Clifford index for reduced curves

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Abstract

We extend the notion of Clifford index to reduced curves with planar singularities by considering rank 1 torsion free sheaves. We investigate the behaviour of the Clifford index with respect to the combinatorial properties of the curve and we show that Green's conjecture holds for certain classes of curves given by the union of two irreducible components.

keyword: Algebraic curve, Clifford index, Green's conjecture

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1 Introduction

Clifford index for smooth curves has been introduced by H. Martens in [15] (see also [12]), and many authors investigated its relation with the geometry of smooth curves. If C is a smooth curve and $\mathcal{L}$ is an invertible sheaf, then the Clifford index of $\mathcal{L}$ is $\mathrm{Cliff}(\mathcal{L}) = \deg(\mathcal{L}) - 2h^0(C, \mathcal{L}) + 2$ and the Clifford index of C is

$$\mathrm{Cliff}(C) = \min_{\mathcal{L}\in \mathrm{Pic}(C)} \{\deg(\mathcal{L}) - 2h^0(C, \mathcal{L}) + 2 \ : \ h^0(C, \mathcal{L}) \geq 2, h^1(C, \mathcal{L}) \geq 2\}.$$

For a smooth curve it is always $\mathrm{Cliff}(C) \geq 0$, with equality holding only for hyperelliptic curves, $\mathrm{Cliff}(C) = 1$ if and only if C is trigonal or plane quintic, and $\mathrm{Cliff}(C) = 2$ if and only if C is tetragonal, or plane sextic (see [16] for a further analysis). Indeed the Clifford index is intimately, but not completely, related to the gonality since it is $\mathrm{Cliff}(C) + 2 \leq \mathrm{gon}(C) \leq \mathrm{Cliff}(C) + 3$ (see [7]).

Caporaso in [4] studied the Clifford index of invertible sheaves on semistable curves finding interesting connections with the combinatorial properties of the curve and pointing out the problems that can arise if the curve has disconnecting nodes. Tenni and the author in [10] proved a generalisation of

Clifford's theorem for singular curves, either reduced with planar singularities or lying on a smooth surface, studying rank one torsion free sheaves of the form $\mathcal{I}_S\omega_C$, where S is a zero dimensional scheme and ω_C is the canonical sheaf.

In this paper we consider reduced curves with planar singularities (e.g., semistable curves) and we study nef torsion free sheaves of rank 1 whose degree is bounded from above by the degree of the canonical sheaf ω_C. We recall that these curves are always Gorenstein.

Notice that, for a curve C with many components the behaviour of the sections of a torsion free sheaf may be rather complicated, hence the Clifford index too. Nevertheless it is possible to find an estimate for the Clifford index and to show some geometric relations with the canonical ring of the curve. Indeed, given a reduced curve $C = C_1 \cup \cdots \cup C_n$, and a rank 1 torsion free sheaves $\mathcal{F}$ on C such that

$$0 \leq \deg[\mathcal{F}_{|C_i}] \leq \deg \omega_{C|C_i} \quad \forall C_i, \ i = 1, \cdots, n \tag{1}$$

we set $\text{Cliff}(\mathcal{F}) := \deg(\mathcal{F}) - 2h^0(C, \mathcal{F}) + 2$. Taking in account all the sheaves that contributes to the Clifford index (see Definition 3.4) we give the following definition of Clifford index for a reduced curve C

$$\begin{aligned} \text{Cliff}(C) := \min\{\text{Cliff}(\mathcal{F}) \ : \ & \mathcal{F} \text{ rank 1 torsion free sheaf s.t.} \\ & \mathcal{F} \text{ verifies (1) }; h^0(\mathcal{F}) \geq 2, \ h^1(\mathcal{F}) \geq 2\}. \end{aligned}$$

In Section 3 we prove that such minimum does exist and we show lower and upper bound for such index, investigating its relation with the combinatorial properties of the curves, in particular *m-connectedness*. Recall that a curve C is *m-connected* if for any proper decomposition $C = A \cup B$, it is $A \cdot B := \deg(\omega_{C|B}) - (2p_a(B) - 2) \geq m$ (cf. [6, §3]).

More precisely we prove firstly that $\text{Cliff}(C)$ can be negative if C is not 4-connected, bounded from below by $-n + 1$, where n is the number of irreducible components of C (see Prop. 3.2). We show also that such bound is sharp providing examples given by chains of curves (see Example 3.11) and we provide an example of a 3-connected curve C with canonical sheaf very ample but $\text{Cliff}(C) = -1$ (see Example 3.12). On the contrary in Thm. 3.5 we show that $\text{Cliff}(C) \geq 0$ if C is 4-connected and in Thm. 3.13 we prove that for every invertible sheaf $\mathcal{L}$ it is $\text{Cliff}(\mathcal{L}) \geq 0$, independently from the connectedness of the curve. Finally in Theorem 3.8 we show the following constraints given by the numerical connectedness of C: if C does not contain rational components and it is m-connected but $(m + 1)$-disconnected (that is, there is a decomposition $C = A \cup B$ with $A \cdot B = m$) then

$$\mathrm{Cliff}(C) \le \min\{m-2, [\frac{p_a(C)-1}{2}]\}.$$

We remark that our results can still be applied to irreducible curves with planar singularities. In particular for an irreducible curve C it is always $\mathrm{Cliff}(C) \ge 0$, with equality holding iff C is hyperelliptic.

To show that the above introduced Clifford index ha a geometrical meaning in Section 4 we give a proof of Green's conjecture for a m-connected curve obtained glueing together two smooth curves. To be more precise we consider a stable curve $C = C_1 \cup C_2$ given by the union of an irreducible smooth general curve C_1 of genus g_1 and an irreducible smooth curve C_2 of genus g_2, meeting transversally in m distinct points $\{x_1, \cdots, x_m\}$. For such curve C if $4 \le m \le \frac{g_1+1}{2}$ and $g_2 \ge 1$ then we show that $\mathrm{Cliff}(C) = m-2$ and Green's conjecture holds for C, i.e., $\mathcal{K}_{p,1}(C, \omega_C) = 0$ iff $p \ge p_a(C) - \mathrm{Cliff}(C) - 1$ (where $\mathcal{K}_{p,1}(C, \omega_C)$ denotes the p-th Koszul group with value in ω_C (see Green's paper [11]). This result is only a modest novelty, since it is based on fundamental results of Voisin in [17, 18] and Aprodu in [1], but we hope it should be helpful in studying curves with many components, e.g., stable curves.

A second application of our results can be found in the paper [3], where the authors, in order to characterize Brill-Noether-Petri curves, analyze the Petri homomorphism for rank 2 vector bundles on a (not necessarily smooth) curve C using some results on the Clifford index.

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2 Notation and preliminary results

We work over an algebraically closed field $\mathbb{K}$ of characteristic 0.

Throughout this paper $C = C_1 \cup \cdots \cup C_n$ will denote a reduced curve with planar singularities. The C_i's are the irreducible components of C.

A subcurve $B \subseteq C$ is a curve of the form $B = C_{i_1} \cup \cdots C_{i_k}$ with $\{i_1, \ldots, i_k\} \subset$

$\{1, \ldots, n\}$. For every subcurve $B \subseteq C$ ω_B denotes the canonical sheaf of B (see [14], Chap. III, §7), K_B denotes a canonical divisor so that $\mathcal{O}_B(K_B) \cong \omega_B$ and $p_a(B)$ the arithmetic genus of B, $p_a(B) = 1 - \chi(\mathcal{O}_B)$.

Notice that by our assumptions every $B \subseteq C$ is Gorenstein (i.e., ω_B is invertible.)

A decomposition $C = A \cup B$ means $A = C_{j_1} \cup \cdots \cup C_{j_h}$, $B = C_{i_1} \cup \cdots \cup C_{i_k}$ such that $\{j_1, \ldots, j_h\} \cup \{i_1, \ldots, i_k\} = \{1, \ldots, n\}$.

For a given decomposition $C = A \cup B$, we will use the following standard exact sequences:

$$0 \to \mathcal{O}_A(-B) \to \mathcal{O}_C \to \mathcal{O}_B \to 0, \tag{2}$$

$$0 \to \omega_A \to \omega_C \to \omega_{C|B} \to 0. \tag{3}$$

where $\mathcal{O}_A(-B) \cong \mathcal{I}_{A \cap B} \cdot \mathcal{O}_A$.

If $C = A \cup B$ is a decomposition of C then the intersection product $A \cdot B$ is defined as follows

$$A \cdot B = \deg_B(\omega_C) - (2p_a(B) - 2).$$

A curve C is *m-connected* if $A \cdot B \geq m$ for every effective decomposition $C = A \cup B$ (cf. [6] for a detailed analysis on Gorentein curves). An m-connected curve C is said to be $(m+1)$-disconnected if there is a decomposition $C = A \cup B$ with $A \cdot B = m$.

For a decomposition $C = A \cup B$ we will use frequently the key formula (cf. [14, Exercise V.1.3])

$$p_a(C) = p_a(A) + p_a(B) + A \cdot B - 1. \tag{4}$$

Let $\mathcal{F}$ be a rank one torsion free sheaf on C. For every subcurve $B \subseteq C$ the degree of $\mathcal{F}$ on B can be defined by the formula $\deg \mathcal{F}_{|B} = \chi(\mathcal{F}_{|B}) - \chi(\mathcal{O}_B)$. A torsion free sheaf $\mathcal{F}$ is said to be nef if $\deg \mathcal{F}_{|B} \geq 0$ for every $B \subseteq C$.

A *cluster* S of *degree* $\deg S = r$ is a 0-dimensional subscheme with length $\mathcal{O}_S = \dim_k \mathcal{O}_S = r$. A cluster $S \subset C$ is *subcanonical* if the space $H^0(C, \mathcal{I}_S\omega_C)$ contains a generically invertible section, i.e. a section s_0 which does not vanish on any subcurve of C. Equivalently S is subcanonical if there exists an ejective map $\mathcal{O}_C \hookrightarrow \mathcal{I}_S\omega_C$ (see [10, §2.3]).

Given a subcanonical cluster S, we define its residual Cluster S^* with respect to a generic invertible section $s_0 \in H^0(C, \omega_C)$ by the following exact sequence

$$0 \longrightarrow \mathcal{H}om(\mathcal{I}_S\omega_C, \omega_C) \xrightarrow{\alpha} \mathcal{H}om(\mathcal{O}_C, \omega_C) \longrightarrow \mathcal{O}_{S^*} \longrightarrow 0$$

where the the map α is defined by $\alpha(\varphi) : 1 \mapsto \varphi(s_0)$. See [10, Section 2] for the definition and main properties.

In the following theorem we summarise some basic results proved in [6] on the relations of m-connectedness with the behaviour of the canonical sheaf ω_C. For a general treatment see [6, §2, §3] and [5].

Theorem 2.1 *Let C be a Gorenstein curve, and ω_C the canonical sheaf of C. Then*

(i) If C is 1-connected then $H^1(C, \omega_C) \cong \mathbb{K}$.

(ii) If C is 2-connected and $C \ncong \mathbb{P}^1$ then $|\omega_C|$ is base point free. To be more precise, P is a base point for $|\omega_C|$ if and only if there exist a decomposition $C = C_1 \cup C_2$ such that $C_1 \cdot C_2 = 1$ and P is a smooth point for each C_i satisfying $\omega_{C|C_i} \cong \omega_{C_i}(P)$.

(iii) If C is 3-connected and C is not honestly hyperelliptic (i.e., there does not exist a finite morphism $\psi\colon C \to \mathbb{P}^1$ of degree 2) then ω_C is very ample.

(cf. [6, Thm. 1.1, Thm. 3.3, Thm. 3.6] and [5, Proposition 2.4]).

Remark 2.2 Notice that for reduced curves the above implications are actually equivalences. Indeed, (i) is obvious; (ii) follows from the fact that a disconnecting point is necessarily a base point for $|\omega_C|$; (iii) follows since, given a decomposition $C = A \cup B$ with $A \cap B = \{P, Q\}$, then $|\omega_C|$ does not separate the 2 points. See also [5, Proposition 2.4] for a detailed analysis of the base points of $|\omega_C|$ on 2-disconnected curves.

3 Clifford index of reduced curves

3.1 Clifford index of rank 1 torsion free sheaves

In this section we extend the notion of Clifford index taking in account nef rank 1 torsion free sheaves whose multidegree is bounded from above by the degree of the canonical sheaf ω_C.

Definition 3.1 *Let $C = C_1 \cup \cdots \cup C_n$ be a connected reduced curve with planar singularities and let $\mathcal{F}$ be a nef rank 1 torsion free sheaf.*

The Clifford index of $\mathcal{F}$ is

$$\mathrm{Cliff}(\mathcal{F}) := \deg(\mathcal{F}) - 2h^0(C, \mathcal{F}) + 2$$

First of all let us consider the case $\mathcal{F} = \mathcal{I}_S\omega_C$, where $S \subset C$ is a subcanonical cluster, i.e S is a 0-dimensional scheme such that $H^0(C, \mathcal{I}_S\omega_C)$ contains a generically invertible section.

Proposition 3.2 *Let $C = C_1 \cup \cdots \cup C_n$ be a connected reduced curve with planar singularities and let S be a subcanonical cluster. Then*

$$\mathrm{Cliff}(\mathcal{I}_S\omega_C) \geq -n + 1.$$

Proof. We argue by induction on the number of irreducible components n.

If the curve C is irreducible or reducible and 2-connected it is a straightforward consequence of [10, Thm. 3.8].

If C is connected but 2-disconnected, then we may take a decomposition $C = A \cup B$, with A, B connected curves such that $A \cdot B = 1$, i.e. $A \cap B = \{P\}$ a point which is smooth for both. Let n_A be the number of irreducible components of A and n_B be the number of irreducible components of B, so that $n = n_A + n_B$.

Let S be a subcanonical cluster, i.e., assume that $H^0(C, \mathcal{I}_S\omega_C)$ contains a section s_0 which does not vanish on any subcurve of C and consider the intersection point P. Notice that P is a smooth point for both curves and it is a base point for the system $|\omega_C|$ by Theorem 2.1. Without loss of generality we may assume that $P \cap S \neq \emptyset$. Indeed, if this is not the case, we may consider a residual cluster $S^* \in H^0(C, \mathcal{I}_S\omega_C)$ with respect to s_0 (see §2). Since P is a base point for $|\omega_C|$, then P must intersect either S or S^*. Serre duality implies that the Clifford index of S and S^* coincide (see [10, Remark 2.13]), thus we may work with the cluster which contains P.

Since P is a smooth point for both the curves and C has planar singularities, we have the isomorphisms of invertible sheaves $\omega_{C|A} \cong \omega_A(P)$ and $\omega_{C|B} \cong \omega_B(P)$. Whence, being $P \cap S \neq \emptyset$, there exists a cluster T_A on A, resp. a cluster T_B on B, such that $(\mathcal{I}_S\omega_C)_{|A} \cong \mathcal{I}_{T_A}\omega_A$, resp. $(\mathcal{I}_S\omega_C)_{|B} \cong \mathcal{I}_{T_B}\omega_B$. Notice that $\deg(\mathcal{I}_{T_A}\omega_A) + \deg(\mathcal{I}_{T_B}\omega_B) = \deg(\mathcal{I}_S\omega_C)$. Moreover they are subcanonical, since a generically invertible section in $H^0(C, \mathcal{I}_S\omega_C)$ restricts to a generically invertible section in $H^0(A, \mathcal{I}_S\omega_{C|A}) = H^0(\mathcal{I}_{T_A}\omega_A)$, and similarly on B. Therefore by induction we may assume $\mathrm{Cliff}(\mathcal{I}_{T_A}\omega_A) \geq -n_A + 1$ and $\mathrm{Cliff}(\mathcal{I}_{T_B}\omega_B) \geq -n_B + 1$.

Consider now the Mayer-Vietoris sequence

$$0 \to \mathcal{I}_S\omega_C \to \begin{matrix} \mathcal{I}_{T_A}\omega_A \\ \oplus \\ \mathcal{I}_{T_B}\omega_B \end{matrix} \to \mathcal{O}_P \to 0. \tag{5}$$

Firstly assume that $H^0(A, \mathcal{I}_{T_A}\omega_A) \oplus H^0(B, \mathcal{I}_{T_B}\omega_B) \to \mathcal{O}_P$ is onto. Notice that this holds when $\mathrm{Cliff}(\mathcal{I}_{T_A}\omega_A)$ is minimum since by [10, Lemma 2.19] the restriction map $H^0(A, \mathcal{I}_{T_A}\omega_A) \to \mathcal{O}_P$ is surjective. In this case it is $h^0(C, \mathcal{I}_S\omega_C) = h^0(A, \mathcal{I}_{T_A}\omega_A) + H^0(B, \mathcal{I}_{T_B}\omega_B) - 1$, hence a straightforward computation yields $\mathrm{Cliff}(\mathcal{I}_S\omega_C) \geq \mathrm{Cliff}(\mathcal{I}_{T_A}\omega_A) + \mathrm{Cliff}(\mathcal{I}_{T_B}\omega_B)$. Therefore, by induction we have

$$\mathrm{Cliff}(\mathcal{I}_S\omega_C) \geq \mathrm{Cliff}(\mathcal{I}_{T_A}\omega_A) + \mathrm{Cliff}(\mathcal{I}_{T_B}\omega_B) \geq -n_A - n_B + 2 = -n + 2$$

If the above map (5) is not surjective on global sections then in particular $H^0(A, \mathcal{I}_{T_A}\omega_A) \to \mathcal{O}_P$ is not onto and by [10, Lemma 2.19] $\mathrm{Cliff}(\mathcal{I}_{T_A}\omega_A)$ is not minimum, i.e., by induction we may assume $\mathrm{Cliff}(\mathcal{I}_{T_A}\omega_A) \geq -n_A + 2$. In this case it is $h^0(C, \mathcal{I}_S\omega_C) = h^0(A, \mathcal{I}_{T_A}\omega_A) + H^0(B, \mathcal{I}_{T_B}\omega_B)$, hence $\mathrm{Cliff}(\mathcal{I}_S\omega_C) \geq \mathrm{Cliff}(\mathcal{I}_{T_A}\omega_A) + \mathrm{Cliff}(\mathcal{I}_{T_B}\omega_B) - 2$ and we get

$$\mathrm{Cliff}(\mathcal{I}_S\omega_C) \geq \mathrm{Cliff}(\mathcal{I}_{T_A}\omega_A) + \mathrm{Cliff}(\mathcal{I}_{T_B}\omega_B) - 2 \geq -n_A + 2 - n_B + 1 - 2 = -n + 1.$$

■

Theorem 3.3 *Let $C = C_1 \cup \cdots \cup C_n$ be a connected reduced curve with planar singularities. Then the following numbers exist and coincide:*

(1) $\min\{\mathrm{Cliff}(\mathcal{F}) :$ *$\mathcal{F}$ rank 1 torsion free sheaf s.t.* $0 \leq \deg[\mathcal{F}_{|C_i}] \leq \deg \omega_{C|C_i}$ *for* $i = 1, \cdots, n$; $h^0(\mathcal{F}) \geq 2,\ h^1(\mathcal{F}) \geq 2\}$

(2) $\min\{\mathrm{Cliff}(\mathcal{I}_S\omega_C) :$ *$S \subset C$ subcanonical cluster s.t.* $h^0(C, \mathcal{I}_S\omega_C) \geq 2,\ h^1(C, \mathcal{I}_S\omega_C) \geq 2\}$

Proof. By Proposition 3.2 the second minimum exists. It is moreover obvious that the second set is included in the first, thus such minimum is bigger than or equal to the infimum of the first set.

To conclude the proof it is enough to prove that for every rank 1 torsion free sheaf $\mathcal{F}$ in the first set attaining the minimal Clifford index there exists a subcanonical cluster T such that $\mathcal{I}_T\omega_C \cong \mathcal{F}$. This is equivalent to prove that a rank 1 torsion free sheaf $\mathcal{F}$ with minimal Clifford index is generically invertible and moreover there exists an inclusion $\mathcal{F} \hookrightarrow \omega_C$ which is generically surjective.

For the first statement, assume for a contradiction that $\mathcal{F}$ itself is not generically invertible and let $B \subset C$ be the maximal subcurve of C such that every section $s \in H^0(C, \mathcal{F})$ vanishes identically on B. Consider the decomposition $C = A \cup B$. Then by the standard exact sequence

$$0 \to \mathcal{F}_{|A}(-B) \to \mathcal{F} \to \mathcal{F}_{|B} \to 0$$

we obtain

$$H^0(A, \mathcal{F}_{|A}(-B)) \cong H^0(C, \mathcal{F})$$

$$0 \to H^0(B, \mathcal{F}) \to H^1(A, \mathcal{F}_{|A}(-B)) \to H^1(C, \mathcal{F}) \to H^1(B, \mathcal{F}) \to 0.$$

We take the sheaf $\mathcal{G} \cong \mathcal{F}_{|A}(-B) \oplus \mathcal{O}_B(A)(-A)$. Notice that $\mathcal{G}$ is a rank 1 torsion free sheaf and moreover it is $0 \leq \deg_{C_i} \mathcal{G} \leq \deg_{C_i} \mathcal{F}$ for every C_i since $\mathcal{F}_{|A}(-B)$ does not vanish on any subcurve of A (see [10, Rem. 4.1] for details).

Then it is immediately seen that $\mathcal{O}_C \hookrightarrow \mathcal{G}$, i.e., $\mathcal{G}$ is generically invertible, and

$$h^0(\mathcal{G}) = h^0(B, \mathcal{O}_B) + h^0(A, \mathcal{F}_{|A}(-B)) \geq h^0(A, \mathcal{F}_{|A}(-B)) + 1 = h^0(C, \mathcal{F}) + 1 \geq 3,$$

$$h^1(\mathcal{G}) = h^1(B, \mathcal{O}_B) + h^1(A, \mathcal{F}_{|A}(-B)) = h^1(B, \mathcal{O}_B) + h^1(C, \mathcal{F}) + h^0(B, \mathcal{F}) - h^1(B, \mathcal{F}) \geq$$

$$\geq h^1(C, \mathcal{F}) + h^1(C, \mathcal{F}) + \deg(\mathcal{F}_{|B}) + h^0(\mathcal{O}_B) \geq 3.$$

Moreover we obtain $\mathrm{Cliff}(\mathcal{G}) < \mathrm{Cliff}(\mathcal{F})$, since $\deg(\mathcal{G}) \leq \deg(\mathcal{F})$ and $h^0(\mathcal{G}) > h^0(\mathcal{F})$, which is absurd. Therefore we have in particular $\mathcal{O}_C \hookrightarrow \mathcal{F}$.

Now we show that $\mathcal{F} \hookrightarrow \omega_C$. The dual sheaf $\mathcal{H}om(\mathcal{F}, \omega_C)$ satisfies the same assumptions $\mathcal{F}$ does and by Serre duality has the same Clifford index. Hence thanks to the previous step $\mathcal{O}_C \hookrightarrow \mathcal{H}om(\mathcal{F}, \omega_C)$. In particular $H^0(C, \mathcal{O}_C) \hookrightarrow H^0(C, \mathcal{H}om(\mathcal{F}, \omega_C)) = \mathrm{Hom}(\mathcal{F}, \omega_C)$, that is, there is a map from $\mathcal{F}$ to ω_C not vanishing on any component. By automatic adjunction ([6, Proposition 2.4]) we conclude that $\mathcal{F} \cong \mathcal{I}_T \omega_C$, for some suitable 0-dimensional scheme T.

■

3.2 Clifford index of curves

The above theorem allows us to introduce the following notion of Clifford index for a reduced curve.

Definition 3.4 *Let* $C = C_1 \cup \cdots \cup C_n$ *be a connected reduced curve with planar singularities. The Clifford index of* C *is*

$$\mathrm{Cliff}(C) := \min\{\mathrm{Cliff}(\mathcal{F}) \;:\; \begin{array}{l} \mathcal{F} \textit{ rank 1 torsion free sheaf s.t.} \\ 0 \leq \deg[\mathcal{F}_{|C_i}] \leq \deg \omega_{C|C_i} \;\; \textit{for every } C_i \;; \\ h^0(\mathcal{F}) \geq 2,\;\; h^1(\mathcal{F}) \geq 2\} \end{array}$$

As in the smooth case, we say that a rank 1 torsion free sheaf $\mathcal{F}$ contributes to the Clifford index of the curve C if $h^0(C, \mathcal{F}) \geq 2$ and $h^1(C, \mathcal{F}) \geq 2$.

For 4-connected curves the Clifford index is always non-negative as can be seen by the following result.

Theorem 3.5 *If* C *is a 4-connected reduced curve with planar singularities then* $\mathrm{Cliff}(C) \geq 0$ *and it is 0 if and only if* C *is honestly hyperelliptic.*

Proof. By [10, Theorem B] if C is 4-connected then for every rank one torsion free sheaf $\mathcal{F}$ we have $h^0(C, \mathcal{F}) \leq \frac{\deg \mathcal{F}}{2} + 1$.

Moreover the above mentioned theorem shows that if equality holds then $\mathcal{F} \cong \mathcal{I}_T \omega_C$, where T is a subcanonical cluster and, as in the smooth case, either $T = 0, \omega_C$ or C is honestly hyperelliptic and T is a multiple of the honest g^1_2.

■

Corollary 3.6 *If* C *is an irreducible curve with planar singularities then* $\mathrm{Cliff}(C) \geq 0$ *and it is 0 if and only if* C *is hyperelliptic.*

If C has many components, numerical m-connectedness plays a relevant role in the the computation of the Clifford index. Indeed we have the following

Proposition 3.7 *Let* $C = C_1 \cup \cdots \cup C_n$ *be a* m*-connected reduced curve with planar singularities.*

Assume there exists a decomposition $C = A \cup B$ *such that* $A \cdot B = m$, $p_a(A) \geq 1$, $p_a(B) \geq 1$. *Then there exists an invertible sheaf* $\mathcal{F}$ *such that* $\mathrm{Cliff}(\mathcal{F}) = m - 2$, $h^0(\mathcal{F}) \geq 2, h^1(\mathcal{F}) \geq 2$.

Proof. Consider the decomposition $C = A \cup B$ with $A \cdot B = m$. Notice that A and B are numerically connected by minimality of m (see [10, Lemma 2.8]) and moreover by our assumptions it is $p_a(A) \geq 1$, $p_a(B) \geq 1$.

If $m = 1$ then by Thm. 2.1 there exists a base point P for the canonical system. Thus $\mathrm{Cliff}(\mathcal{I}_P \omega_C) = -1$ and we may conclude.

From now on we assume that $m \geq 2$ and in particular $|\omega_C|$ is base point free. Choose a generic $s \in H^0(A, \omega_{C|A})$ and take the effective divisor $\Delta = div(s)$. Since $|\omega_C|$ is base point free, Δ is the union of smooth points and moreover by our construction we may assume $\Delta \cap B = \emptyset$. Consider the invertible sheaf $\mathcal{F} := \mathcal{O}_C(\Delta)$. It is

$$\mathcal{F}_{|A} \cong \omega_{C|A}\ , \quad \mathcal{F}_{|B} \cong \mathcal{O}_B.$$

In particular $\mathcal{F}(-B)_{|A} \cong \omega_A$ and we have the exact sequence

$$\begin{array}{l} 0 \to H^0(A, \omega_A) \to H^0(C, \mathcal{F}) \to H^0(B, \mathcal{O}_B) \to \\ \to H^1(A, \omega_A) \to H^1(C, \mathcal{F}) \to H^1(B, \mathcal{O}_B) \to 0. \end{array}$$

But $H^0(C, \mathcal{F})$ does not vanish on B by our construction, whence

$$h^0(C, \mathcal{F}) = h^0(B, \mathcal{O}_B) + h^0(A, \omega_A) = 1 + p_a(A) \geq 2$$

$$h^1(C, \mathcal{F}) = h^1(B, \mathcal{O}_B) + h^1(A, \omega_A) = 1 + p_a(B) \geq 2$$

since both A and B are numerically connected.

Finally by the above computation we get

$$\begin{array}{rcl} \mathrm{Cliff}(\mathcal{F}) & = & \deg_C(\mathcal{O}_C(\Delta)) - 2h^0(C, \mathcal{O}_C(\Delta)) + 2 = \\ & = & 2p_a(A) - 2 + m - 2 \cdot (1 + p_a(A)) + 2 = m - 2. \end{array}$$

■

As an immediate consequence we obtain the following theorem.

Theorem 3.8 *Let $C = C_1 \cup \cdots \cup C_n$ be a connected reduced curve with planar singularities. Assume $C_i \neq \mathbb{P}^1$ for every $i = 1, \cdots, n$.*

If C is m-connected but $(m+1)$-disconnected (that is, there is a decomposition $C = A \cup B$ with $A \cdot B = m$) then

$$\mathrm{Cliff}(C) \leq \min\big\{m - 2, \big[\frac{p_a(C) - 1}{2}\big]\big\}.$$

Proof. First of all let us show that $\mathrm{Cliff}(C) \leq \min[\frac{p_a(C)-1}{2}]$ by a degeneration argument.

Consider a one-parameter degeneration $f : X \to T$, where X is a smooth surface and T an affine curve. Assume that f is flat and proper and there is a point $s_0 \in S$ such that $f^{-1}(t_0) := C_0 \cong C$, whilst for $t \neq 0$ $f^{-1}(t) := C_t$ is a smooth curve of genus $p_a(C)$. For each integer d, let Pic^d_f be the degree-d relative Picard scheme of f parameterizing invertible sheaves of degree

d on the fibers of f (see e.g. [13]). Then, for every invertible sheaf $\mathcal{F}$ in Pic^d_f we have $\mathrm{Cliff}(\mathcal{F}_{|C_t}) \leq \left[\frac{p_a(C)-1}{2}\right]$. Since $h^0(\mathcal{F}_{|C_t})$ and $h^1(\mathcal{F}_{|C_t})$ are semicontinuous function in t, if $\mathcal{F}_{|C_t}$ contributes to the Clifford index for some t, then $\mathcal{F}_{|C_0}$ contributes to the Clifford index of C_0 and moreover by definition it is

$$\mathrm{Cliff}(\mathcal{F}_{|C_0}) \leq \mathrm{Cliff}(\mathcal{F}_{|C_t}) \leq \left[\frac{p_a(C)-1}{2}\right].$$

Now let us show that $\mathrm{Cliff}(C) \leq m-2$, where $m := \min\{A \cdot B : C = A \cup B, A \neq \emptyset, B \neq \emptyset\}$.

Take a proper decomposition $C = A \cup B$ with $A \cdot B = m$. Notice that A and B are numerically connected by minimality of m (see [10, Lemma 2.8]) and moreover by our assumptions it is $p_a(A) \geq 1$, $p_a(B) \geq 1$. Therefore by the above Proposition 3.7 there exists an invertible sheaf $\mathcal{F} = \mathcal{O}_C(\Delta)$ which contributes to the Clifford index of C and whose Clifford index verifies $\mathrm{Cliff}(\mathcal{F}) = m-2$.

■

Remark 3.9 If we restrict our attention to stable curves, it is worth mentioning that the Clifford index, as defined in Definition 3.4, is not the limit of the Clifford index of smooth curves. More precisely, if a curve C is limit of smooth curves C_t with $\mathrm{Cliff}(C_t) \leq \gamma$, then by semicontinuity we still have $\mathrm{Cliff}(C) \leq \gamma$, but the converse does not hold. One can see that with a simple dimensional count. It is easy to compute that the locus of reduced m-connected curves has codimension m in $\overline{\mathcal{M}}_g$, and Theorem 3.8 shows that those curves have Clifford index at most $m-2$. On the contrary, considering the loci $\mathcal{M}^r_{g,d}$ of smooth curves carrying a g^r_d one can see that for small m, the locus of smooth curves having Clifford index at most $m-2$ has a far bigger codimension than m.

Remark 3.10 Let C be a m-connected, but $(m+1)$-disconnected, reduced curve of arithmetic genus $p_a(C) > 0$ and let $C = A \cup B$ be a decomposition of C in two connected curves of arithmetic genus $p_a(A)$, respectively $p_a(B)$, such that $A \cdot B = m$.

By the key formula (4) if $m \leq p_a(A) + p_a(B) + 2$ then it is $m - 2 \leq \left[\frac{p_a(C)-1}{2}\right]$. Therefore it is easy to construct stable curves with given Clifford just by taking m satisfying the above relation.

3.3 Examples of curves with negative Clifford index

In this section we are going to show two examples of curves having negative Clifford index. The first example shows that the inequality of Proposition 3.2 is sharp. The second example shows that for curves not 4-connected the geometric interpretation of the Clifford index is more subtle.

Example 3.11 Let $C = \cup_{i=0}^{n} C_i$ be a chain of smooth curves C_i with positive genus, i.e. $C_i \cdot C_{i+1} = 1$, otherwise $C_i \cdot C_j$ vanishes.

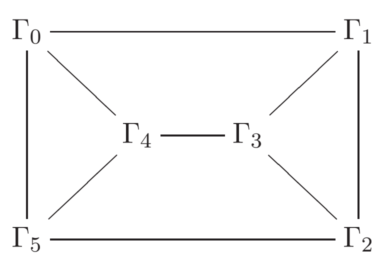
$$\Gamma_0 \text{———} \Gamma_1 \cdots\cdots\cdots \Gamma_{n-1} \text{———} \Gamma_n$$

Let $S = \operatorname{Sing} C$. Then $h^0(C, \mathcal{I}_S\omega_C) = \sum_{i=1}^{n} p_a(C_i) = p_a(C)$. Therefore we obtain

$$\operatorname{Cliff}(\mathcal{I}_S\omega_C) = 2p_a(C) - 2 - \deg S - 2h^0(C, \mathcal{I}_S\omega_C) + 2 = -n + 1.$$

Notice that in the above example every point in S is a base point of $|\omega_C|$. Now, let us point out that this is not always the case. Indeed if C is 3-connected but not 4-connected (i.e., there exists a decomposition $C = A \cup B$ such that $A \cdot B = 3$) then it might happen that $\operatorname{Cliff}(C) < 0$ even thought ω_C is normally generated, as shown in the following example.

Example 3.12 Take $C = \sum_{i=0}^{5} \Gamma_i$ and suppose that $p_a(\Gamma_i) \geq 2$ for every i. Suppose moreover that the intersection products are defined by the following dual graph, where the existence of the simple line means that the intersection product between the two curves is 1.



In this case by [9, Thm. 3.3] ω_C is very ample and normally generated by [9, Thm. 3.3].

For simplicity, assume moreover that for every i, j, k it is $\Gamma_i \cap \Gamma_j \cap \Gamma_k = \emptyset$ and take $S = \bigcup_{i,j}(\Gamma_i \cap \Gamma_j)$, which is a degree 9 cluster. Then it is easy to check $h^0(C, \mathcal{I}_S\omega_C) = \oplus_{i=0}^{5} h^0(\Gamma_i, K_{\Gamma_i})$, which yields $\operatorname{Cliff}(\mathcal{I}_S\omega_C) = -1$. (see [10, Example 5.2]).

3.4 Clifford index of invertible sheaves

The following theorem shows that the Clifford index of an invertible sheaf is always nonnegative.

Theorem 3.13 *Let $C = C_1 \cup \cdots \cup C_n$ be a reduced curve with planar singularities. Let $\mathcal{L}$ be an invertible sheaf such that $0 \leq \deg[\mathcal{L}_{|C_i}] \leq \deg \omega_{C|C_i}$ for $i = 1, \cdots, n$. Then*

$$h^0(C, \mathcal{L}) \leq \frac{1}{2} \deg \mathcal{L} + 1, \text{ i.e., } \quad \mathrm{Cliff}(\mathcal{L}) \geq 0. \tag{6}$$

Proof. First of all we remark that we may assume C to be connected since h^0 and deg are additive with respect to each connected component.

Now notice that we may assume $\mathcal{L} \not\cong \mathcal{O}_C$, $\mathcal{L} \not\cong \omega_C$ and $h^0(C, \mathcal{L}) \neq 0$, since otherwise eq. (6) is obvious. Take $\mathcal{L}$ as above such that $\mathrm{Cliff}(\mathcal{L})$ is minimum. Arguing as in the proof of Theorem 3.3 we conclude that $\mathcal{O}_C \hookrightarrow \mathcal{L} \hookrightarrow \omega_C$, i.e., there exists a subcanonical Cartier divisor S such that $\mathcal{L} \cong \mathcal{I}_S \omega_C$ (see also [10, §2.3]). Hence it is sufficient to show that for every subcanonical Cartier divisor S, $\mathrm{Cliff}(\mathcal{I}_S \omega_C) \geq 0$. We prove this result by induction on the number of irreducible components of C. To simplify the notation we write $K_C - S$ for the divisor such that $\mathcal{O}_C(K_C - S) \cong \mathcal{I}_S \omega_C$.

If C is irreducible, the classical Clifford's theorem holds (see [2, §III:1], or see [10, Theorem A] for the singular case). If C is 2-connected the result follows from [10, Theorem A, case (a)].

Therefore we are left to prove that equation (6) holds for reducible, connected but 2-disconnected curves, i.e., we may assume that there exist connected subcurves C_1 and C_2 such that $C = C_1 \cup C_2$ and $C_1 \cap C_2$ consists of one single point P. In this case P is a smooth point for both curves and for $i = 1, 2$ we can write $K_{C|C_i} \equiv K_{C_i} + P$ as divisors on C_i.

Take the subcanonical Cartier divisor S. Arguing as in Prop. 3.2 we may assume that $P \cap S \neq \emptyset$ since otherwise we can take a residual Cartier divisor S^*.

Let $S_1 := S \cap C_1$ and $S_2 := S \cap C_2$. By the above argument $P \cap S_i \neq \emptyset$ for $i = 1, 2$, and, since P is a smooth point for each C_i, both the divisors $(S_1 - P)$ and $(S_2 - P)$ are Cartier and effective. Moreover they are subcanonical on both the subcurves, since a generically invertible section in $H^0(C, K_C - S)$ restricts to a generically invertible section in $H^0(C_i, K_{C_i}(-(S_i - P)))$.

The exact sequence (3) for the splitting $C = C_1 \cup C_2$ can be written as follows:

$$0 \to \omega_{C_1}(-S_1) \to \omega_C(-S) \to \omega_{C_2}(-(S_2 - P)) \to 0.$$

In particular it gives rise to the inequality

$$h^0(C, K_C - S) \leq h^0(C_1, K_{C_1} - S_1) + h^0(C_2, K_{C_2} - (S_2 - P)). \tag{7}$$

On C_2 we may apply our induction argument obtaining $h^0(C_2, K_{C_2} - (S_2 - P)) \leq \frac{1}{2}\deg(K_{C_2} - (S_2 - P)) + 1$.

Let us consider $H^0(C_1, K_{C_1} - S_1)$. Counting dimensions we have either that $h^0(C_1, K_{C_1} - S_1) = h^0(C_1, K_{C_1} - S_1 + P) - 1$ or that $h^0(C_1, K_{C_1} - S_1) = h^0(C_1, K_{C_1} - (S_1 - P))$.

In the first case eq. (7) becomes

$$h^0(C, K_C - S) \leq h^0(C_1, K_{C_1} - (S_1 - P)) - 1 + h^0(C_2, K_{C_2} - (S_2 - P)).$$

But $S_i - P$ are subcanonical divisors on each subcurve, hence we may apply induction on C_1 and C_2 obtaining

$$\begin{aligned} h^0(C, K_C - S) &\leq \frac{1}{2}\deg(K_{C_1} - (S_1 - P)) + 1 - 1 + \frac{1}{2}\deg(K_{C_2} - (S_2 - P)) + 1 \\ &= \frac{1}{2}\deg(K_C - S) + 1. \end{aligned}$$

In the second case $H^0(C_1, K_{C_1} - S_1) = H^0(C_1, K_{C_1} - (S_1 - P))$ and in particular also S_1 is subcanonical on C_1. Therefore we may apply induction on eq. (7) obtaining

$$\begin{aligned} h^0(C, K_C - S) &\leq h^0(C_1, K_{C_1} - S_1) + h^0(C_2, K_{C_2} - (S_2 - P)) \\ &\leq \frac{1}{2}\deg(K_{C_1} - S_1) + 1 + \frac{1}{2}\deg(K_{C_2} - (S_2 - P)) + 1 \\ &= \frac{1}{2}\deg(K_C - S) + \frac{3}{2}. \end{aligned}$$

To conclude the proof it is enough to show that the above inequality is strict. We argue by contradiction. Assume that $h^0(C, K_C - S) = \frac{1}{2}\deg(K_C - S) + \frac{3}{2}$. Then necessarily

$$\begin{aligned} &h^0(C_1, K_{C_1} - S_1) = \tfrac{1}{2}\deg(K_{C_1} - S_1) + 1 \ ; \\ &h^0(C_2, K_{C_2} - (S_2 - P)) = \tfrac{1}{2}\deg(K_{C_2} - (S_2 - P)) + 1. \end{aligned}$$

In particular $\deg S_1$ must be even and $\deg S_2$ must be odd. But we may switch the roles of C_1 and C_2 and conclude that $\deg S_2$ is even and $\deg S_1$ is odd, which is clearly a contradiction. ■

Remark 3.14 The above result can be extended to non reduced curves, under suitable assumptions. Indeed the above theorem holds for 2-connected curves, whilst in the 2-disconnected case the key point of the proof is the existence of a a decomposition $C = C_1 \cup C_2$ with $C_1.C_2 = 1$ such that:

(a) C_1 and C_2 satisfy Clifford's inequality;

(b) $P = C_1 \cap C_2$ is a base point for $|K_C|$ and P is a smooth point on C_i.

In order to use point (a) we do not really need that C_i are reduced, just that they satisfy Clifford's inequality for some reason. E.g. 2-connected (possibly nonreduced) curves are perfectly fine.

In order to deal with point (b) the key fact is that by Theorem 2.1 P is a base point for $|\omega_C|$ if and only if there exist a subcurve $B \subset C$ such that $\omega_{C|B} \cong \omega_B(P)$ and P is smooth for B.

4 Green's conjecture for certain classes of reduced m-connected curves

Let C be a reduced curve, let $\mathcal{H}$ be an invertible sheaf on C and let $W \subseteq H^0(C, \mathcal{H})$ be a subspace which yields a base point free system of projective dimension r.

The Koszul groups $\mathcal{K}_{p,q}(C, \mathcal{H}, W)$ are defined as the cohomology at the middle of the complex

$$\bigwedge^{p+1} W \otimes H^0(\mathcal{H}^{q-1}) \longrightarrow \bigwedge^{p} W \otimes H^0(\mathcal{H}^{q}) \longrightarrow \bigwedge^{p-1} W \otimes H^0(\mathcal{H}^{q+1})$$

If $W = H^0(C, \mathcal{H})$ they are usually denoted by $\mathcal{K}_{p,q}(C, \mathcal{H})$ (see [11] for the definition and main results). The groups $\mathcal{K}_{p,q}(C, \mathcal{H})$ play a significant role if $\mathcal{H}$ is very ample and normally generated since in this case $\mathcal{K}_{p,q}(C, \mathcal{H}) \otimes \mathcal{O}_{\mathbb{P}^n}(-p-q)$ are the terms of the resolution of the ideal sheaf of the embedded curve (see [11, Thm. 2.a.15]).

If C is a Gorenstein curves with planar singularities, 3-connected and not (honestly) hyperelliptic then by [9, Thm. 3.3] ω_C is very ample and normally generated. Therefore it is worth studying the Koszul groups $\mathcal{K}_{p,q}(C, \omega_C)$. Indeed we have the following result.

Theorem 4.1 *Let C be a Gorenstein curve of aritmetic genus $p_a(C) \geq 3$, with planar singularities, 3-connected and not honestly hyperelliptic. Then*

- $\mathcal{K}_{0,q}(C, \omega_C) = 0$ *for all $q > 0$, i.e., ω_C is normally generated;*

- $\mathcal{K}_{p,q}(C, \omega_C) = 0$ *if* $q \geq 4$*;*
- $\mathcal{K}_{p,3}(C, \omega_C) \cong \mathbb{C}$ *if* $p = g-2$*, and* $\mathcal{K}_{p,3}(C, \omega_C) = 0$ *if* $p \neq g-2$*;*
- $\mathcal{K}_{p,1}(C, \omega_C)^\vee \cong \mathcal{K}_{g-p-2,2}(C, \omega_C)$*;*
- $\mathcal{K}_{p,1}(C, \omega_C) = 0 \Rightarrow \mathcal{K}_{p',1}(C, \omega_C) = 0 \ \forall p' \geq p$*;*
- $\mathcal{K}_{p,2}(C, \omega_C) = 0 \Rightarrow \mathcal{K}_{p',2}(C, \omega_C) = 0 \ \forall p' \leq p$*.*

Proof. $\mathcal{K}_{0,q}(C, \omega_C) = 0$ for all $q > 0$ follows by [9, Thm. 3.3]. The remaining statements follows from the same the arguments used for smooth curves (see [11, Thm. 4.3.1]) and by the duality results given in [8, Prop. 1.4].

■

Taking Definition 3.4 for a generalisation of the usual Clifford index, Green's Conjecture ([11, Conjecture 5.1] can be formulated without changes, i.e., given a 4-connected not hyperelliptic Gorenstein curve C then one may ask if

$$\mathcal{K}_{p,1}(C, \omega_C) = 0 \overset{?}{\iff} p \geq p_a(C) - \mathrm{Cliff}(C) - 1.$$

First of all notice that, as in the smooth case, we have a non vanishing result, and hence an upper bound on the Clifford index of a curve C:

Proposition 4.2 (Green-Lazarsfeld) *Let* $C = C_1 \cup \cdots \cup C_n$ *be a 4-connected, not honestly hyperelliptic, reduced curve with planar singularities. Assume* $C_i \neq \mathbb{P}^1$ *for every* $i = 1, \cdots, n$*.*

Then

$$p \leq p_a(C) - \mathrm{Cliff}(C) - 2 \Longrightarrow \mathcal{K}_{p,1}(C, \omega_C) \neq 0$$

Proof. By Theorem 3.5 it is $\mathrm{Cliff}(C) \geq 0$ and by Theorem 4.1 ω_C is normally generated. Now by Theorem 3.3 there exists a proper subcanonical cluster S such that $\mathcal{I}_S\omega_C$ computes the Clifford index of C.

Let S^* be its residual Cluster with respect to a generic invertible section $s_0 \in H^0(C, \omega_C)$. By definition it is $\mathcal{I}_{S^*}\omega_C \cong \mathcal{H}om(\mathcal{I}_S\omega_C, \omega_C)$ and by Serre duality it is $H^1(C, \mathcal{I}_S K_C) \overset{\mathrm{d}}{=} H^0(C, \mathcal{I}_{S^*} K_C)$, so that $\mathrm{Cliff}(\mathcal{I}_S K_C) = \mathrm{Cliff}(\mathcal{I}_{S^*} K_C)$. Moreover, denoting by $\Lambda := div(s_0)$ the effective divisor corresponding to s_0 we have the following exact sequence

$$0 \to \mathcal{O}_C \cong \mathcal{I}_\Lambda\omega_C \to \mathcal{I}_S\omega_C \to \mathcal{O}_{S^*} \to 0$$

(see [10, §2] for details).

Therefore we can consider $\mathbb{P}(H^0(\mathcal{I}_S\omega_C))$ as a g^r_d, where $d = \deg \mathcal{I}_S\omega_C$ and $h^0(\mathcal{I}_S\omega_C) = r+1$ and $\mathbb{P}(H^0(\mathcal{I}_{S^*}\omega_C))$ as the residual $g^{r'}_{d'}$, where $d' = \deg \mathcal{I}_{S^*}\omega_C$ and $h^0(\mathcal{I}_{S^*}\omega_C) = r'+1$. Setting

$$W_1 = \mathrm{Im}\{H^0(\mathcal{I}_S\omega_C) \hookrightarrow H^0(\omega_C)\}, \quad W_2 = \mathrm{Im}\{H^0(\mathcal{I}_{S^*}\omega_C) \hookrightarrow H^0(\omega_C)\}$$

and

$$\bar{D_1} = \mathrm{Ann}(W_1) \subset H^0(C,\omega_C)^\vee, \quad \bar{D_2} = \mathrm{Ann}(W_2) \subset H^0(C,\omega_C)^\vee$$

we can repeat verbatim the argument adopted by Green and Lazarsfeld in [11, Appendix] obtaining $\mathcal{K}_{r+r'-1,1}(C,\omega_C) \neq 0$.

To conclude it is enough to see that $r + r' - 1 = p_a(C) - \mathrm{Cliff}(C) - 2$ since $d' = 2p_a(C) - 2 - d$ and $\mathrm{Cliff}(\mathcal{I}_S\omega_C) = \mathrm{Cliff}(\mathcal{I}_{S^*}\omega_C) = \mathrm{Cliff}(C)$.

The non vanishing of $\mathcal{K}_{p,1}(C,\omega_C)$ for every $p < p_a(C) - \mathrm{Cliff}(C) - 2$ follows from Theorem 4.1.

■

Corollary 4.3 *Let $C = C_1 \cup \cdots \cup C_n$ be a connected reduced curve with planar singularities. Assume $C_i \neq \mathbb{P}^1$ for every $i = 1, \cdots, n$.*

If C is m-connected but $(m+1)$-disconnected then $\mathrm{Cliff}(C) \leq m-2$ and $\mathcal{K}_{p,1}(C,\omega_C) \neq 0$ if $p \leq p_a(C) - m$.

To show that our notion of Clifford index has a geometrical meaning we show that Green's conjecture holds in the particular case of a stable curve consisting of two smooth components intersecting in m distinct points.

Theorem 4.4 *Let g_1, g_2, m be integers such that $4 \leq m \leq \frac{g_1+1}{2}$ and $g_2 \geq 1$.*

Let $C = C_1 \cup C_2$ be a stable curve given by the union of an irreducible smooth general curve C_1 of genus g_1 and an irreducible smooth curve C_2 of genus g_2, meeting transversally in m distinct points $\{x_1, \cdots, x_m\}$. Then

$$\mathrm{Cliff}(C) = m - 2 \quad \textit{and} \quad \mathcal{K}_{p,1}(C,\omega_C) = 0 \iff p \geq p_a(C) - \mathrm{Cliff}(C) - 1.$$

Proof. Since $p_a(C) = g_1 + g_2 + m - 1$, the theorem follows if we prove that $\mathcal{K}_{p,1}(\omega_C) = 0$ if and only if $p \geq p_a(C) - m + 1 = g_1 + g_2$.

First of all notice that by Thm. 2.1 the linear system $|\omega_C|$ yields an embedding $\varphi : C \hookrightarrow \mathbb{P}^{p_a(C)-1}$ such that $\varphi(C)$ is the union of two curves of genus g_1 (resp. g_2) and degree $2g_1 - 2 + m$ (resp. $2g_2 - 2 + m$) intersecting in m points $\{\varphi(x_1), \cdots, \varphi(x_m)\}$.

Now consider the standard exact sequence

$$0 \to \mathcal{O}_{C_2}(-C_1) \to \mathcal{O}_C \to \mathcal{O}_{C_1} \to 0.$$

Twisting with $\omega_C^{\otimes q}$ and taking cohomology we get the following exact sequence of $S(H^0(C, \omega_C))$-modules

$$0 \to \bigoplus_{q\geq 0} H^0(C_2, \omega_{C|C_2}^{\otimes q}(-C_1)) \to \bigoplus_{q\geq 0} H^0(C, \omega_C^{\otimes q}) \to \bigoplus_{q\geq 0} H^0(C_1, \omega_{C|C_1}^{\otimes q})) \to 0 \tag{8}$$

where the maps preserve the grading.

Let $W := H^0(C, \omega_C)$. We emphasize that $\varphi(C_1)$ and $\varphi(C_2)$ are embedded as degenerate curves in $\mathbb{P}(W^\vee)$, but we can still consider every terms above as $S(W)$-modules. Therefore we will use the notation $\mathcal{K}_{p,q}\big(-,-,W\big)$ to point out that we are finding the resolution of the ideal of such degenerate curves (see [11, Proof of Thm. (3.b.7)] for a similar argument).

By [11, Corollary 1.4.d, Thm. 3.b.1] we have the following exact sequence of Koszul groups :

$$\begin{aligned} \cdots \to \mathcal{K}_{p+1,0}\big(C_1, \omega_{C|C_1}, W\big) \to \mathcal{K}_{p,1}\big(C_2, \mathcal{O}_{C_2}(-C_1), \omega_{C|C_2}, W\big) \to \\ \to \mathcal{K}_{p,1}\big(C, \omega_C\big) \to \mathcal{K}_{p,1}\big(C_1, \omega_{C|C_1}, W\big) \to \cdots \end{aligned} \tag{9}$$

To deal with the above groups we consider the splittings

$W = H^0(C, \omega_C) = H^0(C_2, \omega_{C|C_2}) \bigoplus U$ with $U \cong H^0(C_1, \omega_{C_1})$,

$W = H^0(C, \omega_C) = H^0(C_1, \omega_{C|C_1}) \bigoplus Z$ with $Z \cong H^0(C_2, \omega_{C_2})$.

Setting $s = \max\{0, p - g_1\}$, $t = \max\{0, p - g_2\}$ we have the following decompositions of the Koszul groups appearing in the above exact sequence:

$$\begin{aligned} \mathcal{K}_{p,1}(C_2, \mathcal{O}_{C_2}(-C_1), \omega_{C|C_2}, W) &= \bigoplus_{s\leq p'\leq p} \big[\mathcal{K}_{p',1}(C_2, \mathcal{O}_{C_2}(-C_1), \omega_{C|C_2}) \otimes \bigwedge^{p-p'} U\big] \\ \mathcal{K}_{p,1}(C_1, \omega_{C|C_1}, W) &= \bigoplus_{t\leq p''\leq p} \big[\mathcal{K}_{p'',1}(C_1, \omega_{C|C_1}) \otimes \bigwedge^{p-p''} Z\big] \end{aligned}$$

Let us study at first $\mathcal{K}_{p,1}(C_2, \mathcal{O}_{C_2}(-C_1), \omega_{C|C_2}, W)$.

Fix $p' \leq p$. By duality (cf. [8, Prop. 1.4]) and the shift properties of $\mathcal{K}_{p,q}$ [11, (2.a.17)] we have the following isomorphisms

$$\begin{aligned} \mathcal{K}_{p',1}(C_2, \mathcal{O}_{C_2}(-C_1), \omega_{C|C_2}) &\cong \mathcal{K}_{g_2+m-3-p',1}(C_2, \omega_{C_2} \otimes \mathcal{O}_{C_2}(C_1), \omega_{C|C_2}) \\ &\cong \mathcal{K}_{g_2+m-3-p',2}(C_2, \omega_{C|C_2}). \end{aligned}$$

But $\deg(\omega_{C|C_2}) = 2g_2 - 2 + m$. Whence by [11, Theorem (4.a.1)]

$$\mathcal{K}_{g_2+m-3-p',2}(C_2, \omega_{C|C_2}) = 0 \text{ if } \quad g_2 + m - 3 - p' \leq m - 3.$$

Therefore we get

$$\mathcal{K}_{p,1}\big(C_2, \mathcal{O}_{C_2}(-C_1), \omega_{C|C_2}, W\big) = 0 \text{ if } p \geq g_1 + g_2. \tag{10}$$

Now let us study $\mathcal{K}_{p,1}\big(C_1, \omega_{C|C_1}, W\big)$.

By our assumption C_1 is a general curve of genus $g_1 \geq 2m-1$. For a general curve of genus g_1 we have $\mathrm{Cliff}(C_1) = [\frac{g_1-1}{2}]$ and by the results of Voisin on Green's conjecture for smooth curves with maximal Clifford index ([17], [18]) we have $\mathcal{K}_{p,1}(C_1, \omega_{C_1}) = 0$ if $\quad p \geq g_1 - [\frac{g_1+1}{2}]$.

By our construction $\omega_{C|C_1} \cong \omega_{C_1}(C_2) \cong \omega_{C_1} \otimes \mathcal{O}_{C_1}(x_1 + \ldots + x_m)$, hence by the result of Aprodu on adjoint bundles [1, Thm. 3] we get

$$\mathcal{K}_{p'',1}(C_1, \omega_C) = 0 \text{ if } \quad p'' \geq g_1 + m - [\frac{g_1+1}{2}]$$

and in particular

$$\mathcal{K}_{p,1}\big(C_1, \omega_{C|C_1}, W\big) = 0 \text{ if } \quad p \geq g_2 + g_1 + m - [\frac{g_1+1}{2}]. \tag{11}$$

Therefore, since $m \leq \frac{g_1+1}{2}$ by our assumptions, we obtain

$$\mathcal{K}_{p,1}\big(C_1, \omega_{C|C_1}, W\big) = 0 \text{ if } \quad p \geq g_1 + g_2.$$

Putting our vanishing results (10) and (11) into the exact sequence (9) we deduce that

$$\mathcal{K}_{p,1}(C, \omega_C) = 0 \text{ if } \quad p \geq g_1 + g_2 = p_a(C) - (m-2) - 1.$$

To conclude the proof notice that the above vanishing result implies $\mathrm{Cliff}(C) \geq m-2$ by Proposition 4.2, whereas we have $\mathrm{Cliff}(C) \leq m-2$ by Thm. 3.8 because $m - 2 \leq [\frac{p_a(C)-1}{2}]$ by our numerical assumptions.

Therefore it is $\mathrm{Cliff}(C) = m-2$ and $\mathcal{K}_{p,1}(C, \omega_C) = 0$ if and only if $p \geq p_a(C) - \mathrm{Cliff}(C) - 1$.

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