NON-EXISTENCE OF HOPF-GALOIS STRUCTURES AND BIJECTIVE CROSSED HOMOMORPHISMS

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ABSTRACT. By work of C. Greither and B. Pareigis as well as N. P. Byott, the enumeration of Hopf-Galois structures on a Galois extension of fields with Galois group G may be reduced to that of regular subgroups of $\operatorname{Hol}(N)$ isomorphic to G as N ranges over all groups of order |G|, where $\operatorname{Hol}(-)$ denotes the holomorph. In this paper, we shall give a description of such subgroups of $\operatorname{Hol}(N)$ in terms of bijective crossed homomorphisms $G \longrightarrow N$, and then use it to study two questions related to non-existence of Hopf-Galois structures.

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1. INTRODUCTION

Let G be a finite group, and write Perm(G) for the symmetric group of G. Recall that a subgroup N of Perm(G) is said to be *regular* if the map

$$\xi_N: N \longrightarrow G; \quad \xi_N(\eta) = \eta(1_G)$$

is bijective. Notice that N must have the same order as G in this case. There are two obvious examples, namely $\rho(G)$ and $\lambda(G)$, where

$$\begin{cases} \rho: G \longrightarrow \operatorname{Perm}(G); & \rho(\sigma) = (\tau \mapsto \tau \sigma^{-1}) \\ \lambda: G \longrightarrow \operatorname{Perm}(G); & \lambda(\sigma) = (\tau \mapsto \sigma \tau) \end{cases}$$

are the right and left regular representations of G, respectively. It is easy to see that $\rho(G)$ and $\lambda(G)$ are equal precisely when G is abelian.

Now, consider a finite Galois extension L/K of fields with Galois group G. The group ring K[G] is a Hopf algebra over K and its action on L defines a Hopf-Galois structure on L/K. By C. Greither and B. Pareigis [14], there is a bijection between Hopf-Galois structures on L/K and regular subgroups of Perm(G) normalized by $\lambda(G)$, with the *classical* structure K[G] corresponding to $\rho(G)$. The consideration of the various Hopf-Galois structures, instead of just K[G], has applications in Galois module theory; see [10] for a survey on this subject up to the year 2000.

Therefore, it is of interest to determine the number

 $e(G) = \#\{\text{regular subgroups of Perm}(G) \text{ normalized by } \lambda(G)\}.$

See [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 16, 17] for some known results. In general, it could be difficult to compute e(G) because Perm(G) might have many regular subgroups, and the papers above all make use of the following simplification due to N. P. Byott in [1]. Note that it suffices to compute

$$e(G, N) = \# \begin{cases} \text{regular subgroups of } \operatorname{Perm}(G) \text{ which are} \\ \text{isomorphic to } N \text{ and normalized by } \lambda(G) \end{cases}$$

for each group N of order |G|. Further, define

$$Hol(N) = \{ \pi \in Perm(N) : \pi \text{ normalizes } \lambda(N) \},\$$

called the *holomorph* of N. Then, as shown in [1] or [10, Section 7], we have

(1.1)
$$e(G,N) = \frac{|\operatorname{Aut}(G)|}{|\operatorname{Aut}(N)|} \cdot \# \left\{ \begin{array}{c} \text{regular subgroups in Hol}(N) \\ \text{which are isomorphic to } G \end{array} \right\}$$

which in turn may be rewritten as

$$e(G, N) = \frac{1}{|\operatorname{Aut}(N)|} \cdot \#\operatorname{Reg}(G, \operatorname{Hol}(N)),$$

where we define

$$\operatorname{Reg}(G, \operatorname{Hol}(N)) = \{\beta \in \operatorname{Hom}(G, \operatorname{Hol}(N)) : \beta(G) \text{ is regular}\}.$$

Notice that elements of the above set are automatically injective because N has order |G|. The number (1.1) is much easier to compute because

(1.2)
$$\operatorname{Hol}(N) = \rho(N) \rtimes \operatorname{Aut}(N),$$

by [10, Proposition 7.2], for example. In particular, the set Reg(G, Hol(N)) may be parametrized by certain G-actions on N together with the bijective crossed homomorphisms associated to them; see Proposition 2.1 below.

The purpose of this paper is to use the parametrization of Reg(G, Hol(N))given in Proposition 2.1 to study two questions concerning non-existence of Hopf-Galois structures; see Questions 1.1 and 1.4.

For notation, given a group Γ , we shall write:

$$Z(\Gamma) =$$
 the center of Γ ,
 $[\Gamma, \Gamma] =$ the commutator subgroup of Γ ,
 $Inn(\Gamma) =$ the inner automorphism group of Γ ,
 $Out(\Gamma) =$ the outer automorphism group of Γ .

Also, all groups considered in this paper are finite.

1.1. Isomorphic type. In the case that N = G, notice that $\rho(G)$ and $\lambda(G)$ are regular subgroups of Hol(G) isomorphic to G. We ask:

Question 1.1. Is there a regular subgroup of Hol(G) isomorphic to G other than the obvious ones $\rho(G)$ and $\lambda(G)$?

,

For G abelian, the answer is completely known.

Theorem 1.2. Let A be an abelian group. Then, we have e(A, A) = 1 if and only if $|A| = 2^{\delta} p_1 \cdots p_m$ for distinct odd primes p_1, \ldots, p_m and $\delta \in \{0, 1, 2\}$, where we allow the product of odd primes to be empty.

Most of Theorem 1.2 may be deduced from [6, Theorem 5] and results in [1, 17]. In Section 3.1, we shall give an alternative and independent proof of the backward implication, as well as a proof of the forward implication using only a couple results from [1, 17].

For G non-abelian, the answer is not quite understood. In [9], S. Carnahan and L. N. Childs answered Question 1.1 in the negative when G is non-abelian simple. In Section 3.2, we shall extend their result to all quasisimple groups. Recall that G is said to be *quasisimple* if G = [G, G] and G/Z(G) is simple.

Theorem 1.3. Let Q be a quasisimple group. Then, we have e(Q, Q) = 2.

1.2. Non-isomorphic type. In the case that N has order |G| but $N \not\simeq G$, there is no obvious regular subgroup of $\operatorname{Hol}(N)$ isomorphic to G. We ask:

Question 1.4. Is there a regular subgroup of Hol(N) isomorphic to G?

In [3], N. P. Byott answered Question 1.4 in the negative for every $N \not\simeq G$ when G is non-abelian simple. One key idea in [3] is the use of *characteristic* subgroups of N, that is, the subgroups M of N for which $\varphi(M) = M$ for all $\varphi \in \operatorname{Aut}(N)$. In Section 4.1, we shall reformulate as well as extend this idea in terms of our Proposition 2.1; see Lemma 4.1 below. Then, in Section 4.2, we shall apply Lemma 4.1 to give an alternative proof of the following result due to T. Kohl [16].

Theorem 1.5. Let C_{p^n} be a cyclic group of odd prime power order p^n . Then, we have $e(C_{p^n}, N) = 0$ for all groups N of order p^n with $N \not\simeq C_{p^n}$.

In view of [3], it is natural to ask whether Question 1.4 also has a negative answer for every $N \not\simeq G$ when G is quasisimple. In Section 4.3.1, by applying Lemma 4.1 together with some other techniques from [3], we shall show that this is indeed the case when G is in the following family. **Theorem 1.6.** Let $2A_n$ be the double cover of the alternating group A_n on n letters, where $n \ge 5$. Then, we have $e(2A_n, N) = 0$ for all groups N of order n! with $N \not\simeq 2A_n$.

In order to determine e(G), one has to compute e(G, N) for all groups N of order |G|. This could be difficult because there are lots of such N in general. In the case that e(G, N) = 0 for every $N \not\simeq G$, it suffices to compute e(G, G)and the problem is significantly simplified. However, in most cases, we have $e(G, N) \ge 1$ for at least one $N \not\simeq G$. Nonetheless, it seems very likely that the techniques we develop in Section 4 may be applied to show that e(G, N) = 0for a large family of N, whence reducing the number of N that one needs to consider. As an illustration, in Section 4.3.2, we shall prove:

Theorem 1.7. Let S_n be the symmetric group on n letters, where $n \ge 5$, and let N be a group of order n! with $e(S_n, N) \ge 1$. Then, we have:

(1) N fits into a short exact sequence $1 \longrightarrow A_n \longrightarrow N \longrightarrow \{\pm 1\} \longrightarrow 1$, or

(2) N fits into a short exact sequence $1 \longrightarrow \{\pm 1\} \longrightarrow N \longrightarrow A_n \longrightarrow 1$, or (3) S_n embeds into Out(N).

Moreover, for any proper maximal characteristic subgroup M of N, the quotient N/M is either non-abelian or isomorphic to $\{\pm 1\}$.

Let us note that conditions (1) and (2) in Theorem 1.7 cannot be removed because $e(S_n, S_n) \ge 1$ and $e(S_n, A_n \times \{\pm 1\}) \ge 1$ when $n \ge 5$; see [9] for the exact values of these two numbers.

2. Regular subgroups of the holomorph

Throughout this section, let G and N denote two groups having the same order. Recall that given $\mathfrak{f} \in \operatorname{Hom}(G, \operatorname{Aut}(N))$, a map $\mathfrak{g} \in \operatorname{Map}(G, N)$ is said to be a crossed homomorphism with respect to \mathfrak{f} if

$$\mathfrak{g}(\sigma_1\sigma_2) = \mathfrak{g}(\sigma_1) \cdot \mathfrak{f}(\sigma_1)(\mathfrak{g}(\sigma_2))$$
 for all $\sigma_1, \sigma_2 \in G$.

In general \mathfrak{g} is not a group homomorphism, but for any $\sigma \in G$, we have

(2.1)
$$\mathfrak{g}(\sigma^k) = \prod_{i=0}^{k-1} \mathfrak{f}(\sigma)^i(\mathfrak{g}(\sigma)) \text{ and in particular } \mathfrak{g}(\sigma^{e_\sigma k}) = \mathfrak{g}(\sigma^{e_\sigma})^k$$

for all $k \in \mathbb{N}$, where e_{σ} denotes the order of $\mathfrak{f}(\sigma)$. We shall write $Z^{1}_{\mathfrak{f}}(G, N)$ for the set of all such maps, and $Z^{1}_{\mathfrak{f}}(G, N)^{*}$ for the subset consisting of those which are bijective.

Proposition 2.1. For $\mathfrak{f} \in \operatorname{Map}(G, \operatorname{Aut}(N))$ and $\mathfrak{g} \in \operatorname{Map}(G, N)$, define

$$\beta_{(\mathfrak{f},\mathfrak{g})}: G \longrightarrow \operatorname{Hol}(N); \quad \beta_{(\mathfrak{f},\mathfrak{g})}(\sigma) = \rho(\mathfrak{g}(\sigma)) \cdot \mathfrak{f}(\sigma).$$

Then, we have

$$\begin{aligned} \operatorname{Map}(G, \operatorname{Hol}(N)) &= \{ \beta_{(\mathfrak{f}, \mathfrak{g})} : \mathfrak{f} \in \operatorname{Map}(G, \operatorname{Aut}(N)) \text{ and } \mathfrak{g} \in \operatorname{Map}(G, N) \}, \\ \operatorname{Hom}(G, \operatorname{Hol}(N)) &= \{ \beta_{(\mathfrak{f}, \mathfrak{g})} : \mathfrak{f} \in \operatorname{Hom}(G, \operatorname{Aut}(N)) \text{ and } \mathfrak{g} \in Z^1_{\mathfrak{f}}(G, N) \}, \\ \operatorname{Reg}(G, \operatorname{Hol}(N)) &= \{ \beta_{(\mathfrak{f}, \mathfrak{g})} : \mathfrak{f} \in \operatorname{Hom}(G, \operatorname{Aut}(N)) \text{ and } \mathfrak{g} \in Z^1_{\mathfrak{f}}(G, N)^* \}. \end{aligned}$$

Proof. The first equality is a direct consequence of (1.2). The second equality may be easily verified using the fact that

$$\rho(\eta_1)\varphi_1 \cdot \rho(\eta_2)\varphi_2 = \rho(\eta_1)\varphi_1\rho(\eta_2)\varphi_1^{-1} \cdot \varphi_1\varphi_2 = \rho(\eta_1\varphi_1(\eta_2)) \cdot \varphi_1\varphi_2$$

for $\eta_1, \eta_2 \in N$ and $\varphi_1, \varphi_2 \in Aut(N)$. The third equality is then clear because

$$(\beta_{(\mathfrak{f},\mathfrak{g})}(\sigma))(1_N) = (\rho(\mathfrak{g}(\sigma)) \cdot \mathfrak{f}(\sigma))(1_N) = \rho(\mathfrak{g}(\sigma))(1_N) = \mathfrak{g}(\sigma)^{-1}$$

for $\sigma \in G$. This proves the proposition.

In the rest of this section, let $\mathfrak{f} \in \operatorname{Hom}(G, \operatorname{Aut}(N))$ be fixed, and we shall consider some examples of $\mathfrak{g} \in Z^1_{\mathfrak{f}}(G, N)$.

2.1. The trivial action. Let $\mathfrak{f}_0 \in \operatorname{Hom}(G, \operatorname{Aut}(N))$ be the trivial homomorphism, and note that $Z^1_{\mathfrak{f}_0}(G, N) = \operatorname{Hom}(G, N)$. For $N \not\simeq G$, we then deduce that $Z^1_{\mathfrak{f}_0}(G, N)^* = \emptyset$. As for N = G, we easily see from (1.2) that

(2.2) for
$$\mathfrak{g} \in Z^1_{\mathfrak{f}}(G,G)^*$$
: $\beta_{(\mathfrak{f},\mathfrak{g})}(G) = \rho(G)$ if and only if $\mathfrak{f} = \mathfrak{f}_0$.

Hence, the case when $\mathfrak{f} = \mathfrak{f}_0$ only gives rise to the regular subgroup $\rho(G)$.

2.2. Principal crossed homomorphisms. Given any $\eta \in N$, it is natural to consider its associated *principal* crossed homomorphism, defined by

$$\mathfrak{g}_{\eta} \in Z^{1}_{\mathfrak{f}}(G,N); \quad \mathfrak{g}_{\eta}(\sigma) = \eta^{-1} \cdot \mathfrak{f}(\sigma)(\eta).$$

Unfortunately, this map is never bijective, unless G and N are trivial. Indeed, viewing N as a G-set via the homomorphism \mathfrak{f} , it is easy to check that

 \mathfrak{g}_{η} is injective if and only if $\operatorname{Stab}_{G}(\eta) = \{1_{G}\}.$

In this case, since |G| = |N|, by the orbit-stabilizer theorem, we must have

$$\{\mathfrak{f}(\sigma)(\eta): \sigma \in G\} = N$$
, and so $\mathfrak{f}(\sigma)(\eta) = 1_N$ for some $\sigma \in G$.

This implies that $\eta = 1_N$, but \mathfrak{g}_{1_N} is not bijective unless G and N are trivial.

2.3. Action via inner automorphisms. For $\eta \in N$, we shall write

 $\operatorname{conj}(\eta) = \rho(\eta)\lambda(\eta)$ as well as $\operatorname{conj}(\eta Z(N)) = \operatorname{conj}(\eta)$.

The latter is plainly well-defined, and note that $\operatorname{Inn}(N) \simeq N/Z(N)$ via conj. In the case that $\mathfrak{f}(G) \subset \operatorname{Inn}(N)$, elements in $Z^1_{\mathfrak{f}}(G, N)^*$ turn out to be closely related to certain *fixed point free* pairs of homomorphisms. This connection was first observed by N. P. Byott and L. N. Childs in [5]; see the discussion at the end of Section 2.3.1.

Definition 2.2. A pair (f, g), where $f, g \in \text{Hom}(G, N)$, is fixed point free if

 $f(\sigma) = g(\sigma)$ holds precisely when $\sigma = 1_G$.

Since |G| = |N|, this is easily seen to be equivalent to that the map $G \longrightarrow N$ given by $\sigma \mapsto f(\sigma)g(\sigma)^{-1}$ is bijective; see [5, Proposition 1], for example.

We shall further make the following definition.

Definition 2.3. A pair (f, g), where $f, g \in \text{Hom}(G, N/Z(N))$, is weakly fixed point free if the map $G \longrightarrow N/Z(N)$ given by $\sigma \mapsto f(\sigma)g(\sigma)^{-1}$ is surjective.

2.3.1. Liftable inner actions. In what follows, assume that

there exists $f \in \text{Hom}(G, N)$ with $\mathfrak{f}(\sigma) = \text{conj}(f(\sigma))$ for all $\sigma \in G$.

This implies that $\mathfrak{f}(G) \subset \operatorname{Inn}(N)$ but the converse is false in general. Put

$$\operatorname{Hom}_f(G, N)^* = \{g \in \operatorname{Hom}(G, N) : (f, g) \text{ is fixed point free}\}.$$

Then, we have:

Proposition 2.4. The maps

(2.3)
$$Z^1_{\mathfrak{f}}(G,N) \longrightarrow \operatorname{Hom}(G,N); \quad \mathfrak{g} \mapsto (\sigma \mapsto \mathfrak{g}(\sigma)f(\sigma))$$

 $Z^1_{\mathfrak{f}}(G,N)^* \longrightarrow \operatorname{Hom}_f(G,N)^*; \quad \mathfrak{g} \mapsto (\sigma \mapsto \mathfrak{g}(\sigma)f(\sigma))$

are well-defined bijections.

Proof. First, let $\mathfrak{g} \in Z^1_{\mathfrak{f}}(G, N)$. Then, for $\sigma_1, \sigma_2 \in G$, we have

$$\mathfrak{g}(\sigma_1\sigma_2)f(\sigma_1\sigma_2) = \mathfrak{g}(\sigma_1)\mathrm{conj}(f(\sigma_1))(\mathfrak{g}(\sigma_2)) \cdot f(\sigma_1)f(\sigma_2)$$
$$= \mathfrak{g}(\sigma_1)f(\sigma_1) \cdot \mathfrak{g}(\sigma_2)f(\sigma_2).$$

This shows that the first map is well-defined. Next, let $g \in \text{Hom}(G, N)$, and define $\mathfrak{g}(\sigma) = g(\sigma)f(\sigma)^{-1}$. Then, for $\sigma_1, \sigma_2 \in G$, we have

$$\begin{aligned} \mathfrak{g}(\sigma_1 \sigma_2) &= g(\sigma_1 \sigma_2) f(\sigma_1 \sigma_2)^{-1} \\ &= g(\sigma_1) f(\sigma_1)^{-1} \mathrm{conj}(f(\sigma_1)) (g(\sigma_2) f(\sigma_2)^{-1}) \\ &= \mathfrak{g}(\sigma_1) \cdot \mathfrak{f}(\sigma_1) (\mathfrak{g}(\sigma_2)). \end{aligned}$$

This shows the first map, which is plainly injective, is also surjective. From Definition 2.2, it is clear that \mathfrak{g} is bijective if and only if (f, g) is fixed point free. Hence, the second map is also a well-defined bijection.

Let $\mathfrak{g} \in Z^1_{\mathfrak{f}}(G, N)$ and let $g \in \operatorname{Hom}(G, N)$ be its image under (2.3). Then, for any $\sigma \in G$, we may rewrite

$$\beta_{(\mathfrak{f},\mathfrak{g})}(\sigma) = \rho(g(\sigma)f(\sigma)^{-1}) \cdot \operatorname{conj}(f(\sigma)) = \rho(g(\sigma))\lambda(f(\sigma)) = \beta_{(f,g)}(\sigma),$$

where $\beta_{(f,g)}$ is the homomorphism defined as in [5, Sections 2 and 3]. Hence, we may view Propositions 2.1 and 2.4 as generalizations of [5, Corollary 7].

2.3.2. General inner actions. In what follows, assume that

there exists $f \in \text{Hom}(G, N/Z(N))$ with $\mathfrak{f}(\sigma) = \text{conj}(f(\sigma))$ for all $\sigma \in G$.

This implies that $\mathfrak{f}(G) \subset \operatorname{Inn}(N)$ and the converse is also true. Put

$$\operatorname{Hom}_{f}(G, N/Z(N))^{*} = \{g \in \operatorname{Hom}(G, N/Z(N)) : (f, g) \text{ is }$$

weakly fixed point free}.

Then, essentially the same argument as in Proposition 2.4 shows that:

Proposition 2.5. The maps

(2.4)
$$Z^1_{\mathfrak{f}}(G,N) \longrightarrow \operatorname{Hom}(G,N/Z(N)); \quad \mathfrak{g} \mapsto (\sigma \mapsto \mathfrak{g}(\sigma)Z(N) \cdot f(\sigma))$$

 $Z^1_{\mathfrak{f}}(G,N)^* \longrightarrow \operatorname{Hom}_f(G,N/Z(N))^*; \quad \mathfrak{g} \mapsto (\sigma \mapsto \mathfrak{g}(\sigma)Z(N) \cdot f(\sigma))$

are well-defined.

However, the maps in Proposition 2.5, unlike those in Proposition 2.4, are neither injective nor surjective in general.

Let $\mathfrak{g} \in Z^1_{\mathfrak{f}}(G, N)$ and let $g \in \operatorname{Hom}_f(G, N/Z(N))$ be its image under (2.4). For any $\sigma \in G$, letting $\widetilde{f}(\sigma) \in N$ be an element such that $f(\sigma) = \widetilde{f}(\sigma)Z(N)$, we may then rewrite

$$\beta_{(\mathfrak{f},\mathfrak{g})}(\sigma) = \rho(\mathfrak{g}(\sigma)) \cdot \operatorname{conj}(\widetilde{f}(\sigma)) = \rho(\mathfrak{g}(\sigma)\widetilde{f}(\sigma))\lambda(\widetilde{f}(\sigma)).$$

Observe that $\rho(\eta) = \lambda(\eta)^{-1}$ for $\eta \in Z(N)$, and let $g_0 \in \text{Hom}(G, N/Z(N))$ be the trivial homomorphism. Then, we see that $\beta_{(\mathfrak{f},\mathfrak{g})}(G) \subset \lambda(N)$ when $g = g_0$. Thus, for $N \not\simeq G$, we have $g \neq g_0$ whenever \mathfrak{g} is bijective. As for N = G, it is also easy to verify that

(2.5) for
$$\mathfrak{g} \in Z^1_{\mathfrak{f}}(G,G)^*$$
: $\beta_{(\mathfrak{f},\mathfrak{g})}(G) = \lambda(G)$ if and only if $g = g_0$.

This is analogous to the discussion in Section 2.1.

3. Applications: isomorphic type

3.1. Abelian groups. Let A be an abelian group.

3.1.1. Backward implication. Suppose that

 $|A| = 2^{\delta} p_1 \cdots p_m$, where $p_1 < \cdots < p_m$ are odd primes and $\delta \in \{0, 1, 2\}$.

To prove the backward implication of Theorem 1.2, consider

$$\mathfrak{f} \in \operatorname{Hom}(A, \operatorname{Aut}(A)) \text{ and } \mathfrak{g} \in Z^1_{\mathfrak{f}}(A, A).$$

By (1.1) and Proposition 2.1, it is enough to show that

(3.1)
$$\beta_{(\mathfrak{f},\mathfrak{g})}(A) = \rho(A)$$
 whenever \mathfrak{g} is bijective.

Notice that the hypothesis on A implies $A = A_0 \times A_\sigma$, where $A_\sigma = \langle \sigma \rangle$ with $\sigma \in A$ an element of order $n = p_1 \cdots p_m$, and A_0 is isomorphic to one of

(3.2)
$$\{1\}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Since A_0 and A_{σ} are characteristic subgroups of A, their preimages $\mathfrak{g}^{-1}(A_0)$ and $\mathfrak{g}^{-1}(A_{\sigma})$ are subgroups of A as well. We then deduce that

(3.3)
$$\mathfrak{g}(A_0) = A_0$$
 and $\mathfrak{g}(A_\sigma) = A_\sigma$ whenever \mathfrak{g} is bijective.

Observe that $\operatorname{Aut}(A) = \operatorname{Aut}(A_0) \times \operatorname{Aut}(A_{\sigma})$. We have the following lemmas.

Lemma 3.1. If \mathfrak{g} is bijective, then $\mathfrak{f}(A_0)|_{A_0} = \{Id_{A_0}\}.$

Proof. For $|A_0| \leq 2$, it is clear that $\mathfrak{f}(A_0)|_{A_0} = {\mathrm{Id}_{A_0}}$. For $|A_0| = 4$, suppose that \mathfrak{g} is bijective, and on the contrary that $\mathfrak{f}(\sigma_0)|_{A_0} \neq {\mathrm{Id}}_{A_0}$ for some $\sigma_0 \in A_0$.

First, assume that $A_0 \simeq \mathbb{Z}/4\mathbb{Z}$. Without loss of generality, we may assume that $\sigma_0 \in A_0$ is a generator. Note that $\mathfrak{f}(\sigma_0)(\tau) = \tau^{-1}$ for all $\tau \in A_0$, and so

$$\mathfrak{g}(\sigma_0^2) = \mathfrak{g}(\sigma_0) \cdot \mathfrak{f}(\sigma_0)(\mathfrak{g}(\sigma_0)) = \mathfrak{g}(\sigma_0) \cdot \mathfrak{g}(\sigma_0)^{-1} = 1_{A_0},$$

by (3.3). But this contradicts that \mathfrak{g} is bijective since σ_0 has order four.

Next, assume that $A_0 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Under this identification, applying a change of basis if necessary, we may assume that

$$\mathfrak{f}(\sigma_0)|_{A_0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
, and so $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathfrak{g}(2\sigma_0) = \mathfrak{g}(\sigma_0) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathfrak{g}(\sigma_0)$

by (3.3). We must then have $\mathfrak{g}(\sigma_0) = (1, 1)$. Now, again by (3.3), there exists $\tau_0 \in A_0$ such that $\mathfrak{g}(\tau_0) = (1, 0)$. But then

$$\mathfrak{g}(\sigma_0 + \tau_0) = \mathfrak{g}(\sigma_0) + \mathfrak{f}(\sigma_0)(\mathfrak{g}(\tau_0)) = \begin{pmatrix} 1\\1 \end{pmatrix} + \begin{pmatrix} 0 & 1\\1 & 0 \end{pmatrix} \begin{pmatrix} 1\\0 \end{pmatrix} = \mathfrak{g}(\tau_0)$$

This contradicts that \mathfrak{g} is bijective since plainly σ_0 is not the identity. \Box

Lemma 3.2. If \mathfrak{g} is bijective, then $\mathfrak{f}(A_{\sigma})|_{A_{\sigma}} = \{Id_{A_{\sigma}}\}.$

Proof. Let $d \in \mathbb{Z}$ be coprime to n such that $\mathfrak{f}(\sigma)(\tau) = \tau^d$ for all $\tau \in A_{\sigma}$, and let e denote the multiplicative order of $d \mod n$. For each $1 \leq i \leq m$, write e_i

for the multiplicative order of $d \mod p_i$. Then, we have $e = \operatorname{lcm}(e_1, \ldots, e_m)$. Since e divides n, we may also write

$$e = p_{i_1} \cdots p_{i_r}$$
 for some $1 \le i_1 < \cdots < i_r \le m$,

where the product may be empty. We then deduce from (2.1) and (3.3) that

$$\mathfrak{g}(\sigma^{e(d-1)}) = \mathfrak{g}(\sigma)^{(1+d+\dots+d^{e-1})(d-1)} = \mathfrak{g}(\sigma)^{d^e-1} = 1_{A_\sigma}.$$

Now, suppose that \mathfrak{g} is bijective. The above implies that

$$(3.4) e(d-1) \equiv 0 \pmod{n}.$$

We shall use this to show that $\{i_1, \ldots, i_r\}$ is in fact empty.

For $i \notin \{i_1, \ldots, i_r\}$, we have $e_i = 1$ by (3.4). For $i \in \{i_1, \ldots, i_r\}$, we have

$$d^{e/p_i} \equiv (d^{p_i})^{e/p_i} \equiv 1 \pmod{p_i},$$

whence e_i divides both e/p_i and $p_i - 1$. This implies that e_i divides $p_{i_1} \cdots p_{i_{j-1}}$ when $i = i_j$. But then p_{i_r} cannot divide e, which is a contradiction. It follows that $\{i_1, \ldots, i_r\}$ must be empty. This shows that e = 1 and so $\mathfrak{f}(\sigma)|_{A_{\sigma}} = \mathrm{Id}_{A_{\sigma}}$, as claimed.

Proof of Theorem 1.2: backward implication. Suppose that \mathfrak{g} is bijective. To prove (3.1), by (2.2) as well as Lemmas 3.1 and 3.2, it suffices to show that

(3.5)
$$\mathfrak{f}(A_{\sigma})|_{A_0} = \{ \mathrm{Id}_{A_0} \} \text{ and } \mathfrak{f}(A_0)|_{A_{\sigma}} = \{ \mathrm{Id}_{A_{\sigma}} \}.$$

For any $\sigma_0 \in A_0$ and $\tau \in A_\sigma$, we have

$$\mathfrak{g}(\sigma_0) \cdot \mathfrak{f}(\sigma_0)(\mathfrak{g}(\tau)) = \mathfrak{g}(\sigma_0 \tau) = \mathfrak{g}(\tau \sigma_0) = \mathfrak{g}(\tau) \cdot \mathfrak{f}(\tau)(\mathfrak{g}(\sigma_0)).$$

From (3.3), we then deduce that

$$\mathfrak{g}(\sigma_0) = \mathfrak{f}(\tau)(\mathfrak{g}(\sigma_0)) \text{ and } \mathfrak{g}(\tau) = \mathfrak{f}(\sigma_0)(\mathfrak{g}(\tau)),$$

and in particular (3.5) indeed holds.

3.1.2. Forward implication. Suppose that |A| does not have the form stated in Theorem 1.2. This means that $A = H \times H'$, where H is a subgroup of A isomorphic to one of the groups in the next lemma.

 \square

Lemma 3.3. Suppose that H is isomorphic to one of the following:

(1) Z/pⁿZ or Z/pZ × Z/pZ, where p is an odd prime and n ≥ 2, or
(2) Z/2ⁿZ, where n ≥ 3, or
(3) Z/2Z × Z/2Z × Z/2Z or Z/2Z × Z/4Z, or
(4) Z/4Z × Z/4Z.

Then, there is a regular subgroup of Hol(H) which is isomorphic to H but is not equal to $\rho(H)$.

Proof. For cases (1) and (2), see [1, Lemmas 1 and 2] as well as its corrigendum. For case (3), see [17, Theorem 1.2.5]. For case (4), let us identify H with $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, and define

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Further, define two permutations η_1, η_2 on H by setting

 $\eta_1(\mathbf{x}) = A_1\mathbf{x} + \mathbf{b}_1 \text{ and } \eta_2(\mathbf{x}) = A_2\mathbf{x} + \mathbf{b}_2 \text{ for } \mathbf{x} \in H.$

Note that $\eta_1, \eta_2 \in \text{Hol}(H)$ by (1.2). It is easy to verify that $\langle \eta_1, \eta_2 \rangle \simeq H$ and $\langle \eta_1, \eta_2 \rangle \neq \rho(H)$. A routine calculation also shows that $\langle \eta_1, \eta_2 \rangle$ is regular, and so the claim follows.

Proof of Theorem 1.2: forward implication. By Lemma 3.3, there is a regular subgroup N_H of Hol(H) isomorphic to H and not equal to $\rho(H)$. The image of $N_H \times \rho(H')$ under the natural injective homomorphism

$$\operatorname{Hol}(H) \times \operatorname{Hol}(H') \longrightarrow \operatorname{Hol}(H \times H')$$

is then a regular subgroup isomorphic to $H \times H'$ and not equal to $\rho(H \times H')$. The claim now follows from (1.1).

3.2. Quasisimple groups. Let Q be a quasisimple group. We shall need the next proposition, which is the crucial ingredient for the proof of e(Q, Q) = 2 given in [9] in the special case when Q is non-abelian simple.

Proposition 3.4. Let T be a non-abelian simple group. Then: (a) Schreier's conjecture is true, namely Out(T) is solvable. (b) A pair (f, g) with $f, g \in Aut(T)$ is never fixed point free.

Proof. They are consequences of the classification of finite simple groups; see [13, Theorems 1.46 and 1.48], for example.

We shall also need the following basic properties concerning Q.

Proposition 3.5. The following statements hold.

(a) A proper normal subgroup of Q is contained in Z(Q).

- (b) The kernel of a non-trivial homomorphism $Q \longrightarrow Q/Z(Q)$ is Z(Q).
- (c) The natural homomorphism $\operatorname{Aut}(Q) \longrightarrow \operatorname{Aut}(Q/Z(Q))$ is injective.
- (d) An endomorphism on Q is either trivial or an automorphism.

Proof. This is known, and it is easy to prove (a), which in turn gives (b). For the convenience of reader, we shall give a proof for (c) and (d).

To prove (c), let $\varphi \in \operatorname{Aut}(Q)$ be such that $\varphi(\sigma)\sigma^{-1} \in Z(Q)$ for all $\sigma \in Q$. We easily verify that the map

$$\psi: Q \longrightarrow Z(Q); \quad \psi(\sigma) = \varphi(\sigma)\sigma^{-1}$$

is a homomorphism. But then ψ must be trivial because Z(Q) is abelian and Q = [Q, Q]. This means that $\varphi = \mathrm{Id}_Q$, as desired.

To prove (d), let $\varphi \in \text{Hom}(Q, Q)$ be non-trivial, so then $\ker(\varphi) \subset Z(Q)$ by (a). Put $H = \varphi(Q)$ for brevity. Note that

$$\frac{Q/\ker(\varphi)}{Z(Q)/\ker(\varphi)} \simeq \frac{Q}{Z(Q)}$$
 has trivial center

as well as that $Q/\ker(\varphi) \simeq H$. Hence, we deduce that

$$Z\left(\frac{Q}{\ker(\varphi)}\right) \subset \frac{Z(Q)}{\ker(\varphi)} \text{ and so } |Z(H)| \leq [Z(Q): \ker(\varphi)].$$

It follows that

$$|HZ(Q)| = \frac{|H||Z(Q)|}{|H \cap Z(Q)|} \ge \frac{|H||Z(Q)|}{|Z(H)|} \ge \frac{|H||Z(Q)|}{[Z(Q):\ker(\varphi)]} = |Q|,$$

and so HZ(Q) = Q. Since Q = [Q, Q], we deduce that H = Q. This means that $\varphi \in Aut(Q)$, as desired.

Now, to prove Theorem 1.3, consider

$$\mathfrak{f} \in \operatorname{Hom}(Q, \operatorname{Aut}(Q)) \text{ and } \mathfrak{g} \in Z^1_{\mathfrak{f}}(Q, Q).$$

By (1.1) and Proposition 2.1, it suffices to show that

(3.6)
$$\beta_{(\mathfrak{f},\mathfrak{g})}(Q) \in \{\rho(Q), \lambda(Q)\}$$
 whenever \mathfrak{g} is bijective.

The next lemma is analogous to an argument in [9, p. 84].

Lemma 3.6. We have $f(Q) \subset \text{Inn}(Q)$.

Proof. The group Out(Q/Z(Q)) is solvable by Proposition 3.4 (a). Since we have an injective homomorphism

$$\operatorname{Out}(Q) \longrightarrow \operatorname{Out}(Q/Z(Q))$$

by Proposition 3.5 (c), the group Out(Q) is also solvable. Since Q = [Q : Q], we then see that the homomorphism

$$Q \xrightarrow{f} \operatorname{Aut}(Q) \xrightarrow{\operatorname{quotient}} \operatorname{Out}(Q)$$

must be trivial. This means that $\mathfrak{f}(Q) \subset \operatorname{Inn}(Q)$, as claimed.

In view of Lemma 3.6, we may define

$$f \in \operatorname{Hom}(Q, Q/Z(Q)), f \in \operatorname{Map}(Q, Q), g \in \operatorname{Hom}(Q, Q/Z(Q))$$

as in Subsection 2.3.2. More precisely, we have

$$\mathfrak{f}(\sigma) = \operatorname{conj}(f(\sigma)), \ f(\sigma) = \widetilde{f}(\sigma)Z(Q), \ g(\sigma) = \mathfrak{g}(\sigma)\widetilde{f}(\sigma)Z(Q)$$

for all $\sigma \in Q$. We make the following useful observation.

Lemma 3.7. If f and g are non-trivial, then $\mathfrak{g}(Z(Q)) \subset Z(Q)$.

Proof. Let $\sigma \in Z(Q)$. Then, for any $\tau \in Q$, we have

$$\mathfrak{g}(\sigma)\widetilde{f}(\sigma)\mathfrak{g}(\tau)\widetilde{f}(\sigma)^{-1} = \mathfrak{g}(\sigma\tau) = \mathfrak{g}(\tau\sigma) = \mathfrak{g}(\tau)\widetilde{f}(\tau)\mathfrak{g}(\sigma)\widetilde{f}(\tau)^{-1}.$$

Suppose that f is non-trivial, so then $\ker(f) = Z(Q)$ by Proposition 3.5 (b). This implies that $\tilde{f}(\sigma) \in Z(Q)$, and from the above, we deduce that $\mathfrak{g}(\sigma)$ is centralized by $\mathfrak{g}(\tau)\widetilde{f}(\tau)$. This means that $\mathfrak{g}(\sigma)$ commutes with

$$\bigcup_{\tau \in Q} \mathfrak{g}(\tau) \widetilde{f}(\tau) Z(Q) = \bigcup_{\tau \in Q} g(\tau).$$

Now, suppose further that g is non-trivial. Then, by Proposition 3.5 (b), we have $\ker(g) = Z(Q)$, and in particular g is surjective. Hence, the union above is equal to the entire group Q. This means that $\mathfrak{g}(\sigma) \in Z(Q)$, as desired. \Box

Proof of Theorem 1.3. Suppose that \mathfrak{g} is bijective. In view of (2.2) and (2.5), to prove (3.6), it suffices to show that either f or g is trivial. Suppose for contradiction that they are both non-trivial. Then, they induce automorphisms

$$\overline{f}: Q/Z(Q) \longrightarrow Q/Z(Q) \text{ and } \overline{g}: Q/Z(Q) \longrightarrow Q/Z(Q)$$

by Proposition 3.5 (b). For any $\sigma \in Q$, we have

$$f(\sigma) = g(\sigma) \implies \mathfrak{g}(\sigma) \in Z(Q) \implies \sigma \in Z(Q),$$

by Lemma 3.7 and the bijectivity of \mathfrak{g} . This shows that $(\overline{f}, \overline{g})$ is fixed point free, which is impossible by Proposition 3.4 (b).

4. Applications: Non-Isomorphic type

4.1. Formulation of the main idea. In what follows, let G and N denote two groups having the same order. Notice that by definition, any characteristic subgroup M of N is normal, and we also have a natural homomorphism $\operatorname{Aut}(N) \longrightarrow \operatorname{Aut}(N/M)$.

Lemma 4.1. Let M be a characteristic subgroup of N. Given

 $\mathfrak{f} \in \operatorname{Hom}(G, \operatorname{Aut}(N)) \text{ and } \mathfrak{g} \in Z^1_{\mathfrak{f}}(G, N),$

they induce, respectively, a homomorphism and a map

(4.1)
$$\overline{\mathfrak{f}}: G \longrightarrow \operatorname{Aut}(N) \longrightarrow \operatorname{Aut}(N/M) \text{ and } \overline{\mathfrak{g}}: G \longrightarrow N \longrightarrow N/M.$$

By abuse of notation, define

$$\ker(\overline{\mathfrak{g}}) = \{ \sigma \in G : \overline{\mathfrak{g}}(\sigma) = 1_{N/M} \}.$$

Then, the following are true.

(a) The set $\ker(\overline{\mathfrak{g}})$ is a subgroup of G.

- (b) The map $\overline{\mathfrak{g}}$ induces an injection $G/\ker(\overline{\mathfrak{g}}) \longrightarrow N/M$.
- (c) The map $\overline{\mathfrak{g}}$ restricts to a homomorphism $\ker(\overline{\mathfrak{f}}) \longrightarrow N/M$.
- (d) In the case that N/M is abelian, for any $\sigma \in \ker(\overline{\mathfrak{f}}) \cap Z(G)$, the element $\overline{\mathfrak{g}}(\sigma)$ is fixed by the automorphisms in $\overline{\mathfrak{f}}(G)$.

Proof. Both (a) and (c) are clear. For (b), simply observe that

$$\overline{\mathfrak{g}}(\sigma_1) = \overline{\mathfrak{g}}(\sigma_2) \iff \overline{\mathfrak{g}}(\sigma_1^{-1}\sigma_2) = \overline{\mathfrak{g}}(\sigma_1^{-1}) \cdot \overline{\mathfrak{f}}(\sigma_1^{-1})(\overline{\mathfrak{g}}(\sigma_1))$$
$$\iff \overline{\mathfrak{g}}(\sigma_1^{-1}\sigma_2) = \overline{\mathfrak{g}}(\sigma_1^{-1}\sigma_1)$$
$$\iff \overline{\mathfrak{g}}(\sigma_1^{-1}\sigma_2) = 1_{N/M}$$

for any $\sigma_1, \sigma_2 \in G$. Finally, statement (d) follows from the fact that

$$\overline{\mathfrak{g}}(\sigma)\overline{\mathfrak{g}}(\tau) = \overline{\mathfrak{g}}(\sigma) \cdot \overline{\mathfrak{f}}(\sigma)(\overline{\mathfrak{g}}(\tau)) = \overline{\mathfrak{g}}(\sigma\tau) = \overline{\mathfrak{g}}(\tau\sigma) = \overline{\mathfrak{g}}(\tau) \cdot \overline{\mathfrak{f}}(\tau)(\overline{\mathfrak{g}}(\sigma))$$

for any $\tau \in G$ and $\sigma \in \ker(\overline{\mathfrak{f}}) \cap Z(G)$.

We keep the notation as in Lemma 4.1. To show that e(G, N) = 0, by (1.1) and Proposition 2.1, it is the same as proving that \mathfrak{g} can never be bijective. The idea is that, while we might not understand $\operatorname{Aut}(N)$ or N very well, by passing to $\operatorname{Aut}(N/M)$ and N/M for a suitable characteristic subgroup M of N, we might be able to use Lemma 4.1 to prove the weaker statement that $\overline{\mathfrak{g}}$ can never be surjective.

4.2. Cyclic groups of odd prime power order. Let C_{p^n} be a cyclic group of odd prime power order p^n and let N be a group of order p^n with $N \neq C_{p^n}$. To prove Theorem 1.5, consider

$$\mathfrak{f} \in \operatorname{Hom}(C_{p^n}, \operatorname{Aut}(N)) \text{ and } \mathfrak{g} \in Z^1_{\mathfrak{f}}(C_{p^n}, N).$$

By (1.1) and Proposition 2.1, it is enough to show that \mathfrak{g} cannot be bijective. Take M to be the Frattini subgroup $\Phi(N)$ of N. Then, we know that

$$N/\Phi(N) \simeq (\mathbb{Z}/p\mathbb{Z})^m$$
 and so $\operatorname{Aut}(N/\Phi(N)) \simeq \operatorname{GL}_m(\mathbb{Z}/p\mathbb{Z}),$

where $m \in \mathbb{N}$ is such that $m \geq 2$ because N is non-cyclic. Let $\overline{\mathfrak{f}}$ and $\overline{\mathfrak{g}}$ be as in (4.1). Then, in turn, it suffices to show that $\overline{\mathfrak{g}}$ cannot be surjective.

Fix a generator $\sigma \in C_{p^n}$, and write $|\overline{\mathfrak{f}}(\sigma)| = p^r$, where $0 \leq r \leq n$. The next two lemmas yield, respectively, an upper bound and a lower bound for p^r in terms of m and the index of ker($\overline{\mathfrak{g}}$) in C_{p^n} .

Lemma 4.2. Let $B \in GL_m(\mathbb{Z}/p\mathbb{Z})$ be a matrix of order p^r .

(a) If $m \ge 3$, then $r \le m - 2$. (b) If m = 2, then $r \le 1$, and B is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $\operatorname{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

Proof. The fact that B has order a power of p implies that B is conjugate to a Jordan matrix with $\lambda = 1$ on the diagonal in $\operatorname{GL}_m(\mathbb{Z}/p\mathbb{Z})$. From here, it is easy to see that $B^{p^{m-2}}$ and B^p , respectively, for $m \geq 3$ and m = 2, equal the identity matrix in $\operatorname{GL}_m(\mathbb{Z}/p\mathbb{Z})$. The claim now follows. \Box

Lemma 4.3. The following statements hold.

(a) If $m \geq 3$, then $\langle \sigma^{p^{r+1}} \rangle \subset \ker(\overline{\mathfrak{g}})$ and so $[C_{p^n} : \ker(\overline{\mathfrak{g}})] \leq p^{r+1}$. (b) If m = 2, then $\langle \sigma^p \rangle \subset \ker(\overline{\mathfrak{g}})$ and so $[C_{p^n} : \ker(\overline{\mathfrak{g}})] \leq p$.

Proof. Note that $N/\Phi(N)$ has exponent p. By (2.1), we also have

$$\overline{\mathfrak{g}}(\sigma^{p^{r+1}}) = \overline{\mathfrak{g}}(\sigma^{p^r})^p \text{ and } \overline{\mathfrak{g}}(\sigma^p) = \prod_{i=0}^{p-1} \overline{\mathfrak{f}}(\sigma)^i(\overline{\mathfrak{g}}(\sigma)).$$

The claim for $m \geq 3$ then follows from the first equality. Now, suppose that m = 2. Then, regarding $\overline{\mathfrak{f}}(\sigma)$ an element in $\operatorname{GL}_2(\mathbb{Z}/p\mathbb{Z})$, by Lemma 4.2, it is conjugate to a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \in \mathbb{Z}/p\mathbb{Z}$. Since

(4.2)
$$\sum_{i=0}^{p-1} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^i = \begin{pmatrix} p & \frac{p(p-1)b}{2} \\ 0 & p \end{pmatrix} = \text{zero matrix in } \operatorname{GL}_2(\mathbb{Z}/p\mathbb{Z}),$$

we see that indeed $\sigma^p \in \ker(\overline{\mathfrak{g}})$. This proves the claim.

Proof of Theorem 1.5. Lemmas 4.2 and 4.3 imply that

$$[C_{p^n}: \ker(\overline{\mathfrak{g}})] \le p^{m-1}.$$

We then see from Lemma 4.1 (b) that $\overline{\mathfrak{g}}$ indeed cannot be surjective.

Remark 4.4. Note that the hypothesis that p is odd is required for the second equality in (4.2). In fact, the analogous statement of Theorem 1.5 for p = 2 is false, as shown in [4, Corollary 5.3].

 \square

 \square

4.3. Groups of order *n* factorial. In what follows, let $n \in \mathbb{N}$ with $n \geq 5$. Recall that $2A_n$ is the unique group, up to isomorphism, fitting into a short exact sequence

 $1 \longrightarrow \{\pm 1\} \stackrel{\iota}{\longrightarrow} 2A_n \longrightarrow A_n \longrightarrow 1$

such that $\iota(\{\pm 1\})$ lies in both $Z(2A_n)$ and $[2A_n, 2A_n]$. It is known that $2A_n$ is quasisimple and $Z(2A_n) \simeq \{\pm 1\}$. We then have:

Lemma 4.5. If N = [N, N] and there is a normal subgroup M of N having order two such that $N/M \simeq A_n$, then necessarily $N \simeq 2A_n$.

Proof. This is because a normal subgroup of order two lies in the center. \Box

There are some similarities in the proofs of Theorems 1.6 and 1.7 because:

(i) Both $2A_n$ and S_n have order n!.

(ii) Both $2A_n$ and S_n have only one non-trivial proper normal subgroup.

(iii) The alternating group A_n is a subgroup of S_n and is a quotient of $2A_n$. For (ii), we in particular know that

(4.3)
$$\begin{cases} Z(2A_n) \text{ is the non-trivial proper normal subgroup of } 2A_n, \\ A_n \text{ is the non-trivial proper normal subgroup of } S_n, \end{cases}$$

where the first statement follows from Proposition 3.5 (a). Given a prime pand a group Γ , write $v_p(\Gamma)$ for the non-negative integer such that

 $p^{v_p(\Gamma)}$ = the exact power of p dividing $|\Gamma|$.

Motivated by the arguments in [3], we shall require the next two lemmas.

Lemma 4.6. If A_n has a subgroup of prime power index p^m , then $n = p^m$.

Proof. See [15, (2.2)].

Lemma 4.7. Let $m \in \mathbb{N}$ and let p be a prime. Then, we have (a) $|\operatorname{GL}_m(\mathbb{Z}/p\mathbb{Z})| < \frac{1}{2}(p^m!)$ and $v_p(\operatorname{GL}_m(\mathbb{Z}/p\mathbb{Z})) = m(m-1)/2$, (b) $v_p(S_m) < m$ and $v_p(S_{p^m}) \ge m(m+1)/2$, (c) $v_2(S_{2^{m-1}}) \ge m(m-1)/2 + 2$ for $m \ge 5$. *Proof.* Both claims in (a) and the first claim in (b) follow from

$$|\operatorname{GL}_m(\mathbb{Z}/p\mathbb{Z})| = \prod_{i=0}^{m-1} (p^m - p^i) \text{ and } v_p(S_m) = \sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor,$$

respectively, as in [3, (4.1) and Lemma 3.3]. The two remaining claims hold because $p^m!$ is divisible by $p \cdots p^m$, and $2^{m-1}!$ is divisible by $2 \cdots 2^{m-1} \cdot 6 \cdot 10$ for $m \ge 5$.

Now, for both Theorems 1.6 and 1.7, let N be a group of order n!, and let M be any proper maximal characteristic subgroup of N. The quotient N/M is then a non-trivial and *characteristically simple* group, meaning that it has no non-trivial proper characteristic subgroup. It is then known that

(4.4)
$$N/M \simeq T^m$$
, where T is simple and $m \in \mathbb{N}$.

As shown in [3, Lemma 3.2], we have

$$\operatorname{Aut}(N/M) \simeq \operatorname{Aut}(T)^m \rtimes S_m$$
 when T is non-abelian.

The structure of $\operatorname{Aut}(N/M)$ is of course well-understood when T is abelian.

4.3.1. The double cover of alternating groups. To prove Theorem 1.6, in this section, we shall assume that $N \not\simeq 2A_n$. Consider

$$\mathfrak{f} \in \operatorname{Hom}(2A_n, \operatorname{Aut}(N)) \text{ and } \mathfrak{g} \in Z^1_{\mathfrak{f}}(2A_n, N).$$

By (1.1) and Proposition 2.1, it is enough to show that \mathfrak{g} cannot be bijective. We shall use the same notation as in (4.4), and let $\overline{\mathfrak{f}}, \overline{\mathfrak{g}}$ be as in (4.1). Then, in turn, it suffices to show that $\overline{\mathfrak{g}}$ cannot be surjective.

Lemma 4.8. If N = [N : N], then $\overline{\mathfrak{f}}$ is trivial.

Proof. Suppose that N = [N, N], in which case T is non-abelian. Consider

(4.5)
$$2A_n \xrightarrow{\overline{\mathfrak{f}}} \operatorname{Aut}(N/M) \xrightarrow{\operatorname{identification}} \operatorname{Aut}(T)^m \rtimes S_m \xrightarrow{\operatorname{projection}} S_m.$$

Notice that |T| has an odd prime factor p because a 2-group has non-trivial center. We have $v_p(S_m) < m$ by Lemma 4.7 (b) while

$$v_p(2A_n/Z(2A_n)) = v_p(2A_n) = v_p(N) = m \cdot v_p(T) + v_p(M) \ge m.$$

We then deduce from (4.3) that (4.5) must be trivial. It follows that $\mathfrak{f}(2A_n)$ lies in $\operatorname{Aut}(T)^m$. Note that the homomorphism

$$2A_n \xrightarrow{\overline{\mathfrak{f}}} \operatorname{Aut}(T)^m \xrightarrow{\operatorname{projection}} \operatorname{Out}(T)^m$$

must also be trivial, because $2A_n = [2A_n, 2A_n]$ while $\operatorname{Out}(T)^m$ is solvable by Proposition 3.4 (a). Hence, in fact $\overline{\mathfrak{f}}: 2A_n \longrightarrow \operatorname{Inn}(T)^m \simeq T^m$.

Now, by (4.3), either $\overline{\mathfrak{f}}$ is trivial or $|\ker(\overline{\mathfrak{f}})| \leq 2$. Observe that

$$[2A_n : \ker(\overline{\mathfrak{f}})] \le |T|^m = [N : M] = |2A_n|/|M| \text{ and so } |M| \le |\ker(\overline{\mathfrak{f}})|.$$

If $|\ker(\overline{\mathfrak{f}})| = 1$, then |M| = 1, and we deduce that $2A_n \simeq T^m \simeq N$, which is a contradiction. If $|\ker(\overline{\mathfrak{f}})| = 2$ and |M| = 1, then $\overline{\mathfrak{f}}(2A_n)$ has index two and in particular is normal in $T^m \simeq N$, but this is impossible because N = [N, N]. Finally, if $|\ker(\overline{\mathfrak{f}})| = 2$ and |M| = 2, then $A_n \simeq \overline{\mathfrak{f}}(2A_n) \simeq T^m \simeq N/M$, which contradicts that $2A_n \not\simeq N$ by Lemma 4.5. Thus, indeed $\overline{\mathfrak{f}}$ must be trivial. \Box

Lemma 4.9. If N/M is abelian and $\overline{\mathfrak{g}}$ is surjective, then $\overline{\mathfrak{f}}$ is trivial.

Proof. Recall that $2A_n/Z(2A_n) \simeq A_n$, and note that

$$\left[\frac{2A_n}{Z(2A_n)}:\frac{\ker(\overline{\mathfrak{g}})Z(2A_n)}{Z(2A_n)}\right] = \begin{cases} [2A_n:\ker(\overline{\mathfrak{g}})] & \text{if } Z(2A_n) \subset \ker(\overline{\mathfrak{g}}),\\ \frac{1}{2}[2A_n:\ker(\overline{\mathfrak{g}})] & \text{if } \ker(\overline{\mathfrak{g}}) \cap Z(2A_n) = 1. \end{cases}$$

These are the only cases because $Z(2A_n)$ has order two.

Suppose that N/M is abelian and $\overline{\mathfrak{g}}$ is surjective. Then, we have $T \simeq \mathbb{Z}/p\mathbb{Z}$ for some prime p, and $[2A_n : \ker(\overline{\mathfrak{g}})] = p^m$ by Lemma 4.1 (b). We also have

$$\begin{cases} n = p^m & \text{if } Z(2A_n) \subset \ker(\overline{\mathfrak{g}}), \\ n = 2^{m-1} \text{ with } m \ge 4 & \text{if } \ker(\overline{\mathfrak{g}}) \cap Z(2A_n) = 1, \end{cases}$$

by Lemma 4.6. Recall Lemma 4.7, and observe that

$$\begin{cases} [2A_{p^m} : Z(2A_{p^m})] > |\mathrm{GL}_m(\mathbb{Z}/p\mathbb{Z})| \\ v_2(2A_{2^{m-1}}/Z(2A_{2^{m-1}})) > v_2(\mathrm{GL}_m(\mathbb{Z}/2\mathbb{Z})) \text{ for } m \ge 5. \end{cases}$$

Since $2A_n/\ker(\overline{\mathfrak{f}})$ embeds into $\operatorname{GL}_m(\mathbb{Z}/p\mathbb{Z})$, we then deduce from (4.3) that $\overline{\mathfrak{f}}$

must be trivial, except possibly when

(4.6)
$$\operatorname{ker}(\overline{\mathfrak{g}}) \cap Z(2A_n) = 1 \text{ and } n = 2^{m-1} \text{ for } m = 4.$$

In this last case (4.6), suppose for contradiction that \overline{f} is non-trivial. Since

(4.7)
$$|2A_8| = 40320 \text{ and } |\mathrm{GL}_4(\mathbb{Z}/2\mathbb{Z})| = 20160,$$

necessarily $\overline{\mathfrak{f}}$ is surjective and ker($\overline{\mathfrak{f}}$) = $Z(2A_8)$. For any $\sigma \in Z(2A_8)$, we then deduce from Lemma 4.1 (d) that $\overline{\mathfrak{g}}(\sigma)$ is a fixed point of every automorphism on $N/M \simeq (\mathbb{Z}/2\mathbb{Z})^4$, and so $\overline{\mathfrak{g}}(\sigma) = \mathbb{1}_{N/M}$. This shows that $Z(2A_8) \subset \ker(\overline{\mathfrak{g}})$, which contradicts the first condition in (4.6).

Proof of Theorem 1.6. Suppose for contradiction that $\overline{\mathfrak{g}}$ is surjective. In the case that $N \supseteq [N, N]$, we may choose M to be such that $M \supset [N, N]$, which ensures that N/M is abelian. Then, by Lemmas 4.8 and 4.9, we know that $\overline{\mathfrak{f}}$ is trivial, whence $\overline{\mathfrak{g}}$ is a homomorphism, and so we have $N/M \simeq 2A_n/\ker(\overline{\mathfrak{g}})$. Notice that N/M is non-trivial because M is proper by choice. By (4.3) and the hypothesis that $N \not\simeq 2A_n$, we then deduce that $\ker(\overline{\mathfrak{g}}) = Z(2A_n)$ and so $N/M \simeq A_n$. We now have a contradiction because:

- If $N \supseteq [N, N]$, then N/M is abelian by the choice of M.
- If N = [N, N], then $N \simeq 2A_n$ by Lemma 4.5, but $N \not\simeq 2A_n$ by hypothesis.

Thus, the map $\overline{\mathfrak{g}}$ cannot be surjective, and the theorem now follows.

4.3.2. Symmetric groups. To prove Theorem 1.7, consider

$$\mathfrak{f} \in \operatorname{Hom}(S_n, \operatorname{Aut}(N)) \text{ and } \mathfrak{g} \in Z^1_{\mathfrak{f}}(S_n, N).$$

To prove the first statement, by (1.1) and Proposition 2.1, it suffices to show that one of the three stated conditions holds whenever \mathfrak{g} is bijective.

Proof of Theorem 1.7: first statement. First, suppose that $ker(\mathfrak{f}) = 1$. Since

 $\mathfrak{f}(S_n) \cap \operatorname{Inn}(N)$ is a normal subgroup of $\mathfrak{f}(S_n) \simeq S_n$,

by (4.3) we have $\mathfrak{f}(S_n) \cap \operatorname{Inn}(N) \in {\mathfrak{f}(S_n), \mathfrak{f}(A_n), 1}$. It is easy to see that:

(i) If $\operatorname{Inn}(N) \cap \mathfrak{f}(S_n) = \mathfrak{f}(S_n)$, then $N \simeq S_n$ so condition (1) holds.

(ii) If $\operatorname{Inn}(N) \cap \mathfrak{f}(S_n) = \mathfrak{f}(A_n)$, then A_n embeds into $\operatorname{Inn}(N) \simeq N/Z(N)$ and thus $|Z(N)| \leq 2$. Then, condition (1) holds when |Z(N)| = 1 because a subgroup of index two is always normal, and condition (2) clearly holds when |Z(N)| = 2.

(iii) If
$$\operatorname{Inn}(N) \cap \mathfrak{f}(S_n) = 1$$
, then condition (3) holds.

Note that we do not need \mathfrak{g} to be bijective for the above arguments.

Now, suppose that $\ker(\mathfrak{f}) \neq 1$, so then $\ker(\mathfrak{f}) \in \{A_n, S_n\}$ by (4.3). Suppose also that \mathfrak{g} is bijective. If $\ker(\mathfrak{f}) = S_n$, then $N \simeq S_n$ by (2.2). If $\ker(\mathfrak{f}) = A_n$, then N contains a subgroup isomorphic to A_n by Lemma 4.1 (c), which has index two and hence is normal in N. In both cases, we see that condition (1) holds. This proves the first statement of the theorem. \Box

To prove the second statement, let M be any proper maximal characteristic subgroup of N. We shall use the notation in (4.4), and let $\overline{\mathfrak{f}}, \overline{\mathfrak{g}}$ be as in (4.1). By (1.1) and Proposition 2.1, it suffices to show that

 $N/M \simeq \mathbb{Z}/2\mathbb{Z}$ whenever N/M is abelian and $\overline{\mathfrak{g}}$ is surjective.

The reader should compare our proof below with that of Lemma 4.9.

Proof of Theorem 1.7: second statement. Note that

$$[A_n: A_n \cap \ker(\overline{\mathfrak{g}})] = \begin{cases} [S_n: \ker(\overline{\mathfrak{g}})] & \text{if } \ker(\overline{\mathfrak{g}}) \not\subset A_n, \\ \frac{1}{2}[S_n: \ker(\overline{\mathfrak{g}})] & \text{if } \ker(\overline{\mathfrak{g}}) \subset A_n, \end{cases}$$

and these are the only cases because $[S_n : A_n] = 2$.

Suppose that N/M is abelian and $\overline{\mathfrak{g}}$ is surjective. Then, we have $T \simeq \mathbb{Z}/p\mathbb{Z}$ for some prime p, and $[S_n : \ker(\overline{\mathfrak{g}})] = p^m$ by Lemma 4.1 (b). We also have

$$\begin{cases} n = p^m & \text{if } \ker(\overline{\mathfrak{g}}) \not\subset A_n, \\ n = 2^{m-1} \text{ with } m \ge 4 & \text{if } \ker(\overline{\mathfrak{g}}) \subset A_n, \end{cases}$$

by Lemma 4.6, as well as

$$\begin{cases} v_p(S_{p^m}) > v_p(\operatorname{GL}_m(\mathbb{Z}/p\mathbb{Z})) & \text{for all } m \ge 1, \\ v_2(S_{2^{m-1}}) > v_2(\operatorname{GL}_m(\mathbb{Z}/2\mathbb{Z})) & \text{for all } m \ge 4, \end{cases}$$

by Lemma 4.7 and (4.7). Since $S_n/\ker(\overline{\mathfrak{f}})$ embeds into $\operatorname{GL}_m(\mathbb{Z}/p\mathbb{Z})$, we see that $\ker(\overline{\mathfrak{f}}) \neq 1$, and so $\ker(\overline{\mathfrak{f}}) \supset A_n$ by (4.3). It then follows from Lemma 4.1 (c) that $\overline{\mathfrak{g}}$ restricts to a homomorphism $A_n \longrightarrow N/M$. Since $A_n = [A_n, A_n]$ and N/M is abelian, this implies that $A_n \subset \ker(\overline{\mathfrak{g}})$. But then

$$2 = [S_n : A_n] \ge [S_n : \ker(\overline{\mathfrak{g}})] = [N : M]$$

by Lemma 4.1 (b), and so we must have $N/M \simeq \mathbb{Z}/2\mathbb{Z}$, as claimed.

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