

Equivalence of large- N gauge theories on a group manifold and its coset space

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Abstract

It was shown in arXiv:0912.1456 that the large- N reduction holds on group manifolds in the sense that a large- N gauge theory on a group manifold is realized by a matrix model which is obtained by dimensionally reducing the original theory to zero dimension. In this note, generalizing the above statement, we show that a large- N gauge theory on a group manifold is equivalent to a theory which is obtained by reducing the original theory to its coset space. This is analogous to the statement of the large- N reduction on flat spaces that large- N gauge theories are independent of the volume.

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1 Introduction

The large- N reduction [1] states that large- N gauge theories are independent of the volume of the space-time on which they are defined. (For further developments in the large- N reduction, see [2–22].) In particular, this implies that large- N gauge theories are equivalent to the matrix models called the reduced models that are obtained by dimensionally reducing the original theories to zero dimension. The large- N reduction can give an efficient way to define large- N gauge theories nonperturbatively, and is also conceptually interesting as examples of emergent space-time. The large- N reduction is also called the large- N volume independence when the aspect of volume independence is emphasized.

The large- N reduction had been studied on flat space-time. It was shown in [19, 20] that the large- N reduction holds on group manifolds in the sense that a large- N gauge theory on a group manifold is realized by a matrix model which is obtained by reducing the original theory to zero dimension.

In this note, we examine whether phenomenon analogous to the large- N volume independence occurs on group manifolds. We find that it indeed does in the sense that a large- N gauge theory on a group manifold G is equivalent to the theory obtained by reducing it to a coset space G/H where H is a subgroup of G . Here, we call this phenomenon the large- N equivalence in dimensional reduction on group manifolds.

This paper is organized as follows. In section 2, as a warm up, we examine the large- N volume independence on a torus. In section 3, we review some properties of Lie groups and coset spaces. In section 4, we study the large- N equivalence in dimensional reduction on group manifolds. Section 5 is devoted to conclusion and discussion.

2 Large- N volume independence on torus

In this section, as a warm-up, we examine the large- N volume independence on a D -dimensional torus $T^D \simeq U(1)^D$. We denote coordinates of T^D by x^μ ($\mu = 1, \dots, D$), assuming, for simplicity, the periodicity $x^\mu \sim x^\mu + L$. Using a positive integer K , we define a ‘reduced torus’ $T^D/(Z_K)^D$ whose coordinates are denoted by σ^μ . The periodicity for σ^μ is $\sigma^\mu \sim \sigma^\mu + l$ where $l = L/K$. We have a relation

$$x^\mu = l w^\mu + \sigma^\mu \quad (2.1)$$

with w^μ integers.

To illustrate the large- N volume independence, we consider a scalar matrix field theory on T^D :

$$S = \int d^D x \operatorname{Tr} \left(\frac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi(x) + \frac{m^2}{2} \phi(x)^2 + \frac{\kappa}{3} \phi(x)^3 \right), \quad (2.2)$$

where $\phi(x)$ is a Hermitian matrix-valued field with the matrix size N .

We apply a following reduction rule to the above theory (2.2):

$$\phi(x) \rightarrow e^{iP_\mu x^\mu} \phi(\sigma) e^{-iP_\mu x^\mu} \quad \text{with} \quad P_\mu = \begin{pmatrix} \frac{2\pi n_\mu^{(1)}}{L} & & \\ & \frac{2\pi n_\mu^{(2)}}{L} & \\ & & \ddots \end{pmatrix}, \quad \int d^D x \rightarrow \frac{v}{v'} \int d^D \sigma. \quad (2.3)$$

Here the relation between x^μ and σ^μ is given by (2.1), and P_μ are constant diagonal matrices whose eigenvalues $2\pi n_\mu^{(i)}/L$ ($i = 1, \dots, N$) correspond to the momenta on T^D distributed uniformly in the momentum space. v and v' are given by

$$v = L^D/N, \quad v' = (2\pi/\Lambda)^D, \quad (2.4)$$

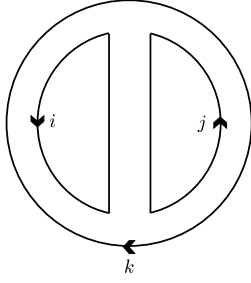


Figure 1: Two-loop planar diagram for the free energy.

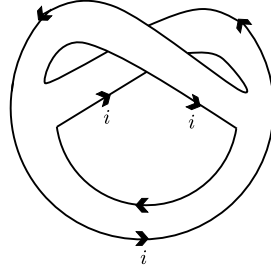


Figure 2: Two-loop non-planar diagram for the free energy.

where Λ is a UV cutoff on $T^D/(Z_K)^D$. (2.4) implies that T^D is divided into N cells with the volume of a unit cell given by v and that $T^D/(Z_K)^D$ is divided into l^D/v' cells with the volume of a unit cell given by v' . Then, we obtain the action of a reduced model defined on $T^D/(Z_K)^D$:

$$S_r = \frac{v}{v'} \int d^D \sigma \text{Tr} \left(\frac{1}{2} (\partial_{\sigma^\mu} \phi(\sigma) + i [P_\mu, \phi(\sigma)])^2 + \frac{m^2}{2} \phi(\sigma)^2 + \frac{\kappa}{3} \phi(\sigma)^3 \right). \quad (2.5)$$

Note that vl^D/v' can be viewed as an effective volume in the reduced model.

We consider the two-loop contribution to the free energy. There are two diagrams. One is planar (Fig.1) and the other non-planar (Fig.2). First, we calculate them in the original theory (2.2). The planar diagram is calculated as

$$\frac{\lambda N^2}{6} \int d^D x d^D x' D(x-x')^3, \quad (2.6)$$

where $\lambda = \kappa^2 N$ is the 't Hooft coupling and $D(x-x')$ is the free propagator of the theory (2.2) with $N=1$:

$$D(x-x') = \frac{1}{L^D} \sum_n \frac{e^{i \frac{2\pi n_\mu}{L} (x_\mu - x'_\mu)}}{\left(\frac{2\pi n_\mu}{L}\right)^2 + m^2}. \quad (2.7)$$

The non-planar diagram is given by (2.6)/ N^2 so that it is suppressed by $1/N^2$ compared to the planar diagram in the $N \rightarrow \infty$ limit.

Next, we calculate them in the reduced model using a bilocal field representation for matrices [19, 20]. We take a coordinate basis $|x\rangle$ in the vector space on which $\phi(\sigma)$ and P_μ act and define a bilocal field

$$\phi(\sigma, x, x') = \langle x | \phi(\sigma) | x' \rangle. \quad (2.8)$$

The reduced model (2.5) is rewritten as

$$S_r = \frac{v}{v'} \int d^D \sigma d^D x d^D x' \frac{1}{2} \phi(\sigma, x', x) \left[-(\partial_{\sigma^\mu} + \partial_\mu + \partial'_\mu)^2 + m^2 \right] \phi(\sigma, x, x') \\ + \frac{\kappa v}{3v'} \int d^D \sigma d^D x d^D x' d^D x'' \phi(\sigma, x, x') \phi(\sigma, x', x'') \phi(\sigma, x'', x). \quad (2.9)$$

We make a change of variables, $\bar{x}^\mu = x^\mu$, $\tilde{x}^\mu = x^\mu - x'^\mu$, $\bar{\sigma}^\mu = \sigma^\mu - x^\mu$, which gives

$$\frac{\partial}{\partial \sigma^\mu} + \frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial \bar{x}^\mu}. \quad (2.10)$$

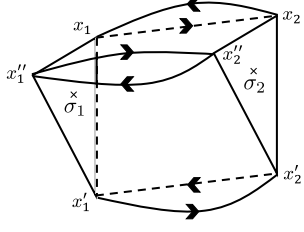


Figure 3: Two-loop planar diagram in the bilocal representation for the free energy.

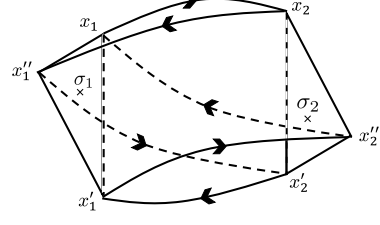


Figure 4: Two-loop non-planar diagram in the bilocal representation for the free energy.

Thus, from (2.10) we obtain the propagator

$$\langle \phi(\sigma_1, x_1, x'_1) \phi(\sigma_2, x'_2, x_2) \rangle = \frac{v'}{v} D(x_1 - x_2) \delta_L^{(D)}((x_1 - x'_1) - (x_2 - x'_2)) \delta_l^{(D)}((\sigma_1 - \sigma_2) - (x_1 - x_2)), \quad (2.11)$$

where $\delta_L^{(D)}$ and $\delta_l^{(D)}$ are periodic delta functions with the period L and l , respectively.

The planar diagram (Fig.3) in the reduced model is calculated as

$$\begin{aligned} & 3 \cdot \frac{1}{2} \left(\frac{\kappa v}{3v'} \right)^2 \int d^D \sigma_1 d^D \sigma_2 d^D x_1 d^D x'_1 d^D x''_1 d^D x_2 d^D x'_2 d^D x''_2 \\ & \quad \times \frac{v'}{v} D(x_1 - x_2) \delta_L^{(D)}((x_1 - x'_1) - (x_2 - x'_2)) \delta_l^{(D)}((\sigma_1 - \sigma_2) - (x_1 - x_2)) \\ & \quad \times \frac{v'}{v} D(x'_1 - x'_2) \delta_L^{(D)}((x'_1 - x''_1) - (x'_2 - x''_2)) \delta_l^{(D)}((\sigma_1 - \sigma_2) - (x'_1 - x'_2)) \\ & \quad \times \frac{v'}{v} D(x''_1 - x''_2) \delta_L^{(D)}((x''_1 - x_1) - (x''_2 - x_2)) \delta_l^{(D)}((\sigma_1 - \sigma_2) - (x''_1 - x''_2)) \\ & = \frac{\kappa^2 v'}{6v} \delta_L^{(D)}(0) L^{2D} \int d^D \sigma_1 d^D \sigma_2 d^D x_1 d^D x_2 D(x_1 - x_2)^3 \delta_l^{(D)}((\sigma_1 - \sigma_2) - (x_1 - x_2))^3 \\ & = \frac{\kappa^2 v'}{6v^2} L^{3D} \int d^D \sigma_1 d^D \sigma_2 d^D \tilde{x} D(\tilde{x})^3 \delta_l^{(D)}((\sigma_1 - \sigma_2) - \tilde{x})^3 \\ & = \frac{\kappa^2 L^{3D} v'}{6v^2} \delta_l^{(D)}(0)^2 \int d^D \sigma_1 d^D \sigma_2 \sum_u D(lu + \sigma_1 - \sigma_2)^3 \\ & = \frac{\kappa^2 L^{3D}}{6v^2 v'} \int d^D \tilde{x} d^D \sigma_2 D(\tilde{x} - \sigma_2)^3 \\ & = \frac{\kappa^2 L^{3D} l^D}{6v^2 v'} \int d^D \tilde{x} D(\tilde{x})^3 \\ & = \frac{\kappa^2 l^D}{6v^2 v'} (Nv)^3 \int d^D \tilde{x} D(\tilde{x})^3 \\ & = \frac{vl^D}{v'} \frac{\lambda N^2}{6} \frac{1}{L^D} \int d^D x d^D x' D(x - x')^3, \end{aligned} \quad (2.12)$$

where we have used $\delta_L(0) = 1/v$ and $\delta_l(0) = 1/v'$.

The non-planar diagram (Fig.4) in the reduced model is calculated as

$$\begin{aligned} & 3 \cdot \frac{1}{2} \left(\frac{\kappa v}{3v'} \right)^2 \int d^D \sigma_1 d^D \sigma_2 d^D x_1 d^D x'_1 d^D x''_1 d^D x_2 d^D x'_2 d^D x''_2 \\ & \quad \times \frac{v'}{v} D(x_1 - x_2) \delta_L^{(D)}((x_1 - x'_1) - (x_2 - x'_2)) \delta_l^{(D)}((\sigma_1 - \sigma_2) - (x_1 - x_2)) \end{aligned}$$

$$\begin{aligned}
& \times \frac{v'}{v} D(x'_1 - x''_2) \delta_L^{(D)}((x'_1 - x''_1) - (x''_2 - x_2)) \delta_l^{(D)}((\sigma_1 - \sigma_2) - (x'_1 - x''_2)) \\
& \times \frac{v'}{v} D(x''_1 - x'_2) \delta_L^{(D)}((x''_1 - x_1) - (x'_2 - x''_2)) \delta_l^{(D)}((\sigma_1 - \sigma_2) - (x''_1 - x'_2)) \\
& = \frac{\kappa^2 v'}{6v} \delta_L^{(D)}(0) \int d^D \sigma_1 d^D \sigma_2 d^D x_1 d^D x'_1 d^D x_2 d^D x''_2 D(x_1 - x_2) D(x'_1 - x''_2) D(x_1 - x''_2) \\
& \quad \times \delta_l^{(D)}((\sigma_1 - \sigma_2) - (x_1 - x_2)) \delta_l^{(D)}((\sigma_1 - \sigma_2) - (x'_1 - x''_2)) \delta_l^{(D)}((\sigma_1 - \sigma_2) - (x_1 - x''_2)) \\
& = \frac{\kappa^2 v'}{6v^2} L^D \int d^D \sigma_1 d^D \sigma_2 \sum_{u, u', u''} D(lu + \sigma_1 - \sigma_2) D(lu' + \sigma_1 - \sigma_2) D(lu'' + \sigma_1 - \sigma_2) \\
& = \left\{ \frac{v l^D \lambda N^2}{v'} \frac{1}{6} \frac{l^D}{L^{2D}} \int d^D \sigma_1 d^D \sigma_2 \sum_{u, u', u''} D(lu + \sigma_1 - \sigma_2) D(lu' + \sigma_1 - \sigma_2) D(lu'' + \sigma_1 - \sigma_2) \right\} \times \left(\frac{v'}{l^D} \right)^2 .
\end{aligned} \tag{2.13}$$

We see again that the non-planar diagram is suppressed compared to the planar diagram in the $v' \rightarrow 0$ limit, because the quantity in the curly bracket in (2.13) has the same order of magnitude as (2.12).

The non-planar diagram is suppressed compared to the planar diagram in both the original and reduced models. By comparing the planar contribution (2.6) and (2.12), we find a relation between the free energy F in the original model and the one F_r in the reduced model in the $N \rightarrow \infty$ limit:

$$\frac{F}{N^2 V} = \frac{F_r}{N^2 v V' / v'} , \tag{2.14}$$

where $V = L^D$ and $V' = l^D$ are the volumes of T^D and $T^D / (Z_K)^D$, respectively, and the LHS and RHS correspond to the planar contribution to the free energy per unit volume divided by N^2 in the original and reduced models, respectively. In a similar manner, by referring the argument in [5], one can show that the relation (2.14) holds to all orders in perturbative expansion. It is also easy to show a correspondence between correlation functions in the $N \rightarrow \infty$ limit [5]:

$$\frac{1}{N^{q/2+1}} \langle \text{Tr}(\phi(x_1) \phi(x_2) \cdots \phi(x_q)) \rangle = \frac{1}{N^{q/2+1}} \left\langle \text{Tr} \left(\hat{\phi}(x_1) \hat{\phi}(x_2) \cdots \hat{\phi}(x_q) \right) \right\rangle_r , \tag{2.15}$$

where $\langle \cdots \rangle$ and $\langle \cdots \rangle_r$ stand for the expectation value in the original and reduced models, respectively, and $\hat{\phi}(x) = e^{iP_\mu x^\mu} \phi(\sigma) e^{-iP_\mu x^\mu}$ with (2.1). Thus, we find that the large- N volume independence holds on a torus in the sense that a theory on T^D is equivalent to a certain theory on $T^D / (Z_K)^D$ in the large- N limit.

Finally, we consider Yang-Mills theory on T^D :

$$S = \frac{1}{4\kappa^2} \int d^D x \text{Tr}(F_{\mu\nu} F_{\mu\nu}) , \tag{2.16}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$. By applying the reduction rule (2.3) to (2.16), we obtain

$$S_r = \frac{v}{v'} \frac{1}{4\kappa^2} \int d^D \sigma \text{Tr} \left(\tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu} \right) , \tag{2.17}$$

where $\tilde{F}_{\mu\nu} = \partial_{\sigma^\mu} \tilde{A}_\nu - \partial_{\sigma^\nu} \tilde{A}_\mu + i[\tilde{A}_\mu, \tilde{A}_\nu]$ with $\tilde{A}_\mu(\sigma) = P_\mu + A_\mu(\sigma)$. Namely, the reduced model agrees with the one that is obtained by dimensionally reducing the original model to $T^D / (Z_K)^D$. If the background $\tilde{A}_\mu = P_\mu$ is stable in the reduced model (2.17), the reduced model is equivalent

to the original model (2.16) in the $N \rightarrow \infty$ with $\kappa^2 N$ fixed in the sense that (2.14) holds and a following relation for Wilson loops also holds:

$$\left\langle \frac{1}{N} P \exp \left(i \int_0^1 d\zeta \frac{dx^\mu(\zeta)}{d\zeta} A_\mu(x(\zeta)) \right) \right\rangle = \left\langle \frac{1}{N} P \exp \left(i \int_0^1 d\zeta \frac{dx^\mu(\zeta)}{d\zeta} \tilde{A}_\mu(\sigma(\zeta)) \right) \right\rangle_r, \quad (2.18)$$

where $x^\mu(\zeta)$ and $\sigma^\mu(\zeta)$ are related as (2.1). Namely, the large- N volume independence holds literally. Note that the stability depends on the dynamics of the model¹.

3 Group manifolds and coset spaces

In this section, we review some basic facts about group manifolds and coset spaces. For more details, see, for instance, [19, 20, 23]. Let G be a compact simply connected Lie group² and H be a Lie subgroup of G . D and d denote the dimensions of G and H , respectively. Then, the dimension of G/H is $D - d$. x^M ($M = 1, \dots, D$), y^m ($m = D - d + 1, \dots, D$) and σ^μ ($\mu = 1, \dots, D - d$) denote the coordinates of G , H and G/H , respectively, while $A, B = 1, \dots, D$, $a, b = D - d + 1, \dots, D$ and $\alpha, \beta = 1, \dots, D - d$ are the corresponding local Lorentz indices.

Let t_A be a basis for the Lie algebra of G in which t_a are a basis for the Lie algebra of H . t_A satisfy commutation relations $[t_A, t_B] = i f_{ABC} t_C$ with f_{ABC} completely anti-symmetric and $f_{ab\alpha} = 0$. $g(x) \in G$ is factorized locally as $g(x) = L(\sigma)h(y)$, where $h(y) \in H$. The isometry of G is the $G \times G$ symmetry, where one corresponds to the left translation and the other the right translation. Only the left translation survives as the isometry of G/H .

A $D \times D$ matrix $Ad(g)$ for $g \in G$ is defined by $g t_A g^{-1} = t_B Ad(g)_{BA}$. It is easy to show that $Ad(g)_{AB} Ad(g)_{AC} = \delta_{BC}$. Note that if h is an element of H , $Ad(h)_{\alpha\alpha} = Ad(h)_{\alpha\alpha} = 0$, which implies that $Ad(h)_{\alpha\beta} Ad(h)_{\alpha\gamma} = \delta_{\beta\gamma}$, $Ad(h)_{ab} Ad(h)_{ac} = \delta_{bc}$.

The right invariant 1-form E_M^A and the left invariant 1-form S_M^A are defined by

$$\partial_M g(x) g^{-1}(x) = -i E_M^A(x) t_A, \quad g^{-1}(x) \partial_M g(x) = i S_M^A(x) t_A. \quad (3.1)$$

They satisfy the Maurer-Cartan equation

$$\partial_M E_N^A - \partial_N E_M^A - f_{ABC} E_M^B E_N^C = 0, \quad \partial_M S_N^A - \partial_N S_M^A - f_{ABC} S_M^B S_N^C = 0. \quad (3.2)$$

Defining e_μ^A , \tilde{e}_m^a , s_μ^A and \tilde{s}_m^a by

$$\begin{aligned} \partial_\mu L(\sigma) L^{-1}(\sigma) &= -i e_\mu^A(\sigma) t_A, & \partial_m h(y) h^{-1}(y) &= -i \tilde{e}_m^a(y) t_a, \\ L^{-1}(\sigma) \partial_\mu L(\sigma) &= i s_\mu^A(\sigma) t_A, & h^{-1}(y) \partial_m h(y) &= i \tilde{s}_m^a(y) t_a, \end{aligned} \quad (3.3)$$

we obtain the relations:

$$\begin{aligned} E_\mu^\alpha(x) &= e_\mu^\alpha(\sigma), & E_\mu^a(x) &= e_\mu^a(\sigma), & E_m^\alpha(x) &= Ad(L)_{\alpha b}(\sigma) \tilde{e}_m^b(y), & E_m^a(x) &= Ad(L)_{ab}(\sigma) \tilde{e}_m^b(y), \\ S_\mu^\alpha(x) &= Ad(h^{-1})_{\alpha\beta}(y) s_\mu^\beta(\sigma), & S_\mu^a(x) &= Ad(h^{-1})_{ab}(y) s_\mu^b(\sigma), & S_m^\alpha(x) &= 0, & S_m^a(x) &= \tilde{s}_m^a(y). \end{aligned} \quad (3.4)$$

A metric of G , $G_{MN} = E_M^A E_N^A = S_M^A S_N^A$, is right and left invariant. By using (3.4), we obtain $ds_G^2 = s_\mu^\beta s_\nu^\beta d\sigma^\mu d\sigma^\nu + (Ad(h^{-1})_{ba} \tilde{s}_m^b dy^m + s_\mu^a d\sigma^\mu)^2$, where the invariant metric of G/H , $g_{\mu\nu}$, is given by $g_{\mu\nu} = s_\mu^\alpha s_\nu^\alpha$. The Haar measure of G is given by $dg = d^D x \sqrt{G(x)}$, where $G(x)$ is $\det G_{MN}(x)$.

¹ Instability corresponds to SSB of the so-called $U(1)^D$ symmetry or the center invariance.

² If G is not simply connected, the reduced model is not obtained in a globally consistent way.

It is factorized as $dg = d^{D-d}\sigma d^d y \sqrt{g(\sigma)} \det \tilde{s}_m^a(y)$. We denote the invariant measure of G/H , $d^{D-d}\sigma \sqrt{g}$, by dL .

We define the right invariant Killing vectors \mathcal{L}_A and the left invariant Killing vectors \mathcal{R}_A by

$$\mathcal{L}_A = -iE_A^M \frac{\partial}{\partial x^M}, \quad \mathcal{R}_A = -iS_A^M \frac{\partial}{\partial x^M}, \quad (3.5)$$

where E_A^M and S_A^M are the inverses of E_M^A and S_M^A , respectively. \mathcal{L}_A and \mathcal{R}_A generate the left translation and right translation, respectively, and obey the following commutation relations:

$$[\mathcal{L}_A, \mathcal{L}_B] = if_{ABC}\mathcal{L}_C, \quad [\mathcal{R}_A, \mathcal{R}_B] = if_{ABC}\mathcal{R}_C, \quad [\mathcal{L}_A, \mathcal{R}_B] = 0. \quad (3.6)$$

By using (3.4), we obtain

$$\begin{aligned} \mathcal{L}_\alpha &= is_\beta^\mu Ad(L)_{\beta\alpha} \frac{\partial}{\partial \sigma^\mu} - i\tilde{e}_b^m \left(Ad(L)_{b\alpha} - s_\rho^b s_\beta^\rho Ad(L)_{\beta\alpha} \right) \frac{\partial}{\partial y^m}, \\ \mathcal{L}_a &= is_\beta^\mu Ad(L)_{\beta a} \frac{\partial}{\partial \sigma^\mu} - i\tilde{e}_b^m \left(Ad(L)_{ba} - s_\rho^b s_\beta^\rho Ad(L)_{\beta a} \right) \frac{\partial}{\partial y^m}, \\ \mathcal{R}_\alpha &= -iAd(h^{-1})_{\alpha\beta} s_\beta^\mu \frac{\partial}{\partial \sigma^\mu} + iAd(h^{-1})_{bc} Ad(h^{-1})_{\alpha\beta} \tilde{s}_b^m s_\nu^c s_\beta^\nu \frac{\partial}{\partial y^m}, \\ \mathcal{R}_a &= -i\tilde{s}_a^m \frac{\partial}{\partial y^m}, \end{aligned} \quad (3.7)$$

where s_α^μ , \tilde{s}_a^m and \tilde{e}_a^m are the inverses of s_μ^α , \tilde{s}_m^a and \tilde{e}_m^a , respectively. $\mathcal{L}'_A = is_\beta^\mu Ad(L)_{\beta A} \frac{\partial}{\partial \sigma^\mu}$ are the Killing vectors on G/H and are indeed independent of y .

Let us consider a scalar matrix field theory on G given by³

$$\begin{aligned} S &= \int d^D x \sqrt{G(x)} \text{Tr} \left[\frac{1}{2} G^{MN} \partial_M \phi(x) \partial_N \phi(x) + \frac{m^2}{2} \phi(x)^2 + \frac{\kappa}{3} \phi(x)^3 \right] \\ &= \int d^{D-d} \sigma d^d y \sqrt{g(\sigma)} \det \tilde{s}_m^a(y) \text{Tr} \left[-\frac{1}{2} (\mathcal{L}_A \phi(x))^2 + \frac{m^2}{2} \phi(x)^2 + \frac{\kappa}{3} \phi(x)^3 \right], \end{aligned} \quad (3.8)$$

where $\phi(x)$ is an $N \times N$ hermitian matrix. This theory has the $G \times G$ symmetry. Namely, (3.8) is invariant under $\delta\phi = \epsilon \mathcal{L}_A \phi$ or $\delta\phi = \epsilon \mathcal{R}_A \phi$. We impose a constraint $\mathcal{R}_a \phi(x) = 0$, which implies from (3.7) that ϕ is independent of y . Then, the theory (3.8) is reduced to the theory on G/H as

$$\begin{aligned} S &= \int d^{D-d} \sigma d^d y \sqrt{g} \det \tilde{s}_m^a \text{Tr} \left[-\frac{1}{2} (\mathcal{L}'_A \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\kappa}{3} \phi^3 \right] \\ &= V_H \int d^{D-d} \sigma \sqrt{g} \text{Tr} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{m^2}{2} \phi^2 + \frac{\kappa}{3} \phi^3 \right), \end{aligned} \quad (3.9)$$

where V_H is the volume of H . The theory (3.9) has the left G symmetry. Note that this is a consistent truncation in the sense that every solution to the equation of motion in (3.9) is also a solution to the equation of motion in (3.8).

As an example, we consider $SU(2) \simeq S^3$ and $SU(2)/U(1) \simeq S^2$. We have

$$g = e^{-i\varphi\sigma_3/2} e^{-i\theta\sigma_2/2} e^{-i\psi\sigma_3/2}, \quad L = e^{-i\varphi\sigma_3/2} e^{-i\theta\sigma_2/2}, \quad h = e^{-i\psi\sigma_3/2}, \quad (3.10)$$

³The invariant 1-forms, the Killing vectors, the integral measures and the delta functions are all well-defined globally. Hence, all expressions on group manifolds and their coset spaces make sense globally, although they look dependent on coordinate patches.

where θ , φ and ψ are the Euler angles, and σ_i ($i = 1, 2, 3$) are the Pauli matrices. Here $\mu = (\theta, \varphi)$, $m = \psi$, $\alpha = (1, 2)$, and $a = 3$. \mathcal{L}_A are given by

$$\begin{aligned}\mathcal{L}_1 &= -i \left(-\sin \varphi \frac{\partial}{\partial \theta} - \cot \theta \cos \varphi \frac{\partial}{\partial \varphi} + \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \psi} \right), \\ \mathcal{L}_2 &= -i \left(-\cos \varphi \frac{\partial}{\partial \theta} - \cot \theta \sin \varphi \frac{\partial}{\partial \varphi} + \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \psi} \right), \\ \mathcal{L}_3 &= -i \frac{\partial}{\partial \varphi}.\end{aligned}\tag{3.11}$$

\mathcal{R}_A are given by

$$\begin{aligned}\mathcal{R}_1 &= -i \left(-\sin \psi \frac{\partial}{\partial \theta} + \frac{\cos \psi}{\sin \theta} \frac{\partial}{\partial \varphi} - \cot \theta \cos \psi \frac{\partial}{\partial \psi} \right), \\ \mathcal{R}_2 &= -i \left(-\cos \psi \frac{\partial}{\partial \theta} - \frac{\sin \psi}{\sin \theta} \frac{\partial}{\partial \varphi} + \cot \theta \sin \psi \frac{\partial}{\partial \psi} \right), \\ \mathcal{R}_3 &= i \frac{\partial}{\partial \psi}.\end{aligned}\tag{3.12}$$

The right and left invariant metric of $SU(2)$ is given by $ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2 + (d\psi + \cos \theta d\varphi)^2$. The first and second terms in the RHS give the metric of $SU(2)/U(1)$. The Haar measure takes the form $dg = \sin \theta d\theta d\varphi d\psi$.

4 Bilocal representation for the reduced model on G/H

We consider a coordinate basis $|g\rangle$ for G as in the case of torus. We define the generators of the left translation \hat{L}_A by $e^{i\epsilon \hat{L}_A} |g\rangle = |e^{i\epsilon t_A} g\rangle^4$. It is easy to show that $\hat{L}_A |g\rangle = -\mathcal{L}_A |g\rangle$, $\langle g | \hat{L}_A = \mathcal{L}_A \langle g |$. We denote the volumes of G and G/H by V and V' , respectively. To obtain a reduced model of (3.8) defined on G/H , we apply the following rule

$$\phi(g) \rightarrow \phi(L), \quad \mathcal{L}_A \rightarrow [\hat{L}_A, \quad], \quad \int dg \rightarrow \frac{v}{v'} \int dL = \frac{v}{v'} \int d^{D-d} \sigma \sqrt{g}.\tag{4.1}$$

The first rule is realized by imposing $\mathcal{R}_a \phi = 0$. Thus, by introducing the bilocal representation for $\phi(L)$

$$\phi(L, g, g') = \langle g | \phi(L) | g' \rangle,\tag{4.2}$$

we obtain a bilocal representation of the reduced model:

$$\begin{aligned}S_r &= \frac{v}{v'} \int dL dg dg' \frac{1}{2} \phi(L, g', g) \left[\left(\mathcal{L}'_A{}^L + \mathcal{L}_A{}^g + \mathcal{L}_A{}^{g'} \right)^2 + m^2 \right] \phi(L, g, g') \\ &\quad + \frac{\kappa v}{3v'} \int dL dg dg' dg'' \phi(L, g, g') \phi(L, g', g'') \phi(L, g'', g).\end{aligned}\tag{4.3}$$

As in the case of a torus, we make a change of variables $w = g$, $\xi = g'^{-1}g$, $\rho = g^{-1}L$ and obtain a relation

$$\left(\mathcal{L}'_A{}^L + \mathcal{L}_A{}^g + \mathcal{L}_A{}^{g'} \right) \phi(L, g, g') = \mathcal{L}_A{}^w \phi(L, g, g').\tag{4.4}$$

Thus, the propagator is read off as

$$\langle \phi(L_1, g_1, g'_1) \phi(L_2, g'_2, g_2) \rangle = \frac{v'}{v} \Delta(g_1 g_2^{-1}) \delta(g_1'^{-1} g_1, g_2'^{-1} g_2) \delta_{G/H}(g_1^{-1} L_1, g_2^{-1} L_2),\tag{4.5}$$

⁴ \hat{L}_A are the generators in the regular representation [19, 20].

where $\Delta(g_1 g_2^{-1})$ is the propagator of the original model (3.8) with $N = 1$, $\delta(g_1, g_2)$ is the right and left invariant delta function on G , and $\delta_{G/H}(L_1, L_2)$ is the left invariant delta function on G/H .

We consider the two-loop contribution to the free energy in the original and reduced models. The planar diagram (Fig.3) in the original theory is calculated as

$$\frac{\lambda N^2}{6} \int dg_1 dg_2 \Delta(g_1 g_2^{-1})^3 . \quad (4.6)$$

The non-planar diagram (Fig.4) is calculated as (4.6)/ N^2 . The planar diagram (Fig.3) in the reduced model is calculated as

$$\begin{aligned} & 3 \cdot \frac{1}{2} \left(\frac{\kappa v}{3v'} \right)^2 \left(\frac{v'}{v} \right)^3 \int dL_1 dL_2 dg_1 dg'_1 dg''_1 dg_2 dg'_2 dg''_2 \\ & \quad \times \Delta(g_1 g_2^{-1}) \delta(g_1'^{-1} g_1, g_2'^{-1} g_2) \delta_{G/H}(g_1^{-1} L_1, g_2^{-1} L_2) \\ & \quad \times \Delta(g'_1 g_2'^{-1}) \delta(g_1''^{-1} g'_1, g_2''^{-1} g'_2) \delta_{G/H}(g_1'^{-1} L_1, g_2'^{-1} L_2) \\ & \quad \times \Delta(g''_1 g_2''^{-1}) \delta(g_1^{-1} g''_1, g_2^{-1} g''_2) \delta_{G/H}(g_1''^{-1} L_1, g_2''^{-1} L_2) . \end{aligned} \quad (4.7)$$

A change of variables $g_2' = g_2 g_1^{-1} g_1'$ and $g_2'' = g_2 g_1^{-1} g_1''$ leads to

$$\begin{aligned} & \frac{\kappa^2 v'}{6v} \delta(0) \int dL_1 dL_2 dg_1 dg'_1 dg''_1 dg_2 \Delta(g_1 g_2^{-1})^3 \delta_{G/H}(g_1^{-1} L_1, g_2^{-1} L_2) \\ & \quad \times \delta_{G/H}(g_1'^{-1} L_1, g_1'^{-1} g_1 g_2^{-1} L_2) \delta_{G/H}(g_1''^{-1} L_1, g_1''^{-1} g_1 g_2^{-1} L_2) . \end{aligned} \quad (4.8)$$

Making a further change of variables $g_1'^{-1} g_1 \rightarrow g_1'^{-1}$ and $g_1''^{-1} g_1 \rightarrow g_1''^{-1}$, we obtain

$$\begin{aligned} & \frac{\kappa^2 v'}{6v} \delta(0) \int dL_1 dL_2 dg_1 dg'_1 dg''_1 dg_2 \Delta(g_1 g_2^{-1})^3 \delta_{G/H}(g_1^{-1} L_1, g_2^{-1} L_2) \\ & \quad \times \delta_{G/H}(g_1'^{-1} g_1^{-1} L_1, g_1'^{-1} g_2^{-1} L_2) \delta_{G/H}(g_1''^{-1} g_1^{-1} L_1, g_1''^{-1} g_2^{-1} L_2) \\ & = \frac{\kappa^2 v'}{6v} \delta(0) \int dL_1 dL_2 dg_1 dg'_1 dg''_1 dg_2 \Delta(g_1 g_2^{-1})^3 \delta_{G/H}(g_1^{-1} L_1, g_2^{-1} L_2)^3 \\ & = \frac{\kappa^2 v'}{6v} \delta(0) V^2 \delta_{G/H}(0)^2 V' \int dg_1 dg_2 \Delta(g_1 g_2^{-1})^3 \\ & = \frac{v V'}{v'} \frac{\lambda N^2}{6V} \int dg_1 dg_2 \Delta(g_1 g_2^{-1})^3 . \end{aligned} \quad (4.9)$$

In the above calculation $\delta(0) = 1/v$, $\delta_{G/H}(0) = 1/v'$, $V = Nv$ and $\lambda = \kappa^2 N$ have been used. The non-planar diagram (Fig.4) in the reduced model is calculated as

$$\begin{aligned} & 3 \cdot \frac{1}{2} \left(\frac{\kappa v}{3v'} \right)^2 \left(\frac{v'}{v} \right)^3 \int dL_1 dL_2 dg_1 dg'_1 dg''_1 dg_2 dg'_2 dg''_2 \\ & \quad \times \Delta(g_1 g_2^{-1}) \delta(g_1'^{-1} g_1, g_2'^{-1} g_2) \delta_{G/H}(g_1^{-1} L_1, g_2^{-1} L_2) \\ & \quad \times \Delta(g'_1 g_2''^{-1}) \delta(g_1''^{-1} g'_1, g_2^{-1} g''_2) \delta_{G/H}(g_1'^{-1} L_1, g_2''^{-1} L_2) \\ & \quad \times \Delta(g''_1 g_2'^{-1}) \delta(g_1^{-1} g''_1, g_2'^{-1} g'_2) \delta_{G/H}(g_1''^{-1} L_1, g_2'^{-1} L_2) . \end{aligned} \quad (4.10)$$

A change of variables $g_1'' = g_1' g_2''^{-1} g_2$ and $g_2' = g_2 g_1^{-1} g_1'$ gives rise to

$$\begin{aligned} & \frac{\kappa^2 v'}{6v} \int dL_1 dL_2 dg_1 dg_1' dg_2 dg_2'' \Delta(g_1 g_2^{-1}) \Delta(g_1' g_2''^{-1}) \Delta(g_1' g_2''^{-1} g_2 g_1^{-1} g_1 g_2^{-1}) \delta(g_1^{-1} g_1' g_2''^{-1} g_2, g_2''^{-1} g_2 g_1^{-1} g_1') \\ & \times \delta_{G/H}(g_1^{-1} L_1, g_2^{-1} L_2) \delta_{G/H}(g_1'^{-1} L_1, g_2''^{-1} L_2) \delta_{G/H}(g_2^{-1} g_2'' g_1'^{-1} L_1, g_1'^{-1} g_1 g_2^{-1} L_2) . \end{aligned} \quad (4.11)$$

A further change of variables $g_1 g_2^{-1} = \tilde{g}_2^{-1}$ and $g_1' g_2''^{-1} = \tilde{g}_1'$ leads to

$$\begin{aligned} & \frac{\kappa^2 v'}{6v} \int dL_1 dL_2 dg_1 d\tilde{g}_1' d\tilde{g}_2 dg_2'' \Delta(\tilde{g}_2^{-1}) \Delta(\tilde{g}_1') \Delta(\tilde{g}_1' \tilde{g}_2 g_1 g_2''^{-1} \tilde{g}_1'^{-1} \tilde{g}_2^{-1}) \delta(g_1^{-1} \tilde{g}_1' \tilde{g}_2 g_1, g_2''^{-1} \tilde{g}_2 \tilde{g}_1' g_2'') \\ & \times \delta_{G/H}(L_1, \tilde{g}_2^{-1} L_2) \delta_{G/H}(L_1, \tilde{g}_1' L_2) \delta_{G/H}(L_1, \tilde{g}_1' \tilde{g}_2 g_1 g_2''^{-1} \tilde{g}_1'^{-1} \tilde{g}_2^{-1} L_2) \\ & = \frac{\kappa^2 v'}{6v} \int dL_1 dL_2 dg_1 d\tilde{g}_1' d\tilde{g}_2 dg_2'' \Delta(\tilde{g}_2^{-1}) \Delta(\tilde{g}_1') \Delta(g_1 g_2''^{-1}) \delta(\tilde{g}_1' \tilde{g}_2 g_1, g_1 g_2''^{-1} \tilde{g}_2 \tilde{g}_1' g_2'') \\ & \times \delta_{G/H}(L_1, \tilde{g}_2^{-1} L_2) \delta_{G/H}(L_1, \tilde{g}_1' L_2) \delta_{G/H}(L_1, g_1 g_2''^{-1} L_2) . \end{aligned} \quad (4.12)$$

By making a change of variables $g_1 g_2''^{-1} = \tilde{g}$, we obtain

$$\begin{aligned} & \frac{\kappa^2 v'}{6v} V \int dL_1 dL_2 d\tilde{g}_1' d\tilde{g}_2 d\tilde{g} \Delta(\tilde{g}_2^{-1}) \Delta(\tilde{g}_1') \Delta(\tilde{g}) \\ & \times \delta(\tilde{g}_1' \tilde{g}_2 \tilde{g}, \tilde{g}_2 \tilde{g}_1') \delta_{G/H}(L_1, \tilde{g}_2^{-1} L_2) \delta_{G/H}(L_1, \tilde{g}_1' L_2) \delta_{G/H}(L_1, \tilde{g} L_2) . \end{aligned} \quad (4.13)$$

Since $\delta(\tilde{g}_1' \tilde{g}_2 \tilde{g}, \tilde{g}_2 \tilde{g}_1') \leq \delta(0) = 1/v$,

Absolute value of (4.13)

$$\begin{aligned} & \leq \frac{\kappa^2 v'}{6v} \delta(0) V \int dL_1 dL_2 dh dh' dh'' |\Delta(L_2^{-1} L_1 h) \Delta(L_2^{-1} L_1 h') \Delta(L_2^{-1} L_1 h'')| \\ & = \left\{ \frac{v V'}{v'} \frac{\lambda N^2 V'}{6V} \frac{V'}{V} \int dL_1 dL_2 dh dh' dh'' |\Delta(L_2^{-1} L_1 h) \Delta(L_2^{-1} L_1 h') \Delta(L_2^{-1} L_1 h'')| \right\} \times \left(\frac{v'}{V'} \right)^2 . \end{aligned} \quad (4.14)$$

We see that the above quantity is analogous to the one (2.13) and suppressed by $(v'/V')^2$ in the $v' \rightarrow 0$ limit compared to (4.9). Thus, the non-planar diagram is suppressed compared to the planar diagram in both the original and reduced model. By comparing (4.6) and (4.9), we again obtain the relation (2.14). As in the case of torus, one can show that (2.14) holds to all orders in perturbative expansion.

Defining $\hat{\phi}(g)$ by $\hat{\phi}(g) = e^{iL_A \theta_A} \phi(L) e^{-iL_A \theta_A}$, where $g = e^{i\theta_A t_A}$, we see that the relation (2.15) also holds in this case. Thus, we find that the large- N equivalence in dimensional reduction holds on group manifolds.

Finally, we consider $U(N)$ Yang-Mills theory on G :

$$S = \frac{1}{4\kappa^2} \int d^D x \sqrt{G} G^{AC} G^{BD} \text{Tr}(F_{AB} F_{CD}) , \quad (4.15)$$

where $F_{AB} = \partial_A A_B - \partial_B A_A + i[A_A, A_B]$. We expand the gauge field A_A in terms of the right invariant 1-form as $A_M = E_M^A X_A$. By using (3.2), we rewrite (4.15) as

$$S = -\frac{1}{4\kappa^2} \int d^D x \sqrt{G} \text{Tr}(\mathcal{L}_A X_B - \mathcal{L}_B X_A - i f_{ABC} X_C + [X_A, X_B])^2 . \quad (4.16)$$

By imposing $\mathcal{R}_a X_A = 0$ on (4.15), we obtain Yang-Mills theory on G/H . The reduced model on G/H is given by

$$S_r = -\frac{v}{v'} \frac{1}{4\kappa^2} \int d^{D-d} \sigma \sqrt{g} \operatorname{Tr} \left(\mathcal{L}_A \tilde{X}_B - \mathcal{L}_B \tilde{X}_A - i f_{ABC} X_C + [\tilde{X}_A, \tilde{X}_B] \right)^2, \quad (4.17)$$

where $\tilde{X}_A(\sigma) = L_A + X_A(\sigma)$. If G is simple, the gauge theory is massive due to the $f_{ABC} X_C$ term so that there is no moduli for the background $\tilde{X}_A = L_A$. Thus, since the background $\tilde{X}_A = L_A$ is stable, the large- N equivalence in dimensional reduction on group manifolds holds. Namely, the reduced model is equivalent to the original model (4.16) in the $N \rightarrow \infty$ with $\kappa^2 N$ fixed in the sense that (2.14) holds and a following relation for Wilson loops [24] holds:

$$\left\langle \frac{1}{N} P \exp \left(i \int_0^1 d\zeta \frac{dx^M(\zeta)}{d\zeta} E_M^A(x(\zeta)) X_A(x(\zeta)) \right) \right\rangle = \left\langle \frac{1}{N} P \exp \left(i \int_0^1 d\zeta \frac{dx^M(\zeta)}{d\zeta} E_M^A(x(\zeta)) \tilde{X}_A(\sigma(\zeta)) \right) \right\rangle_r, \quad (4.18)$$

where $x^M(\zeta)$ and $\sigma^\mu(\zeta)$ are related through $g(x) = L(\sigma)h(y)$.

5 Conclusion and Discussion

In this paper, we showed that a theory on a group manifold G is equivalent to the corresponding theory on G/H with H a subgroup of G in the large- N limit. The degrees of freedom on G are retrieved by the degrees of freedom of matrices in a consistent way with the dimensional reduction to G/H . An advantage of reduction to G/H with a finite volume compared to reduction to a matrix model is that one does not need to introduce k multiplicity and take the $k \rightarrow \infty$ limit to extract only planar contribution as in the latter case [19, 20], since the UV cutoff V'/v' plays the role of extracting planar contribution. While we showed the equivalence perturbatively, we can show it non-perturbatively based on the continuum Schwinger-Dyson equations as in [5], by assuming the stability of the background, which is a counterpart of the center symmetry.

An interesting application of the large- N equivalence in dimensional reduction on group manifolds is that the $SU(2|4)$ symmetric gauge theory on $R \times S^2$ is equivalent to $\mathcal{N} = 4$ super Yang-Mills theory on $R \times S^3$ in the large- N limit. (For another large- N equivalence between these two theories, see [17, 25].) Both of the theories have gravity duals, so that the above equivalence would be seen on the gravity side. It is interesting to search for gravity duals of other large- N equivalences in dimensional reduction [26–30].

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