

On certain sums concerning the gcd's and lcm's of k positive integers

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Abstract

We use elementary arguments to prove results on the order of magnitude of certain sums concerning the gcd's and lcm's of k positive integers, where $k \geq 2$ is fixed. We refine and generalize an asymptotic formula of Bordellès (2007), and extend certain related results of Hilberdink and Tóth (2016). We also formulate some conjectures and open problems.

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1 Introduction

Consider the gcd-sum function

$$G(n) := \sum_{k=1}^n (k, n) = \sum_{d|n} d\varphi(n/d) \quad (n \in \mathbb{N}),$$

where $\varphi(n)$ is Euler's totient function. The function $G(n)$ is multiplicative and the asymptotic formula

$$\sum_{n \leq x} G(n) = \frac{x^2}{2\zeta(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + \mathcal{O}(x^{1+\theta+\varepsilon}), \quad (1.1)$$

holds for every $\varepsilon > 0$, where γ is Euler's constant, and θ is the exponent appearing in Dirichlet's divisor problem. See the survey paper [8] by the third author.

The function

$$G^{(-1)}(n) := \sum_{k=1}^n \frac{1}{(k, n)} = \sum_{d|n} \frac{\varphi(n/d)}{d} \quad (n \in \mathbb{N}),$$

is also multiplicative. Bordellès [1, Th. 5.1] deduced that

$$\sum_{n \leq x} G^{(-1)}(n) = \frac{\zeta(3)}{2\zeta(2)} x^2 + O\left(x(\log x)^{2/3}(\log \log x)^{4/3}\right). \quad (1.2)$$

The error term of estimate (1.2) comes from the classical result of Walfisz [9, Satz 1, p. 144],

$$R(x) := \sum_{n \leq x} \varphi(n) - \frac{1}{2\zeta(2)}x^2 = O\left(x(\log x)^{2/3}(\log \log x)^{4/3}\right). \quad (1.3)$$

We remark that recently (1.3) was improved by Liu [4] into

$$R(x) = O\left(x(\log x)^{2/3}(\log \log x)^{1/3}\right), \quad (1.4)$$

therefore, this serves as the remainder of (1.2). Also see the preprint by Suzuki [7].

The lcm-sum function

$$L(n) := \sum_{k=1}^n [k, n] = \frac{n}{2} \left(1 + \sum_{d|n} d\varphi(d) \right) \quad (n \in \mathbb{N}).$$

was investigated by Bordellès [1], Ikeda and Matsuoka [3], and others. The function $L(n)$ is not multiplicative and one has, see [1, Th. 6.3],

$$\sum_{n \leq x} L(n) = \frac{\zeta(3)}{8\zeta(2)}x^4 + O\left(x^3(\log x)^{2/3}(\log \log x)^{4/3}\right). \quad (1.5)$$

By using (1.4), the exponent of the $\log \log x$ factor in the error of (1.5) can be improved into $1/3$.

Now let

$$L^{(-1)}(n) := \sum_{k=1}^n \frac{1}{[k, n]} \quad (n \in \mathbb{N}).$$

Bordellès [1, Th. 7.1] proved that

$$\sum_{n \leq x} L^{(-1)}(n) = \frac{1}{\pi^2}(\log x)^3 + A(\log x)^2 + O(\log x), \quad (1.6)$$

with an explicitly given constant A .

By the general identity

$$\sum_{m, n \leq x} \psi(m, n) = 2 \sum_{n \leq x} \sum_{m=1}^n \psi(m, n) - \sum_{n \leq x} \psi(n, n),$$

valid for any function $\psi : \mathbb{N}^2 \rightarrow \mathbb{C}$, which is symmetric in the variables, (1.1), (1.2), (1.5) and (1.6), together with the remark on (1.4) lead to the asymptotic formulas

$$\sum_{m, n \leq x} (m, n) = \frac{x^2}{\zeta(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + O\left(x^{1+\theta+\varepsilon}\right), \quad (1.7)$$

$$\sum_{m, n \leq x} \frac{1}{(m, n)} = \frac{\zeta(3)}{\zeta(2)}x^2 + O\left(x(\log x)^{2/3}(\log \log x)^{1/3}\right), \quad (1.8)$$

$$\sum_{m,n \leq x} [m, n] = \frac{\zeta(3)}{4\zeta(2)} x^4 + O\left(x^3(\log x)^{2/3}(\log \log x)^{1/3}\right), \quad (1.9)$$

and

$$\sum_{m,n \leq x} \frac{1}{[m, n]} = \frac{2}{\pi^2} (\log x)^3 + A_1 (\log x)^2 + O(\log x), \quad (1.10)$$

respectively, where $A_1 = 2A$.

It is easy to generalize (1.7) and (1.8) for sums with k variables by using the general identity

$$\sum_{n_1, \dots, n_k \leq x} f((n_1, \dots, n_k)) = \sum_{d \leq x} (\mu * f)(d) [x/d]^k,$$

where f is an arbitrary arithmetic function, μ is the Möbius function and $*$ stands for the Dirichlet convolution of arithmetic functions. For example, we have the next result: For any $k \geq 3$,

$$\sum_{n_1, \dots, n_k \leq x} \frac{1}{(n_1, \dots, n_k)} = \frac{\zeta(k+1)}{\zeta(k)} x^k + O\left(x^{k-1}\right).$$

However, it is more difficult to derive asymptotic formulas for similar sums involving the lcm $[n_1, \dots, n_k]$. As corollaries of more general results concerning a large class of functions f , the first and third authors [2, Cor 1] proved that for any $k \geq 3$ and any real number $r > -1$,

$$\sum_{n_1, \dots, n_k \leq x} [n_1, \dots, n_k]^r = A_{r,k} x^{k(r+1)} + O\left(x^{k(r+1) - \frac{1}{2} \min(r+1, 1) + \varepsilon}\right) \quad (1.11)$$

and

$$\sum_{n_1, \dots, n_k \leq x} \left(\frac{[n_1, \dots, n_k]}{n_1 \cdots n_k} \right)^r = A_{r,k} x^k + O\left(x^{k - \frac{1}{2} \min(r+1, 1) + \varepsilon}\right),$$

where $A_{k,r}$ are explicitly given constants. Here, (1.11) is the k dimensional generalization of (1.9). Furthermore, [2, Cor 2] shows that for any $k \geq 3$ and any real number $r > 0$,

$$\sum_{n_1, \dots, n_k \leq x} \left(\frac{[n_1, \dots, n_k]}{(n_1, \dots, n_k)} \right)^r = B_{r,k} x^{k(r+1)} + O\left(x^{k(r+1) - \frac{1}{2} + \varepsilon}\right),$$

with explicitly given constants $B_{k,r}$. The proofs use the fact that (n_1, \dots, n_k) and $[n_1, \dots, n_k]$ are multiplicative functions of k variables and the associated multiple Dirichlet series factor over the primes into Euler products. The proofs given in [2] cannot be applied in the case $r = -1$.

It is the goal of the present paper to investigate the order of magnitude of the sums

$$S_k(x) := \sum_{n_1, \dots, n_k \leq x} \frac{1}{[n_1, \dots, n_k]}, \quad (1.12)$$

$$T_k(x) := \sum_{n_1, \dots, n_k \leq x} \frac{(n_1, \dots, n_k)}{[n_1, \dots, n_k]}, \quad (1.13)$$

$$U_k(x) := \sum_{\substack{n_1, \dots, n_k \leq x \\ (n_1, \dots, n_k) = 1}} \frac{1}{[n_1, \dots, n_k]}, \quad (1.14)$$

$$V_k(x) := \sum_{n_1, \dots, n_k \leq x} \frac{n_1 \cdots n_k}{[n_1, \dots, n_k]}, \quad (1.15)$$

where $k \geq 2$ is fixed, by using elementary arguments. Theorem 2.1, concerning the sum $S_2(x)$, refines formulas (1.6) and (1.10) of Bordellès [1]. Theorems 2.3 and 3.1 give the exact order of magnitude of the sums $S_k(x)$ and $U_k(x)$, respectively, for $k \geq 3$. Theorem 4.1 concerns the sums $V_k(x)$, while Theorem 5.2 provides an asymptotic formula with remainder term for $T_k(x)$, for any fixed $k \geq 2$. Some conjectures and open problems are formulated as well.

2 The sums $S_k(x)$

First consider the sums $S_k(x)$ defined by (1.12). In the case $k = 2$ we use Dirichlet's hyperbola method to prove the next result, which improves formulas (1.6) and (1.10).

Theorem 2.1.

$$\sum_{n \leq x} L^{(-1)}(n) = \frac{1}{\pi^2} (\log x)^3 + A (\log x)^2 + B \log x + C + O\left(x^{-1/2} (\log x)^2\right), \quad (2.1)$$

that is,

$$\sum_{m, n \leq x} \frac{1}{[m, n]} = \frac{2}{\pi^2} (\log x)^3 + A_1 (\log x)^2 + B_1 \log x + C_1 + O\left(x^{-1/2} (\log x)^2\right),$$

where the constants A, B, C can be explicitly computed, and $A_1 = 2A$, $B_1 = 2B - 1$, $C_1 = C - \gamma$.

Proof. We have

$$L^{(-1)}(n) = \sum_{k=1}^n \frac{(k, n)}{kn} = \frac{1}{n} \sum_{d|n} d \sum_{\substack{k=1 \\ (k, n) = d}}^n \frac{1}{k} = \frac{1}{n} \sum_{d|n} \sum_{\substack{t=1 \\ (t, n/d) = 1}}^{n/d} \frac{1}{t} = \frac{1}{n} \sum_{d|n} h(d), \quad (2.2)$$

where

$$\begin{aligned} h(n) &:= \sum_{\substack{m=1 \\ (m, n) = 1}}^n \frac{1}{m} = \sum_{m=1}^n \frac{1}{m} \sum_{d|(m, n)} \mu(d) = \sum_{d|n} \frac{\mu(d)}{d} \sum_{j=1}^{n/d} \frac{1}{j} \\ &= \sum_{d|n} \frac{\mu(d)}{d} \left(\log \frac{n}{d} + \gamma + O\left(\frac{d}{n}\right) \right) = \sum_{d|n} \frac{\mu(d)}{d} \log \frac{n}{d} + \gamma \frac{\varphi(n)}{n} + O\left(\frac{2^{\omega(n)}}{n}\right). \end{aligned}$$

Hence,

$$H(x) := \sum_{n \leq x} h(n) = \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{m \leq x/d} \log m + \gamma \sum_{n \leq x} \frac{\varphi(n)}{n} + O\left(\sum_{n \leq x} \frac{2^{\omega(n)}}{n}\right).$$

By using the known estimates

$$\begin{aligned}\sum_{n \leq x} \log n &= x \log x - x + O(\log x), \\ \sum_{n \leq x} \frac{\varphi(n)}{n} &= \frac{6}{\pi^2} x + O(\log x), \\ \sum_{n \leq x} \frac{2^{\omega(n)}}{n} &= O((\log x)^2),\end{aligned}$$

we deduce that

$$\begin{aligned}H(x) &= (x \log x - x) \sum_{d \leq x} \frac{\mu(d)}{d^2} - x \sum_{d \leq x} \frac{\mu(d) \log d}{d^2} + \frac{6}{\pi^2} \gamma x + O((\log x)^2) \\ &= \frac{6}{\pi^2} (x \log x + cx) + O((\log x)^2),\end{aligned}\tag{2.3}$$

with a certain constant c . Let $\mathbf{1}(n) = 1$ ($n \in \mathbb{N}$), and let $*$ denote the Dirichlet convolution. By Dirichlet's hyperbola method,

$$\begin{aligned}\sum_{n \leq x} (\mathbf{1} * h)(n) &= \sum_{n \leq \sqrt{x}} (H(x/n) + h(n) \lfloor x/n \rfloor) - \lfloor \sqrt{x} \rfloor H(\sqrt{x}) \\ &= \sum_{n \leq \sqrt{x}} H(x/n) + x \sum_{n \leq \sqrt{x}} \frac{h(n)}{n} - \sqrt{x} H(\sqrt{x}) + O(H(\sqrt{x})).\end{aligned}$$

By partial summation,

$$x \sum_{n \leq \sqrt{x}} \frac{h(n)}{n} = \sqrt{x} H(\sqrt{x}) + x \int_1^{\sqrt{x}} \frac{H(t)}{t^2} dt,$$

and using (2.3) we deduce

$$\begin{aligned}\sum_{n \leq x} (\mathbf{1} * h)(n) &= \frac{6}{\pi^2} \sum_{n \leq \sqrt{x}} \left(\frac{x}{n} \log \left(\frac{x}{n} \right) + c \left(\frac{x}{n} \right) \right) + \frac{6x}{\pi^2} \int_1^{\sqrt{x}} \left(\frac{\log t}{t} + c \right) \frac{dt}{t} + O(\sqrt{x}(\log x)^2) \\ &= x \left(\frac{3}{\pi^2} (\log x)^2 + a \log x + b \right) + O(\sqrt{x}(\log x)^2),\end{aligned}$$

for some constants a, b , which can be explicitly calculated.

Here $(\mathbf{1} * h)(n) = nL^{(-1)}(n)$, according to (2.2), and we obtain (2.1) by partial summation. \square

It is more difficult to handle the sums $S_k(x)$ in the case $k \geq 3$. We will apply the following general result proved by the second and third authors [5], using elementary arguments.

Theorem 2.2. ([5]) *Let k be a positive integer and let $f : \mathbb{N} \rightarrow \mathbb{C}$ be a multiplicative function satisfying the following properties:*

(i) $f(p) = k$ for every prime p ,

(ii) $f(p^\nu) = \nu^{O(1)}$ for every prime p and every integer $\nu \geq 2$, where the constant implied by the O symbol is uniform in p .

Then

$$\sum_{n \leq x} \frac{f(n)}{n} = \frac{1}{k!} C_f (\log x)^k + D_f (\log x)^{k-1} + O\left((\log x)^{k-2}\right),$$

where C_f and D_f are constants,

$$C_f = \prod_p \left(1 - \frac{1}{p}\right)^k \left(\sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^\nu}\right).$$

We have the following result.

Theorem 2.3. *Let $k \geq 3$ be a fixed integer. Then*

$$S_k(x) \asymp (\log x)^{2^k-1} \quad \text{as } x \rightarrow \infty.$$

Proof. Since $[n_1, \dots, n_k] \leq n_1 \cdots n_k \leq x^k$, we can write

$$S_k(x) = \sum_{n \leq x^k} \frac{1}{n} \sum_{\substack{n_1, \dots, n_k \leq x \\ [n_1, \dots, n_k] = n}} 1 \quad (2.4)$$

Let

$$a_k(n) := \sum_{\substack{n_1, \dots, n_k \in \mathbb{N} \\ [n_1, \dots, n_k] = n}} 1.$$

Now if $n \leq x$, then the inner sum in (2.4) is just $a_k(n)$ (since $n \leq x$ forces $n_1, \dots, n_k \leq x$), while in any case it is at most $a_k(n)$. Thus

$$\sum_{n \leq x} \frac{a_k(n)}{n} \leq S_k(x) \leq \sum_{n \leq x^k} \frac{a_k(n)}{n}. \quad (2.5)$$

To see the properties of the function $a_k(n)$ write

$$\sum_{d|n} a_k(d) = \sum_{d|n} \sum_{[n_1, \dots, n_k] = d} 1 = \sum_{[n_1, \dots, n_k] | n} 1 = \sum_{n_1 | n, \dots, n_k | n} 1 = \tau(n)^k.$$

Therefore, by Möbius inversion, we have $a_k = \mu * \tau^k$. This shows that $a_k(n)$ is multiplicative and its values at the prime powers p^ν are given by $a_k(p^\nu) = (\nu + 1)^k - \nu^k$ ($\nu \geq 1$). In particular, $a_k(p) = 2^k - 1$.

Applying Theorem 2.2 for the function $f(n) = a_k(n)$, with $2^k - 1$ instead of k , we get that

$$\sum_{n \leq x} \frac{a_k(n)}{n} \sim \alpha_k (\log x)^{2^k-1} \quad \text{as } x \rightarrow \infty, \quad (2.6)$$

for some constant α_k . Now, from (2.5) and (2.6) the result follows. \square

Remark 2.4. It is natural to expect that $S_k(x) \sim c_k(\log x)^{2^k-1}$ as $x \rightarrow \infty$, with a certain constant c_k . In fact, in view of Theorem 2.1, the plausible conjecture is that

$$S_k(x) = P_{2^k-1}(\log x) + O(x^{-r}), \quad (2.7)$$

where $P_{2^k-1}(t)$ is a polynomial in t of degree $2^k - 1$ and r is a positive real number. We pose as an open problem to find the constants c_k and to prove (2.7).

3 The sums $U_k(x)$

Next consider the sums $U_k(x)$ defined by (1.14). In the case $k = 2$,

$$U_2(x) \sim \frac{6}{\pi^2}(\log x)^2 \quad \text{as } x \rightarrow \infty,$$

and it is not difficult to deduce a more precise asymptotic formula.

We have the following general result.

Theorem 3.1. *Let $k \geq 3$ be a fixed integer. Then*

$$U_k(x) \asymp (\log x)^{2^k-2} \quad \text{as } x \rightarrow \infty.$$

Proof. Similar to the proof of Theorem 2.3. We have

$$U_k(x) = \sum_{\substack{n_1, \dots, n_k \leq x \\ (n_1, \dots, n_k) = 1}} \frac{1}{[n_1, \dots, n_k]} = \sum_{n \leq x^k} \frac{1}{n} \sum_{\substack{n_1, \dots, n_k \leq x \\ [n_1, \dots, n_k] = n \\ (n_1, \dots, n_k) = 1}} 1. \quad (3.1)$$

Let

$$b_k(n) = \sum_{\substack{n_1, \dots, n_k \in \mathbb{N} \\ [n_1, \dots, n_k] = n \\ (n_1, \dots, n_k) = 1}} 1.$$

Now if $n \leq x$, then the inner sum in (3.1) is exactly $b_k(n)$, while in any case it is at most $b_k(n)$. Thus

$$\sum_{n \leq x} \frac{b_k(n)}{n} \leq U_k(x) \leq \sum_{n \leq x^k} \frac{b_k(n)}{n}. \quad (3.2)$$

Write

$$\begin{aligned} \sum_{d|n} b_k(d) &= \sum_{d|n} \sum_{\substack{[n_1, \dots, n_k] = d \\ (n_1, \dots, n_k) = 1}} 1 = \sum_{\substack{[n_1, \dots, n_k] | n \\ (n_1, \dots, n_k) = 1}} 1 \\ &= \sum_{n_1 | n, \dots, n_k | n} \sum_{\delta | (n_1, \dots, n_k)} \mu(\delta) = \sum_{\delta a_1 b_1 = n, \dots, \delta a_k b_k = n} \mu(\delta) \\ &= \sum_{\delta t = n} \mu(\delta) \sum_{a_1 b_1 = t} 1 \cdots \sum_{a_k b_k = t} 1 = \sum_{\delta t = n} \mu(\delta) \tau(t)^k. \end{aligned}$$

Therefore, by Möbius inversion $b_k = \mu * \mu * \tau^k$. This shows that $b_k(n)$ is multiplicative and its values at the prime powers p^ν are given by $b_k(p^\nu) = (\nu + 1)^k - 2\nu^k + (\nu - 1)^k$ ($\nu \geq 1$). In particular, $b_k(p) = 2^k - 2$.

Applying now Theorem 2.2 for the function $f(n) = b_k(n)$, with $2^k - 2$ instead of k , we deduce that

$$\sum_{n \leq x} \frac{b_k(n)}{n} \sim \alpha'_k (\log x)^{2^k - 2} \quad \text{as } x \rightarrow \infty \quad (3.3)$$

for some constant α'_k . Now, from (3.2) and (3.3) we have $U_k(x) \asymp (\log x)^{2^k - 2}$. \square

Remark 3.2. We conjecture that $U_k(x) \sim d_k (\log x)^{2^k - 2}$ as $x \rightarrow \infty$, with a certain constant d_k . The sums $S_k(x)$ and $U_k(x)$ are strongly related. Namely, by grouping the terms according to the values $(n_1, \dots, n_k) = d$ one obtains

$$S_k(x) = \sum_{d \leq x} \frac{1}{d} U_k(x/d), \quad (3.4)$$

and conversely,

$$U_k(x) = \sum_{d \leq x} \frac{\mu(d)}{d} S_k(x/d). \quad (3.5)$$

If $U_k(x) \sim d_k (\log x)^{2^k - 2}$ holds, then by (3.4) it follows that $S_k(x) \sim \frac{d_k}{2^k - 1} (\log x)^{2^k - 1}$. Conversely, assume that the asymptotic formula (2.7) is true, where c_k is the leading coefficient of the polynomial $P_{2^k - 1}(t)$. Then (3.5), together with the well known results

$$\sum_{n \leq x} \frac{\mu(n)}{n} = O((\log x)^{-1}), \quad \sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n} = -1,$$

and Shapiro's estimates [6, Th. 4.1]

$$\sum_{n \leq x} \frac{\mu(n)}{n} \left(\log \left(\frac{x}{n} \right) \right)^m = m (\log x)^{m-1} + \sum_{j=1}^{m-2} c_j^{(m)} (\log x)^j + O(1),$$

valid for any integer $m \geq 2$, where $c_i^{(m)}$ are constants, imply that

$$U_k(x) = (2^k - 1)c_k (\log x)^{2^k - 2} + b_{2^k - 3} (\log x)^{2^k - 3} + \dots + b_1 \log x + O(1),$$

with some constants b_i .

4 The sums $V_k(x)$

The sums $V_k(x)$ defined by (1.15) are sums of integers. In the case $k = 2$ we have, according to (1.7),

$$V_2(x) = \sum_{m, n \leq x} (m, n) \sim \frac{6}{\pi^2} x^2 \log x. \quad (4.1)$$

Theorem 4.1. *Let $k \geq 3$ be a fixed integer. Then*

$$x^k \ll V_k(x) \ll x^k (\log x)^{2^k-2} \quad \text{as } x \rightarrow \infty.$$

Proof. The lower bound is trivial by $n_1 \cdots n_k \geq [n_1, \dots, n_k]$. Also, by grouping the terms according to the values $(n_1, \dots, n_k) = d$, and by denoting $M = \max(m_1, \dots, m_k)$ we have

$$\begin{aligned} V_k(x) &= \sum_{\substack{dm_1, \dots, dm_k \leq x \\ (m_1, \dots, m_k)=1}} \frac{dm_1 \cdots dm_k}{[dm_1, \dots, dm_k]} = \sum_{\substack{m_1, \dots, m_k \leq x \\ (m_1, \dots, m_k)=1}} \frac{m_1 \cdots m_k}{[m_1, \dots, m_k]} \sum_{d \leq x/M} d^{k-1} \\ &\ll x^k \sum_{\substack{m_1, \dots, m_k \leq x \\ (m_1, \dots, m_k)=1}} \frac{m_1 \cdots m_k}{[m_1, \dots, m_k] M^k} \leq x^k \sum_{\substack{m_1, \dots, m_k \leq x \\ (m_1, \dots, m_k)=1}} \frac{1}{[m_1, \dots, m_k]} = x^k U_k(x), \end{aligned}$$

and the upper bound follows from Theorem 3.1. \square

Remark 4.2. We conjecture that $V_k(x) \sim \lambda_k x^k (\log x)^{2^k-k-1}$ as $x \rightarrow \infty$, with a certain constant λ_k , in accordance with (4.1) for the case $k = 2$. We pose as another open problem to prove this and to find the constants λ_k .

5 The sums $T_k(x)$

Finally, we investigate the sums $T_k(x)$ defined by (1.13) and establish an asymptotic formula with remainder term for it. We give a short direct proof in the case $k = 2$. Then for any fixed $k \geq 2$ we use multiple Dirichlet series to get the result.

Let

$$F(n) := \sum_{k=1}^n \frac{(k, n)}{[k, n]} \quad (n \in \mathbb{N}). \quad (5.1)$$

Theorem 5.1.

$$\sum_{n \leq x} F(n) = 2x + O((\log x)^2), \quad (5.2)$$

that is,

$$\sum_{m, n \leq x} \frac{(m, n)}{[m, n]} = 3x + O((\log x)^2).$$

Proof. Let $\phi_2(n) = \sum_{d|n} d^2 \mu(n/d)$ be the Jordan function of order 2. We have

$$\begin{aligned} F(n) &= \sum_{k=1}^n \frac{(k, n)^2}{kn} = \frac{1}{n} \sum_{k=1}^n \frac{1}{k} \sum_{d|(k, n)} \phi_2(d) = \frac{1}{n} \sum_{d|n} \phi_2(d) \sum_{\substack{k=1 \\ d|k}}^n \frac{1}{k} \\ &= \frac{1}{n} \sum_{d|n} \frac{\phi_2(d)}{d} \sum_{j=1}^{n/d} \frac{1}{j} = \frac{1}{n} \sum_{d|n} \frac{\phi_2(d)}{d} H_{n/d}, \end{aligned}$$

where $H_m = \sum_{j=1}^m 1/j$ is the harmonic sum. Therefore, using that

$$\sum_{n \leq x} \frac{\phi_2(n)}{n^2} = \frac{x}{\zeta(3)} + O(1),$$

we deduce

$$\begin{aligned} \sum_{n \leq x} F(n) &= \sum_{dm \leq x} \frac{\phi_2(d)}{d^2 m} H_m = \sum_{m \leq x} \frac{H_m}{m} \sum_{d \leq x/m} \frac{\phi_2(d)}{d^2} \\ &= \sum_{m \leq x} \frac{H_m}{m} \left(\frac{x}{\zeta(3)m} + O(1) \right) = \frac{x}{\zeta(3)} \sum_{m \leq x} \frac{H_m}{m^2} + O \left(\sum_{m \leq x} \frac{H_m}{m} \right) \\ &= \frac{x}{\zeta(3)} \sum_{m=1}^{\infty} \frac{H_m}{m^2} + O \left(x \sum_{m > x} \frac{H_m}{m^2} \right) + O \left(\sum_{m \leq x} \frac{H_m}{m} \right) \\ &= \frac{x}{\zeta(3)} \cdot 2\zeta(3) + O \left(x \sum_{m > x} \frac{\log m}{m^2} \right) + O \left(\sum_{m \leq x} \frac{\log m}{m} \right) = 2x + O((\log x)^2), \end{aligned}$$

by using that

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3), \quad (5.3)$$

which is Euler's result. \square

Theorem 5.2. *If $k \geq 2$, then*

$$T_k(x) = \beta_k x + O \left((\log x)^{2^k - 2} \right),$$

where

$$\beta_k := \sum_{\substack{n_1, \dots, n_k=1 \\ (n_1, \dots, n_k)=1}}^{\infty} \frac{1}{[n_1, \dots, n_k] \max(n_1, \dots, n_k)} = \frac{1}{\zeta(2)} \sum_{n_1, \dots, n_k=1}^{\infty} \frac{1}{[n_1, \dots, n_k] \max(n_1, \dots, n_k)}.$$

Proof. By grouping the terms according to $(n_1, \dots, n_k) = d$, where $n_j = dm_j$ ($1 \leq j \leq k$), $(m_1, \dots, m_k) = 1$, we have

$$\begin{aligned} T_k(x) &= \sum_{\substack{dm_1, \dots, dm_k \leq x \\ (m_1, \dots, m_k)=1}} \frac{d}{[dm_1, \dots, dm_k]} = \sum_{\substack{dm_1, \dots, dm_k \leq x \\ (m_1, \dots, m_k)=1}} \frac{1}{[m_1, \dots, m_k]} \\ &= \sum_{\substack{m_1, \dots, m_k \leq x \\ (m_1, \dots, m_k)=1}} \frac{1}{[m_1, \dots, m_k]} \sum_{d \leq x/M} 1 = \sum_{\substack{m_1, \dots, m_k \leq x \\ (m_1, \dots, m_k)=1}} \frac{\lfloor x/M \rfloor}{[m_1, \dots, m_k]}, \end{aligned}$$

where $M = \max(m_1, \dots, m_k)$. Let

$$h(n_1, \dots, n_k) := \begin{cases} \frac{1}{[n_1, \dots, n_k]}, & \text{if } (n_1, \dots, n_k) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$T_k(x) = x \sum_{n_1, \dots, n_k \leq x} \frac{h(n_1, \dots, n_k)}{\max(n_1, \dots, n_k)} + O\left(\sum_{n_1, \dots, n_k \leq x} h(n_1, \dots, n_k)\right) \quad (5.4)$$

and we estimate the right-hand sums in turn. Here $h(n_1, \dots, n_k)$ is a symmetric and multiplicative function of k variables and for prime powers $p^{\nu_1}, \dots, p^{\nu_k}$ ($\nu_1, \dots, \nu_k \geq 0$) one has

$$h(p^{\nu_1}, \dots, p^{\nu_k}) = \begin{cases} \frac{1}{p^{\max(\nu_1, \dots, \nu_k)}}, & \text{if } \min(\nu_1, \dots, \nu_k) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Consider its Dirichlet series

$$H(s_1, \dots, s_k) := \sum_{n_1, \dots, n_k=1}^{\infty} \frac{h(n_1, \dots, n_k)}{n_1^{s_1} \dots n_k^{s_k}} = \prod_p \sum_{\substack{\nu_1, \dots, \nu_k=0 \\ \min(\nu_1, \dots, \nu_k)=0}}^{\infty} \frac{1}{p^{\max(\nu_1, \dots, \nu_k) + \nu_1 s_1 + \dots + \nu_k s_k}}.$$

By grouping the terms according to the values of $r = \max(\nu_1, \dots, \nu_k)$ we deduce

$$H(s_1, \dots, s_k) = \prod_p \frac{1}{p^r} \sum_{r=0}^{\infty} \sum_{\substack{\nu_1, \dots, \nu_k=0 \\ \max(\nu_1, \dots, \nu_k)=r \\ \min(\nu_1, \dots, \nu_k)=0}} \frac{1}{p^{\nu_1 s_1 + \dots + \nu_k s_k}},$$

which converges absolutely for $\Re s_j > 0$ ($1 \leq j \leq k$).

We shall need an estimate for $H_k(\varepsilon, \dots, \varepsilon)$ for $\varepsilon > 0$ (small). We have

$$H(\varepsilon, \dots, \varepsilon) = \prod_p \left(1 + \frac{1}{p} \sum_{j=1}^{k-1} \binom{k}{j} \frac{1}{p^{j\varepsilon}} + O\left(\frac{1}{p^2}\right) \right).$$

Therefore,

$$\log H(\varepsilon, \dots, \varepsilon) = \sum_p \frac{1}{p} \sum_{j=1}^{k-1} \binom{k}{j} \frac{1}{p^{j\varepsilon}} + O(1) = \sum_{j=1}^{k-1} \binom{k}{j} \sum_p \frac{1}{p^{1+j\varepsilon}} + O(1).$$

But $\sum_p p^{-1-\varepsilon} = \log \frac{1}{\varepsilon} + O(1)$ as $\varepsilon \rightarrow 0$. Thus,

$$H(\varepsilon, \dots, \varepsilon) = \exp\left(\sum_{j=1}^{k-1} \binom{k}{j} \log \frac{1}{\varepsilon} + O(1)\right) \asymp \left(\frac{1}{\varepsilon}\right)^{2^k - 2}. \quad (5.5)$$

Furthermore, for any $\varepsilon > 0$, we have

$$\begin{aligned} \sum_{n_1, \dots, n_k \leq x} h(n_1, \dots, n_k) &= \sum_{n_1, \dots, n_k \leq x} \frac{h(n_1, \dots, n_k)}{(n_1 \cdots n_k)^{\varepsilon/k}} (n_1 \cdots n_k)^{\varepsilon/k} \\ &\leq x^\varepsilon \sum_{n_1, \dots, n_k \leq x} \frac{h(n_1, \dots, n_k)}{(n_1 \cdots n_k)^{\varepsilon/k}} \leq x^\varepsilon H(\varepsilon/k, \dots, \varepsilon/k). \end{aligned} \quad (5.6)$$

Next, note that $\max(n_1, \dots, n_k) \geq (n_1 \cdots n_k)^{1/k}$, so that

$$\sum_{n_1, \dots, n_k \leq x} \frac{h(n_1, \dots, n_k)}{\max(n_1, \dots, n_k)} \leq \sum_{n_1, \dots, n_k \leq x} \frac{h(n_1, \dots, n_k)}{(n_1 \cdots n_k)^{1/k}} \leq H(\varepsilon/k, \dots, \varepsilon/k),$$

which converges. Hence,

$$\beta_k = \sum_{n_1, \dots, n_k=1}^{\infty} \frac{h(n_1, \dots, n_k)}{\max(n_1, \dots, n_k)}$$

is finite and $\beta_k \leq H(\varepsilon/k, \dots, \varepsilon/k)$. Also,

$$\begin{aligned} \beta_k - \sum_{n_1, \dots, n_k \leq x} \frac{h(n_1, \dots, n_k)}{\max(n_1, \dots, n_k)} &= \sum_{\substack{n_1, \dots, n_k \in \mathbb{N} \\ \text{some } n_i > x}} \frac{h(n_1, \dots, n_k)}{\max(n_1, \dots, n_k)} \\ &\leq k \sum_{\substack{n_1 \geq n_2, \dots, n_k \\ n_1 > x}} \frac{h(n_1, \dots, n_k)}{n_1} \leq k \sum_{\substack{n_1 \geq n_2, \dots, n_k \\ n_1 > x}} \frac{h(n_1, \dots, n_k)}{n_1^{1-\varepsilon} (n_1 n_2 \cdots n_k)^{\varepsilon/k}} \\ &\leq \frac{k}{x^{1-\varepsilon}} \sum_{n_1, \dots, n_k=1}^{\infty} \frac{h(n_1, \dots, n_k)}{(n_1 \cdots n_k)^{\varepsilon/k}} = kx^{\varepsilon-1} H(\varepsilon/k, \dots, \varepsilon/k). \end{aligned} \quad (5.7)$$

Hence, (5.4) and the estimates (5.6), (5.7) give

$$T_k(x) = \beta_k x + O(x^\varepsilon H(\varepsilon/k, \dots, \varepsilon/k)).$$

Now we choose $\varepsilon = 1/\log x$ and use the bound (5.5). The proof is complete. \square

Remark 5.3. For $k = 2$, Theorem 5.2 recovers Theorem 5.1. Note that

$$\beta_2 = \frac{1}{\zeta(3)} \sum_{m, n=1}^{\infty} \frac{1}{mn \max(m, n)} = \frac{2}{\zeta(3)} \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=1}^m \frac{1}{n} - 1 = 3,$$

by Euler's result (5.3). Is it possible to evaluate the constants β_k for any $k \geq 2$?

The sums $T_k(x)$ and $U_k(x)$ are related by the formulas

$$T_k(x) = \sum_{d \leq x} U_k(x/d), \quad U_k(x) = \sum_{d \leq x} \mu(d) T_k(x/d).$$

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