int d^3x \left\{ \varepsilon^{ijk} \sum_{a=1}^2 A_i^a F_{jk}^a \right\} - \Gamma$$
 (13)

where

$$\Gamma = \frac{-i}{8\pi^2} \int d^3x \ \varepsilon^{ijk} \left\{ J_i^1 J_j^2 J_k^3 - J_i^{-1} J_j^{-2} J_k^{-3} \right\} \ . \tag{14}$$

The Hopf invariant (13) is nonlocal, since A_{μ}^{a} cannot be written in terms of the fields \mathfrak{n}_{a} , and therefore (13) does not possess local $U(1)^{2}$ symmetry. Note that both the Abelian Chern-Simons (CS) terms and Γ are not topological. The Hopf invariant can be constructed by means of Novikov's procedure [31] via the isomorphism between $\pi_{3}\left(SU(3)/U(1)^{2}\right)$ and $\pi_{3}\left(SU(3)\right)$, which indicates $H_{F_{2}}=Q[U]$, where $Q[U]=\frac{1}{24\pi^{2}}\int \mathrm{Tr}\left(U^{\dagger}\mathrm{d}U\right)^{3}$ is the winding number of the map $U:S^{3}\to SU(3)$.

The winding number is equivalent to the CS term for the SU(3) flat connection $U^{\dagger}dU$. Therefore, similar to the CP^1 Hopf invariant (3) discussed in [30, 32], the F_2 Hopf invariant (13) can also be given by the non-Abelian CS term with the SU(3) flat connection.

The Kähler form can be defined as

$$\lambda = \frac{i}{2\pi} \sum_{p=1}^{3} B_p J^p \wedge J^{-p} \tag{15}$$

where the coefficients B_p are real constants [23]. For non-symmetric manifolds, like the flag space F_2 , the Kähler form λ in general is not closed, i.e., $d\lambda \neq 0$. The so-called skew torsion $T = d\lambda$ is given by the form

$$T = \frac{1}{2\pi} \sum_{p} B_{p} \left(J^{1} \wedge J^{2} \wedge J^{3} + J^{-1} \wedge J^{-2} \wedge J^{-3} \right) . \quad (16)$$

Under the local $U(1)^2$ transformation, both the Kähler form and the torsion are invariant. Note that in the 2-dimensional F_2 nonlinear σ -model, the solutions of the Euler-Lagrange equation make the Kähler form closed, and the torsion then disappears identically. By analogy, in this paper we consider a class of configurations that satisfies the torsion-free condition T=0.

B. Parametrization

In order to make the analysis transparent, let us parametrize the SU(3) matrix U in terms of complex scalar fields which are equivalent to the local coordinates of the target space F_2 . The coordinates can be introduced naturally via the inverse of a generalization of stereographic projection; i.e., by the mapping $SL(3,\mathbb{C})/B_+ \to SU(3)/U(1)^2$, where B_+ is the Borel subgroup of upper triangular matrices (see, e.g., [33]). However, we need two additional degrees of freedom in order to describe the Hopf invariant, because it is nonlocal and requires $8 = \dim SU(3)$ degrees of freedom rather than $6 = \dim F_2$. Therefore, we begin the parametrization not with a 3×3 lower triangular matrix in which all the diagonal

components are unity which is an element of $SL(3,\mathbb{C})/B_+$ but instead with a lower triangular matrix in $SL(3,\mathbb{C})$ of the form

$$X = \begin{pmatrix} \chi_1 & 0 & 0 \\ \chi_2 & \chi_4 & 0 \\ \chi_3 & \chi_5 & (\chi_1 \chi_4)^{-1} \end{pmatrix} \in SL(3, \mathbb{C})$$
 (17)

where the χ_i are complex functions, with χ_1 and χ_4 being finite. Note that the matrix (17) has ten degrees of freedom.

The parametrization can then be obtained by using the Gram-Schmidt orthogonalization process. We write X in terms of column vectors as $X = (\vec{c}_1, \vec{c}_2, \vec{c}_3)$, and we introduce the mutually orthogonal vectors \vec{v}_j :

$$\vec{v}_{1} = \vec{c}_{1} ,
\vec{v}_{2} = \vec{c}_{2} - \frac{(\vec{c}_{2}, \vec{v}_{1})}{(\vec{v}_{1}, \vec{v}_{1})} \vec{v}_{1} ,
\vec{v}_{3} = \vec{c}_{3} - \frac{(\vec{c}_{3}, \vec{v}_{2})}{(\vec{v}_{2}, \vec{v}_{2})} \vec{v}_{2} - \frac{(\vec{c}_{3}, \vec{v}_{1})}{(\vec{v}_{1}, \vec{v}_{1})} \vec{v}_{1} .$$
(18)

Normalization of the vectors \vec{v}_j is achieved under the two conditions

$$|\chi_1|^2 + |\chi_2|^2 + |\chi_3|^2 = 1, |\chi_1|^2 (|\chi_4|^2 + |\chi_5|^2) + |\chi_3\chi_4 - \chi_2\chi_5|^2 = 1.$$
 (19)

Then we write

$$U = (\vec{v}_1, \vec{v}_2, \vec{v}_3). \tag{20}$$

This is a unitary matrix with eight degrees of freedom, because the vectors \vec{v}_j form a complete basis set, and they are described by the five complex scalars χ_i with the two constraints (19). Finally, we parametrize the \vec{v}_i in terms of three complex scalar fields, which correspond to the local coordinates of the flag manifold. We introduce them as $(u_1, u_2, u_3) = (\chi_2/\chi_1, \chi_3/\chi_1, \chi_5/\chi_4)$, where we also write $\arg(\chi_\alpha) = \vartheta_\alpha$ for $\alpha = 1, 4$. Then the SU(3) matrix (20) can be written as $U = \left(Z_{\rm A}e^{i\vartheta_1}, Z_{\rm B}e^{i\vartheta_4}, Z_{\rm C}e^{-i(\vartheta_1+\vartheta_4)}\right)$ where

$$Z_{A} = \frac{1}{\sqrt{\Delta_{1}}} \begin{pmatrix} 1\\ u_{1}\\ u_{2} \end{pmatrix},$$

$$Z_{B} = \frac{1}{\sqrt{\Delta_{1}\Delta_{2}}} \begin{pmatrix} -u_{1}^{*} - u_{2}^{*}u_{3}\\ 1 - u_{1}u_{2}^{*}u_{3} + |u_{2}|^{2}\\ -u_{1}^{*}u_{2} + u_{3} + u_{3}|u_{1}|^{2} \end{pmatrix}, \qquad (21)$$

$$Z_{C} = \frac{1}{\sqrt{\Delta_{2}}} \begin{pmatrix} u_{1}^{*}u_{3}^{*} - u_{2}^{*}\\ -u_{3}^{*}\\ 1 \end{pmatrix}$$

with

$$\Delta_1 = 1 + |u_1|^2 + |u_2|^2,$$

$$\Delta_2 = 1 + |u_3|^2 + |u_1u_3 - u_2|^2.$$
(22)

The unitarity of U is guaranteed by the orthonormalization condition and the completeness relation between the complex vectors:

$$Z_a^{\dagger} Z_b = \delta_{ab} , \qquad (23)$$

$$Z_{\rm A} \otimes Z_{\rm A}^{\dagger} + Z_{\rm B} \otimes Z_{\rm B}^{\dagger} + Z_{\rm C} \otimes Z_{\rm C}^{\dagger} = \mathbf{1}_3.$$
 (24)

These identities are helpful for all the computations in this study. Note that the triplet $\{Z_{\rm A}, Z_{\rm B}, Z_{\rm C}\}$ plays the role of an

order parameter if the model is viewed as an effective theory of the SU(3) antiferromagnetic Heisenberg model [20, 27].

As mentioned earlier, we require eight degrees of freedom to describe the Hopf invariant. However, the two degrees of freedom corresponding to ϑ_1 and ϑ_4 are canceled out in the energy, the Euler-Lagrange equation and so on, although these variables make the calculations much more complicated. Therefore it is useful to introduce the matrix W without the phase factors; i.e.,

$$W = (Z_{\rm A}, Z_{\rm B}, Z_{\rm C}).$$
 (25)

This can also be written as $W=U\exp\left[i\Theta^ah_a\right]$ with $\Theta^1=-\frac{\vartheta_1-\vartheta_4}{\sqrt{2}}$ and $\Theta^2=-\frac{\sqrt{3}(\vartheta_1+\vartheta_4)}{\sqrt{2}}$. By virtue of the local symmetry, the Lagrangian satisfies $\mathcal{L}\left[\mathfrak{n}_a\right]=\mathcal{L}\left[\mathfrak{m}_a\right]$ where $\mathfrak{m}_a=Wh_aW^\dagger$. In addition, the static energy can be written in terms of the off-diagonal components of the Murer-Cartan form $W^\dagger\partial_\mu W$. We again decompose it as

$$W^{\dagger} \partial_{\mu} W = i C_{\mu}^{a} h_{a} + i K_{\mu}^{p} e_{p} \tag{26}$$

and (10) then yields the relations

$$C_{\mu}^{a} = A_{\mu}^{a} + \partial_{\mu}\Theta^{a}, \quad K_{\mu}^{p} = J_{\mu}^{p}e^{-i\alpha_{a}^{p}\Theta^{a}}.$$
 (27)

Since $J_{\mu}^{-p}=(J_{\mu}^{p})^{*}$, we can replace J_{i}^{p} in the energy (12) with K_{i}^{p} : The static energy thus can also be written as

$$E = \int d^3x \sum_{q=1}^{3} \left[K_i^q K_i^{-q} - \frac{1}{4} \left(K_{[i}^q K_{j]}^{-q} - K_{[i}^{q+1} K_{j]}^{-(q+1)} \right)^2 \right].$$
(28)

Also, the KK form, Kähler form and skew torsion can be written as

$$F^{a} = -2i\sum_{p=1}^{3} \alpha_{a}^{p} K^{p} \wedge K^{-p}, \tag{29}$$

$$\lambda = \frac{i}{2\pi} \sum_{p=1}^{3} B_p K^p \wedge K^{-p},\tag{30}$$

$$T = \frac{1}{2\pi} \sum_{p=1}^{3} B_p \left(K^1 \wedge K^2 \wedge K^3 + K^{-1} \wedge K^{-2} \wedge K^{-3} \right) . \tag{31}$$

Hereinafter we use W rather than U, except for the Hopf invariant.

III. EQUATION OF MOTION AND HOPFIONS

First we derive the formal Euler-Lagrange equation, and we then consider two classes of configurations that satisfy the torsion-free condition T=0. The Euler-Lagrange equation is equivalent to the conservation law for the Noether current \mathcal{J}_{μ} associated with the global SU(3) transformation; i.e., $\partial_{\mu}\mathcal{J}^{\mu}=0$. The current takes the form

$$\mathcal{J}_{\mu} = \sum_{a=1}^{2} \left(\left[\mathfrak{m}_{a}, \partial_{\mu} \mathfrak{m}_{a} \right] - i \sum_{b=1}^{2} F_{\mu\nu}^{a} \left[\mathfrak{m}_{a}, \left[\mathfrak{m}_{b}, \partial^{\nu} \mathfrak{m}_{b} \right] \right] \right). \tag{32}$$

where $\mathfrak{m}_a = W h_a W^{\dagger}$. If we factorize the current as $\mathcal{J}_{\mu} = W \mathcal{B}_{\mu} W^{\dagger}$, the equations of motion can be written as

$$\partial_{\mu}\mathcal{B}^{\mu} + \left[W^{\dagger} \partial_{\mu} W, \mathcal{B}^{\mu} \right] = 0. \tag{33}$$

The current \mathcal{B}_{μ} consists of just the off-diagonal components of the Murer-Cartan form:

$$\mathcal{B}_{\mu} = i \sum_{p} \left(K_{\mu}^{p} - i \sum_{a=1}^{2} \alpha_{a}^{p} F_{\mu\nu}^{a} K^{p\nu} \right) e_{p} . \tag{34}$$

To simplify the notation, we introduce $R^p_{\mu} = \sum_a \alpha^p_a C^a_{\mu}$ and $G^p_{\mu\nu} = \sum_a \alpha^p_a F^a_{\mu\nu}$. Then, equation (33) can be written explicitly as

$$\partial^{\mu} \left(K_{\mu}^{q} - i G_{\mu\nu}^{q} K^{q\nu} \right) + i R^{q\mu} \left(K_{\mu}^{q} - i G_{\mu\nu}^{q} K^{q\nu} \right) + G^{q\mu\nu} K_{\mu}^{-q-1} K_{\nu}^{-q+1} = 0$$
(35)

for $\forall q \equiv 1, 2, 3 \pmod{3}$ and their complex conjugations.

The normal route to confirm the existence of knot solitons consists of solving equation (35) with some symmetric ansatz for the complex scalar fields u_i . However, since (35) is highly nonlinear and very complicated, this seems quite a hard task. Here, we instead employ a different strategy. That is, we first introduce configurations that satisfy the torsion-free condition T=0 and then, as we shall see, the Euler-Lagrange equation (35) simplify into a solvable form.

A. Trivial CP^1 reduction

The first class we consider is a trivially embedded configuration; i.e., an $F_1 = CP^1$ Hopfion into F_2 space. It can be obtained by requiring two of the three scalar fields to be trivial. Without loss of generality, we here set $u_1 = u_3 = 0$ and write $u_2 = u$. Then, the complex vectors Z_a can be written in terms of the function u(x) as

$$Z_{\rm A} = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} 1\\0\\u \end{pmatrix}, \ Z_{\rm B} = \begin{pmatrix} 0\\1\\0 \end{pmatrix}, \ Z_{\rm C} = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} -u^*\\0\\1 \end{pmatrix}$$
(36)

where $\Delta = 1 + |u|^2$. The currents K_{μ}^p are given by

$$K_{\mu}^{1} = K_{\mu}^{3} = 0, \qquad K_{\mu}^{2} = \frac{i}{\Lambda} \partial_{\mu} u$$
 (37)

and the skew torsion T vanishes. It can be checked directly that the equations of motion (35) for $q \equiv 1$ and 3 are automatically satisfied and that for $q \equiv 2$ reduces to

$$\partial^{\mu} \left[\partial_{\mu} u - i G_{\mu\nu} \partial^{\nu} u \right] + \left(i R_{\mu} - \partial_{\mu} \log \Delta \right) \left(\partial^{\mu} u - i G^{\mu\nu} \partial_{\nu} u \right) = 0$$
(38)

where for convenience we have introduced R_{μ} and $G_{\mu\nu}$, which take the forms

$$R_{\mu} \equiv \frac{i}{\Lambda} \left(u^* \partial_{\mu} u - u \partial_{\mu} u^* \right) \,, \tag{39}$$

$$G_{\mu\nu} \equiv -\frac{2i}{\Delta^2} \left(\partial_{\mu} u \partial_{\nu} u^* - \partial_{\mu} u^* \partial_{\nu} u \right) . \tag{40}$$

The static energy for the configuration (36) is given by

$$E_{\rm tri} = \int d^3x \left(\frac{\partial_i u \partial_i u^*}{\Delta^2} - \frac{\left(\partial_i u \partial_j u^* - \partial_i u^* \partial_j u\right)^2}{2\Delta^4} \right). \tag{41}$$

Both the equation of motion (38) and the energy (41) are exactly the same as those of the CP^1 SFN model (1) with (2). Next we determine the Hopf invariant. First we plug

the configuration (36) into the SU(3) matrix U and define $\mathcal{Z}_0 = e^{i\vartheta_1}/\sqrt{\Delta}$, $\mathcal{Z}_1 = ue^{i\vartheta_1}/\sqrt{\Delta}$ with $\vartheta_4 = 0$. Then we obtain

$$A^{1} = -\frac{1}{\sqrt{2}}\mathcal{A}, \quad A^{2} = \sqrt{\frac{3}{2}}\mathcal{A},$$
 (42)

where

$$\mathcal{A} = i\vec{\mathcal{Z}}^{\dagger} d\vec{\mathcal{Z}}, \quad \vec{\mathcal{Z}} = \begin{pmatrix} \mathcal{Z}_0 \\ \mathcal{Z}_1 \end{pmatrix}.$$
 (43)

Therefore we find that the F_2 Hopf invariant (13) of the embedding configuration coincides with the \mathbb{CP}^1 version (3), i.e.,

$$H_{\rm tri} = \frac{1}{4\pi^2} \int \mathcal{A} \wedge d\mathcal{A}. \tag{44}$$

This coincidence is obviously due to the fact that (36) is just a trivial embedding configuration. Next we examine another class of configuration which has a more nontrivial nature, and see what happens to the Euler-Lagrange equation, the energy, and the Hopf invariant.

B. Nontrivial CP^1 reduction

For the trivial embedding (36), we observed that two pairs of the currents K_{μ}^{p} vanished; i.e., $K_{\mu}^{\pm 1}=K_{\mu}^{\pm 3}=0$. Here we relax these conditions and examine the case where just one pair of the currents vanishes; i.e., $K_{\mu}^{\pm 2}=0$, while both $K_{\mu}^{\pm 1}$ and $K_{\mu}^{\pm 3}$ remain finite. This automatically satisfies the torsion-free condition. Note that the result is independent of the choice of the components; for a different pair, one just repeats the same prescription by permuting the vectors Z_a . The condition $K_{\mu}^{\pm 2}=0$ reads

$$u_3 \partial_{\mu} u_1 - \partial_{\mu} u_2 = 0, \qquad \mu = 0, 1, 2, 3.$$
 (45)

This is satisfied if u_2 is a function of u_1 — i.e., $u_2 = f(u_1)$ — and u_3 is given by $u_3 = f'(u_1)$ where the prime stands for the derivative with respect to u_1 . This means that the only independent field is u_1 , so that the Euler-Lagrange equation seems to be an overdetermined system. In order to reduce the number of independent equations, we consider the case where the Euler-Lagrange equations for $q \equiv 1$ and 3 are proportional to each other. This is the case when the ratio Δ_1/Δ_2 is a constant. Note that we leave the equation for $q \equiv \pm 2$ intact because $q \equiv 2$ is now special due to the constraint $K_\mu^2 = 0$. By comparing the order of u_1 in Δ_i 's, one finds that this condition requires

$$|u_1|^2 = |f'(u_1)|^2, \qquad |f|^2 = |u_1f' - f|^2.$$
 (46)

Since we are not interested in embedding solutions here, we omit the case where u_1 is a constant and obtain

$$f(u) = \frac{1}{2}u^2 e^{i\varphi} \tag{47}$$

where $\varphi \in [0, 2\pi]$ is a constant. Note that due to U(1) symmetries, the constant φ can take an arbitrary value. For simplicity, we choose $\varphi = \pi$ and write $u_1 = \sqrt{2}u$. Then, the triplet vectors become

$$Z_{\rm A} = \frac{1}{\Delta} \left(1, \sqrt{2}u, -u^2 \right)^T, \ Z_{\rm C} = \frac{1}{\Delta} \left(-u^{*2}, \sqrt{2}u^*, 1 \right)^T,$$
$$Z_{\rm B} = \frac{1}{\Delta} \left(-\sqrt{2}u^*, 1 - |u|^2, -\sqrt{2}u \right)^T. \tag{48}$$

It is worth noting that the three vectors are linked by the Bäcklund transformation, i.e.,

$$Z_{\rm B} = \frac{P_{+}Z_{\rm A}}{|P_{+}Z_{\rm A}|}, \qquad Z_{\rm C} = \frac{P_{+}Z_{\rm B}}{|P_{+}Z_{\rm B}|}$$
 (49)

where $P_+Z_a = \partial_u Z_a - \left(Z_a^{\dagger}\partial_u Z_a\right)Z_a$. Similar relations among the triplet vectors are observed for the non-embedding solutions of the two-dimensional F_2 nonlinear σ -model [21, 22]. The currents K_{μ}^p are given by the form

$$K_{\mu}^{1} = \frac{\sqrt{2}i}{\Delta}\partial_{\mu}u^{*}, \quad K_{\mu}^{2} = 0, \quad K_{\mu}^{3} = -\frac{\sqrt{2}i}{\Delta}\partial_{\mu}u^{*}.$$
 (50)

With the forms (50), the Euler-Lagrange equation (35) for $q \equiv 2$ is automatically satisfied. In addition, we obtain $R_{\mu}^1 = R_{\mu}^3 = -R_{\mu}$ and $G_{\mu\nu}^1 = G_{\mu\nu}^3 = -G_{\mu\nu}$, and one then observes easily that equations (35) for both $q \equiv 1$ and 3 reduce to the complex conjugates of (38). To see this, one can use the fact that R_{μ} and $G_{\mu\nu}$ are real. This yields a somewhat surprising observation: These results clearly mean that it is not only in the trivial embedding case but also in the non-embedding case that all the known Hopfion solutions u in the CP^1 SFN model solve the Euler-Lagrange equation.

Though the Euler-Lagrange equations in both classes are solved by the same function, they are clearly inequivalent. The configuration (48) possesses a static energy that is exactly four times greater than (36); i.e.,

$$E_{\text{nontri}}[u] = 4E_{\text{tri}}[u]. \tag{51}$$

To evaluate the relevant Hopf number, we write $u=\mathcal{Z}_1/\mathcal{Z}_0,\ e^{i\vartheta_1}=\mathcal{Z}_0^2/|\mathcal{Z}_0|^2,$ and $\vartheta_4=0.$ This yields $A^1=-\sqrt{2}\mathcal{A},\ A^2=\sqrt{6}\mathcal{A},$ and therefore

$$H_{\text{nontri}} = \frac{1}{\pi^2} \int \mathcal{A} \wedge d\mathcal{A} = 4H_{\text{tri}}.$$
 (52)

It is worth noting that the F_2 Hopf invariant is equivalent to a non-Abelian CS term with an SU(3) flat connection, but that of the solutions is given by the sum of just the Abelian CS terms, because the configuration (36) satisfies $\Gamma = 0$.

According to (51) and (52), the F_2 nontrivial Hopfion with $H_{\text{nontri}} = 4n$ (where n is an integer) can be viewed as a molecular state of four embedding solutions with the Hopf number H = n which are sitting on top of each other with no binding energy. Such a situation has been observed in an SU(N) Skyrme model [34]. Since our solutions of equation (38) are not of the BPS type, it is probably impossible to remove one of them from the others without changing the energy. However, this should be confirmed by studying the moduli parameters of the solutions. This will be reported in subsequent papers.

IV. ISO-SPINNING HOPFIONS

We have seen that the Euler-Lagrange equations are solved by the same function u both for the trivial embedding and for the non-embedding ansatz. Their energies and Hopf invariants are respectively proportional, as shown in (51) and (52). In the previous sections, we saw that they are inequivalent, although they look similar. Here we shall show that their quantum natures are quite different. In this section, we give a brief analysis to demonstrate the notable differences in some quantum aspects based on the collective coordinate quantization of the zero modes. We consider an adiabatic iso-rotation associated with the SU(3) global symmetry, i.e., with the time-dependent transformation $\mathfrak{m}_a(\vec{x}) \to \mathfrak{m}_a(t, \vec{x}) = \beta(t)\mathfrak{m}_a(\vec{x})\beta^{\dagger}(t)$, where $\beta(t) \in SU(3)$. Then the Lagrangian can be written as

$$L = -E_{cl} + r_0^2 \int d^3x \left[\text{Tr} \left(\left[\beta^{\dagger} \dot{\beta}, \mathfrak{m}_a \right] \left[\beta^{\dagger} \dot{\beta}, \mathfrak{m}_a \right] \right) + 2F_{0i}^a F_{0i}^a \right]$$
 (53)

where $E_{\rm cl}$ is the static energy of the Hopfion, the dot denotes the time derivative — i.e., $\dot{\beta} = d\beta/dt$ — and

$$F_{0j}^{a} = -\frac{i}{2} \operatorname{Tr} \left(\mathfrak{m}_{a} \left[\left[\beta^{\dagger} \dot{\beta}, \mathfrak{m}_{b} \right], \partial_{j} \mathfrak{m}_{b} \right] \right). \tag{54}$$

The energy collection depends on the length scale $r_0 = (Me)^{-1}$.

In order for the integral in (53) to be finite, $\beta^{\dagger}\dot{\beta}$ and \mathfrak{m}_a must commute with each other at spatial infinity. Since the fields $\mathfrak{m}_a(\vec{x})$ approach constant elements of $u(1) \times u(1)$ as x goes to infinity, $\beta^{\dagger}\dot{\beta}$ must also be in $u(1) \times u(1)$ and therefore can be written as

$$\beta^{\dagger}\dot{\beta} = \sqrt{2}i\left(\frac{\omega_1}{2}h_1 + \frac{\omega_2}{\sqrt{3}}h_2\right) \tag{55}$$

where ω_a denotes the angular velocity in iso-space. We chose the coefficients in (55) to be consistent with the definition of the SU(3) Euler angle [35].

The quantum Lagrangian (53) can be written as a quadratic form of the angular velocities,

$$L = -E_{\rm cl} + \frac{1}{2} \boldsymbol{\omega}^T \mathcal{I} \boldsymbol{\omega} \tag{56}$$

where $\boldsymbol{\omega}^T = (\omega_1, \omega_2)$ and

$$\mathcal{I} = \begin{pmatrix} I_{11} & I_{12} \\ I_{12} & I_{22} \end{pmatrix}. \tag{57}$$

The moments of inertia are explicitly obtained as follows:

• For the non-trivial reduction case,

$$I_{11} = 2r_0^2 \int d^3x \, \frac{1}{\Delta^4} \left[\left(10 - 7|u|^2 + 10|u|^4 \right) |u|^2 + 4 \left(7 - 13|u|^2 + 7|u|^4 \right) \left(\partial_i \log \Delta \right)^2 \right]$$

$$I_{12} = -4r_0^2 \int d^3x \, \frac{1}{\Delta^4} \left(3 - \Delta \right) \left(3 - 2\Delta \right) \times \left[|u|^2 + 4 \left(\partial_i \log \Delta \right)^2 \right]$$

$$I_{22} = 8r_0^2 \int d^3x \, \frac{1}{\Delta^4} \left[\left(2 + |u|^2 + 2|u|^4 \right) |u|^2 + 4 \left(1 - |u|^2 + |u|^4 \right) \left(\partial_i \log \Delta \right)^2 \right].$$
(58)

• For the trivial reduction case,

$$I_{11} = \frac{I_{12}}{2} = \frac{I_{22}}{4} = 2r_0^2 \int d^3x \ \frac{|u|^2 + (\partial_i \log \Delta)^2}{\Delta^2} \,.$$
 (59)

Using a Legendre transformation of the Lagrangian (53), we obtain the Hamiltonian $\mathcal{H} = \omega_i P_i - L$ with the canonical momentum defined by

$$P_i \equiv \frac{\partial L}{\partial \omega_i} = I_{ij}\omega_j, \qquad i, j = 1, 2.$$
 (60)

In the nontrivial reduction case, the Hamiltonian is obtained straightforwardly as

$$\mathcal{H} = E_{\rm cl} + \frac{1}{2} \frac{1}{\rm Det} \mathcal{I} \left\{ I_{22} P_1^2 - 2I_{12} P_1 P_2 + I_{11} P_2^2 \right\}$$
 (61)

where we have used the commutation relation $[P_1, P_2] = 0$, because the operators are associated the Abelian subgroup of SU(3). Since the operators are already diagonalized, the Hopfions can be assigned two quantum numbers associated with the two zero-modes when the Hamiltonian operates a relevant wave function. On the other hand, in the embedding case, we are allowed to define only one operator because the SU(3) matrix W satisfies the commutation relation $\left[W, h_1 - \frac{1}{\sqrt{3}}h_2\right] = 0$ and therefore

$$\left[\dot{\beta}\beta,\mathfrak{m}_{a}\right] = \frac{i\left(\omega_{1} + 2\omega_{2}\right)}{\sqrt{2}}\left[h_{1},\mathfrak{m}_{a}\right]. \tag{62}$$

This implies that the embedding Hopfions can rotate around only one axis in isospace. Actually, we can obtain from (60) only one operator, which has the form

$$P_1 = \frac{P_2}{2} = I(\omega_1 + 2\omega_2) \equiv P$$
 (63)

where we have written $I_{11} = I$. Therefore the Hamiltonian becomes

$$\mathcal{H} = E_{\rm cl} + \frac{P^2}{2I} \,. \tag{64}$$

Consequently, Hopfions of the embedding type inherit the quantum properties of the \mathbb{CP}^1 Hopfions; they can possess at most one quantum number after (64) acts on a proper wave function. The quantum properties of the two types of Hopfion solutions seem quite different, at least qualitatively, which is a reflection of their different symmetries.

V. CONCLUSION

We have studied Hopfions in the SFN model on the target space $F_2 = SU(3)/U(1)^2$ which is an SU(3) generalization of the standard SFN model, for which the target space is $CP^1 = SU(2)/U(1)$. By analogy with the 2-dimensional F_2 nonlinear σ -model, we introduced two classes of configurations that satisfy the torsion-free condition; i.e., a trivial embedding of the CP^1 Hopfions and the SU(3) genuine one, which can be constructed through the Bäcklund transformation. For both cases, the Euler-Lagrange equation reduces to that of the CP^1 SFN model. In addition, though the Hopf invariant is equivalent to the CS term for the SU(3) flat connection, we showed that the invariant of the solutions is also given by the CS terms for the Abelian components of the flat connection.

The most important open problem is probably the stability of the genuine solutions. Their energy and Hopf invariant are exactly four times greater than those of the embeddings, comparing the two configurations given by the same scalar function. On the other hand, since the embedding Hopfions are essentially equivalent to the CP^1 Hopfions, their energy with H = 4n is less than four times the energy of one with H = n; i.e., $E_{H=4n} < 4 \times E_{H=n}$ [14]. Therefore, the genuine solutions with H = 4n are likely to decay into the embedding solution with H = 4n, rather than into four solutions with

H=n. It is well worth confirming whether or not the genuine solutions are stable, and if not, what they decay into. Note that even if they are unstable, these solutions probably play an important role in some branch of physics, like the known saddle-point solutions: the electro-weak sphaleron, and the meron in the pure Yang-Mills theory. However, the stability may restrict the potential for applications. It is also important to understand the mathematical implications of the torsion-free condition in detail and to confirm whether there exist Hopfions outside this condition.

We also examined some quantum aspects of the Hopfions based on collective coordinate quantization and found that they are quite different for different ansätze as a result of the difference in their symmetries.

To determine the physical spectra of glueballs, we need to

perform a more complete analysis of the collective coordinate quantization, including rotational modes, and also to discuss their statistical properties. The analysis of this subject is now in progress and the results will be reported in a subsequent paper.

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