Small Deformations of Kinks and Walls

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A Rayleigh-Schrödinger type of perturbation scheme is employed to study weak self-interacting scalar potential perturbations occurring in scalar field models describing 1D domain kinks and 3D domain walls. The solutions for the unperturbed defects are modified by the perturbing potentials. An illustration is provided by adding a cubic potential to the familiar quartic kink potential and solving for the first order correction to the kink solution, using a "slab approximation". A result is the appearance of an asymmetric scalar potential with different, nondegenerate, vacuum values and the subsequent formation of vacuum bubbles.

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1. INTRODUCTION

Exact solutions describing 1D domain kinks and 3D domain walls for φ^4 scalar field theory with a symmetric potential of the form $V_0(\varphi) = \frac{1}{4}\lambda(\varphi^2 - a^2)^2$ are well known, with vacuum values located by $\varphi = \pm a$, and static solutions assume the form $\varphi(x) = \pm a \tanh(kx) = \pm a \tanh(x/w)$, where w is a "width parameter" for the kink/wall [1]-[4]. However, the addition of a small perturbing potential $V_1(\varphi)$ will, in general, distort the simple $\tanh(kx)$ solutions in some way that depends upon the form of $V_1(\varphi)$ [5]-[9].

An effort here is made to focus upon a Rayleigh-Schrödinger type of perturbation scheme resulting in corrections to the unperturbed solutions, the corrections being due to the perturbing potential $V_1(\varphi)$. This method involves an expansion of the solution $\varphi(x)$ in terms of powers of an expansion parameter g, along with an expansion of the full potential $V(\varphi) = V_0(\varphi) + V_1(\varphi)$ about the zeroth order solution $\varphi_0(x)$ which solves the unperturbed equation of motion.

This perturbation scheme differs from the excellent one introduced by Almeida, Bazeia, Losano, and Menezes [7] for (1+1) dimensional topological defects, wherein the unperturbed action $S_0(\varphi)$ is supplemented by an additional perturbing action $S_1(\varphi) = \alpha \int d^2x F(\varphi, X)$, where α is a very small parameter controlling the perturbative expansion, $X = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi$, and $F(\varphi, X)$ is, in principle, an arbitrary function of φ and X describing the perturbation to a kink-like defect in $(1+1)$ dimensions. Additionally, an example of the perturbation method that is presented here differs from the examples of those provided in [7], where the perturbations in [7] leave the total potential symmetric about $\varphi = 0$, with degenerate vacuum states.

The scheme posed here is illustrated with an example where the perturbing potential is chosen to be $V_1(\varphi) = \frac{1}{3}\mu\varphi^3$, with μ being a small mass parameter. An approximate solution for the first order correction is obtained for this potential, with the use of a "*slab approximation*" for the unperturbed kink/wall defect [10]. (For a very thin domain wall described by a scalar field φ with an associated energy scale $E \sim \sqrt{\lambda}a \sim 1$ GeV, the wall thickness is $\lesssim 1$ fermi.) The additional contribution $V_1(\varphi)$ causes the total potential $V(\varphi)$ to become *asymmetric*, with two slightly different vacuum states φ_+ and φ_-, and two different, and *nondegenerate*, associated vacuum values $V^+(\varphi_+)$ and $V^-(\varphi_-)$, so that $\Delta V = V^+ - V^- \neq 0$ for $\mu \neq 0$. Stress-energy components for the two different sides of the wall are calculated, with $T^+_{\mu\nu} \neq T^-_{\mu\nu}$, indicating an instability against bending, ending in a formation of a network of vacuum bubbles. Without an efficient stabilizing mechanism, the bubbles subsequently collapse, releasing radiation in the form of φ boson particles of mass $m \approx \sqrt{2\lambda}a$.

2. PERTURBATION SCHEME

2.1. Potential and motion

An expansion parameter g is introduced so that we formally write

$$V(\varphi) = V_0(\varphi) + gV_1(\varphi) \tag{1}$$

where g is an expansion, or control, parameter with $0 \leq g \leq 1$. (In computing final corrections, we take the limit $g \to 1$.) When $g = 0$ the potential is the unperturbed potential V_0 and when $g = 1$ then V is the full potential $V = V_0 + V_1$. To compute the set of corrections $\{\varphi_n(x)\}$, $n = 1, 2, 3, ...$, to the unperturbed solution $\varphi_0(x)$, a final setting $g = 1$ is chosen, but to obtain a set of equations describing the various orders of corrections $\{\varphi_n(x)\}$ the value of g is temporarily left arbitrary with $g \in [0, 1]$.

The Lagrangian for the (real) scalar field is

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi) \tag{2}$$

and units are chosen for which $\hbar = c = 1$. The metric is mostly negative with diag $\eta_{\mu\nu} = (+,-,-,-)$. Furthermore, we define the function $F(\varphi)$ as the derivative of $V(\varphi)$:

$$F(\varphi) = F_0(\varphi) + gF_1(\varphi) \equiv \frac{\partial V(\varphi)}{\partial \varphi} = V'(\varphi) = V_0'(\varphi) + gV_1'(\varphi) \tag{3}$$

with the prime denoting differentiation with respect to the argument of the function, i.e., $F_0(\varphi) = V_0'(\varphi) = dV_0(\varphi)/d\varphi$, $F_1(\varphi) = V_1'(\varphi)$, $F'(\varphi) = V''(\varphi)$, etc. The quantity $F(\varphi_0)$ denotes $F(\varphi)$ evaluated at $\varphi = \varphi_0$:

$$F(\varphi_0) = V'(\varphi_0) = \frac{\partial V(\varphi)}{\partial \varphi}\Big|_{\varphi=\varphi_0} \tag{4}$$

The equation of motion that follows from $\mathcal{L}$ is

$$\Box\varphi + F(\varphi) = 0 \tag{5}$$

where $\Box = \partial_t^2 - \nabla^2$.

2.2. Expansion scheme

As with ordinary Rayleigh-Schrödinger perturbation theory in quantum mechanics, when the perturbing potential V_1 is modulated by the expansion parameter g, as in (1), the field φ also becomes dependent upon g until a particular final setting for g is chosen (say, $g = 1$), and we therefore have $\varphi(x, g) = \varphi_0(x) + \delta\varphi(x, g)$ where $\varphi_0(x)$ satisfies the unperturbed equation of motion

$$\Box\varphi_0(x) + F_0(\varphi_0) = 0 \tag{6}$$

with $gV_1 \to 0$ in (1), and it is assumed that φ_0 dominates $\delta\varphi$, $|\delta\varphi| \ll |\varphi_0|$ (with the possible exception where $\varphi_0 \approx 0$, but $\delta\varphi$ is assumed to remain "small" in some well defined sense).

We expand $\varphi(x, g)$ in powers of g,

$$\begin{aligned} \varphi(x,g) &= \textstyle\sum_{n=0}^{\infty} g^n \varphi_n(x) = \varphi_0(x) + \delta\varphi(x,g) \\ \delta\varphi(x,g) &= \textstyle\sum_{n=1}^{\infty} g^n \varphi_n(x) = g\varphi_1(x) + g^2\varphi_2(x) + g^3\varphi_3(x) + \cdots \end{aligned} \tag{7}$$

and $V(\varphi)$ and $F(\varphi) = V'(\varphi)$ are Taylor expanded about the zeroth order solution φ_0:

$$F(\varphi) = F(\varphi_0) + F'(\varphi_0)(\delta\varphi) + \frac{1}{2!}F''(\varphi_0)(\delta\varphi)^2 + \cdots \tag{8}$$

Noting that $F(\varphi) = F_0(\varphi) + gF_1(\varphi) = V_0'(\varphi) + gV_1'(\varphi)$, the expansion for $F(\varphi)$ takes the form

$$\begin{aligned} F(\varphi) &= F_0(\varphi) + gF_1(\varphi) \\ &= \left[F_0(\varphi_0) + F_0'(\varphi_0)(\delta\varphi) + \tfrac{1}{2!}F_0''(\varphi_0)(\delta\varphi)^2 + \cdots\right] \\ &\quad + g\left[F_1(\varphi_0) + F_1'(\varphi_0)(\delta\varphi) + \tfrac{1}{2!}F_1''(\varphi_0)(\delta\varphi)^2 + \cdots\right] \end{aligned} \tag{9}$$

The full system, given by (5) can be rewritten with the help of (7) and (9) as

$$\begin{aligned} &\Box(\varphi_0 + \delta\varphi) + \left[F_0(\varphi_0) + F_0'(\varphi_0)(\delta\varphi) + \tfrac{1}{2!}F_0''(\varphi_0)(\delta\varphi)^2 + \cdots\right] \\ &+ g\left[F_1(\varphi_0) + F_1'(\varphi_0)(\delta\varphi) + \tfrac{1}{2!}F_1''(\varphi_0)(\delta\varphi)^2 + \cdots\right] = 0 \end{aligned} \tag{10}$$

Using (7) for the expansion for $\delta\varphi$, and keeping only up to $O(g^3)$ terms, we have, approximately,

$$\begin{aligned} &\Box(\varphi_0 + g\varphi_1 + g^2\varphi_2 + g^3\varphi_3) + \left[F_0(\varphi_0) + F_0'(\varphi_0)(g\varphi_1 + g^2\varphi_2 + g^3\varphi_3)\right] \\ &+ \left[gF_1(\varphi_0) + F_1'(\varphi_0)(g^2\varphi_1 + g^3\varphi_2) + \tfrac{1}{2!}F_1''(\varphi_0)(g^3\varphi_1^2)\right] = 0 \end{aligned} \tag{11}$$

The various g^n terms can be collected to give a set of equations for the φ_n:

$$\begin{aligned} &g^0 : \ \Box\varphi_0 + F_0(\varphi_0) = 0 \\ &g^1 : \ \Box\varphi_1 + F_0'(\varphi_0)\varphi_1 + F_1(\varphi_0) = 0 \\ &g^2 : \ \Box\varphi_2 + F_0'(\varphi_0)\varphi_2 + F_1'(\varphi_0)\varphi_1 + \tfrac{1}{2}F_0''(\varphi_0)\varphi_1^2 = 0 \\ &g^3 : \ \Box\varphi_3 + F_0'(\varphi_0)\varphi_3 + F_1'(\varphi_0)\varphi_2 + \tfrac{1}{2}F_1''(\varphi_0)\varphi_1^2 + F_0''(\varphi_0)\varphi_1\varphi_2 = 0 \end{aligned} \tag{12}$$

3. KINK AND DOMAIN WALL POTENTIAL

3.1. Zeroth order solution for the unperturbed system

We now consider perturbations to time independent topological defects that depend upon a single coordinate x. In particular, attention is given to one dimensional (1D) kinks and three dimensional (3D) planar domain walls. These configurations are described by (2) and (5) with a φ^4 double well potential,

$$V_0(\varphi) = \frac{\lambda}{4}(\varphi^2 - a^2)^2, \qquad F_0(\varphi) = \lambda\varphi(\varphi^2 - a^2) \tag{13}$$

The unperturbed system, with $\varphi = \varphi_0$, satisfies $\Box\varphi_0 + F_0(\varphi_0) = 0$ and admits a φ^4 kink/wall solution

$$\varphi_0(x) = a\tanh(kx) = a\tanh\left(\frac{x}{w}\right) = a\tanh\left(\frac{x}{2\delta}\right) \tag{14}$$

(The antikink/antiwall solution is given by $\varphi_{\bar{K}}(x) = -\varphi_K(x) = -\varphi_0(x)$.) The parameters k, w, and δ are given by

$$k = \frac{1}{w} = \frac{1}{2\delta} = \sqrt{\frac{\lambda a^2}{2}}, \quad w = \sqrt{\frac{2}{\lambda a^2}}, \quad \delta = \frac{1}{\sqrt{2\lambda a^2}} \tag{15}$$

where w is a "width" parameter for the kink/wall and δ is a "half-width" parameter. The field φ_0 interpolates between the vacuum values $\varphi_0 = \pm a$ where $V_0(\pm a) = 0$ with $V_0(0) = \frac{1}{4}\lambda a^4$.

3.2. The slab approximation

The stress-energy of a planar φ^4 domain wall is [2, 4]

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\mathcal{L}, \quad T^\mu_\nu = f(x) \text{ diag } (1, 0, 1, 1), \quad f(x) = \frac{1}{2}\lambda a^4 \text{sech}^4\left(\frac{x}{w}\right) \tag{16}$$

with an energy density $T_{00} = f(x) = \frac{1}{2}\lambda a^4 \text{sech}^4\left(\frac{x}{w}\right)$. The surface energy density (energy/unit area) of a domain wall is [2]

$$\sigma = \frac{1}{A}\int T_{00} d^3x = \int T_{00} dx = \frac{2}{3}\sqrt{2\lambda} a^3 \tag{17}$$

The energy density T_{00} of a wall is concentrated within the wall's core, which is centered at $x = 0$. The energy density T_{00} peaks at $x = 0$ and then rapidly falls off to zero outside of the wall. It is therefore useful, making calculations tractable, to employ a "*slab approximation*"[10] for the solution $\varphi_0(x)$ wherein $\varphi_0(x)$ is taken to be zero inside a slab of "*effective thickness*" $W = 2\Delta$, where Δ is considered to be an "*effective half width*" of the wall (rather than the "half width" parameter δ) enclosing most of the wall's energy. (Note that if we imposed boundaries to be located at $x = \pm\delta$, then at a boundary $kx = \pm k\delta = \pm\frac{1}{2}$ which would give $\varphi_0/a \sim \tanh(\pm k\delta) = \pm\tanh(\frac{1}{2}) \sim \pm\frac{1}{2}$, and φ_0 would fall short of its vacuum values for which $\varphi_0/a \sim \pm 1$.) The base solution $\varphi_0(x)$ assumes its asymptotic values of $\pm a$ outside of the slab. The effective half thickness Δ is to be determined by the application of boundary conditions at the edges of the slab. The wall's energy is therefore envisioned as being in the form of a slab of thickness $W = 2\Delta$, centered at $x = 0$, with $\varphi \approx 0$ inside the slab and $\varphi_0 \approx \pm a$ outside.

For an energy scale with $\sqrt{\lambda}a \sim O(\text{GeV})$, we expect the slab thickness to be quite small, with $\Delta \sim O(\delta) \sim O(\text{GeV}^{-1})$, i.e., $\Delta \sim O(.2 \text{ fm})$. (The Nambu action for a domain wall approximates the wall as a sheet of zero thickness. See, for example, [11] for a description of the Nambu action for a domain wall, and [12] for thickness corrections to the Nambu action.)

Specifically, within the slab approximation, we make the approximations $\tanh(kx) \approx 0$ for $kx \in (-k\Delta, k\Delta)$ and $|\tanh(kx)| \approx 1$ for $k|x| > k\Delta$, i.e.,

$$\tanh(kx) \approx \left\{ \begin{array}{lll} 0, & x \in (-\Delta, \Delta) & \text{(inside wall)} \\ -1, & x < -\Delta & \text{(outside wall)} \\ +1, & x > \Delta & \text{(outside wall)} \end{array} \right\} \tag{18a}$$

$$|\tanh(kx)| \approx \left\{ \begin{array}{lll} 0, & x \in (-\Delta, \Delta) & \text{(inside wall)} \\ 1, & |x| > \Delta & \text{(outside wall)} \end{array} \right\} \tag{18b}$$

Constructing correction solutions for $\varphi_n(x)$ using the slab approximation is then somewhat analogous to finding quantum mechanical solutions $\psi(x)$ for a finite square well potential, where boundary conditions include continuity of the wave function $\psi(x)$ and its first derivative $\psi'(x)$ at the discontinuous boundaries of the well, along with appropriate asymptotic behaviors. Likewise, at least for the complimentary solutions φ_c of the homogeneous DEs (where $F_1 \to 0$), we assume continuity of $\varphi_c(x)$ and $\varphi_c'(x)$ at $x = \pm\Delta$, along with appropriate asymptotic behaviors for φ as $x \to \pm\infty$ in order to obtain quick estimates for the effective width Δ.

3.3. First order corrections

The first order correction $\varphi_1(x)$ (the g^1 equation in (12)), i.e., $\Box\varphi_1 + F_0'(\varphi_0)\varphi_1 + F_1(\varphi_0) = 0$, reads as

$$\partial_x^2\varphi_1 - F_0'(\varphi_0)\varphi_1 = F_1(\varphi_0) \tag{19}$$

where $\partial_x^2 = \partial^2/\partial x^2$ and $F_0'(\varphi_0) = (\partial F_0(\varphi)/\partial\varphi)|_{\varphi_0}$. The inhomogeneous term for this second order ordinary differential equation (DE) is $F_1(\varphi_0) = V_1'(\varphi)|_{\varphi_0}$, which is left unspecified for the time being. The solution $\varphi_1(x)$ consists of a "complimentary function" $\varphi_{1,c}(x) \equiv \psi_c(x)$ along with a "particular solution" $\psi_p(x)$, so that $\varphi_1(x) = \psi_c(x) + \psi_p(x)$. It is assumed that, in principle, the DE (19) possesses, in reality, a sufficiently smooth, continuous inhomogeneous function $F_1(\varphi_0)$ for the smooth function $\varphi_0 \sim \tanh(kx)$, along with a smooth, continuous homogeneous solution ψ_c. However, for a given function $F_1(\varphi)$, an implementation of the slab approximation for φ_0 and $F(\varphi_0)$ can, in general, destroy the global continuity of F_1 within this approximation, requiring instead a piecewise continuity of $F_1(\varphi_0)$.

The base solution is $\varphi_0(x) = a\tanh(kx)$ and $F_0(\varphi)$ is given by (13) with $F_0'(\varphi) = \lambda(3\varphi^2 - a^2)$. We therefore have

$$F_0'(\varphi_0) = \lambda a^2 \left[3\tanh^2(kx) - 1\right] = 2k^2 \left[3\tanh^2(kx) - 1\right] \tag{20}$$

with $2k^2 = \lambda a^2$. The DE (19) therefore becomes

$$\partial_x^2 \varphi_1 - 2k^2 \left[3\tanh^2(kx) - 1\right] \varphi_1 = F_1(\varphi_0) \tag{21}$$

Setting $F_1 = 0$ gives the homogeneous DE for the complimentary function $\varphi_{1,c} = \psi_c$:

$$\partial_x^2 \psi_c - 2k^2 \left[3\tanh^2(kx) - 1\right] \psi_c = 0 \tag{22}$$

Since $\varphi_0(x) = a\tanh(kx)$ is a smooth, continuous function of x, we expect that for a smooth, continuous function $F_1(\varphi_0(x))$ there exists a smooth, continuous solution $\varphi_1(x)$ that is compatible with boundary conditions. However, if we implement the slab approximation, discontinuities are introduced which may show up in the functions $\psi_p(x)$ and therefore $\varphi_1(x)$. These discontinuities may then require the consideration of piecewise continuous functions for ψ_p and φ_1.

Eq.(22) is difficult to solve in terms of simple elementary functions. However, it is interesting to comment that it resembles a Schrödinger equation for a potential $U(x) = 3k^2\tanh^2(kx)$, i.e.,

$$-\frac{1}{2}\frac{\partial^2 \psi_c}{\partial x^2} + 3k^2\tanh^2(kx)\psi_c = k^2\psi_c \tag{23}$$

with an "energy" eigenvalue $E = k^2$ (with a mass m set equal to unity). If the $\tanh^2$ potential is approximated by a finite square well of height $U_0 = 3E$ and width $W = 2\Delta$, then there exist eigenstates of the system which are bound states, and the eigenfunctions are subjected to boundary conditions involving continuity of $\psi_c(x)$ and $\psi_c'(x)$ at the edges of the potential well, i.e., at $x = \pm\Delta$. In addition, ψ_c must remain finite at $x = \pm\infty$. (For the field theory case, we do not require that the function ψ_c normalize to unity.)

We now proceed to invoke the slab approximation for the $\tanh^2(kx)$ function in (22), in which case, by (18), (22) is represented by the slab DEs

$$\psi_c''(x) + 2k^2\psi_c(x) = 0, \quad x \in (-\Delta, \Delta) \tag{24a}$$

$$\psi_c''(x) - 4k^2\psi_c(x) = 0, \quad |x| > \Delta \tag{24b}$$

The differential operator $\partial^2/\partial x^2$ and the slab potential for $\tanh^2(kx)$ are even in x, so that the solutions $\psi_c(x)$ must have definite parity with $\psi_c(x) = \pm\psi_c(-x)$. Also, it is required that ψ_c be finite at $x = \pm\infty$. The solution set for (24) is therefore given by

$$\psi_c(x) = \left\{ \begin{array}{cc} \left\{ \begin{array}{c} \psi_c^{(even)} = D\cos(\sqrt{2}kx) \\ \psi_c^{(odd)} = C\sin(\sqrt{2}kx) \end{array} \right\}, & x \in (-\Delta, \Delta) \\ Ae^{2kx}\ , & x < -\Delta \\ Be^{-2kx}\ , & x > \Delta \end{array} \right\} \tag{25}$$

with

$$\psi_c'(x) = \left\{ \begin{array}{cc} \left\{ \begin{array}{c} \psi_c'^{(even)} = -\sqrt{2}kD\sin(\sqrt{2}kx) \\ \psi_c'^{(odd)} = \sqrt{2}kC\cos(\sqrt{2}kx) \end{array} \right\}, & x \in (-\Delta, \Delta) \\ 2kAe^{2kx} \ , & x < -\Delta \\ -2kBe^{-2kx} \ , & x > \Delta \end{array} \right\} \tag{26}$$

Upon imposing boundary conditions that ψ_c and ψ_c' be continuous at $x = \pm\Delta$, we get the following results for the even and odd solutions for a determination of $k\Delta$.

Even solutions: $\psi_c(x) = \psi_c(-x)$, $(A = B)$: (See (25).)

$$\begin{array}{l} D\cos(\sqrt{2}k\Delta) = Ae^{-2k\Delta} \\ D\sin(\sqrt{2}k\Delta) = \sqrt{2}Ae^{-2k\Delta} \\ \Longrightarrow \ \tan(\sqrt{2}k\Delta) = \sqrt{2} \end{array} \tag{27}$$

From (27) we have $\tan(\sqrt{2}k\Delta) = \sqrt{2}$ which is solved by $k\Delta_n^{(even)} = \frac{1}{\sqrt{2}}\left[\tan^{-1}(\sqrt{2}) + n\pi\right]$, $n \in \mathbb{Z}$ $(n = ..., -2, -1, 0, 1, 2, ...)$. For values of $k\Delta > 0$ we require $n = 0, 1, 2, ...$, with $k\Delta_{n=0}^{(even)} = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}) \approx .675$. Therefore, the physical solution is given by $k\Delta_n^{(even)} = \frac{1}{\sqrt{2}}\left[\tan^{-1}(\sqrt{2}) + n\pi\right]$, $n \geq 0$. This allows the determination of $k\Delta_n^{(even)} \geq 0$ for a given value of $k = \sqrt{\lambda a^2/2}$, satisfying the condition $\tan(\sqrt{2}k\Delta) = \sqrt{2}$ for **even** solutions $\psi_c(x)$.

Odd solutions: $\psi_c(x) = -\psi_c(-x)$, $(A = -B)$: (See (25).)

$$\begin{array}{l} C\sin(\sqrt{2}k\Delta) = Be^{-2k\Delta} \\ C\cos(\sqrt{2}k\Delta) = -\sqrt{2}Be^{-2k\Delta} \\ \Longrightarrow \ \tan(\sqrt{2}k\Delta) = -1/\sqrt{2} \end{array} \tag{28}$$

From (28) we have $\tan(\sqrt{2}k\Delta) = -\frac{1}{\sqrt{2}}$ which is solved by $k\Delta_n^{(odd)} = \frac{1}{\sqrt{2}}\left[-\tan^{-1}(\frac{1}{\sqrt{2}}) + n\pi\right] = -\frac{1}{\sqrt{2}}\left[\tan^{-1}(\frac{1}{\sqrt{2}}) - n\pi\right]$, $n \in \mathbb{Z}$. For $k\Delta_n^{(odd)} > 0$ we require $n \geq 1$. Therefore, $k\Delta_n^{(odd)} \geq k\Delta_{n=1}^{(odd)} \approx 1.78$. This allows the determination of $k\Delta_n^{(odd)} \geq 0$ for a given value of $k = \sqrt{\lambda a^2/2}$, satisfying the condition $\tan(\sqrt{2}k\Delta) = -1/\sqrt{2}$ for **odd** solutions $\psi_c(x)$.

In summary, it is found that $k\Delta_n = (\sqrt{\lambda a^2/2}\ )\Delta_n$ takes values[1]

$$k\Delta_n = \left\{ \begin{array}{ll} k\Delta_n^{(even)} = \frac{1}{\sqrt{2}}\left[\tan^{-1}(\sqrt{2}) + n\pi\right] \gtrsim .675, & (n = 0, 1, 2, ...) \ \text{(even solutions)} \\ k\Delta_n^{(odd)} = \frac{1}{\sqrt{2}}\left[-\tan^{-1}(\frac{1}{\sqrt{2}}) + n\pi\right] \gtrsim 1.78, & (n = 1, 2, ...) \ \text{(odd solutions)} \end{array} \right\} \tag{29}$$

[1] $\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}) \approx .676$, and $-\frac{1}{\sqrt{2}}\tan^{-1}(\frac{1}{\sqrt{2}}) \approx -.435$

allowing a rough approximation for a determination of an effective slab width $W_n = 2\Delta_n$ for even or odd solutions $\psi_c(x)$. (It is noted later that a more exact determination of $k\Delta$ with an inclusion of the inhomogeneous term $F_1(\varphi_0)$ and the particular solution $\psi_p(x)$ requires solving a much more complicated algebra problem. However, a qualitative justification for the approximate accuracy of the estimates obtained above is provided in an illustration.)

These approximate values of $k\Delta$ emerging from the complimentary functions $\psi_c(x)$ illustrate the following salient features.

(i) The *effective width* Δ is *quantized* in order to admit either even or odd solutions ψ_c to (24). This is analogous to the quantum mechanical result for a finite square well where the *energy eigenvalues* for even and odd wavefunctions are quantized for a given well width.

(ii) There is always an even solution ψ_c for $k\Delta_n$ with $n = 0$, but there are no odd solutions ψ_c for $k\Delta_n$ unless $n \geq 1$.

(iii) It is found that $k\Delta_n^{(even)} \geq k\Delta_{n=0}^{(even)} \approx .675$, $(n \geq 0)$ and $k\Delta_n^{(odd)} \geq k\Delta_{n=1}^{(odd)} \approx 1.78$, $(n \geq 1)$. These can be compared to the value $k\delta = .5$ for the "half width" parameter δ, showing that $\Delta \gtrsim \delta$, with $\Delta \sim O(\delta)$.

(iv) The fact that $k\Delta_{\min}^{(odd)} > k\Delta_{\min}^{(even)}$ can be understood in analogy to the solutions for the square well problem: in order to accommodate the boundary conditions in (28), one must be able to fit at least $\frac{1}{4}$ of a wavelength between $x = 0$ and $x = \Delta$. In other words, $\theta_{\min} \equiv k\Delta_{\min}^{(odd)} \gtrsim \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{4}\right) = \frac{\pi}{2} = 1.57$, which is near the value of $k\Delta_{\min}^{(odd)} \approx 1.78$ determined above.

4. ILLUSTRATION

4.1. The potential

As an example of the application of the perturbation method, along with the use of the slab approximation, we consider a perturbation given by

$$V_1(\varphi) = \frac{1}{3}\mu\varphi^3, \quad V_1'(\varphi) = F_1(\varphi) = \mu\varphi^2 \tag{30}$$

where μ is a small constant with dimensions of mass, and we take $\mu > 0$ for definiteness. It is endeavored to determine the first order correction $\varphi_1(x)$ to the unperturbed solution $\varphi_0(x)$. (We now adopt the setting $g = 1$.) The total solution is then $\varphi(x) = \varphi_0(x) + \varphi_1(x)$. Furthermore, the complimentary functions $\psi_c(x)$ have been determined, given by (25).

The total potential (with $g \to 1$) is

$$V(\varphi) = V_0(\varphi) + V_1(\varphi) = \frac{1}{4}\lambda(\varphi^2 - a^2)^2 + \frac{1}{3}\mu\varphi^3 \tag{31}$$

and it is assumed that $\mu \ll \lambda a$. The V_1 term causes the potential V to become slightly asymmetric, with $V^+ \equiv V(a) = \frac{1}{3}\mu a^3$ and $V^- \equiv V(-a) = -\frac{1}{3}\mu a^3$ so that in vacuum we have, approximately, $\Delta V = V^+ - V^- = \frac{2}{3}\mu a^3$. The locations of the vacuum states are determined by $F(\varphi) = V'(\varphi) = 0$, where

$$F(\varphi) = F_0(\varphi) + F_1(\varphi) = \varphi\left[\lambda(\varphi^2 - a^2) + \mu\varphi\right] \tag{32}$$

The condition $F(\varphi)|_{\varphi_{vac}} = 0$ yields

$$\varphi_{vac} = \pm a\left(1 + \frac{\mu^2}{4\lambda^2 a^2}\right)^{1/2} - \left(\frac{\mu}{2\lambda a}\right)a \approx \pm a - \left(\frac{\mu}{2\lambda a}\right)a \approx \pm a \tag{33}$$

Since $\frac{\mu}{\lambda a} \ll 1$, for the sake of convenience we approximate $\varphi_{vac} \approx \pm a$ for the evaluation of $V(\varphi)$ in the vacuum states, so that as above, $V^\pm = V(\varphi_{vac} = \pm a) = \pm\frac{1}{3}\mu a^3$.

Now notice that vacuum states located by $\varphi_{vac} = \pm a - \left(\frac{\mu}{2\lambda a}\right)a$ of (33) can also be obtained by using the slab approximation. Writing $\varphi_1(x) = \psi_c(x) + \psi_p(x)$ along with (24) for the DEs for ψ_c, the slab approximation for $\varphi_0(x)$ allows (21) to be expressed by the interior and exterior DEs for the particular solution ψ_p:

$$\left\{\begin{array}{ll} \psi_p'' + 2k^2\psi_p = F_1(\varphi_0) = 0, & x \in (-\Delta, \Delta), \quad \varphi_0 = 0 \\ \psi_p'' - 4k^2\psi_p = F_1(\varphi_0) = \mu a^2, & |x| > \Delta, \quad \varphi_0^2 = a^2 \end{array}\right\} \tag{34}$$

Since the inhomogeneous terms are constants, we take $\psi_p = \text{const}$, in which case $\psi_p'' = 0$. Eq.(34) is then solved by

$$\psi_p = \left\{\begin{array}{ccc} 0, & x \in (-\Delta, \Delta), & \varphi_0 = 0 \\ -\frac{1}{2}\left(\frac{\mu}{\lambda a}\right)a, & |x| > \Delta, & |\varphi_0| = a \end{array}\right\} \tag{35}$$

The first order correction is then given by

$$\varphi_1 = \psi_c + \psi_p = \left\{\begin{array}{l} \left\{\begin{array}{c} D\cos(\sqrt{2}kx) \\ C\sin(\sqrt{2}kx) \end{array}\right\}, \; x \in (-\Delta, \Delta), \quad \varphi_0 = 0 \\ Ae^{2kx} - \frac{1}{2}\left(\frac{\mu}{\lambda a}\right)a, \quad x < -\Delta, \quad \varphi_0 = -a \\ Be^{-2kx} - \frac{1}{2}\left(\frac{\mu}{\lambda a}\right)a, \; x > \Delta, \quad \varphi_0 = +a \end{array}\right\} \tag{36}$$

and the full solution $\varphi = \varphi_0 + \varphi_1$ is

$$\varphi = \varphi_0 + \varphi_1 = \left\{ \begin{array}{ll} \left\{ \begin{array}{l} D\cos(\sqrt{2}kx) \\ C\sin(\sqrt{2}kx) \end{array} \right\}, & x \in (-\Delta, \Delta), \quad \varphi_0 = 0 \\ Ae^{2kx} - a - \frac{1}{2}\left(\frac{\mu}{\lambda a}\right) a, & x < -\Delta, \quad \varphi_0 = -a \\ Be^{-2kx} + a - \frac{1}{2}\left(\frac{\mu}{\lambda a}\right) a, & x > \Delta, \quad \varphi_0 = +a \end{array} \right\} \tag{37}$$

The constants are left undetermined here, but $\psi_c(x)$ and $\psi_c'(x)$ have previously been subjected to a matching of interior and exterior solutions at the boundaries $x = \pm\Delta$ (for the case $\mu = 0$) which, along with an account of solution parity, allows for a reduction of the constants (say B, C, and D in terms of A), as in the quantum mechanical case of a finite square well. (See, e.g., (27)-(29) for the cases $B = \pm A$. The smallness of ψ_p is expected to have negligible impact on the full solution φ, and therefore little impact upon the determination of effective slab width Δ.)

Matching the full slab approximated solutions for $\varphi_1(x) = \psi_c(x) + \psi_p$ (with $\psi_p \neq 0$) at the boundaries $x = \pm\Delta$ presents a more convoluted algebraic problem than for the $\mu = 0$, $\psi_p = 0$ case. Since we do not expect a small perturbation V_1 to make a significant change to the unperturbed solution φ_0, we content ourselves with the estimates given in (29) as approximate representations of the effective slab widths using the slab approximation, as these estimates are independent of the constants A, B, C, and D appearing in the complimentary function part ψ_c of the correction φ_1.

4.2. Vacuum bubbles

The stress-energy of the domain wall, from (16), is

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi + \eta_{\mu\nu}\left[\frac{1}{2}(\partial_x\varphi)^2 + V(\varphi)\right] \tag{38}$$

Multiplication of the equation of motion $\partial_x^2\varphi = \partial_\varphi V$ by $\partial_x\varphi$ and an integration leads to

$$\frac{1}{2}(\partial_x\varphi)^2 = V(\varphi) + K \tag{39}$$

where K is a constant of integration. The stress-energy tensor is then

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi + \eta_{\mu\nu}\left[2V(\varphi) + K\right] \tag{40}$$

This gives, by (39),

$$T_{00} = 2V + K, \quad T_{xx} = K, \quad T_{yy} = T_{zz} = -T_{00} = -(2V + K) \tag{41}$$

However, due to the asymmetry of the potential, as $x \to \pm\infty$ and $\varphi \to \pm a$, then the potential takes asymptotic values $V(\varphi) \to V^{\pm}$ with $V^{+} = +\frac{1}{3}\mu a^3$ and $V^{-} = -\frac{1}{3}\mu a^3$. From (39) we have positive and negative domains $(\partial_x\varphi)^2_{\pm}$, for which there are two different integration constants $K^{\pm}$, and therefore (39) gets modified to

$$\frac{1}{2}(\partial_x\varphi)^2_{\pm} = V(\varphi) + K^{\pm} \tag{42}$$

Since $\partial_x\varphi \to 0$ in vacuum, then $V^{\pm} = -K^{\pm} = \pm\frac{1}{3}\mu a^3$, and the asymptotic stress-energy splits into two parts, $T_{\mu\nu}(\pm\infty) \to T^{\pm}_{\mu\nu}$. We then have (40) taking the form

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi + \eta_{\mu\nu}\left[2V(\varphi) + K^{\pm}\right] = \partial_\mu\varphi\partial_\nu\varphi + \eta_{\mu\nu}\left[2V(\varphi) - V^{\pm}\right] \tag{43}$$

In vacuum, $\partial_\mu\varphi \to 0$, $V(\varphi) \to V^{\pm}$, $K^{\pm} = -V^{\pm}$, and asymptotically, as $x \to \pm\infty$, the stress-energy becomes

$$T_{\mu\nu} \to T^{\pm}_{\mu\nu} = \eta_{\mu\nu}V^{\pm} = \pm\eta_{\mu\nu}\left(\tfrac{1}{3}\mu a^3\right), \qquad |x| \to \infty \tag{44}$$

The components of $T^{\pm}_{\mu\nu}$ are

$$T^{\pm}_{00} = \pm V^{\pm} = \pm\tfrac{1}{3}\mu a^3, \quad T^{\pm}_{xx} = T^{\pm}_{yy} = T^{\pm}_{zz} = -T^{\pm}_{00} = \mp V^{\pm} = \mp\tfrac{1}{3}\mu a^3 \tag{45}$$

This stress-energy $T^{\pm}_{\mu\nu} = T_{\mu\nu}(x = \pm\infty)$ is quickly acquired outside the core of the domain wall. In the slab approximation we have, roughly, $T_{\mu\nu} \approx T^{\pm}_{\mu\nu} = \pm\eta_{\mu\nu}\left(\frac{1}{3}\mu a^3\right)$ for $|x| > \Delta$ just outside of the slab.

At any rate, effectively $T_{\mu\nu} \approx \pm\eta_{\mu\nu}\left(\frac{1}{3}\mu a^3\right)$ on the outer edges of the domain wall, and the tangential stresses $T_{yy} = T_{zz} = -T_{00}$ are higher on one side than the other, with $|\Delta T_{yy}| = |T^{+}_{yy} - T^{-}_{yy}| \approx \frac{2}{3}\mu a^3$, etc., leading to an instability against bending. The bending tends to occur toward the lower T_{yy}, T_{zz} side, i.e., toward the higher T_{00} side. Because of this instability we conclude that walls collapse [13], and that a network of bubbles eventually forms [14]. A bubble encloses a region of slightly higher energy density and is surrounded by a region of the lower energy density - the true vacuum. Such a network formation can occur due to self intersecting trajectories of a domain wall, and is enhanced by collisions of bending and/or vibrating walls and antiwalls. Without some efficient stabilization mechanism, the bubbles collapse with the release of radiation in the form of φ boson particles of mass $m \approx \sqrt{2\lambda}a$.

5. SUMMARY

A Rayleigh-Schrödinger type of perturbation scheme is used to study a self-interacting scalar field with weak perturbations to a potential which admits known analytic solutions. In particular, the φ^4 double well potential $V_0(\varphi) = \frac{1}{4}\lambda(\varphi^2 - a^2)^2$ occurring in models describing 1D domain kinks and 3D domain walls is investigated. The exact solutions for the unperturbed domain defects, described by $\varphi_0(x) = a\tanh(kx)$, are modified by the perturbing potentials.

This method is illustrated by adding a $V_1(\varphi) = \frac{1}{3}\mu\varphi^3$ cubic potential perturbation to the familiar φ^4 quartic kink potential V_0. A "slab approximation" is employed and the first order corrections $\varphi_1(x)$ to the unperturbed solution are found, allowing an approximate representation of the solution $\varphi(x) = \varphi_0(x) + \varphi_1(x)$ for the scalar field theory with potential $V(\varphi) = V_0(\varphi)+V_1(\varphi)$. A result is the appearance of an asymmetric scalar potential $V(\varphi)$ with slightly different, nondegenerate, vacuum values. Consequently, the domain walls become unstable against bending, with the subsequent formation of a network of vacuum bubbles. Within the context of this single scalar field model, the vacuum bubbles collapse, releasing radiation in the form of φ boson particles.

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