BOUNDING COHOMOLOGY ON A SMOOTH PROJECTIVE SURFACE

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ABSTRACT. The following conjecture arose out of discussions between B. Harbourne, J. Roé, C. Cilberto and R. Miranda: for a smooth projective surface X there exists a positive constant c_X such that $h^1(\mathcal{O}_X(C)) \leq c_X h^0(\mathcal{O}_X(C))$ for every prime divisor C on X. We show that the conjecture is true for some smooth projective surfaces with Picard number 2.

1. INTRODUCTION

In this note we work over the field $\mathbb{C}$ of complex numbers. By a $(negative)$ $curve$ on a surface we will mean a reduced, irreducible curve (with negative self-intersection). By a $(-k)$-$curve$, we mean a negative curve C with $C^2 = -k < 0$.

The bounded negativity conjecture (BNC for short) is one of the most intriguing problems in the theory of projective surfaces and can be formulated as follows.

Conjecture 1.1. [3, Conjecture 1.1] *For a smooth projective surface X there exists an integer $b(X) \geq 0$ such that $C^2 \geq -b(X)$ for every curve $C \subseteq X$.*

Let us say that a smooth projective surface X has

$$b(X) > 0$$

if there is at least one negative curve on X.

In [7], T. Bauer, P. Pokora and D. Schmitz established the following theorem.

Theorem 1.2. [7, Theorem] *For a smooth projective surface X over an an algebraic closed field the following two statements are equivalent:*

(i) X has bounded Zariski denominators.
(ii) X satisfies the BNC.

Here, X has bounded Zariski denominators (cf. [7]) if there exists an integer $d(X) \geq 1$ such that for every pseudo-effective integral divisor D the denominators in the Zariski decomposition of D are bounded from above by $d(X)$ (cf. [19, 10]).

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The main aim of this note is to study the following conjecture, which implies Conjecture 1.1 (cf. [9, Proposition 14]).

Conjecture 1.3. [2, Conjecture 2.5.3] *Let X be a smooth projective surface. Then there exists a positive constant c_X such that $h^1(\mathcal{O}_X(C)) \leq c_X h^0(\mathcal{O}_X(C))$ for every curve C on X.*

In [2], the authors disproved Conjecture 1.3 by giving a counterexample of surface of general type (cf. [2, Corollary 3.1.2]). However, they pointed out that it could still be true that Conjecture 1.3 holds when restricted to *rational* surfaces, in any characteristic. Indeed, the smooth projective rational surfaces with an effective anticanoncial divisor satisfy Conjecture 1.3 (cf. [2, Proposition 3.1.3]). In particular, if X is the blow-up of $\mathbb{P}^2$ at n generic points and $c_X = 0$, then Conjecture 1.3 for this X is an equivalent version of the SHGH conjecture as follows.

Conjecture 1.4. [2, Conjecture 2.5.1] *Let X be the blow-up of $\mathbb{P}^2$ at n generic points. Then $h^1(X, \mathcal{O}_X(C)) = 0$ for every curve C on X.*

In order to give our main result, we now recall the following question posed in [9].

Question 1.5. [9, Question 4] Does there exist a constant $m(X)$ such that $\frac{(K_X \cdot D)}{D^2} < m(X)$ for any effective divisor D with $D^2 > 0$ on a smooth projective surface X?

If Conjecture 1.3 is true for a smooth projective surface X, then X is affirmative for Question 1.5 (cf. [9, Proposition 15]). This motivates us to give the following definition.

Definition 1.6. Let X be a smooth projective surface.

(1) For every $\mathbb{R}$-divisor D with $D^2 \neq 0$ on X, we define a value of D as follows:
$$l_D := \frac{(K_X \cdot D)}{\max\left\{1, D^2\right\}}.$$
(2) For every $\mathbb{R}$-divisor D with $D^2 = 0$ on X, we define a value of D as follows:
$$l_D := \frac{(K_X \cdot D)}{\max\left\{1, h^0(\mathcal{O}_X(D))\right\}}.$$
(3) X satisfies $\mathbf{Hyp(A)}$ if $\overline{\mathrm{NE}}(X) = \sum_{i=1}^{\rho(X)} \mathbb{R}_{\geq 0}[C_i]$ such that each C_i is a curve. Here, $\rho(X)$ is the Picard number of X.
(4) X satisfies $\mathbf{Hyp(B)}$ if there exists a positive constant $m(X)$ such that $l_C \leq m(X)$ for every curve $C^2 \neq 0$ on X.
(5) X satisfies $\mathbf{Hyp(C)}$ if there exists a positive constant $m(X)$ such that $l_C \leq m(X)$ for every curve C on X.

To solve Conjecture 1.3 partially, for the case when $\rho(X)=2$, we give the main result as follows.

Theorem 1.7. *Let X be a smooth projective surface. The following statements hold.*

(1) If X satisfies the BNC and there exists a positive constant $m(X)$ such that $l_C \leq m(X)$ for every curve C on X and $D^2 \leq m(X)h^0(\mathcal{O}_X(D))$ for every curve D with $l_D > 1$ and $D^2 > 0$ on X, then X satisfies Conjecture 1.3.

(2) Suppose $\kappa(X)=0$ and the canonical divisor K_X is nef. Then X satisfies Conjecture 1.3.

(3) Suppose $\rho(X)=2$ and $\kappa(X)=-\infty$. Then X satisfies $\mathbf{Hyp}(\mathbf{B})$. In particular, every ruled surface with invariant $e>0$ satisfies Conjecture 1.3.

(4) Suppose $\rho(X)=2$ and X has two negative curves. Then X satisfies Conjecture 1.3.

(5) Supoose $\rho(X)=2, \kappa(X)=1$ and $b(X)>0$. Then X satisfies Conjecture 1.3.

(6) Suppose $\rho(X)=2$ and X satisfies $\mathbf{Hyp}(\mathbf{A})$. Then X satisfies $\mathbf{Hyp}(\mathbf{B})$.

Remark 1.8. (1) It is hard to establish that there exists a positive constant $m(X)$ such that $l_D \leq m(X)$ for $D \in |nC|$ with the Iitaka dimension $\kappa(X,C)=1$ and $n \gg 0$, where C is a curve on X. This is related to effectivity of Iitaka fibrations, which are known for the pluricanonical system $|mK_X|$ of every smooth projective variety X in arbitrary dimension (cf. [12, 18, 8]). Therefore, we have to consider a weaker hypothesis $\mathbf{Hyp}(\mathbf{B})$.

(2) In [16, Claim 2.11], we give a classification of the smooth projective surfaces X with $\rho(X)=2$ and two negative curves C_1 and C_2. Here, the closed Mori cone $\overline{\mathrm{NE}}(X)=\mathbb{R}_{\geq 0}[C_1]+\mathbb{R}_{\geq 0}[C_2]$, i.e, X satisfies $\mathbf{Hyp}(\mathbf{A})$. Moreover, see Remark 2.11 about Theorem 1.7(6).

2. The Proof of Theorem 1.7

In this section, we divide our proof of Theorem 1.7 into some steps.

Proof of Theorem 1.7 (1). Take a curve C on X. Note that by Serre duality (cf. [11, Corollary III.7.7 and III.7.12]), $h^2(\mathcal{O}_X(C))=h^0(\mathcal{O}_X(K_X-C)) \leq p_g(X)$. As a result,

$$h^2(\mathcal{O}_X(C))-\chi(\mathcal{O}_X) \leq q(X)-1. \tag{2.1}$$

Here, $p_g(X)$ and $q(X)$ are the geometric genus of X and the irregularity of X respectively. Our main condition is the following:

(*) There exists a positive constant $m(X)$ such that $l_C \leq m(X)$ for every curve C on X and $D^2 \leq m(X)h^0(\mathcal{O}_X(D))$ for every curve D with $l_D>1$ and $D^2>0$ on X.

We divide the proof into the following three cases.

Case (i). Suppose $C^2>0$. Then by Riemann-Roch theorem (cf. [11, Theorem V.1.6]),

$$h^1(\mathcal{O}_X(C))=h^0(\mathcal{O}_X(C))+h^2(\mathcal{O}_X(C))-\chi(\mathcal{O}_X)+\frac{C^2(l_C-1)}{2}. \tag{2.2}$$

If $l_C \leq 1$, then Equation (2.1) and (2.2) imply that $h^1(\mathcal{O}_X(C)) \leq h^0(\mathcal{O}_X(C)) + q(X) - 1$, which is the desired result by $c_X := q(X)$. If $l_C > 1$, then Equation (2.1) and (2.2) and the condition (*) imply that $2h^1(\mathcal{O}_X(C)) \leq (m^2(X) - m(X) + 2)h^0(\mathcal{O}_X(C)) + 2(q(X) - 1)$, which is the desired result by $2c_X := m^2(X) - m(X) + q(X)$.

Case (ii). Suppose $C^2 = 0$. Then by Riemann-Roch theorem,

$$2h^1(\mathcal{O}_X(C)) = 2h^2(\mathcal{O}_X(C)) - 2\chi(\mathcal{O}_X) + h^0(\mathcal{O}_X(C))(l_C + 2), \tag{2.3}$$

which, Equation (2.1) and the condition (*) imply that

$$2h^1(\mathcal{O}_X(C)) \leq 2(q(X) - 1) + h^0(\mathcal{O}_X(C))(m(X) + 2),$$

which is the desired result by $2c_X := m(X) + 2q(X)$.

Case (iii). Suppose $C^2 < 0$. Then $h^0(\mathcal{O}_X(C)) = 1$. Since X satisfies the BNC, there exists a positive constant $b(X)$ such that every curve C on X has $C^2 \geq -b(X)$. By Riemann-Roch theorem,

$$2h^1(\mathcal{O}_X(C)) = 2 + 2h^2(\mathcal{O}_X(C)) - 2\chi(\mathcal{O}_X) + l_C - C^2, \tag{2.4}$$

which, Equation (2.1) and the condition (*) imply that $2h^1(\mathcal{O}_X(C)) \leq 2q(X) + m(X) + b(X)$, which is the desired result by $2c_X := 2q(X) + m(X) + b(X)$.

In all, we complete the proof of Theorem 1.7(1). □

Remark 2.1. (1) Suppose X satisfies Conjecture 1.3. Then by [9, Proposition 15], Equation (2.3) and (2.4), X satisfies $\mathbf{Hyp}(\mathbf{B})$ and $\mathbf{Hyp}(\mathbf{C})$.

(2) The condition of Theorem 1.7(1) may be not necessary. Let $c_X \gg 1$. Take a curve C with $C^2 > 0$ on X. Suppose X satisfies Conjecture 1.3. Then Equation (2.1) and (2.2) imply that

$$\begin{aligned}\frac{C^2(l_C - 1)}{2} &= h^1(\mathcal{O}_X(C)) - h^0(\mathcal{O}_X(C)) - h^2(\mathcal{O}_X(C)) + \chi(\mathcal{O}_X)\\ &\leq (c_X - 1)h^0(\mathcal{O}_X(C)) + \chi(\mathcal{O}_X).\end{aligned}$$

If we can find a sequence $\{l_{C_i}\}$ on X such that $C_i^2 > 0, l_{C_i} > 1$ and $\lim_{i\to\infty} l_{C_i} = 1$, then it is unknown that there exists a positive constant $m(X)$ such that $C^2 \leq m(X)h^0(\mathcal{O}_X(C))$ for every curve C with $C^2 > 0$ and $l_C > 1$ on X. Therefore, the following question is asked.

Question 2.2. Let X be a smooth projective surface. Suppose X satisfies Conjecture 1.3. Is there a positive constant $m(X)$ such that $C^2 \leq m(X)h^0(\mathcal{O}_X(C))$ for every curve C with $C^2 > 0$ and $l_C > 1$ on X?

Proof of Theorem 1.7(2). Since $\kappa(X) = 0$ and K_X is nef, $K_X \equiv 0$ (numerical). As a result, $l_C = 0$ for every curve C on X. By the adjunction formula, $C^2 \geq -2$. By Riemann-Roch theorem, $2h^1(\mathcal{O}_X(C)) = 2h^0(\mathcal{O}_X(C)) + 2h^2(\mathcal{O}_X(C)) - 2\chi(\mathcal{O}_X) - C^2$, which and Equation (2.1) imply that $h^1(\mathcal{O}_X(C)) \leq (q(X) + 1)h^0(\mathcal{O}_X(C))$. Therefore, X satisfies Conjecture 1.3. □

Lemma 2.3. *Every ruled surface satisfies* $\mathbf{Hyp}(\mathbf{B})$. *In particular, every ruled surface with either invariant* $e > 0$ *or* $e = 0$ *over a curve of genus* $g \leq 1$ *satisfy Conjecture Conjecture 1.3.*

Proof. Let $\pi : X \to B$ be a ruled surface over a smooth curve B of genus g, with invariant e. Let $C \subseteq X$ be a section, and let f be a fibre. By [11, Proposition V.2.3 and 2.9],

$$\mathrm{Pic}\, X \cong \mathbb{Z}C \oplus \pi^*\mathrm{Pic}B, C \cdot f = 1, f^2 = 0, C^2 = -e, K_X \equiv -2C + (2g-2-e)f.$$

Let $D \equiv aC + bf$ with $a, b \in \mathbb{Z}$ be a curve on X. Now we divide the remaining proof into the following four cases.

Case 1. Suppose $e > 0$. Then by [11, Proposition V.2.20(a)], $a > 0, b \geq ae$. As a result, every curve $D(\neq C, f)$ has $D^2 > 0$, $(C \cdot D) \geq 0$ and $(f \cdot D) > 0$. Thus,

$$\begin{aligned} l_D &= \frac{(K_X \cdot D)}{D^2} \\ &\leq \frac{2(C\cdot D) + |2g-2-e|(f\cdot D)}{a(C\cdot D) + b(f \cdot D)} \\ &\leq \max\left\{\frac{2}{a}, \frac{|2g-2-e|}{b}\right\}. \end{aligned}$$

Here, a and b are positive integers. Therefore, X satisfies $\mathbf{Hyp}(\mathbf{B})$ and $\mathbf{Hyp}(\mathbf{C})$. If $b \geq 2g-2-e$, then

$$(K_X - D)D = -(2+a)(C\cdot D) + (2g-2-e-b)(f\cdot D) \leq 0. \tag{2.5}$$

By Riemann-Roch theorem, Equation (2.1) and (2.5) imply that

$$\begin{aligned} h^1(\mathcal{O}_X(D)) &= h^0(\mathcal{O}_X(D)) + h^2(\mathcal{O}_X(D)) + \frac{(K_X - D)D}{2} - \chi(\mathcal{O}_X) \\ &\leq q(X)h^0(\mathcal{O}_X(D)). \end{aligned}$$

If $b < 2g-2-e$, then $a < (2g-2-e)e^{-1}$ by $b \geq ae$. As a result, $D^2 < 2(2g-2-e)^2e^{-1}$. Hence, by Theorem 1.7(i), X satisfies Conjecture 1.3.

Case 2. Suppose $e = 0$ and $g \leq 1$. Then by [11, Proposition V.2.20(a)], $a > 0$ and $b \geq 0$. As a result, $(K_X - D)D = -(2+a)b + a(2g-2-b) \leq 0$, which and Equation (2.1) imply that

$$\begin{aligned} h^1(\mathcal{O}_X(D)) &= h^0(\mathcal{O}_X(D)) + h^2(\mathcal{O}_X(D)) + \frac{(K_X - D)D}{2} - \chi(\mathcal{O}_X) \\ &\leq q(X)h^0(\mathcal{O}_X(D)). \end{aligned}$$

Therefore, X satisfies Conjecture 1.3. In particular, by Remark 2.1(1), X satisfies $\mathbf{Hyp}(\mathbf{B})$.

Case 3. Suppose $e = 0$ and $g \geq 2$. Then by [11, Proposition V.2.20(a)], $a > 0$ and $b \geq 0$. As a result, every curve $D(\neq aC, f)$ has $D^2 > 0$, $(C \cdot D) > 0$ and $(f \cdot D) > 0$. Note that every curve D has zero self-intersection if and only if either $D \equiv aC$ or $D \equiv f$. Suppose $b > 0$. Then

$$\begin{aligned} l_D &= \frac{(K_X \cdot D)}{D^2} \\ &\leq \frac{2(C \cdot D) + |2g-2|(f \cdot D)}{a(C \cdot D) + b(f \cdot D)} \\ &\leq \max\left\{\frac{2}{a}, \frac{|2g-2|}{b}\right\}. \end{aligned}$$

Here, a and b are positive integers. Therefore, X satisfies $\mathbf{Hyp}(\mathbf{B})$.

Case 4. Suppose $e < 0$. Then by [11, Proposition V.2.21], every curve D has either $D^2 = 0$ or $D^2 > 0$. Moreover, $D^2 > 0$ implies that D is ample and $a > 0, b > \frac{1}{2}ae$. Now suppose $D^2 > 0$. Then $D \cdot C > 0$ and $D \cdot f > 0$. Take $C' = C + \frac{1}{2}ef$ and then $D \cdot C' > 0$ and

$$\begin{aligned} l_D &= \frac{(K_X \cdot D)}{D^2} \\ &= \frac{-2(C \cdot D) + (2g-2-e)(f \cdot D)}{a(C' \cdot D) + (b - \frac{1}{2}ae)(f \cdot D)} \\ &\leq \frac{|4g-4-2e|}{2b-ae}. \end{aligned}$$

Here, $2b - ae$ is a positive integer. Therefore, X satisfies $\mathbf{Hyp}(\mathbf{B})$.

In all, we complete the proof of Lemma 2.3. □

It is well-known that the smooth projective surfaces satisfy the minimal model conjecture (cf. [13, 5]) as follows.

Lemma 2.4. *Let X be a smooth projective surface. If the canonical divisor K_X is pseudo-effective, then the Kodaira dimension $\kappa(X) \geq 0$.*

Lemma 2.5. *Let X be a smooth projective surface with $\rho(X) = 2$. If $\kappa(X) = -\infty$, then X satisfies $\mathbf{Hyp}(\mathbf{B})$. In particular, every ruled surface with either invariant $e > 0$ or $e = 0$ over a curve of genus $g \leq 1$ and one point blow-up of $\mathbb{P}^2$ satisfy Conjecture 1.3.*

Proof. Let S be a relatively minimal model of X. A smooth projective surface S is relatively minimal if it has no (-1)-rational curves. By the Enrique-Kodaira classification of relatively minimal surfaces (cf. [11, 6, 13]), it must be one of the following cases: a surface with nef canonical divisor, a ruled surface or $\mathbb{P}^2$. Since $\kappa(X) = -\infty$, by Lemma 2.4, K_X is not nef. Therefore, S is either a ruled surface or $\mathbb{P}^2$. As a result, $\rho(X) = 2$ implies that X is either a ruled surface or one point blow-up of $\mathbb{P}^2$. By Lemma 2.3, every ruled surface satisfies $\mathbf{Hyp}(\mathbf{B})$. In particular, every ruled surface with invariant $e > 0$ and every ruled surface with $e = 0$ over a curve of genus $g \leq 1$ satisfy Conjecture 1.3. Now suppose $\pi : X \to \mathbb{P}^2$ is one point blow-up of $\mathbb{P}^2$ with a exceptional curve E and $\mathrm{Pic}(\mathbb{P}^2) = \mathbf{Z}[H]$, where $H = \mathcal{O}_{\mathbb{P}^2}(1)$. Then $K_X = \pi^*(-3H) + E$ and $C = \pi^*(dH) - mE$, where $m := \mathrm{mult}_p(\pi_* C)$ and C is a curve on X. Note that $d \geq m$ since $\pi_* C$ is a plane projective curve. Thus, every curve C (not E) on X has $C^2 \geq 0$ and then C is nef. Since $-K_X$ is ample, $C - K_X$ is ample. Therefore, by Kadaira vanishing theorem, $h^1(\mathcal{O}_X(C)) = 0$. Therefore, X satisfies Conjecture 1.3. By Remark 2.1(1), X satisfies $\mathbf{Hyp}(\mathbf{B})$. □

Lemma 2.6. *Let X be a smooth projective surface with $\rho(X) = 2$. Then the following statements hold.*

(i) $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}[f_1] + \mathbb{R}_{\geq 0}[f_2]$*,* $f_1^2 \leq 0, f_2^2 \leq 0$ *and* $f_1 \cdot f_2 > 0$*. Here,* f_1, f_2 *are extremal rays.*

(ii) If a curve C has $C^2 \leq 0$, then $C \equiv af_1$ or $C \equiv bf_2$ for some $a, b \in \mathbb{R}_{>0}$.

(iii) Suppose a divisor $D \equiv a_1 f_1 + a_2 f_2$ with $a_1, a_2 > 0$ in (i). Then D is big. Moreover, if D is a curve, then D is nef and big and $D^2 > 0$.

Proof. By [13, Lemma 1.22], (i) and (ii) are clear since $\rho(X) = 2$. For (iii), $D \equiv a_1 f_1 + a_2 f_2$ with $a_1, a_2 > 0$ is an interior point of Mori cone, then by [15, Theorem 2.2.26], D is big. Moreover, if D is a curve, then D is nef. As a result, $D^2 > 0$. □

Lemma 2.7. *Let X be a smooth projective surface with $\rho(X) = 2$. If X has two negative curves C_1 and C_2, then the nef cone $\mathrm{Nef}(X)$ is*

$$\mathrm{Nef}(X) = \left\{ a_1 C_1 + a_2 C_2 \,\middle|\, a_1(C_1 \cdot C_2) \geq a_2(-C_2^2), a_2(C_1 \cdot C_2) \geq a_1(-C_1^2), a_1 > 0, a_2 > 0 \right\}.$$

Proof. Since $\rho(X) = 2$, $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}[C_1] + \mathbb{R}_{\geq 0}[C_2]$ by Lemma 2.6(ii). As a result, an effective $\mathbf{R}$- divisor $D \equiv a_1 C_1 + a_2 C_2$ is nef if and only if $D \cdot C_1 \geq 0$ and $D \cdot C_2 \geq 0$, which imply the desired result. □

Lemma 2.8. *Let X be a smooth projective surface with $\rho(X) = 2$. Suppose X has two negative curves C_1 and C_2. Then X satisfies Conjecture 1.3.*

Proof. Note that $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}[C_1] + \mathbb{R}_{\geq 0}[C_2]$ by Lemma 2.6(ii). We first show that X satisfies $\mathbf{Hyp}(\mathbf{B})$. By [16, Claim 2.11], $\kappa(X) \geq 0$, i.e., there exists a positive integral number m such that $h^0(X, \mathcal{O}_X(mK_X)) \geq 0$. Therefore, K_X is $\mathbb{Q}$-effective divisor. As a result, $K_X \equiv aC_1 + bC_2$ with $a, b \in \mathbb{R}_{\geq 0}$. Take a curve $D \equiv a_1 C_1 + a_2 C_2$ with $a_1, a_2 > 0$, then by Lemma 2.6(iii), $D^2 > 0$. As a result, $D \cdot C \geq 0$ and X has no any curves with zero self-intersection. $D^2 \geq 1$ implies that either $D \cdot C_1 \geq 1$ and $D \cdot C_2 \geq 0$ or $D \cdot C_1 \geq 0$ and $D \cdot C_2 \geq 1$. Without loss of generality, suppose that $D \cdot C_2 \geq 0$ and $D \cdot C_1 \geq 1$. Then $a_1 \geq (C_1^2 + (C_1 \cdot C_2)^2(-C_2^2)^{-1})^{-1}$. Here, $C_1^2 + (C_1 \cdot C_2)^2(-C_2^2)^{-1} > 0$ since $\rho(X) = 2$. By symmetry and Lemma 2.7,

$$a_i \geq c := \min \left\{ (C_i^2 + \frac{(C_1 \cdot C_2)^2}{-C_j^2})^{-1}, \frac{-C_j^2}{(C_1 \cdot C_2)} (C_i^2 + \frac{(C_1 \cdot C_2)^2}{-C_j^2})^{-1} \right\},$$

where $i \neq j \in \{1,2\}$. Therefore,

$$\begin{aligned} l_D &= \frac{a(D\cdot C_1)+b(D\cdot C_2)}{a_1(D\cdot C_1)+a_2(D\cdot C_2)} \\ &\leq \max\left\{\frac{a}{c},\frac{b}{c}\right\}. \end{aligned}$$

So X satisfies $\mathbf{Hyp}(\mathbf{B})$. If $a_1 > a$ and $a_2 > b$, then

$$(K_X - D)D = (a-a_1)(D\cdot C_1) + (b-a_2)(D\cdot C_2) < 0.$$

This and Equation (2.1) imply that

$$\begin{aligned} h^1(\mathcal{O}_X(D)) &= h^0(\mathcal{O}_X(D)) + h^2(\mathcal{O}_X(D)) + \frac{(K_X\cdot D)-D^2}{2} - \chi(\mathcal{O}_X) \\ &\leq q(X)h^0(\mathcal{O}_X(D)). \end{aligned}$$

If $a_1 \leq a$ or $a_2 \leq b$, then by Lemma 2.7, $a_2 \leq a(C_1\cdot C_2)(-C_2^2)^{-1}$ or $a_1 \leq b(C_1\cdot C_2)(-C_1^2)^{-1}$. As a result,

$$D^2 \leq \max\left\{2a^2(C_1\cdot C_2)^2(-C_2^2)^{-1}, 2b^2(C_1\cdot C_2)^2(-C_1^2)^{-1}\right\}.$$

Therefore, X satisfies Conjecture 1.3 by Theorem 1.7(1). □

Lemma 2.9. *Let X be a smooth projective surface with $\rho(X) = 2$. If $\kappa(X) = 1$ and $b(X) > 0$, then X satisfies Conjecture 1.3.*

Proof. Since $\kappa(X) = 1, \rho(X) = 2$ and $\kappa(X)$ is a birational invariant, K_X is nef and semi-ample. By [4, Proposition IX.2], we have $K_X^2 = 0$ and there is a surjective morphism $p : X \to B$ over a smooth curve B, whose general fibre F is an elliptic curve. Note that X has exactly one negative curve C by $b(X) > 0$ and [16, Claim 2.14]. In fact, p is an Iitaka fibration of X. In [12], S. Iitaka proved that if m is any natural number divisible by 12 and $m \geq 86$, then $|mK_X|$ defines the Iitaka fibration. Hence, there exists a curve F as a general fiber of p such that $F \equiv mK_X$. Then by Lemma 2.6(i)(ii), $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}[F] + \mathbb{R}_{\geq 0}[C]$. Note that $(F\cdot C) > 0$ since $\rho(X) = 2$. Take a curve $D \equiv a_1F + a_2C$ with $a_1, a_2 \geq 0$. By Lemma 2.6(iii), $D^2 > 0$ if and only if $a_1, a_2 > 0$, $D^2 = 0$ if and only if $D \equiv a_1F$. Now suppose $D \equiv a_1F$. Then $l_D = 0$. Note that $h^1(\mathcal{O}_X(D)) \leq q(X)h^0(\mathcal{O}_X(D))$ by Riemann-Roch theorem and Equation (2.1). Now suppose $D^2 > 0$. Then $(F\cdot D) \geq 1$ and $(C\cdot D) \geq 0$, which imply that

$$a_2 \geq (F\cdot C)^{-1}, a_1 \geq a_2(-C^2)(F\cdot C)^{-1}. \tag{2.6}$$

Therefore, by Equation (2.6),

$$\begin{aligned} l_D &= \frac{(F\cdot D)}{m(a_1(F\cdot D)+a_2(C\cdot D))} \\ &\leq (F\cdot C)^2(-mC^2)^{-1}. \end{aligned}$$

Hence, X satisfies $\mathbf{Hyp}(\mathbf{C})$. If $ma_1 \geq 1$, then $(K_X - D)D = (1-ma_1)(K_X\cdot D) - a_2(C\cdot D) \leq 0$. As a result, $h^1(\mathcal{O}_X(D)) = q(X)h^0(\mathcal{O}_X(C))$ by Riemann-Roch theorem and Equation (2.1). If $ma_1 < 1$, then by Equation (2.6), $a_2 < (F \cdot C)(-mC^2)^{-1}$. So $D^2 < 2m^{-2}(F \cdot C)^2(-C^2)^{-1}$. Hence, by Theorem 1.7(1), X satisfies Conjecture 1.3. □

Lemma 2.10. *Let X be a smooth projective surface with $\rho(X) = 2$. Suppose X satisfies* $\mathbf{Hyp}(\mathbf{A})$. *Then X satisfies* $\mathbf{Hyp}(\mathbf{B})$. *Moreover, if $\overline{\mathrm{NE}}(X)$ is generated by two curves with zero Iitaka dimension, then X satisfies* $\mathbf{Hyp}(\mathbf{C})$.

Proof. By Lemma 2.5, we can assume that $\kappa(X) \geq 0$, i.e., there exists a positive integral number m such that $h^0(X, \mathcal{O}_X(mK_X)) \geq 0$. Therefore, K_X is $\mathbb{Q}$-effective divisor. Since X satisfies $\mathbf{Hyp}(\mathbf{A})$, by Lemma 2.6(ii)(i), $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}[C_1] + \mathbb{R}_{\geq 0}[C_2]$. Here, C_1 and C_2 are two curves and $C_1^2, C_2^2 \leq 0$. As a result, $K_X \equiv aC_1 + bC_2$ with $a, b \geq 0$.

If $C_1^2 < 0$ and $C_2^2 < 0$, it follows from Lemma 2.8 and Remark 2.1(1).

Now suppose $C_1^2 = 0$. Then X has at most one negative curve. By Lemma 2.6(iii), $D^2 > 0$ if and only if $D \equiv a_1C_1 + a_2C_2$ with $a_1, a_2 > 0$, $D^2 > 0$ if and only if D is nef and big. As a result, $D^2 > 0$ implies that $(D \cdot C_1) \geq 1$, i.e., $a_2 \geq (C_1 \cdot C_2)^{-1}$. Now we divide the remaining proof into the following two cases.

Case (i). Suppose $C_2^2 = 0$. Then X has no negative curve. If $D^2 > 0$, then $(D \cdot C_2) \geq 1$, which implies that $a_1 \geq (C_1 \cdot C_2)^{-1}$. Therefore,

$$\begin{aligned} l_D &= \frac{a(D \cdot C_1) + b(D \cdot C_2)}{a_1(D \cdot C_1) + a_2(D \cdot C_2)} \\ &\leq \max\Big\{a(C_1 \cdot C_2), b(C_1 \cdot C_2)\Big\}. \end{aligned}$$

Therefore, X satisfies $\mathbf{Hyp}(\mathbf{B})$. Moreover, if $\kappa(X, C_1) = \kappa(X, C_2) = 0$, then X has only two curves with zero self-intersection. Hence, X satisfies $\mathbf{Hyp}(\mathbf{C})$.

Case (ii). Suppose $C_2^2 < 0$. X has only one negative curve C_2. If $D^2 > 0$, then $(D \cdot C_2) \geq 0$, which implies that $a_1 \geq a_2(-C_2^2)(C_1 \cdot C_2)^{-1}$. Therefore,

$$\begin{aligned} l_D &= \frac{a(D \cdot C_1) + b(D \cdot C_2)}{a_1(D \cdot C_1) + a_2(D \cdot C_2)} \\ &\leq \max\Big\{a(-C_2^2)^{-1}(C_1 \cdot C_2)^2, b(C_1 \cdot C_2)\Big\}. \end{aligned}$$

Therefore, X satisfies $\mathbf{Hyp}(\mathbf{B})$. Moreover, if $\kappa(X, C_1) = 0$, then X has only one curve with zero self-intersection. Hence, X satisfies $\mathbf{Hyp}(\mathbf{C})$. □

Remark 2.11. For the examples of Lemma 2.10, there exists a K3 surface with two negative curves (cf. [14, Theorem 2], [16, Claim 2.12]), Lemma 2.9 is an example of X with $\kappa(X, C_1) = 1$ and $\kappa(X, C_2) = 0$. The remaining case is that $\kappa(X) \geq 0, C_1^2 = 0$ and $\kappa(X, C_1) = \kappa(X, C_2) = 0$. J. Roé told us that there is an example for the case as follows.

Example 2.12. (J. Roé's example) Let $\pi : X \to Y$ be one-point blow-up of a smooth projective surface Y with $\rho(Y) = 1$. Then one extremal ray in $\overline{NE}(X)$ is the exceptional curve $E = C_2$. Since $\rho(Y) = 1$, the other extremal ray is determined by the Seshadri constant (cf. [15, Definition 5.1.1]). Let A be the ample generator of $\mathrm{Pic}\ (Y)$. Because of the duality between $\mathrm{Nef}(X)$ and $\overline{\mathrm{NE}}(X)$ (cf. [15, Proposition 1.4.28]), the class of $[C_1]$ is $\pi^*A - mE$, where $m = \frac{A^2}{e}$ and e is the Seshadri constant of A at the point that is blown-up. The assumption $C_1^2 = 0$ implies that $m = e = \sqrt{A^2}$. This does happen for the general smooth surfaces of degree e^2 in $\mathbb{P}^3$ (cf. [17, 1]). The easiest case would be a general point p on a general quartic surface $Y \subseteq \mathbb{P}^3$. Blow it up. A is the pullback to Y of the hyperplane class in $\mathbb{P}^3$. C_1 has class $\pi^*A - 2E$, which is given by the strict transform of the nodal quartic curve obtained as intersection of the quartic surface with its tangent plane at p. So C_1 is a plane nodal quartic, and it has genus $g \geq 2$. The restriction of $n(\pi^*A - 2E)$ to C_1 has degree zero. To prove that $\kappa(X, C_1) = 0$, we need to check that nC_1 is the only section of $n(\pi^*A - 2E)$. Note that the degree zero intersection divisor is not torsion in $\mathrm{Pic}\ (C_1)$, which is true if p and Y are general. Assume that another section D such that $D \sim_{\mathbb{Q}} nC_1 + T$, where effective $\mathbb{Q}$-divisor T is algebraically equivalent to zero. As a result, $T|_{C_1} = 0$, a contradiction.

Proof of Theorem 1.7(3)∼(6). (3) follows from Lemma 2.5. (4) follows from Lemma 2.8. (5) follows from Lemma 2.9. (6) follows from Lemma 2.10. □

We end by posing the following problem.

Problem 2.13. Classify all algebraic surfaces with $\mathbf{Hyp(B)}$.

Remark 2.14. For every smooth projective surface X, we conjecture that X satisfies $\mathbf{Hyp(B)}$ if X satisfies $\mathbf{Hyp(A)}$.

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