Sylvester double sums, subresultants and symmetric multivariate Hermite interpolation

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Abstract

Sylvester doubles sums, introduced first by Sylvester (see [Sylvester, 1840, Sylvester, 1853]), are symmetric expressions of the roots of two polynomials P and Q. Sylvester's definition of double sums makes no sense if P an Q have multiple roots, since the definition involves denominators that vanish when there are multiple roots. The aims of this paper are to give a new definition for Sylvester double sums making sense if P and Q have multiple roots, which coincides with the definition by Sylvester in the case of simple roots, to prove the fundamental property of Sylvester double sums, i.e. that Sylvester double sums indexed by (k, ℓ) are equal up to a constant if they share the same value for $k + \ell$, and to prove the relationship between double sums and subresultants, i.e. that they are equal up to a constant. In the simple root case, proofs of these properties are already known (see [Lascoux and Pragacz, 2002, d'Andrea et al., 2007, Roy and Szpirglas, 2011]). The more general proofs given here are using generalized Vandermonde determinants and a new symmetric multivariate Hermite interpolation as well as an induction on the length of the remainder sequence of P and Q.

Keywords: subresultants, Sylvester double sums, multivariate Hermite interpolation, generalized Vandermonde determinants

Introduction

The first aim of this paper is to provide a definition for Sylvester double sums making sense if P and Q have multiple roots, which is done using quotients of Vandermonde determinants involving variables, and substitutions. When the structure of the multiplicities of the roots of P and Q is known, we obtain a direct expresion of the Sylvester double sums in terms of generalized Vandermonde determinants.

The second aim of the paper is to prove, in the general case, the fundamental property for Sylvester double sums, i.e. that Sylvester double sums indexed by (k, ℓ) are equal up to a constant if they share the same value for $k+\ell$. In order to prove this fundamental property, it is convenient to define more general objects, the *multi Sylvester double sums*. We introduce a new multivariate symmetric Hermite interpolation and use it to study the properties of multi Sylvester double sums. The strategy then consists in proving the fundamental property for multi Sylvester double sums and obtaining the result for Sylvester double sums as a corollary by identifying coefficients.

The third aim of the paper is to prove the relationship between double sums and subresultants, i.e. that they are equal up to a constant. Our strategy is based on an induction on the length of the remainder sequence of P and Q.

Our more general proofs are new even in the special case when the roots of the polynomials are simple.

The idea of introducing a multivariate symmetric Hermite interpolation and using multi Sylvester double sums was inspired by [Krick et al., 2016]'s use of multivariate symmetric Lagrange interpolation and introduction of multi Sysvtester double sums in the context of simple roots.

The content of the paper is the following.

In Section 1 we give a general definition for Sylvester double sums, valid also when there are multiple roots, and prove that it coincides with Sylvester's definition in the special case where all roots are simple (Proposition 6).

In Section 2 we consider generalized Vandermonde determinants and use them to give a new formula for Sylvester double sums when the structure of multiplicities is known (Proposition 13).

In Section 3, we introduce an Hermite interpolation for multivariate symmetric polynomials (Proposition 20).

In Section 4 we study multi Sylvester double sums. We give their definition in subsection 4.1 In subsection 4.2 we compute the multi Sylvester double sums and Sylvester double sums for indices (k, ℓ) with $k + \ell \geq \deg(Q)$. In subsection 4.3 we prove that multi Sylvester double sums and Sylvester double sums indexed by k, ℓ, depend only (up to a constant) on $j = k + \ell$ (Theorem 32 and Theorem 31).

In Section 5 we give a relationship between Sylvester double sums of (P, Q) and Sylvester double sums of (Q, R) where R is the opposite of the remainder of P by Q in the Euclidean division (Proposition 38).

Finally we prove in Section 6 that Sylvester double sums coincide (up to a constant) with subresultants, by an induction on the length of the remainder sequence of P and Q (Theorem 40).

1 Sylvester double sums

We give a general definition for Sylvester double sums, valid also when the polynomials have multiple roots, and prove that it coincides with Sylvester's definition in the special case where all roots are simple (Proposition 6).

1.1 Basic notations and definitions

Let $\mathbb{K}$ be a field of characteristic 0.
Let $\mathbf{A}$ be a finite list of elements of $\mathbb{K}$.
We denote $\mathbf{A}' \subset_a \mathbf{A}$ when $\mathbf{A}'$ is a sublist of $\mathbf{A}$ with a elements (i.e. the list $\mathbf{A}'$ is ordered by the restriction of the order on the list $\mathbf{A}$).
Let $\mathbf{B}$ be another finite list of elements of $\mathbb{K}$.
We denote

$$\Pi(\mathbf{A},\mathbf{B}) = \prod_{\substack{x\in\mathbf{A}\\ y\in\mathbf{B}}} (x-y).$$

Note that $\Pi(\mathbf{A},\mathbf{B})$ is independant on the order of $\mathbf{A}$ and $\mathbf{B}$.
We abbreviate $\Pi(\{x\},\mathbf{B})$ and $\Pi(\mathbf{A},\{y\})$ to $\Pi(x,\mathbf{B})$ and $\Pi(\mathbf{A},y)$ respectively.
Note that $\Pi(\mathbf{A},\mathbf{B})$ is the classical resultant of the monic polynomials $\Pi(X,\mathbf{A})$ and $\Pi(X,\mathbf{B})$.

Definition 1. *The* Vandermonde vector *of length i of $x \in \mathbb{K}$, denoted by $v_i(x)$, is*

$$v_i(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ \vdots \\ x^{i-1} \end{bmatrix}. \tag{1}$$

Let $\mathbf{A} = (x_1,\ldots,x_i)$ be a finite ordered list of elements of $\mathbb{K}$. The Vandermonde matrix $\mathcal{V}(\mathbf{A})$ is the $i\times i$ matrix having as column vectors $v_i(x_1),\ldots,v_i(x_i)$. The Vandermonde determinant $V(\mathbf{A})$ is the determinant of the Vandermonde matrix $\mathcal{V}(\mathbf{A})$. It is well known that

$$V(\mathbf{A}) = \prod_{i\geq k>j\geq 1} (x_k - x_j).$$

By $\mathbf{B}\|\mathbf{A}$ the we denote the list obtained by concatening $\mathbf{B}$ and $\mathbf{A}$.
The following result is obvious.

Lemma 2.

$$V(\mathbf{B}\|\mathbf{A}) = V(\mathbf{A})\Pi(\mathbf{A},\mathbf{B})V(\mathbf{B}). \tag{2}$$

and, as a special case, given a variable U,

$$V(\mathbf{B}\|U) = \Pi(U,\mathbf{B})V(\mathbf{B}).$$

1.2 Definition of Sylvester double sums

Let $\mathbf{P} = (x_1,\ldots,x_p)$ and $\mathbf{Q} = (y_1,\ldots,y_q)$ be two finite ordered sets of element of $\mathbb{K}$ and $P = \Pi(X,\mathbf{P})$, $Q = \Pi(X,\mathbf{Q})$
The Sylvester double sum of (P,Q) of index $k\in\mathbb{N}, \ell\in\mathbb{N}$ is usually defined as the following polynomial in $\mathbb{K}[U]$:

$$\sum_{\substack{\mathbf{K}\subset_k \mathbf{P}\\ \mathbf{L}\subset_\ell \mathbf{Q}}} \Pi(U,\mathbf{K})\Pi(U,\mathbf{L})\frac{\Pi(\mathbf{K},\mathbf{L})\Pi(\mathbf{P}\setminus\mathbf{K},\mathbf{Q}\setminus\mathbf{L})}{\Pi(\mathbf{K},\mathbf{P}\setminus\mathbf{K})\Pi(\mathbf{L},\mathbf{Q}\setminus\mathbf{L})} \tag{3}$$

(see [Sylvester, 1840, Sylvester, 1853]).
This definition of Sylvester double sums makes no sense if P and Q have multiple roots, since some of the quantities $\Pi(\mathbf{K},\mathbf{P}\setminus\mathbf{K})$ (resp. $\Pi(\mathbf{L},\mathbf{Q}\setminus\mathbf{L})$) at the denominator are equal to 0.
In this section, we give a general definition of Sylvester double sums, valid even if P and Q have multiple roots and prove that it coincides with the classical one when all these roots are simple.
Let $\mathbf{X}=(X_1,\ldots,X_p)$ and $\mathbf{Y}=(Y_1,\ldots,Y_q)$ be two ordered sets of indeterminates.
Given $\mathbf{X}'\subset_k \mathbf{X}$ (resp. $\mathbf{Y}'\subset_\ell \mathbf{Y}$), we denote $s_{\mathbf{X}'}$ (resp. $s_{\mathbf{Y}'}$) the signature of the permutation $\sigma_{\mathbf{X}'}$ (resp. $\sigma_{\mathbf{Y}'}$) putting the elements of $\mathbf{X}$ (resp. $\mathbf{Y}$) in the order $(\mathbf{X}\setminus\mathbf{X}')\|\mathbf{X}'$ (resp.$(\mathbf{Y}\setminus\mathbf{Y}')\|\mathbf{Y}'$).
For any $k\in\mathbb{N},\ell\in\mathbb{N}$, we define the polynomial $F^{k,\ell}(\mathbf{X},\mathbf{Y})(U)$ in $K[\mathbf{X},\mathbf{Y},U]$

$$F^{k,\ell}(\mathbf{X},\mathbf{Y})(U)=\sum_{\substack{\mathbf{X}'\subset_k \mathbf{X}\\ \mathbf{Y}'\subset_\ell \mathbf{Y}}} s_{\mathbf{X}'}s_{\mathbf{Y}'}V((\mathbf{Y}\setminus\mathbf{Y}')\|(\mathbf{X}\setminus\mathbf{X}'))V(\mathbf{Y}'\|\mathbf{X}'\|U) \tag{4}$$

Note that if $k>p$ or $\ell>q$ then $F^{k,\ell}(\mathbf{X},\mathbf{Y})(U)=0$.

Proposition 3. *The polynomial $F^{k,\ell}(\mathbf{X},\mathbf{Y})(U)$ is antisymmetric in the variables $\mathbf{X}$ and in the variables $\mathbf{Y}$.*

Proof. For any permutation σ of the ordered set $\mathbf{X}$, we call also σ the action of σ on a polynomial F in $K[\mathbf{X},\mathbf{Y},U]$, i.e $\sigma(F)(\mathbf{X},\mathbf{Y})=F(\sigma(\mathbf{X}),\mathbf{Y})$. Denoting s the signature of σ we want to prove

$$\sigma(F^{k,\ell})(\mathbf{X},\mathbf{Y})(U)=sF^{k,\ell}(\mathbf{X},\mathbf{Y})(U). \tag{5}$$

It is enough to prove (5) for a transposition exchanging two sucessive elements, of signature -1.
So, let τ be the transposition exchanging X_i and X_{i+1}. We want to prove

$$\tau(F^{k,\ell})(\mathbf{X},\mathbf{Y})(U)=-F^{k,\ell}(\mathbf{X},\mathbf{Y})(U). \tag{6}$$

We denote by $\tau(\mathbf{X})$ the ordered set obtained from $\mathbf{X}$ by exchanging X_i and X_{i+1}. Given $\mathbf{X}'\subset_k \mathbf{X}$, we denote by $\tau(\mathbf{X}')$ the ordered set $\tau(\mathbf{X}')\subset_k \tau(\mathbf{X})$ (i.e. $\tau(\mathbf{X}')$ is ordered by the restriction of the order on $\tau(\mathbf{X})$) and by $\bar{\mathbf{X}}'$ the ordered set $\tau(\mathbf{X}')\subset_k \mathbf{X}$ (i.e. $\bar{\mathbf{X}}'$ and $\tau(\mathbf{X}')$ have the same elements but $\bar{\mathbf{X}}'$ is ordered by the restriction of the order on $\mathbf{X}$).
Denote

$$F^{\mathbf{X}',\mathbf{Y}'}=s_{\mathbf{X}'}s_{\mathbf{Y}'}V((\mathbf{Y}\setminus\mathbf{Y}')\|(\mathbf{X}\setminus\mathbf{X}'))V(\mathbf{Y}'\|\mathbf{X}'\|U).$$

We have 3 cases to consider.

- If $X_i \in \mathbf{X}'$ and $X_{i+1} \in \mathbf{X}'$ then $\tau(\mathbf{X} \setminus \mathbf{X}') = \mathbf{X} \setminus \mathbf{X}'$ and

$$\begin{aligned}
\tau(F^{\mathbf{X}',\mathbf{Y}'}) &= s_{\mathbf{X}'} s_{\mathbf{Y}'} V((\mathbf{Y} \setminus \mathbf{Y}') \| \tau(\mathbf{X} \setminus \mathbf{X}')) V(\mathbf{Y}' \| \tau(\mathbf{X}') \| U) \\
&= s_{\mathbf{X}'} s_{\mathbf{Y}'} V((\mathbf{Y} \setminus \mathbf{Y}') \| (\mathbf{X} \setminus \mathbf{X}')) V(\mathbf{Y}' \| \tau(\mathbf{X}') \| U) \\
&= -F^{\mathbf{X}',\mathbf{Y}'}.
\end{aligned}$$

- If $X_i \notin \mathbf{X}'$ and $X_{i+1} \notin \mathbf{X}'$ then $\tau(\mathbf{X}') = \mathbf{X}'$ and

$$\begin{aligned}
\tau(F^{\mathbf{X}',\mathbf{Y}'}) &= s_{\mathbf{X}'} s_{\mathbf{Y}'} V((\mathbf{Y} \setminus \mathbf{Y}') \| \tau(\mathbf{X} \setminus \mathbf{X}')) V(\mathbf{Y}' \| \tau(\mathbf{X}') \| U) \\
&= s_{\mathbf{X}'} s_{\mathbf{Y}'} V((\mathbf{Y} \setminus \mathbf{Y}') \| \tau(\mathbf{X} \setminus \mathbf{X}')) V(\mathbf{Y}' \| \mathbf{X}' \| U) \\
&= -F^{\mathbf{X}',\mathbf{Y}'}.
\end{aligned}$$

- If $X_i \in \mathbf{X}'$ and $X_{i+1} \notin \mathbf{X}'$, or $X_i \notin \mathbf{X}'$ and $X_{i+1} \in \mathbf{X}'$, then $\sigma_{\bar{\mathbf{X}}'} = \tau \circ \sigma_{\mathbf{X}'}$, $\tau(\mathbf{X}') = \bar{\mathbf{X}}'$ and $\tau(\mathbf{X} \setminus \mathbf{X}') = \mathbf{X} \setminus \bar{\mathbf{X}}'$ so that

$$\begin{aligned}
\tau(F^{\mathbf{X}',\mathbf{Y}'}) &= s_{\mathbf{X}'} s_{\mathbf{Y}'} V((\mathbf{Y} \setminus \mathbf{Y}') \| \tau(\mathbf{X} \setminus \mathbf{X}')) V(\mathbf{Y}' \| \tau(\mathbf{X}') \| U) \\
&= -s_{\bar{\mathbf{X}}'} s_{\mathbf{Y}'} V((\mathbf{Y} \setminus \mathbf{Y}') \| (\mathbf{X} \setminus \bar{\mathbf{X}}')) V(\mathbf{Y}' \| \bar{\mathbf{X}}' \| U) \\
&= -F^{\bar{\mathbf{X}}',\mathbf{Y}'}
\end{aligned}$$

  and

$$\begin{aligned}
\tau(F^{\bar{\mathbf{X}}',\mathbf{Y}'}) &= s_{\bar{\mathbf{X}}} s_{\mathbf{Y}'} V((\mathbf{Y} \setminus \mathbf{Y}') \| \tau(\mathbf{X} \setminus \bar{\mathbf{X}}')) V(\mathbf{Y}' \| \tau(\bar{\mathbf{X}}') \| U) \\
&= -s_{\mathbf{X}} s_{\mathbf{Y}'} V((\mathbf{Y} \setminus \mathbf{Y}') \| (\mathbf{X} \setminus \mathbf{X}')) V(\mathbf{Y}' \| \mathbf{X}' \| U) \\
&= -F^{\mathbf{X}',\mathbf{Y}'},
\end{aligned}$$

  From which we deduce

$$\tau\left(F^{\mathbf{X}',\mathbf{Y}'} + F^{\bar{\mathbf{X}}',\mathbf{Y}'}\right) = -\left(F^{\bar{\mathbf{X}}',\mathbf{Y}'} + F^{\mathbf{X}',\mathbf{Y}'}\right).$$

  So, we get (6).

The exchange between two elements of $\mathbf{Y}$ can be treated similarly. □

Lemma 4. *If $A(\mathbf{X}, \mathbf{Y})$ in $K[\mathbf{X}, \mathbf{Y}]$ is antisymmetric with respect to the variables $\mathbf{X}$, then $A(\mathbf{X}, \mathbf{Y}) = S(\mathbf{X}, \mathbf{Y})V(\mathbf{X})$ where $S \in K[\mathbf{X}, \mathbf{Y}]$ is symmetric with respect to the variables $\mathbf{X}$.*

Proof. If $A(\mathbf{X}, \mathbf{Y})$ is antisymmetric with respect to $\mathbf{X}$ then, for any $j < k$, denote $\tau_{j,k}(\mathbf{X})$ the ordered set of variables obtained by transposing X_j and X_k.

$$\frac{A(\mathbf{X}, \mathbf{Y}) - A(\tau_{j,k}(\mathbf{X}, \mathbf{Y}))}{X_j - X_k} = 2\frac{A(\mathbf{X}, \mathbf{Y})}{X_j - X_k}$$

is a polynomial. So $A(\mathbf{X}, \mathbf{Y}) = S(\mathbf{X}, \mathbf{Y})V(\mathbf{X})$ and $S(\mathbf{X}, \mathbf{Y})$ is a symmetric polynomial with respect to $\mathbf{X}$. □

Applying Lemma 4 and Proposition 3 we denote $S^{k,\ell}(\mathbf{X},\mathbf{Y})(U)$ the symmetric polynomial with respect to the indeterminates $\mathbf{X}$ and with respect to the indeterminates $\mathbf{Y}$ satisfying

$$S^{k,\ell}(\mathbf{X},\mathbf{Y})(U) = \frac{F^{k,\ell}(\mathbf{X},\mathbf{Y})(U)}{V(\mathbf{X})V(\mathbf{Y})}. \tag{7}$$

Given two monic univariate polynomials P and Q of degree p and q we denote $\mathbf{P} = (x_1,\ldots,x_p)$ and $\mathbf{Q} = (y_1,\ldots,y_q)$ ordered lists of the roots of P and Q in an algebraic closure $\mathbf{C}$ of $\mathbb{K}$, counted with multiplicities.

Definition 5. *The* generalized Sylvester double sum of (P,Q) *for the exponents* $k,\ell \in \mathbb{N}\times\mathbb{N}$ *is defined by*

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = S^{k,\ell}(\mathbf{P},\mathbf{Q})(U).$$

Note that this definition does not depend on the order given for the roots of P and Q.

This definition of generalized Sylvester double sums for monic polynomials coincides with the usual definition of Sylvester double sums when the polynomials P and Q have no multiple roots, as we see now in Proposition 6.

Proposition 6. *If P,Q have only simple roots,*

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = \sum_{\substack{\mathbf{K}\subset_k \mathbf{P}\\ \mathbf{L}\subset_\ell \mathbf{Q}}} \Pi(U,\mathbf{K})\Pi(U,\mathbf{L})\frac{\Pi(\mathbf{K},\mathbf{L})\Pi(\mathbf{P}\setminus\mathbf{K},\mathbf{Q}\setminus\mathbf{L})}{\Pi(\mathbf{K},\mathbf{P}\setminus\mathbf{K})\Pi(\mathbf{L},\mathbf{Q}\setminus\mathbf{L})}$$

Proof of Proposition 6.

$$\sum_{\substack{\mathbf{K}\subset_k\mathbf{P}\\ \mathbf{L}\subset_\ell\mathbf{Q}}} \Pi(U,\mathbf{K})\Pi(U,\mathbf{L})\frac{\Pi(\mathbf{K},\mathbf{L})\Pi(\mathbf{P}\setminus\mathbf{K},\mathbf{Q}\setminus\mathbf{L})}{\Pi(\mathbf{K},\mathbf{P}\setminus\mathbf{K})\Pi(\mathbf{L},\mathbf{Q}\setminus\mathbf{L})} =$$

$$\begin{aligned}
&= \sum_{\substack{\mathbf{K}\subset_k\mathbf{P}\\ \mathbf{L}\subset_\ell\mathbf{Q}}} \Pi(U,\mathbf{K})\Pi(U,\mathbf{L})\frac{V(\mathbf{L})\Pi(\mathbf{K},\mathbf{L})V(\mathbf{K})V(\mathbf{Q}\setminus\mathbf{L})\Pi(\mathbf{P}\setminus\mathbf{K},\mathbf{Q}\setminus\mathbf{L})V(\mathbf{P}\setminus\mathbf{K})}{V(\mathbf{K})\Pi(\mathbf{K},\mathbf{P}\setminus\mathbf{K})V(\mathbf{P}\setminus\mathbf{K})V(\mathbf{L})\Pi(\mathbf{L},\mathbf{Q}\setminus\mathbf{L})V(\mathbf{Q}\setminus\mathbf{L})}\\
&= \sum_{\substack{\mathbf{K}\subset_k\mathbf{P}\\ \mathbf{L}\subset_\ell\mathbf{Q}}} s_{\mathbf{K}}s_{\mathbf{L}}\frac{V(\mathbf{L}\|\mathbf{K}\|U)V((\mathbf{Q}\setminus\mathbf{L})\|(\mathbf{P}\setminus\mathbf{K}))}{V(\mathbf{P})V(\mathbf{Q})}\\
&= \frac{F^{k,\ell}(\mathbf{P},\mathbf{Q})(U)}{V(\mathbf{P})V(\mathbf{Q})} = S^{k,\ell}(\mathbf{P},\mathbf{Q})(U)\\
&= \mathrm{Sylv}^{k,\ell}(P,Q)(U)
\end{aligned}$$

applying Lemma 2. □

2 Generalized Vandermonde determinants and Sylvester double sums

We consider generalized Vandermonde determinants (also called sometimes confluent Vandermonde determinants, see [Lancaster and M.Tismenetsky, 1985, Horn and Johnson, 1991]) and connect them with the Sylvester double sums (Proposition 13).

Notation 7. *Let P be a polynomial of degree p with coefficients in a field $\mathbb{K}$. Let $(x_1,\ldots,x_m)$ be an ordered set of the distinct roots of P in an algebraic closure $\mathbf{C}$ of $\mathbb{K}$, with x_i of multiplicity μ_i, and let $\mathbf{P}$ be the multiset of roots of P, represented by the ordered set*

$$\mathbf{P} = (x_{1,0},\ldots,x_{1,\mu_1-1},\ldots,x_{m,0},\ldots,x_{m,\mu_m-1}),$$

with $x_{i,j}=(x_i,j)$ for $0\le j\le \mu_i-1$, $\sum_{i=1}^m \mu_i = p$.

Let Q be a polynomial of degree q with coefficients in $\mathbb{K}$. Let $(y_1,\ldots,y_n)$ be an ordered set of the distinct roots of Q in $\mathbf{C}$ with y_i of multiplicity ν_i, , for $i=1,\ldots,n$. Let $\mathbf{Q}$ be the ordered multiset of its root, represented by the ordered set

$$\mathbf{Q} = (y_{1,0},\ldots,y_{1,\nu_1-1},\ldots,y_{n,0},\ldots,y_{n,\nu_n-1}),$$

with $y_{i,j}=(y_i,j)$ for $0\le j\le \nu_i-1$, $\sum_{i=1}^n \nu_i = q$.
We introduce an ordered set of variables $X_{\mathbf{P}} = (X_{1,0},\ldots,X_{1,\mu_1-1},\ldots,X_{m,0},\ldots,X_{m,\mu_m-1})$ and an ordereed set of variables $Y_{\mathbf{Q}} = (Y_{1,0},\ldots,Y_{1,\nu_1-1},\ldots,Y_{n,0},\ldots,Y_{n,\nu_n-1})$.
For a polynomial $f(X_{\mathbf{P}},X_{\mathbf{Q}})$ we denote $f(\mathbf{P},\mathbf{Q})$ the result of the substitution of $X_{i,j}$ by x_i and $Y_{i,j}$ by y_i.

Notation 8. *Given f a polynomial depending on the variable U, we denote*

$$f^{[i]} = \frac{1}{i!}\frac{\partial^i f}{\partial U^i}. \tag{8}$$

Definition 9 (Generalized Vandermonde determinant)**.** *Let* $\mathbf{K} \subset_k \mathbf{P}$*,* $\mathbf{L} \subset_\ell \mathbf{Q}$ *and* $\mathbf{U} = (U_1, \ldots, U_u)$ *an ordered set of* u *indeterminates.*

The generalized Vandermonde matrix $\mathcal{V}[\mathbf{L}\|\mathbf{K}\|\mathbf{U})$ *is the* $(\ell+k+u) \times (\ell+k+u)$ *matrix having as column vectors the* ℓ *columns* $v^{[j]}_{k+\ell+u}(y_i)$ *for* $y_{i,j} \in \mathbf{L}$ *followed by the* k *columns* $v^{[j]}_{k+\ell+u}(x_i)$ *for* $x_{i,j} \in \mathbf{K}$ *followed by the* u *columns* $v_{k+\ell+u}(U_i)$ *(using notation (1) and notation (8)).*

The generalized Vandermonde determinant $V[\mathbf{L}\|\mathbf{K}\|\mathbf{U})$ *is the determinant of* $\mathcal{V}[\mathbf{L}\|\mathbf{K}|\mathbf{U})$*.*

- *In the particular case* $u = 0$ *we denote* $V[\mathbf{L}\|\mathbf{K}]$ *the corresponding determinant.*
- *In the particular case* $k = p, \ell = u = 0$ *we denote* $V[\mathbf{P}]$ *the corresponding determinant .*
- *Similarly, in the particular case* $k = 0, \ell = q, u = 0$ *we denote* $V[\mathbf{Q}]$ *the corresponding determinant.*

Remark 10. The peculiar notation $V[\mathbf{L}\|\mathbf{K}\|\mathbf{U})$ with one square bracket to the left and one parenthesis to the right is here to indicate that the column $v^{[j]}_{k+\ell+u}(x_i)$ indexed by $x_{i,j} \in \mathbf{K}$ and $v^{[j]}_{k+\ell+u}(y_i)$ indexed by $y_{i,j} \in \mathbf{L}$ have been derivated, while there is no derivation with respect to the columns indexed by the variables in $\mathbf{U}$.

While the classical Vandermonde determinant $V(\mathbf{P})$ is null when P has multiple roots, we have the following result for the generalized Vandermonde determinant.

Lemma 11. *The generalized Vandermonde determinant* $V[\mathbf{P}]$ *is equal to*

$$V[\mathbf{P}] = \prod_{1 \le i < j \le m} (x_j - x_i)^{\mu_i \mu_j}.$$

Proof. The proof is done by induction on p.
If $p = 1$, $V[\mathbf{P}] = 1$.
Suppose that

$$V[\mathbf{P}] = \prod_{1 \le i < j \le m} (x_j - x_i)^{\mu_i \mu_j}.$$

The polynomial $F(U) = V[\mathbf{P}\|U)$ is of degree p, with leading coefficient $V[\mathbf{P}]$ and satisfies the property

$$\text{for all } 1 \le i \le m, \text{ for all } 0 \le j < \mu_i, \;\; F^{[j]}(x_i) \;=\; 0,$$

So,

$$F(U) = V[\mathbf{P}] \prod_{i=1}^{m} (U - x_i)^{\mu_i} = \prod_{1 \le i < j \le m} (x_j - x_i)^{\mu_i \mu_j} \prod_{i=1}^{m} (U - x_i)^{\mu_i}.$$

Consider $T(U) = (U - x)P(U)$.

$-$ First case: x is not a root of P. Let $\mathbf{T}$ the ordered set (obtained by adding x at the end of $\mathbf{P}$) of roots of the polynomial T, so $x = x_{m+1}$ is a root of T with multiplicity 1. Then

$$V[\mathbf{T}] = F(x) = \prod_{1 \le i < j \le m} (x_j - x_i)^{\mu_i \mu_j} \prod_{i=1}^{m} (x - x_i)^{\mu_i} = \prod_{1 \le i < j \le m+1} (x_j - x_i)^{\mu_i \mu_j}.$$

$-$ Second case: x is a root of P. So there exists $1 \le j \le m$ such that $x = x_j$, and x_j is a root of multiplicity $\mu_j + 1$ of T.
Let $\mathbf{T}$ the ordered set of roots of the polynomial T obtained by inserting $x_{j,\mu_j} = (x_j, \mu_j)$ after x_{j,μ_j-1} in $\mathbf{P}$. Then

$$\begin{aligned} V[\mathbf{T}] &= (-1)^{\mu_{j+1}+\cdots+\mu_m} F^{[\mu_j]}(x_j) \\ &= (-1)^{\mu_{j+1}+\cdots+\mu_m} V[\mathbf{P}] \prod_{\substack{i=1 \\ i \neq j}}^{m} (x_j - x_i)^{\mu_i} \\ &= \prod_{1 \le i < j \le m} (x_j - x_i)^{\mu_i(\mu_j+1)} \end{aligned}$$

□

Remark 12. If $\mathbf{K} \subset_k \mathbf{P}$, it can happen that $V[\mathbf{K}] = 0$. Taking for example $\mathbf{P} = (x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1})$ and $\mathbf{K} = (x_{1,1}, x_{2,1})$, it is easy to check that $V[\mathbf{K}] = 0$.

From now on, and till Section 5, P and Q are monic polynomials,

The following proposition makes the link between generalized Vandermonde determinants and Sylvester double sums.
We denote $s_{\mathbf{K}}$ (resp. $s_{\mathbf{L}}$) the signature of the permutation $\sigma_{\mathbf{K}}$ (resp. $\sigma_{\mathbf{L}}$) obtained by putting the elements of $\mathbf{P}$ (resp. $\mathbf{Q}$) in the order $(\mathbf{P} \setminus \mathbf{K}) \| \mathbf{K}$ (resp. $(\mathbf{Q} \setminus \mathbf{L}) \| \mathbf{L}$).

Proposition 13.

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = \sum_{\substack{\mathbf{K} \subset_k \mathbf{P} \\ \mathbf{L} \subset_\ell \mathbf{Q}}} s_{\mathbf{K}} s_{\mathbf{L}} \frac{V[(\mathbf{Q} \setminus \mathbf{L}) \| (\mathbf{P} \setminus \mathbf{K})] V[\mathbf{L} \| \mathbf{K} \| U)}{V[\mathbf{P}] V[\mathbf{Q}]}$$

In the proof of Proposition 13, we use the following notation 14 and Lemma 15.

Notation 14. *For any polynomial f depending on the variables $X_{\mathbf{P}}$, and $\mathbf{K} \subset_k \mathbf{P}$, denote $\partial^{[\mathbf{K}]} f$ the polynomial defined by induction on r as follows.*

$$\partial^{[\emptyset]} f = f$$

If $\mathbf{K} = \mathbf{K}' \| (x_{i,j})$,

$$\partial^{[\mathbf{K}]} f = \frac{1}{j!} \frac{\partial^j \partial^{[\mathbf{K}']} f}{\partial X_{i,j}^j}.$$

Similarly, for any polynomial f depending on the variables $Y_{\mathbf{Q}}$, and $\mathbf{L} \subset_\ell \mathbf{Q}$, denote $\partial^{[\mathbf{L}]} f$ the polynomial defined by induction on s as follows.

$$\partial^{[\emptyset]} f = f$$

If $\mathbf{L} = \mathbf{L}' \| (y_{i,j})$,

$$\partial^{[\mathbf{L}]} f = \frac{1}{j!} \frac{\partial^j \partial^{[\mathbf{L}']} f}{\partial X_{i,j}^j}.$$

Note that

$$V[\mathbf{L} \| \mathbf{K} \| \mathbf{U}) = f(\mathbf{K}, \mathbf{L}, \mathbf{U})$$

with $f(X_{\mathbf{K}}, Y_{\mathbf{L}}, \mathbf{U}) = \partial^{[\mathbf{L}]} \partial^{[\mathbf{K}]} V(Y_{\mathbf{L}} \| X_{\mathbf{K}} \| \mathbf{U})$.

Lemma 15.

$$\partial^{[\mathbf{P}]} \left(V(X_{\mathbf{P}}) f(X_{\mathbf{P}}) \right) (\mathbf{P}) = V[\mathbf{P}] f(\mathbf{P})$$

Proof. We first note that

$$\partial^{[\mathbf{P}]} \left(V(X_{\mathbf{P}}) f(X_{\mathbf{P}}) \right) = \partial^{[\mathbf{P}]} (V(X_{\mathbf{P}})) f(X_{\mathbf{P}}) + \sum_r V_r(X_{\mathbf{P}}) f_r(X_{\mathbf{P}})$$

where $V_r(X_{\mathbf{P}})$ (resp. $f_r(X_{\mathbf{P}})$) is obtained from $V(X_{\mathbf{P}})$ (resp. from $f(X_{\mathbf{P}})$) by partial derivations, one variable $X_{i,j}$ at least being derived less than j times (resp. at least one time). Denoting $X_{i,j}$ the first variable which is being derived less than j times in $V_r(X_{\mathbf{P}})$, we define j' as the order of derivation of $X_{i,j}$ in $V_r(X_{\mathbf{P}})$. We notice that $V_r(X_{\mathbf{P}})$ is the determinant of a matrix with two equal columns, the one indexed by i, j' and the one indexed by i, j. Hence $V_r(\mathbf{P}) = 0$. This proves the claim. □

Proof of Proposition 13. Since

$$F^{k,\ell}(X_{\mathbf{P}}, Y_{\mathbf{Q}})(U) = V(X_{\mathbf{P}}) V(Y_{\mathbf{Q}}) S^{k,\ell}(X_{\mathbf{P}}, Y_{\mathbf{Q}})(U),$$

using Lemma 15 we obtain

$$\partial^{[\mathbf{Q}]} \partial^{[\mathbf{P}]} F^{k,\ell}(\mathbf{P}, \mathbf{Q})(U) = V[\mathbf{P}] V[\mathbf{Q}] S^{k,\ell}(\mathbf{P}, \mathbf{Q})(U) = V[\mathbf{P}] V[\mathbf{Q}] \mathrm{Sylv}^{k,\ell}(P, Q)(U).$$

On the other hand, denoting

$$h_{\mathbf{K},\mathbf{L}}(X_{\mathbf{P}}, Y_{\mathbf{Q}})(U) = V(Y_{\mathbf{Q} \setminus \mathbf{L}} \| X_{\mathbf{P} \setminus \mathbf{K}}) V(Y_{\mathbf{L}} \| X_{\mathbf{K}} \| U),$$

we have

$$\partial^{[\mathbf{Q}]} \partial^{[\mathbf{P}]} h_{\mathbf{K},\mathbf{L}}(\mathbf{P}, \mathbf{Q})(U) = V[(\mathbf{Q} \setminus \mathbf{L}) \| (\mathbf{P} \setminus \mathbf{K})] V[\mathbf{L} \| \mathbf{K} \| U).$$

Since

$$F^{k,\ell}(X_{\mathbf{P}}, Y_{\mathbf{Q}})(U) = \sum_{\substack{\mathbf{K} \subset_k \mathbf{P} \\ \mathbf{L} \subset_\ell \mathbf{Q}}} s_{\mathbf{K}} s_{\mathbf{L}} h_{\mathbf{K},\mathbf{L}}(X_{\mathbf{P}}, Y_{\mathbf{Q}})(U)$$

we get

$$\partial^{[\mathbf{Q}]} \partial^{[\mathbf{P}]} F^{k,\ell}(\mathbf{P}, \mathbf{Q})(U) = \sum_{\substack{\mathbf{K} \subset_k \mathbf{P} \\ \mathbf{L} \subset_\ell \mathbf{Q}}} s_{\mathbf{K}} s_{\mathbf{L}} V[(\mathbf{Q} \setminus \mathbf{L}) \| (\mathbf{P} \setminus \mathbf{K})] V[\mathbf{L} \| \mathbf{K} \| U)$$ □

The following lemma will be useful later.

Lemma 16.

1. *For* $\mathbf{L} \subset_\ell \mathbf{Q}$*, defining*

$$f(Y_{\mathbf{Q}\setminus\mathbf{L}}) = (-1)^{p(q-\ell)}\partial^{[\mathbf{Q}\setminus\mathbf{L}]}\left(V(Y_{\mathbf{Q}\setminus\mathbf{L}})\prod_{Y\in Y_{\mathbf{Q}\setminus\mathbf{L}}} P(Y)\right),$$

   we have

$$V[(\mathbf{Q}\setminus\mathbf{L})\|\mathbf{P}] = V[\mathbf{P}]f(\mathbf{Q}\setminus\mathbf{L}).$$

2. *For* $\mathbf{K} \subset_k \mathbf{P}$*, defining*

$$g(X_{\mathbf{P}\setminus\mathbf{K}}) = \partial^{[\mathbf{P}\setminus\mathbf{K}]}\left(V(X_{\mathbf{P}\setminus\mathbf{K}})\prod_{X\in X_{\mathbf{P}\setminus\mathbf{K}}} Q(X)\right),$$

   we have

$$V[\mathbf{Q}\|(\mathbf{P}\setminus\mathbf{K})] = V[\mathbf{Q}]g(\mathbf{P}\setminus\mathbf{K})$$

Proof. Defining

$$\begin{aligned} h(X_{\mathbf{P}}, Y_{\mathbf{Q}\setminus\mathbf{L}}) &= \partial^{[\mathbf{Q}\setminus\mathbf{L}]}V(Y_{\mathbf{Q}\setminus\mathbf{L}}\|X_{\mathbf{P}}) \\ &= \partial^{[\mathbf{Q}\setminus\mathbf{L}]}\left(V(X_{\mathbf{P}})\Pi(X_{\mathbf{P}}, Y_{\mathbf{Q}\setminus\mathbf{L}})V(Y_{\mathbf{Q}\setminus\mathbf{L}})\right) \\ &= V(X_{\mathbf{P}})\partial^{[\mathbf{Q}\setminus\mathbf{L}]}\left(\Pi(X_{\mathbf{P}}, Y_{\mathbf{Q}\setminus\mathbf{L}})V(Y_{\mathbf{Q}\setminus\mathbf{L}})\right) \end{aligned}$$

and applying Lemma 15, we get

$$\begin{aligned} \partial^{[\mathbf{P}]}h(\mathbf{P}, Y_{\mathbf{Q}\setminus\mathbf{L}}) &= V[\mathbf{P}]\partial^{[\mathbf{Q}\setminus\mathbf{L}]}\left(V(Y_{\mathbf{Q}\setminus\mathbf{L}})\Pi(\mathbf{P}, Y_{\mathbf{Q}\setminus\mathbf{L}})\right) \\ &= V[\mathbf{P}]\partial^{[\mathbf{Q}\setminus\mathbf{L}]}\left(V(Y_{\mathbf{Q}\setminus\mathbf{L}})\prod_{Y\in Y_{\mathbf{Q}\setminus\mathbf{L}}}(-1)^p P(Y)\right) \\ &= V[\mathbf{P}]f(Y_{\mathbf{Q}\setminus\mathbf{L}}). \end{aligned}$$

and finally

$$V[(\mathbf{Q}\setminus\mathbf{L})\|\mathbf{P}] = \partial^{[\mathbf{P}]}h(\mathbf{P}, \mathbf{Q}\setminus\mathbf{L}) = V[\mathbf{P}]f(\mathbf{Q}\setminus\mathbf{L}).$$

Which is Lemma 16.1.
The proof for Lemma 16.2 is similar. □

Lemma 16 has the following corollary.

Corollary 17. *If* Q *divides* P*, then*

$$\mathrm{Sylv}^{0,j}(P,Q)(U) = 0$$

Proof. In this case, $f(\mathbf{Q}\setminus\mathbf{L}) = 0$ as any root of Q is a root of P with at least the same multiplicity. So, applying Lemma 16.1, $V[(\mathbf{Q}\setminus\mathbf{L})\|\mathbf{P}] = 0$. It follows

$$\mathrm{Sylv}^{0,j}(P,Q)(U) = \sum_{\mathbf{L}\subset_j\mathbf{Q}} s_{\mathbf{L}}\frac{V[(\mathbf{Q}\setminus\mathbf{L})\|\mathbf{P}]V[\mathbf{L}\|U)}{V[\mathbf{P}]V[\mathbf{Q}]} = 0$$

□

3 Hermite Interpolation for multivariate symetric polynomials

We now introduce an Hermite interpolation for multivariate symmetric polynomials.

We consider an ordered set of $p-k$ variables $\mathbf{U}$.

Proposition 18. *The set*

$$\mathcal{B}_{\mathbf{P},k}(\mathbf{U}) = \left\{ \frac{V[\mathbf{K}\|\mathbf{U})}{V[\mathbf{P}]V(\mathbf{U})} \mid \mathbf{K} \subset_k \mathbf{P} \right\}$$

is a basis of the vector-space of symmetric polynomials in $\mathbf{U}$ *of multidegree at most* $k, \ldots, k$.

The proof of Proposition 18 uses the following Lemma.

Lemma 19.

1. $V[\mathbf{K}\|(\mathbf{P} \setminus \mathbf{K})] = (-1)^{k(p-k)} s_{\mathbf{K}} V[\mathbf{P}] \neq 0.$
2. *If* $\mathbf{K}' \neq \mathbf{K}$, $V[\mathbf{K}'\|(\mathbf{P} \setminus \mathbf{K})] = 0.$

Proof.

1. It is clear that $V[\mathbf{K}\|(\mathbf{P} \setminus \mathbf{K})] = (-1)^{k(p-k)} s_{\mathbf{K}} V[\mathbf{P}] \neq 0$, since $s_{\mathbf{K}}$ is the signature of the permutation putting $\mathbf{P}$ in the order $(\mathbf{P} \setminus \mathbf{K})\|\mathbf{K}$.
2. The fact that $V[\mathbf{K}'\|(\mathbf{P} \setminus \mathbf{K})] = 0$ when $\mathbf{K}' \neq \mathbf{K}$ follows from the fact that the matrix $\mathcal{V}[\mathbf{K}'\|(\mathbf{P} \setminus \mathbf{K})]$ has two equal columns. □

Proof of Proposition 18. Since the number of subsets of cardinality k of $\mathbf{P}$ is $\binom{p}{k}$ and that $\binom{p}{k}$ is also the dimension of the vector space of symmetric polynomials in $\mathbf{U}$ of multidegree at most $k, \ldots, k$, it is enough to prove that

$$\sum_{\mathbf{K}' \subset_k \mathbf{P}} c_{\mathbf{K}'} \frac{V[\mathbf{K}'\|\mathbf{U})}{V[\mathbf{P}]V(\mathbf{U})} = 0$$

implies $c_{\mathbf{K}} = 0$ for all $\mathbf{K} \subset_k \mathbf{P}$.
Let us fix $\mathbf{K} \subset_k \mathbf{P}$. Since

$$\sum_{\mathbf{K}' \subset_k \mathbf{P}} c_{\mathbf{K}'} V[\mathbf{K}'\|\mathbf{U}) = 0,$$

it follows by substitution and derivation that

$$\sum_{\mathbf{K}' \subset_k \mathbf{P}} c_{\mathbf{K}'} \partial^{[\mathbf{P} \setminus \mathbf{K}]} V[\mathbf{K}'\| X_{\mathbf{P} \setminus \mathbf{K}}) = 0.$$

When replacing $X_{\mathbf{P}\setminus\mathbf{K}}$ by $\mathbf{P}\setminus\mathbf{K}$ we obtain

$$\sum_{\mathbf{K}'\subset_k\mathbf{P}} c_{\mathbf{K}'} V[\mathbf{K}'\|(\mathbf{P}\setminus\mathbf{K})] = 0.$$

Using Lemma 19, we get $c_{\mathbf{K}} = 0$. □

The following Proposition gives the connection between a symetric polynomial in $\mathbf{U}$ of multidegree at most $k,\ldots,k$ and its coordinates in the basis $\mathcal{B}_{\mathbf{P},k}(\mathbf{U})$.

Proposition 20. (Multivariate symmetric Hermite Interpolation) *Let g be a symetric polynomial in $\mathbf{U}$ of multidegree at most $k,\ldots,k$. Writing*

$$g(\mathbf{U}) = \sum_{\mathbf{K}\subset_k\mathbf{P}} g_{\mathbf{K}} \frac{V[\mathbf{K}\|\mathbf{U})}{V[\mathbf{P}]V(\mathbf{U})}$$

then

$$g_{\mathbf{K}} = (-1)^{k(p-k)} s_{\mathbf{K}} h(\mathbf{P}\setminus\mathbf{K})$$

with

$$h(X_{\mathbf{P}\setminus\mathbf{K}}) = \partial^{[\mathbf{P}\setminus\mathbf{K}]}(V(X_{\mathbf{P}\setminus\mathbf{K}})g(X_{\mathbf{P}\setminus\mathbf{K}})).$$

Proof. We have

$$\sum_{\mathbf{K}\subset_k\mathbf{P}} g_{\mathbf{K}} V[\mathbf{K}\|\mathbf{U}) = V[\mathbf{P}]g(\mathbf{U})V(\mathbf{U}).$$

Derivating both sides by $\partial^{[\mathbf{P}\setminus\mathbf{K}']}$ and substituting $\mathbf{P}\setminus\mathbf{K}'$ for $\mathbf{U}$ we get, using Lemma 19

$$g_{\mathbf{K}'} V[\mathbf{K}'\|(\mathbf{P}\setminus\mathbf{K}')] = g_{\mathbf{K}'} s_{\mathbf{K}'} (-1)^{k(p-k)} V[\mathbf{P}] = V[\mathbf{P}]h(\mathbf{P}\setminus\mathbf{K}'),$$

and finally

$$g_{\mathbf{K}'} = (-1)^{k(p-k)} s_{\mathbf{K}'} h(\mathbf{P}\setminus\mathbf{K}').$$

□

Remark 21. Proposition 18 generalizes a result in [Chen and Louck, 1996] given for Lagrange interpolation of symmetric multivariate polynomials.

As a corollary of Proposition 20, we recover the classical Hermite Interpolation

Proposition 22. (Hermite Interpolation) *Given an ordered list*

$$\mathbf{q} = (q_{1,0},\ldots,q_{1,\mu_1-1},\ldots,q_{m,0},\ldots,q_{m,\mu_m-1})$$

of p numbers, there is one and only one polynomial of degree at most $p-1$ satisfying the property

$$\text{for all } 1\le i\le m, \text{ for all } 0\le j<\mu_i,\ Q^{[j]}(x_i) \ = \ q_{i,j}.$$

Proof. If $k = p - 1$ in Proposition 18, then

$$\mathcal{B}_{\mathbf{P},p-1}(U) = \left\{ \frac{V[\mathbf{P} \setminus \{x_{i,j}\} \| U)}{V[\mathbf{P}]} \mid x_{i,j} \in \mathbf{P} \right\}$$

is a basis of the vector space of univariate polynomials in U of degree at most $\leq p - 1$.
Note that $(-1)^{p-1} s_{\mathbf{P} \setminus \{x_{i,j}\}} = (-1)^{\mu_i + \cdots + \mu_m - j - 1}$. So, the family

$$\left\{ (-1)^{\mu_i + \cdots + \mu_m - j - 1} q_{i,j} \mid i = 1, \ldots, m, j = 0, \ldots \mu_i - 1 \right\}$$

is the coordinates in the basis $\mathcal{B}_{\mathbf{P},p-1}(U)$ of a polynomial $Q(U)$ (necessarily unique) of degree at most $p - 1$ such that $Q^{[j]}(x_i) = q_{i,j}$, applying Proposition 20. □

4 Multi Sylvester double sums

We introduce in subsection 4.1 multi Sylvester double sums and study their properties, using the Hermite interpolation for symmetric multivariate polynomials. In subsection 4.2 we compute the multi Sylvester double sums and Sylvester double sums for indices (k, ℓ) with $k + \ell \geq q$. In subsection 4.3 we prove the fundamental property of Sylvester double sums, i.e. that Sylvester double sums indexed by k, ℓ, depend only (up to a constant) on $j = k + \ell < p$. This was already known in the simple roots case but even in this case our proof is new.

4.1 Definition of multi Sylvester double sums

The idea of replacing the variable U by a block of indeterminates to define multi Sylvester double sums is directly inspired from [Krick et al., 2016].

Definition 23. *The* multi Sylvester double sum*, for (k, ℓ) a pair of natural numbers with $k + \ell = j$, is the polynomial* $\mathrm{MSylv}^{k,\ell}(P, Q)(\mathbf{U})$*, where* $\mathbf{U}$ *is a block of indeterminates of cardinality $p - j$,*

$$\mathrm{MSylv}^{k,\ell}(P, Q)(\mathbf{U}) = \sum_{\substack{\mathbf{K} \subset_k \mathbf{P} \\ \mathbf{L} \subset_\ell \mathbf{Q}}} s_{\mathbf{K}} s_{\mathbf{L}} \frac{V[(\mathbf{Q} \setminus \mathbf{L}) \| (\mathbf{P} \setminus \mathbf{K})] V[\mathbf{L} \| \mathbf{K} \| \mathbf{U})}{V[\mathbf{P}] V[\mathbf{Q}] V(\mathbf{U})} \tag{9}$$

In particular

$$\mathrm{MSylv}^{j,0}(P, Q)(\mathbf{U}) = \sum_{\mathbf{K} \subset_j \mathbf{P}} s_{\mathbf{K}} \frac{V[\mathbf{Q} \| (\mathbf{P} \setminus \mathbf{K})]}{V[\mathbf{Q}]} \frac{V[\mathbf{K} \| \mathbf{U})}{V[\mathbf{P}] V(\mathbf{U})} \tag{10}$$

The following proposition gives the relationship between multi Sylvester double sums and Sylvester double sums.

Proposition 24. *Denoting* $\mathbf{U} = U \| \mathbf{U}'$ *with* $\mathbf{U}'$ *a block of* $p - j - 1$ *indeterminates,*

$\mathrm{Sylv}^{k,\ell}(P,Q)(U)$ *is the coefficient of* $\prod_{U' \in \mathbf{U}'} U'^j$ *in* $\mathrm{MSylv}^{k,\ell}(P,Q)(\mathbf{U})$.

The proof of Proposition 24 is based on the following Lemma.

Lemma 25. $V[\mathbf{K}\|U]$ *is the coefficient of* $\prod_{U' \in \mathbf{U}'} U'^k$ *in* $\dfrac{V[\mathbf{K}\|U\|\mathbf{U}')}{V(U\|\mathbf{U}')}$.

Proof.

$$\begin{aligned}\frac{\partial^{[\mathbf{K}]} V(X_{\mathbf{K}}\|U\|\mathbf{U}')}{V(U\|\mathbf{U}')} &= \frac{\partial^{[\mathbf{K}]}(V(X_{\mathbf{K}}\|U)\Pi(\mathbf{U}',X_{\mathbf{K}})\Pi(\mathbf{U}',U)V(\mathbf{U}'))}{\Pi(\mathbf{U}',U)V(\mathbf{U}')} \\ &= \partial^{[\mathbf{K}]}(V(X_{\mathbf{K}}\|U)\Pi(\mathbf{U}',X_{\mathbf{K}}))\end{aligned}$$

Noting that

$$\partial^{[\mathbf{K}]}\left(V(X_{\mathbf{K}}\|U)\Pi(\mathbf{U}',X_{\mathbf{K}})\right) = \partial^{[\mathbf{K}]} V(X_{\mathbf{K}}\|U) \times \Pi(\mathbf{U}',X_{\mathbf{K}}) + \sum_r V_r(X_{\mathbf{K}},U)\Pi_r(\mathbf{U}',X_{\mathbf{K}})$$

where each $\Pi_r(\mathbf{U}',X_{\mathbf{K}})$ is obtained by partial derivation of $\Pi(\mathbf{U}',X_{\mathbf{K}})$ with respect to at least one variable in $X_{\mathbf{K}}$, it is clear that the degree of some $U' \in \mathbf{U}'$ in $\Pi_r(\mathbf{U}',X_{\mathbf{K}})$ is less than k. The claim follows, substituting $\mathbf{K}$ to $X_{\mathbf{K}}$. □

Proof of Proposition 24. The coefficient of $\prod_{U' \in \mathbf{U}'} U'^j$ in $\dfrac{V[\mathbf{L}\|\mathbf{K}\|U\|\mathbf{U}')}{V(U\|\mathbf{U}')}$ is $V[\mathbf{L}\|\mathbf{K}\|U)$ by Lemma 25. The coefficient of $\prod_{U' \in \mathbf{U}'} U'^j$ in $\mathrm{MSylv}^{k,\ell}(P,Q)(\mathbf{U})$ is

$$\sum_{\substack{\mathbf{K} \subset_k \mathbf{P} \\ \mathbf{L} \subset_\ell \mathbf{Q}}} s_{\mathbf{K}} s_{\mathbf{L}} \frac{V[(\mathbf{Q} \setminus \mathbf{L})\|(\mathbf{P} \setminus \mathbf{K})]V[\mathbf{L}\|\mathbf{K}\|U)}{V[\mathbf{P}]V[\mathbf{Q}]} = \mathrm{Sylv}^{k,\ell}(P,Q)(U)$$

by Proposition 13. □

4.2 Computation of (multi) Sylvester double sums for $j \geq q$

Proposition 26. *If* $q \leq j < p$

$$\mathrm{MSylv}^{j,0}(P,Q)(\mathbf{U}) = (-1)^{j(p-j)} \prod_{U \in \mathbf{U}} Q(U)$$

Proof. The polynomial $\prod_{U \in \mathbf{U}} Q(U)$ is a symmetric polynomial in $\mathbf{U}$ of multidegree $q, \ldots, q$, so at most $j, \ldots, j$. Its coordinates in the basis $\mathcal{B}_{\mathbf{P},j}(\mathbf{U})$ are, for $\mathbf{K} \subset_j \mathbf{P}$, $(-1)^{j(p-j)} s_{\mathbf{K}} h(\mathbf{P} \setminus \mathbf{K})$ where

$$h(X_{\mathbf{P}\setminus\mathbf{K}}) = \partial^{[\mathbf{P}\setminus\mathbf{K}]}\left(V(X_{\mathbf{P}\setminus\mathbf{K}}) \prod_{X \in X_{\mathbf{P}\setminus\mathbf{K}}} Q(X)\right)$$

by Proposition 20, and moreover

$$h(\mathbf{P} \setminus \mathbf{K}) = \frac{V[\mathbf{Q} \| (\mathbf{P} \setminus \mathbf{K})]}{V[\mathbf{Q}]}$$

by Lemma 16.2.

So, the polynomials $\mathrm{MSylv}^{j,0}(P,Q)(\mathbf{U})$ and $(-1)^{j(p-j)} \prod_{U \in \mathbf{U}} Q(U)$ have the same coordinates in the basis $\mathcal{B}_{\mathbf{P},k}(\mathbf{U})$ and are equal. □

As a corollary

Proposition 27.

1. $\mathrm{Sylv}^{p-1,0}(P,Q)(U) = (-1)^{p-1} Q(U)$
2. *For any* $q < j < p-1$, $\mathrm{Sylv}^{j,0}(P,Q)(U) = 0$
3. $\mathrm{Sylv}^{q,0}(P,Q)(U) = (-1)^{q(p-q)} Q(U)$

Proof.

1. For $j = p-1$ Proposition 26 is exactly

$$\mathrm{Sylv}^{p-1,0}(P,Q)(U) = (-1)^{p-1} Q(U).$$

.

2. If $q < j < p-1$, denoting $\mathbf{U} = U \| \mathbf{U}'$ with $\mathbf{U}'$ a block of $p-j-1$ indeterminates, the coefficient of $\prod_{U' \in \mathbf{U}'} U'^j$ in $\prod_{U' \in \mathbf{U}} Q(U')$ is equal to 0, so $\mathrm{Sylv}^{j,0}(P,Q)(U) = 0$ applying Proposition 26.

3. From Proposition 26 and Proposition 24, denoting $\mathbf{U} = U \| \mathbf{U}'$ with $\mathbf{U}'$ a block of $p-q-1$ indeterminates, we know that $\mathrm{Sylv}^{q,0}(P,Q)(U)$ is equal to the coefficient of $\prod_{U' \in \mathbf{U}'} U'^q$ in $(-1)^{q(p-q)} Q(U) \prod_{U' \in \mathbf{U}'} Q(U')$. This coefficient is exactly $(-1)^{q(p-q)} Q(U)$. □

Proposition 28. *If* $\ell \le q \le k + \ell = j < p$ *then*

$$\mathrm{MSylv}^{k,\ell}(P,Q,\mathbf{U}) = (-1)^{\ell(p-j)} \binom{q}{\ell} \mathrm{MSylv}^{j,0}(P,Q,\mathbf{U})$$

Proof. Let $\mathbf{L} \subset_\ell \mathbf{Q}$ and $\mathbf{U}' = (U'_1, \ldots, U'_{p-k})$; the polynomial $\dfrac{V[(\mathbf{Q} \setminus \mathbf{L}) \| \mathbf{U}')}{V(\mathbf{U}')}$ is a symmetric polynomial in the indeterminates $\mathbf{U}'$ of degree at most $q - \ell \le k$ in each indeterminate $U'_i, 1 \le i \le p-k$. So, we can write this polynomial in the basis $\mathcal{B}_{\mathbf{P},k}(\mathbf{U}')$

$$\frac{V[(\mathbf{Q} \setminus \mathbf{L}) \| \mathbf{U}')}{V(\mathbf{U}')} = \sum_{\mathbf{K} \subset_k \mathbf{P}} g_{\mathbf{K}} \frac{V[\mathbf{K} \| \mathbf{U}')}{V[\mathbf{P}] V(\mathbf{U}')}$$

where, by Proposition 20

$$g_{\mathbf{K}} = (-1)^{k(p-k)} s_{\mathbf{K}} V[(\mathbf{Q} \setminus \mathbf{L}) \| (\mathbf{P} \setminus \mathbf{K})].$$

We deduce from this

$$V[(\mathbf{Q} \setminus \mathbf{L}) \| \mathbf{U}') = \sum_{\mathbf{K} \subset_k \mathbf{P}} (-1)^{k(p-k)} s_{\mathbf{K}} V[(\mathbf{Q} \setminus \mathbf{L}) \| (\mathbf{P} \setminus \mathbf{K})] \frac{V[\mathbf{K} \| \mathbf{U}')}{V[\mathbf{P}]}$$

We replace $\mathbf{U}'$ by $\mathbf{U}' = Y_{\mathbf{L}} \| \mathbf{U}$, where $\mathbf{U}$ is a set of $p-j$ indeterminates, derivate with respect to $\partial^{[\mathbf{L}]}$ and replace $Y_{\mathbf{L}}$ by $\mathbf{L}$; we obtain

$$\begin{aligned} V[(\mathbf{Q} \setminus \mathbf{L}) \| \mathbf{L} \| \mathbf{U}) &= \sum_{\mathbf{K} \subset_k \mathbf{P}} (-1)^{k(p-k})s_{\mathbf{K}} V[(\mathbf{Q} \setminus \mathbf{L}) \| (\mathbf{P} \setminus \mathbf{K})] \frac{V[\mathbf{K} \| \mathbf{L} \| \mathbf{U})}{V[\mathbf{P}]} \\ &= (-1)^{k\ell} (-1)^{k(p-k)} \sum_{\mathbf{K} \subset_k \mathbf{P}} s_{\mathbf{K}} V[(\mathbf{Q} \setminus \mathbf{L}) \| (\mathbf{P} \setminus \mathbf{K})] \frac{V[\mathbf{L} \| \mathbf{K} \| \mathbf{U})}{V[\mathbf{P}]} \end{aligned}$$

As

$$V[(\mathbf{Q} \setminus \mathbf{L}) \| \mathbf{L} \| \mathbf{U}) = s_{\mathbf{L}} V[\mathbf{Q} \| \mathbf{U}),$$

we have

$$V[\mathbf{Q} \| \mathbf{U}) = \sum_{\mathbf{K} \subset_k \mathbf{P}} (-1)^{k(p-j)} s_{\mathbf{K}} s_{\mathbf{L}} V[(\mathbf{Q} \setminus \mathbf{L}) \| (\mathbf{P} \setminus \mathbf{K})] \frac{V[\mathbf{L} \| \mathbf{K} \| \mathbf{U})}{V[\mathbf{P}]}$$

and

$$\frac{V[\mathbf{Q} \| \mathbf{U})}{V(\mathbf{U})} = \sum_{\mathbf{K} \subset_k \mathbf{P}} (-1)^{k(p-j)} s_{\mathbf{K}} s_{\mathbf{L}} V[(\mathbf{Q} \setminus \mathbf{L}) \| (\mathbf{P} \setminus \mathbf{K})] \frac{V[\mathbf{L} \| \mathbf{K} \| \mathbf{U})}{V[\mathbf{P}] V(\mathbf{U})}.$$

The polynomial $\dfrac{V[\mathbf{Q} \| \mathbf{U})}{V(\mathbf{U})}$ is a symmetric polynomial in the indeterminates $\mathbf{U}$ of degree at most $q \leq j$ in each indeterminate $U_i, 1 \leq i \leq p-j$. So, we can write it in the basis $\mathcal{B}_{\mathbf{P},j}(\mathbf{U})$

$$\frac{V[\mathbf{Q} \| \mathbf{U})}{V(\mathbf{U})} = \sum_{\mathbf{W} \subset_j \mathbf{P}} g_{\mathbf{W}} \frac{V[\mathbf{W} \| \mathbf{U})}{V[\mathbf{P}] V(\mathbf{U})}$$

where by Proposition 20

$$g_{\mathbf{W}} = (-1)^{j(p-j)} s_{\mathbf{W}} V[\mathbf{Q} \| (\mathbf{P} \setminus \mathbf{W})].$$

So

$$\sum_{\mathbf{W} \subset_j \mathbf{P}} s_{\mathbf{W}} V[\mathbf{Q} \| (\mathbf{P} \setminus \mathbf{W})] V[\mathbf{W} \| \mathbf{U}) = \sum_{\mathbf{K} \subset_k \mathbf{P}} (-1)^{\ell(p-j)} s_{\mathbf{K}} s_{\mathbf{L}} V[(\mathbf{Q} \setminus \mathbf{L}) \| (\mathbf{P} \setminus \mathbf{K})] V[\mathbf{L} \| \mathbf{K} \| \mathbf{U}).$$

It follows

$$\sum_{\substack{\mathbf{W}\subset_j\mathbf{P}\\ \mathbf{L}\subset_\ell\mathbf{Q}}} s_{\mathbf{W}} V[\mathbf{Q}\|(\mathbf{P}\backslash\mathbf{W})]V[\mathbf{W}\|\mathbf{U}) = (-1)^{\ell(p-j)} \sum_{\substack{\mathbf{K}\subset_k\mathbf{P}\\ \mathbf{L}\subset_\ell\mathbf{Q}}} s_{\mathbf{K}} s_{\mathbf{L}} V[(\mathbf{Q}\backslash\mathbf{L})\|(\mathbf{P}\backslash\mathbf{K})]V[\mathbf{L}\|\mathbf{K}\|\mathbf{U})$$

$$\binom{q}{\ell} \sum_{\mathbf{W}\subset_j\mathbf{P}} s_{\mathbf{W}} V[\mathbf{Q}\|(\mathbf{P}\backslash\mathbf{W})]V[\mathbf{W}\|\mathbf{U}) = (-1)^{\ell(p-j)} \sum_{\substack{\mathbf{K}\subset_k\mathbf{P}\\ \mathbf{L}\subset_\ell\mathbf{Q}}} s_{\mathbf{K}} s_{\mathbf{L}} V[(\mathbf{Q}\backslash\mathbf{L})\|(\mathbf{P}\backslash\mathbf{K})]V[\mathbf{L}\|\mathbf{K}\|\mathbf{U})$$

$$\binom{q}{\ell} \frac{\sum_{\mathbf{W}\subset_j\mathbf{P}} s_{\mathbf{W}} V[\mathbf{Q}\|(\mathbf{P}\setminus\mathbf{W})]V[\mathbf{W}\|\mathbf{U})}{V[\mathbf{P}]V[\mathbf{Q}]V(\mathbf{U})} = (-1)^{\ell(p-j)} \sum_{\substack{\mathbf{K}\subset_k\mathbf{P}\\ \mathbf{L}\subset_\ell\mathbf{Q}}} s_{\mathbf{K}} s_{\mathbf{L}} \frac{V[(\mathbf{Q}\setminus\mathbf{L})\|(\mathbf{P}\setminus\mathbf{K})]V[\mathbf{L}\|\mathbf{K}\|\mathbf{U})}{V[\mathbf{P}]V[\mathbf{Q}]V(\mathbf{U})}$$

and

$$\mathrm{MSylv}^{k,\ell}(P,Q,\mathbf{U}) = (-1)^{\ell(p-j)} \binom{q}{\ell} \mathrm{MSylv}^{j,0}(P,Q,\mathbf{U}). \qquad \square$$

Corollary 29. *For $q \le j < p$,*

$$\mathrm{Sylv}^{k,\ell}(P,Q) = (-1)^{\ell(p-j)} \binom{q}{\ell} \mathrm{Sylv}^{j,0}(P,Q)$$

Proof. Immediate using Proposition 28 and Proposition 24. □

Proposition 30.

1. *For any (k,ℓ) with $q = k+\ell$,*

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = (-1)^{k(p-q)} \binom{q}{k} Q$$

2. *For any (k,ℓ) with $\ell \le q, j = k+\ell$ with $q < j < p-1$,*

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = 0$$

3. *For any (k,ℓ) with $\ell \le q, k+\ell = p-1$,*

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = (-1)^{k} \binom{q}{\ell} Q(U)$$

Proof. Follows from Corollary 29 and Proposition 27. □

4.3 Fundamental property of (multi) Sylvester double sums

This section is essentially devoted to the proof of Theorem 31, which is a fundamental property of Sylvester double sums: up to a constant Sylvester double sums $\mathrm{Sylv}^{r,j-r}(P,Q)$ depend only on j. Such a result has been already given for $q \le j < p$ by Corollary 29.

Theorem 31. *If $k \in \mathbb{N}$, $\ell \in \mathbb{N}$, $k+\ell = j < q < p$*

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = (-1)^{\ell(p-j)} \begin{pmatrix} j \\ \ell \end{pmatrix} \mathrm{Sylv}^{j,0}(P,Q)(U).$$

We, in fact, prove Theorem 31 as a corollary of a multivariate version (Theorem 32). The proof of Theorem 32 uses in an essential way the Exchange Lemma coming from [Krick et al., 2016].

Theorem 32. *If $k \in \mathbb{N}$, $\ell \in \mathbb{N}$, $k+\ell = j < q < p$, and $\mathbf{U}$ a set of $p-j$ indeterminates,*

$$\mathrm{MSylv}^{k,\ell}(P,Q)(\mathbf{U}) = (-1)^{\ell(p-j)} \begin{pmatrix} j \\ \ell \end{pmatrix} \mathrm{MSylv}^{j,0}(P,Q)(\mathbf{U}).$$

To prove Theorem 32 for $j < q$, we need a lemma

Lemma 33. *Let $\mathbf{K} \subset_k \mathbf{P}, \mathbf{L} \subset_\ell \mathbf{Q}$ and $\mathbf{U} = U_1, \ldots, U_u$ an ordered set of variables. Then $(-1)^{u(u-1)/2} V[\mathbf{L} \| \mathbf{K}]$ is the coefficient of the leading monomial*

$$\prod_{i=1}^{u} U_i^{k+\ell+u-i}$$

of $V[\mathbf{L}\|\mathbf{K}\|\mathbf{U})$ with respect to the lexicographical ordering.

Proof.

$$\begin{aligned} V[\mathbf{L}\|\mathbf{K}\|\mathbf{U}) &= \partial^{[\mathbf{K}]}\partial^{[\mathbf{L}]}(V(Y_{\mathbf{L}}\|X_{\mathbf{K}}\|\mathbf{U}))(\mathbf{K},\mathbf{L},\mathbf{U}) \\ &= V(\mathbf{U})\partial^{[\mathbf{K}]}\partial^{[\mathbf{L}]}(V(Y_{\mathbf{L}}\|X_{\mathbf{K}})\Pi(\mathbf{U}, Y_{\mathbf{L}}\|X_{\mathbf{K}}))(\mathbf{K},\mathbf{L}). \end{aligned}$$

The coefficient of $\prod_{i=1}^{u} U_i^{k+\ell+u-i}$ in $V[\mathbf{L}\|\mathbf{K}\|\mathbf{U})$ is $(-1)^{u(u-1)/2}$ multiplied by the coefficient of $\prod_{i=1}^{u} U_i^{k+\ell}$ in $\partial^{[\mathbf{K}]}\partial^{[\mathbf{L}]}(V(Y_{\mathbf{L}}\|X_{\mathbf{K}})\Pi(\mathbf{U}, X_{\mathbf{K}}\|Y_{\mathbf{L}})(\mathbf{K},\mathbf{L})$. This coefficient is

$$\partial^{[\mathbf{K}]}\partial^{[\mathbf{L}]}V(Y_{\mathbf{L}}\|X_{\mathbf{K}})(\mathbf{K},\mathbf{L}) = V[\mathbf{L}\|\mathbf{K}];$$

indeed if any derivation is done on $\Pi(\mathbf{U}, X_{\mathbf{K}}\|Y_{\mathbf{L}})$, with respect to $\mathbf{K}$ or $\mathbf{L}$, the degree in at least one indeterminate $U_i \in \mathbf{U}$ decreases strictly. □

Proof of Theorem 32 . We have $j < q$. Let $\mathbf{U}'$ be a block of $p-\ell$ indeterminates. On one hand,

$$\begin{aligned}\sum_{\mathbf{T}\subset_\ell \mathbf{P}} \Pi(X_{\mathbf{P}\setminus\mathbf{T}}, Y_{\mathbf{Q}}) \frac{\Pi(\mathbf{U}', X_{\mathbf{T}})}{\Pi(X_{\mathbf{P}\setminus\mathbf{T}}, X_{\mathbf{T}})} &= \sum_{\mathbf{T}\subset_\ell \mathbf{P}} \frac{V(Y_{\mathbf{Q}}\|X_{\mathbf{P}\setminus\mathbf{T}})V(X_{\mathbf{T}}\|\mathbf{U}')}{V(Y_{\mathbf{Q}})V(\mathbf{U}')V(X_{\mathbf{T}}\|X_{\mathbf{P}\setminus\mathbf{T}})} \\ &= (-1)^{\ell(p-\ell)} \sum_{\mathbf{T}\subset_\ell \mathbf{P}} s_{\mathbf{T}} \frac{V(Y_{\mathbf{Q}}\|X_{\mathbf{P}\setminus\mathbf{T}})V(X_{\mathbf{T}}\|\mathbf{U}')}{V(Y_{\mathbf{Q}})V(\mathbf{U}')V(X_{\mathbf{P}})}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{\mathbf{L}\subset_\ell \mathbf{Q}} \Pi(X_{\mathbf{P}}, Y_{\mathbf{Q}\setminus\mathbf{L}}) \frac{\Pi(\mathbf{U}', Y_{\mathbf{L}})}{\Pi(Y_{\mathbf{L}}, Y_{\mathbf{Q}\setminus\mathbf{L}})} &= \sum_{\mathbf{L}\subset_\ell \mathbf{Q}} \frac{V(Y_{\mathbf{Q}\setminus\mathbf{L}}\|X_{\mathbf{P}})V(Y_{\mathbf{L}}\|\mathbf{U}')}{V(X_{\mathbf{P}})V(\mathbf{U}')V(Y_{\mathbf{Q}\setminus\mathbf{L}}\|Y_{\mathbf{L}})} \\ &= \sum_{\mathbf{L}\subset_\ell \mathbf{Q}} s_{\mathbf{L}} \frac{V(Y_{\mathbf{Q}\setminus\mathbf{L}}\|X_{\mathbf{P}})V(Y_{\mathbf{L}}\|\mathbf{U}')}{V(X_{\mathbf{P}})V(\mathbf{U}')V(Y_{\mathbf{Q}})}\end{aligned}$$

From the Exchange Lemma in [Krick et al., 2016], we can write

$$\sum_{\mathbf{T}\subset_\ell \mathbf{P}} \Pi(X_{\mathbf{P}\setminus\mathbf{T}}, Y_{\mathbf{Q}}) \frac{\Pi(\mathbf{U}', X_{\mathbf{T}})}{\Pi(X_{\mathbf{P}\setminus\mathbf{T}}, X_{\mathbf{T}})} = \sum_{\mathbf{L}\subset_\ell \mathbf{Q}} \Pi(X_{\mathbf{P}}, Y_{\mathbf{Q}\setminus\mathbf{L}}) \frac{\Pi(\mathbf{U}', Y_{\mathbf{L}})}{\Pi(Y_{\mathbf{L}}, Y_{\mathbf{Q}\setminus\mathbf{L}})} \tag{11}$$

So, we deduce from (11)

$$\sum_{\mathbf{T}\subset_\ell \mathbf{P}} s_{\mathbf{T}} \frac{V(Y_{\mathbf{Q}}\|X_{\mathbf{P}\setminus\mathbf{T}})V(X_{\mathbf{T}}\|\mathbf{U}')}{V(Y_{\mathbf{Q}})V(\mathbf{U}')V(X_{\mathbf{P}})} = (-1)^{\ell(p-\ell)} \sum_{\mathbf{L}\subset_\ell \mathbf{Q}} s_{\mathbf{L}} \frac{V(Y_{\mathbf{Q}\setminus\mathbf{L}}\|X_{\mathbf{P}})V(Y_{\mathbf{L}}\|\mathbf{U}')}{V(X_{\mathbf{P}})V(\mathbf{U}')V(Y_{\mathbf{Q}})} \tag{12}$$

$$\sum_{\mathbf{T}\subset_\ell \mathbf{P}} s_{\mathbf{T}} V(Y_{\mathbf{Q}}\|X_{\mathbf{P}\setminus\mathbf{T}})V(X_{\mathbf{T}}\|\mathbf{U}') = (-1)^{\ell(p-\ell)} \sum_{\mathbf{L}\subset_\ell \mathbf{Q}} s_{\mathbf{L}} V(Y_{\mathbf{Q}\setminus\mathbf{L}}\|X_{\mathbf{P}})V(Y_{\mathbf{L}}\|\mathbf{U}') \tag{13}$$

Hence, derivating with respect to $\mathbf{Q}$ and substituting $\mathbf{Q}$ to $Y_{\mathbf{Q}}$,

$$\sum_{\mathbf{T}\subset_\ell \mathbf{P}} s_{\mathbf{T}} V[\mathbf{Q}\|X_{\mathbf{P}\setminus\mathbf{T}})V(X_{\mathbf{T}}\|\mathbf{U}') = (-1)^{\ell(p-\ell)} \sum_{\mathbf{L}\subset_\ell \mathbf{Q}} s_{\mathbf{L}} V[(\mathbf{Q}\setminus\mathbf{L})\|X_{\mathbf{P}})V[\mathbf{L}\|\mathbf{U}') \tag{14}$$

We fix $\mathbf{K} \subset_k \mathbf{P}$. The total degree with respect to $X_{\mathbf{K}}$ of $V[\mathbf{Q}\setminus\mathbf{L}\|X_{\mathbf{P}}]V[\mathbf{L}\|\mathbf{U}')$ is

$$d_1 = \binom{k}{2} + k(p+q-j).$$

Denoting, for any $\mathbf{T} \subset_\ell \mathbf{P}$, c the cardinality of $\mathbf{K}\cap\mathbf{T}$, we note that the cardinality of $(\mathbf{P}\setminus\mathbf{T})\cap\mathbf{K}$ is $k-c$.
So, the total degree with respect to $X_{\mathbf{K}}$ of $V[\mathbf{Q}\|X_{\mathbf{P}\setminus\mathbf{T}})V(X_{\mathbf{T}}\|\mathbf{U}')$ is

$$d_{2,c} = \binom{k-c}{2} + (k-c)(p+q-j+c) + \binom{c}{2} + c(p-c),$$

i.e.

$$d_{2,c} = \begin{pmatrix} k \\ 2 \end{pmatrix} + k(p+q-j) - c(q-j+c)$$

and

$$d_1 - d_{2,c} = c(q-j+c)$$

So $d_{2,c} < d_1$ if $c > 0$ and $d_{2,c} = d_1$ if $c = 0$ i.e. if $\mathbf{T} \subset \mathbf{P} \setminus \mathbf{K}$. This implies that subsets $\mathbf{T}$ which intersect $\mathbf{K}$ don't contribute to the homogeneous part of total degree d_1 in $X_{\mathbf{K}}$ on the left side of (14).
Note that, if $\mathbf{T} \subset_\ell \mathbf{P} \setminus \mathbf{K}$,

$$V[\mathbf{Q}\|X_{\mathbf{P}\setminus\mathbf{T}}) = r_{\mathbf{K},\mathbf{T}} V[\mathbf{Q}\|X_{\mathbf{P}\setminus(\mathbf{K}\cup\mathbf{T})}\|X_{\mathbf{K}}),$$

where $r_{\mathbf{K},\mathbf{T}}$ is the signature of the permutation $\rho_{\mathbf{K},\mathbf{T}}$ taking the ordered set $\mathbf{P} \setminus \mathbf{T}$ to the ordered set $(\mathbf{P} \setminus (\mathbf{K} \cup \mathbf{T}))\|\mathbf{K}$.

We can also write

$$V[(\mathbf{Q} \setminus \mathbf{L})\|X_{\mathbf{P}}) = s_{\mathbf{K}} V[(\mathbf{Q} \setminus \mathbf{L})\|X_{\mathbf{P}\setminus\mathbf{K}}\|X_{\mathbf{K}})$$

If $X_{\mathbf{K}} = X_{u_1}, \ldots, X_{u_k}$, taking the coefficient of $\prod_{i=1}^{k} X_{u_k}^{k-i+p+q-j}$ in both sides of (14) gives, by Lemma 33

$$\sum_{\mathbf{T}\subset_\ell \mathbf{P}\setminus\mathbf{K}} r_{\mathbf{K},\mathbf{T}} s_{\mathbf{T}} V[\mathbf{Q}\|X_{\mathbf{P}\setminus(\mathbf{K}\cup\mathbf{T})}) V(X_{\mathbf{T}}\|\mathbf{U}') = (-1)^{\ell(p-\ell)} \sum_{\mathbf{L}\subset_\ell \mathbf{Q}} s_{\mathbf{K}} s_{\mathbf{L}} V[(\mathbf{Q}\setminus\mathbf{L})\|X_{\mathbf{P}\setminus\mathbf{K}}) V[\mathbf{L}\|\mathbf{U}'). \tag{15}$$

Derivating both sides of (15) with respect to $\partial^{[\mathbf{P}\setminus\mathbf{K}]}$ and replacing $X_{\mathbf{P}\setminus\mathbf{K}}$ by $\mathbf{P} \setminus \mathbf{K}$, followed by replacing $\mathbf{U}'$ by $X_{\mathbf{K}}\|\mathbf{U}$, where $\mathbf{U}$ is a set of $p-j$ indeterminates, derivating with respect to $\partial^{[\mathbf{K}]}$ and replacing $X_{\mathbf{K}}$ by $\mathbf{K}$ gives

$$\sum_{\mathbf{T}\subset_\ell \mathbf{P}\setminus\mathbf{K}} r_{\mathbf{K},\mathbf{T}} s_{\mathbf{T}} V[\mathbf{Q}\|(\mathbf{P}\setminus(\mathbf{K}\cup\mathbf{T}))] V[\mathbf{T}\|\mathbf{K}\|\mathbf{U}) = (-1)^{\ell(p-\ell)} \sum_{\mathbf{L}\subset_\ell \mathbf{Q}} s_{\mathbf{K}} s_{\mathbf{L}} V[(\mathbf{Q}\setminus\mathbf{L})\|(\mathbf{P}\setminus\mathbf{K})] V[\mathbf{L}\|\mathbf{K}\|\mathbf{U}). \tag{16}$$

Summing with respect to $\mathbf{K}$, ve get

$$\sum_{\substack{\mathbf{K}\subset_k \mathbf{P} \\ \mathbf{T}\subset_\ell \mathbf{P}\setminus\mathbf{K}}} r_{\mathbf{K},\mathbf{T}} s_{\mathbf{T}} V[\mathbf{Q}\|(\mathbf{P}\setminus(\mathbf{K}\cup\mathbf{T}))] V[\mathbf{T}\|\mathbf{K}\|\mathbf{U}) = (-1)^{\ell(p-\ell)} \mathrm{MSylv}^{k,\ell}(P,Q)(U) V[\mathbf{P}] V[\mathbf{Q}] V(\mathbf{U}).$$

Denote $\mathbf{W}$ the set $\mathbf{K} \cup \mathbf{T}$ ordered by the induced order on $\mathbf{P}$. Let $\tau_{\mathbf{K},\mathbf{T}}$ be the permutation sending the ordered set $\mathbf{P} \setminus \mathbf{W}\|\mathbf{W}$ to the ordered set $\mathbf{P} \setminus \mathbf{W}\|\mathbf{T}\|\mathbf{K}$ and $t_{\mathbf{K},\mathbf{T}}$ its signature. We deduce

$$\sum_{\substack{\mathbf{K}\subset_k \mathbf{P} \\ \mathbf{T}\subset_\ell \mathbf{P}\setminus\mathbf{K}}} t_{\mathbf{K},\mathbf{T}} r_{\mathbf{K},\mathbf{T}} s_{\mathbf{T}} V[\mathbf{Q}\|(\mathbf{P}\setminus\mathbf{W})] V[\mathbf{W}\|\mathbf{U}) = (-1)^{\ell(p-\ell)} \mathrm{MSylv}^{k,\ell}(P,Q)(U) V[\mathbf{P}] V[\mathbf{Q}] V(\mathbf{U}).$$

We remark that $t_{\mathbf{K},\mathbf{T}} r_{\mathbf{K},\mathbf{T}} s_{\mathbf{T}} = (-1)^{k\ell} s_{\mathbf{W}}$. Indeed, denoting $\iota_{\mathbf{K},\mathbf{T}}$ the permutation sending the ordered set $(\mathbf{P} \setminus \mathbf{W})\|\mathbf{T}\|\mathbf{K}$ to the ordered set $(\mathbf{P} \setminus \mathbf{W})\|\mathbf{K}\|\mathbf{T}$, with signature $(-1)^{k\ell}$, and by $\rho'_{\mathbf{K},\mathbf{T}}$ the permutation sending the ordered set $(\mathbf{P} \setminus \mathbf{T})\|\mathbf{T}$ to the ordered set $(\mathbf{P} \setminus (\mathbf{K} \cup \mathbf{T}))\|\mathbf{K}\|\mathbf{T}$, with signature $r_{\mathbf{K},\mathbf{T}}$, we have the follwing sequence of permutations

$$\begin{array}{rrcl}
\sigma_{\mathbf{W}}: & \mathbf{P} & \longleftrightarrow & (\mathbf{P} \setminus \mathbf{W})\|\mathbf{W} \\
\tau_{\mathbf{K},\mathbf{T}}: & (\mathbf{P} \setminus \mathbf{W})\|\mathbf{W} & \longleftrightarrow & (\mathbf{P} \setminus \mathbf{W})\|\mathbf{T}\|\mathbf{K} \\
\iota_{\mathbf{K},\mathbf{T}}: & (\mathbf{P} \setminus \mathbf{W})\|\mathbf{T}\|\mathbf{K} & \longleftrightarrow & (\mathbf{P} \setminus \mathbf{W})\|\mathbf{K}\|\mathbf{T} \\
{\rho'}_{\mathbf{K},\mathbf{T}}^{-1}: & (\mathbf{P} \setminus \mathbf{W})\|\mathbf{K}\|\mathbf{T} & \longleftrightarrow & (\mathbf{P} \setminus \mathbf{T})\|\mathbf{T} \\
\sigma_{\mathbf{T}}^{-1}: & (\mathbf{P} \setminus \mathbf{T})\|\mathbf{T} & \longleftrightarrow & \mathbf{P} \qquad ,
\end{array}$$

with $\sigma_{\mathbf{T}} \circ {\rho'}_{\mathbf{K},\mathbf{T}}^{-1} \circ \iota_{\mathbf{K},\mathbf{T}} \circ \tau_{\mathbf{K},\mathbf{T}} \circ \sigma_{\mathbf{W}}^{-1} = \mathrm{Id}$.

Noting that there are $\binom{j}{\ell}$ ways of decomposing $\mathbf{W} \subset_j \mathbf{P}$ as $\mathbf{W} = \mathbf{K} \cup \mathbf{T}$, we get

$$\mathrm{MSylv}^{k,\ell}(P,Q)(\mathbf{U}) = (-1)^{\ell(p-j)} \binom{j}{\ell} \mathrm{MSylv}^{j,0}(P,Q)(\mathbf{U}). \quad \square$$

Proof of Theorem 31. Theorem 31 is an immediate consequence of Theorem 32, by applying Proposition 24. □

5 Sylvester double sums and remainders

In Section 5 we give a relationship between the Sylvester double sums of P, Q and those of Q, R where R is the opposite of the remainder of P by Q in the Euclidean division.

We are now dealing with not necessarily monic polynomials.

Definition 34. *Let P be a polynomial of degree p which leading coefficient is denoted* $\mathrm{lc}(P)$. *Let Q be a polynomial of degree q which leading coefficient is denoted* $\mathrm{lc}(Q)$.
Let (k,ℓ) with $j = k + \ell \le p$ be a pair of natural numbers. We define

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = \mathrm{lc}(P)^{q-j}\mathrm{lc}(Q)^{p-j}\mathrm{Sylv}^{k,\ell}\left(\frac{P}{\mathrm{lc}(P)}, \frac{Q}{\mathrm{lc}(Q)}\right)(U)$$

Remark 35. Note that if $k \in \mathbb{N}$, $\ell \in \mathbb{N}, \ell \le q$, $k + \ell = j < q$

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = (-1)^{\ell(p-j)} \binom{j}{\ell} \mathrm{Sylv}^{j,0}(P,Q)(U)$$

follows immediately from Theorem 31 and Definition 34.

We use Notation 7 to define the ordered sets $\mathbf{P}$ and $\mathbf{Q}$ representing the multisets of roots of P and Q.

Rewriting Lemma 16 in the non monic case, we get Lemma 36.

Lemma 36.

1. *For* $\mathbf{L} \subset_\ell \mathbf{Q}$*, defining*

$$f(Y_{\mathbf{Q}\setminus\mathbf{L}}) = (-1)^{p(q-\ell)}\partial^{[\mathbf{Q}\setminus\mathbf{L}]}\left(V(Y_{\mathbf{Q}\setminus\mathbf{L}})\prod_{Y\in Y_{\mathbf{Q}\setminus\mathbf{L}}} P(Y)\right),$$

we have

$$f(\mathbf{Q}\setminus\mathbf{L}) = \mathrm{lc}(P)^{q-\ell}\frac{V[(\mathbf{Q}\setminus\mathbf{L})\|\mathbf{P})]}{V[\mathbf{P}]}$$

2. *For* $\mathbf{K} \subset_k \mathbf{P}$*, defining*

$$g(X_{\mathbf{P}\setminus\mathbf{K}}) = \partial^{[\mathbf{P}\setminus\mathbf{K}]}\left(V(X_{\mathbf{P}\setminus\mathbf{K}})\prod_{X\in X_{\mathbf{P}\setminus\mathbf{K}}} Q(X)\right),$$

we have

$$g(\mathbf{P}\setminus\mathbf{K}) = \mathrm{lc}(Q)^{p-k}\frac{V[\mathbf{Q}\|(\mathbf{P}\setminus\mathbf{K})]}{V[\mathbf{Q}]}$$

Similarly, reewriting Proposition 27 in the non monic case, we get Proposition 37.

Proposition 37.

1. $\mathrm{Sylv}^{p-1,0}(P,Q)(U) = (-1)^{p-1}\mathrm{lc}(P)^{q-p+1}Q(U)$
2. *For any* $q < j < p-1$, $\mathrm{Sylv}^{j,0}(P,Q)(U) = 0$
3. $\mathrm{Sylv}^{q,0}(P,Q)(U) = (-1)^{q(p-q)}\mathrm{lc}(Q)^{p-q-1}Q(U)$

We proceed now to the proof of Proposition 38 wich is the main result of Section 5.

Proposition 38. *Let* $R = -\mathrm{Rem}(\mathrm{P},\mathrm{Q})$*. If* $j \in \mathbb{N}$*,* $j < q$

- *If* $R = 0$, $\mathrm{Sylv}^{j,0}(P,Q)(U) = 0$.
- *If* $R \neq 0$, $\mathrm{Sylv}^{j,0}(P,Q)(U) = (-1)^{q(p-q)}\mathrm{lc}(Q)^{p-r}\mathrm{Sylv}^{j,0}(Q,R)(U)$

The following elementary lemma plays a key role in the proof of Proposition 38.

Lemma 39. *Let* $R = -\mathrm{Rem}(\mathrm{P},\mathrm{Q})$*. For every* $y_{i,j} \in \mathbf{Q}$*,* $0 \le j' < j$*,*

$$P^{[j']}(y_i) = -R^{[j']}(y_i)$$

Proof. Write $P = CQ - R$, derivate j' times and evaluate at y_i. $\square$

Proof of Proposition 38. If $R = 0$, $\mathrm{Sylv}^{0,j}\left(\dfrac{P}{\mathrm{lc}(P)}, \dfrac{Q}{\mathrm{lc}(Q)}\right)(U) = 0$ follows from Corollary 17. So,

$$\mathrm{Sylv}^{j,0}(P,Q)(U) = (-1)^{j(p-j)}\mathrm{Sylv}^{0,j}(P,Q)(U) = 0$$

.

If $R \neq 0$, let r be the degree of R. Let $(z_1, \ldots, z_v)$ be an ordered set of the distinct roots of R in an algebraic closure $\mathbf{C}$ of $\mathbb{K}$, with z_i of multiplicity ξ_i, and, as in Notation 7, let $\mathbf{R}$ be the multiset of roots of R, represented by the ordered set

$$\mathbf{R} = (z_{1,0}, \ldots, z_{1,\xi_1 - 1}, \ldots, z_{v,0}, \ldots, z_{v,\xi_v - 1}),$$

with $z_{i,j} = (z_i, j)$ for $0 \le j \le \xi_i - 1$, $\sum_{i=1}^{v} \xi_i = r$.
If $j \le q$, define for $\mathbf{L} \subset_j \mathbf{Q}$

$$\begin{array}{rcl} f(Y_{\mathbf{Q}\setminus\mathbf{L}}) & = & (-1)^{p(q-j)}\partial^{[\mathbf{Q}\setminus\mathbf{L}]}\left(V(Y_{\mathbf{Q}\setminus\mathbf{L}})\prod_{Y\in Y_{\mathbf{Q}\setminus\mathbf{L}}} P(Y)\right) \\ h(Y_{\mathbf{Q}\setminus\mathbf{L}}) & = & \partial^{[\mathbf{Q}\setminus\mathbf{L}]}\left(V(Y_{\mathbf{Q}\setminus\mathbf{L}})\prod_{Y\in Y_{\mathbf{Q}\setminus\mathbf{L}}} R(Y)\right) \end{array}$$

Note that

$$f(\mathbf{Q}\setminus\mathbf{L}) = (-1)^{(p+1)(q-j)}h(\mathbf{Q}\setminus\mathbf{L})$$

from Lemma 39.
So

$$\begin{aligned}
\mathrm{Sylv}^{0,j}(P,Q)(U) &= \frac{\mathrm{lc}(P)^{q-j}\mathrm{lc}(Q)^{p-j}}{V[\mathbf{P}]V[\mathbf{Q}]}\sum_{\mathbf{L}\subset_j\mathbf{Q}} s_{\mathbf{L}} V[(\mathbf{Q}\setminus\mathbf{L})\|\mathbf{P}]V[\mathbf{L}\|U] \\
&= \frac{\mathrm{lc}(Q)^{p-j}}{V[\mathbf{Q}]}\sum_{\mathbf{L}\subset_j\mathbf{Q}} s_{\mathbf{L}} f(\mathbf{Q}\setminus\mathbf{L})V[\mathbf{L}\|U] \text{ applying Lemma 36.1} \\
&= (-1)^{(p+1)(q-j)}\frac{\mathrm{lc}(Q)^{p-j}}{V[\mathbf{Q}]}\sum_{\mathbf{L}\subset_j\mathbf{Q}} s_{\mathbf{L}} h(\mathbf{Q}\setminus\mathbf{L})V[\mathbf{L}\|U] \\
&= (-1)^{(p+1)(q-j)}\frac{\mathrm{lc}(Q)^{p-j}}{V[\mathbf{Q}]}\sum_{\mathbf{L}\subset_j\mathbf{Q}} s_{\mathbf{L}} \frac{\mathrm{lc}(R)^{q-j}V[\mathbf{R}\|(\mathbf{Q}\setminus\mathbf{L})]}{V[\mathbf{R}]}V[\mathbf{L}\|U] \text{ applying Lemma 36.2} \\
&= (-1)^{(p+1)(q-j)}\frac{\mathrm{lc}(Q)^{p-j}\mathrm{lc}(R)^{q-j}}{V[\mathbf{Q}]V[\mathbf{R}]}\sum_{\mathbf{L}\subset_j\mathbf{Q}} s_{\mathbf{L}} V[\mathbf{R}\|(\mathbf{Q}\setminus\mathbf{L})]V[\mathbf{L}\|U] \\
&= (-1)^{(p+1)(q-j)}\frac{\mathrm{lc}(Q)^{p-r}\mathrm{lc}(Q)^{r-j}\mathrm{lc}(R)^{q-j}}{V[\mathbf{Q}]V[\mathbf{R}]}\sum_{\mathbf{L}\subset_j\mathbf{Q}} s_{\mathbf{L}} V[\mathbf{R}\|(\mathbf{Q}\setminus\mathbf{L})]V[\mathbf{L}\|U] \\
&= (-1)^{(p+1)(q-j)}\mathrm{lc}(Q)^{p-r}\mathrm{Sylv}^{j,0}(Q,R)
\end{aligned}$$

The claim follows since

$$\mathrm{Sylv}^{j,0}(P,Q)(U) = (-1)^{j(p-j)}\mathrm{Sylv}^{0,j}(P,Q)(U)$$

by Theorem 31 and

$$(-1)^{j(p-j)}(-1)^{(p+1)(q-j)} = (-1)^{q(p-q)}.$$

□

6 Sylvester double sums and subresultants

Finally we prove in this section that Sylvester double sums coincide (up to a constant) with subresultants, by an induction on the length of the remainder sequence of P and Q.
This section is devoted to the proof of the link between double sums and subresultants which is known in the simple case (see [Lascoux and Pragacz, 2002, d'Andrea et al., 2007, Roy and Szpirglas, 2011]).

Notation. $\varepsilon_k = (-1)^{k(k-1)/2}$. The sign ε_k is the signature of the permutation reversing the order i.e. sending $1, 2, \ldots, k-1, k$ to $k, k-1, \ldots, 2, 1$. We have also $\varepsilon_k = 1$ if $k \equiv 0, 1 \mod 4$, $\varepsilon_k = -1$ if $k \equiv 2, 3 \mod 4$. As a consequence

$$\varepsilon_{i+1} = (-1)^i \varepsilon_i. \tag{17}$$

We have also

$$\varepsilon_{i+j} = (-1)^{ij} \varepsilon_i \varepsilon_j, \tag{18}$$

which follows from the fact that reversing $i+j$ numbers can be done in three steps: reversing the first i ones, then the last j one and placing the last j numbers in front of the i first.
The main theorem of this section is the following.

Theorem 40. *Let $k \in \mathbb{N}$, $\ell \in \mathbb{N}$, $\ell \leq q$, $k+\ell = j < p-1$*

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = (-1)^{k(p-j)} \varepsilon_{p-j} \begin{pmatrix} j \\ k \end{pmatrix} \mathrm{Sres}_j(P,Q)(U).$$

Remark 41. When $j = p-1$, $\mathrm{Sres}_{p-1}(P,Q)(U) = Q(U)$ by convention; so, as

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = (-1)^k \begin{pmatrix} q \\ \ell \end{pmatrix} \mathrm{lc}(Q)^{q-p+1} Q(U) \text{ for } k+\ell = p-1, \text{ we get}$$

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = (-1)^k \begin{pmatrix} q \\ \ell \end{pmatrix} \mathrm{lc}(Q)^{q-p+1} \mathrm{Sres}_{p-1}(P,Q)(U).$$

In order to prove Theorem 40, we use an induction on the length of the remainder sequence of P and Q based on Proposition 38.
Before proving Theorem 40, we recall the following properties of subresultants.

Lemma 42. *Let $R = -\mathrm{Rem}(\mathrm{P}, \mathrm{Q})$.*

1. $q < j < p-1$	$\mathrm{Sres}_j(P,Q)(U)$	$= 0$
2. $j = q$	$\mathrm{Sres}_q(P,Q)(U)$	$= \varepsilon_{p-q} \mathrm{lc}(Q)^{p-q-1} Q(U)$
3. $j = q-1$	$\mathrm{Sres}_{q-1}(P,Q)(U)$	$= \varepsilon_{p-q} \mathrm{lc}(Q)^{p-q+1} R(U)$
4. $j < q-1, R \neq 0$	$\mathrm{Sres}_j(P,Q)(U)$	$= \varepsilon_{p-q} \mathrm{lc}(Q)^{p-r} \mathrm{Sres}_j(Q,R)(U)$
5. $j < q-1, R = 0$	$\mathrm{Sres}_j(P,Q)(U)$	$= 0$

Proof. All items follow from [Basu et al., 2003] except the computation of $\mathrm{Sres}_{q-1}(P,Q)(U)$. $\mathrm{Sres}_{q-1}(P,Q)(U)$ is clearly equal to $\varepsilon_{p-q+2}\mathrm{lc}(Q)^{p-q+1}(-R(U))$ by replacing the row of P by a row of $-R$ in the Sylvester-Habicht matrix, and reversing the order of its $p-q+2$ rows. Notice now that $\varepsilon_{p-q+2} = -\varepsilon_{p-q}$. □

We also recall the following similar properties of Sylvester double sums.

Lemma 43. *Let* $R = -\mathrm{Rem}(\mathrm{P},\mathrm{Q})$. *Let* $j \in \mathbb{N}$, $j < p-1$

1. $q < j < p-1$	$\mathrm{Sylv}^{j,0}(P,Q)(U)$	$= 0,$
2. $j = q$	$\mathrm{Sylv}^{q,0}(P,Q)(U)$	$= (-1)^{q(p-q)}\mathrm{lc}(Q)^{p-q-1}Q(U)$
3. $j = q-1$	$\mathrm{Sylv}^{q-1,0}\,(P,Q)\,(U)$	$= (-1)^{(q-1)(p-q+1)+p-q}\mathrm{lc}(Q)^{p-q+1}R(U)$
4. $j < q-1, R \neq 0$	$\mathrm{Sylv}^{j,0}(P,Q)(U)$	$= (-1)^{q(p-q)}\mathrm{lc}(Q)^{p-r}\mathrm{Sylv}^{j,0}(Q,R)(U)$
5. $j < q-1, R = 0$	$\mathrm{Sylv}^{j,0}(P,Q)(U)$	$= 0,$

Proof. All items follow from Proposition 37 except for the computation of $\mathrm{Sylv}^{q-1,0}(P,Q)(U)$. Using Proposition 38 and Proposition 37 for Q, R, we obtain

$$\mathrm{Sylv}^{q-1,0}(P,Q)(U) = (-1)^{q(p-q)}\mathrm{lc}(Q)^{p-r}\mathrm{Sylv}^{q-1,0}(Q,R)(U) = (-1)^{q-1+q(p-q)}\mathrm{lc}(Q)^{p-r+r-q+1}R(U)$$

It remains to remark that $(q-1)(p-q+1)+p-q = q(p-q)+q-1$. □

Proof of Theorem 40. The statement for $q \leq j < p-1$ follows from Lemma 42 1,2, Lemma 43 1,2 and Theorem 31.
The statement for $j = q-1$ follows from Lemmma 42 3, Lemma 43 3, Theorem 31 and (17) since $\varepsilon_{p-q+1} = (-1)^{p-q}\varepsilon_{p-q}$.
For $j < q-1$ we first prove the special case

$$\mathrm{Sylv}^{j,0}(P,Q) = (-1)^{j(p-j)}\varepsilon_{p-j}\mathrm{Sres}_j(P,Q) \tag{19}$$

The proof is by induction on the length of the remainder sequence of P, Q.
The basic case is when Q divides P, i.e. $R = 0$, and the claim is true by Lemma 42 5, Lemma 43 5.
Otherwise suppose, by induction hypothesis that

$$\mathrm{Sylv}^{j,0}(Q,R) = (-1)^{j(q-j)}\varepsilon_{q-j}\mathrm{Sres}_j(Q,R) \tag{20}$$

Using Lemma 42 5 and Lemma 43 5 it remains to note that

$$(-1)^{j(p-j)}(-1)^{j(q-j)}(-1)^{q(p-q)} = (-1)^{(q-j)(p-q)}$$

and conclude by (18) since $\varepsilon_{p-j} = (-1)^{(q-j)(p-q)}\varepsilon_{p-q}\varepsilon_{q-j}$.
The general case for k, ℓ now follows from Theorem 31.

□

The authors thank the referees for their relevant remarks. Special thanks to Daniel Perrucci for a very careful rereading.

References

[Basu et al., 2003] Basu, S., Pollack, R., and Roy, M.-F. (2003). *Algorithms in real algebraic geometry*. Springer.

[Chen and Louck, 1996] Chen, W. and Louck, J. (1996). Interpolation for symmetric functions. *Advances in Mathematics*, 117.

[d'Andrea et al., 2007] d'Andrea, C., Hong, H., Krick, T., and Szanto, A. (2007). An elementary proof of sylvester's double sums for subresultants. *Journal of Symbolic Computation*, 42-3.

[Horn and Johnson, 1991] Horn, R. and Johnson, C. (1991). *Topics in matrix analysis*. Cambridge University Press.

[Krick et al., 2016] Krick, T., Szanto, A., and Valdetarro, M. (2016). Symmetric interpolation, exchange lemma and double sums. *Communications in Algebra*, 45.

[Lancaster and M.Tismenetsky, 1985] Lancaster, P. and M.Tismenetsky (1985). *The theory of matrices*. Academic Press.

[Lascoux and Pragacz, 2002] Lascoux, A. and Pragacz, P. (2002). Double sylvester sums for subresultants and multi-schur fonctions. *Journal of Symbolic Computation*, 35-6.

[Roy and Szpirglas, 2011] Roy, M.-F. and Szpirglas, A. (2011). Sylvester double sums and subresultants. *Journal of Symbolic Computation*, 46.

[Sylvester, 1840] Sylvester, J. J. (1840). Note on elimination. *Philosophical Magazine*, XVII.

[Sylvester, 1853] Sylvester, J. J. (1853). A theory of the syzygetic relations of two rational integral functions. *Philosophical Transactions of the Royal Society of London*, CXLIII-III.