GETZLER RELATION AND VIRASORO CONJECTURE FOR GENUS ONE

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ABSTRACT. We derive explicit universal equations for primary Gromov-Witten invariants by applying Getzler's genus one relation to quantum powers of Euler vector field. As an application, we provide some evidences for the genus-1 Virasoro conjecture.

1. INTRODUCTION

The Virasoro conjecture of Eguchi-Hori-Xiong [5] predicts some mysterious relations between Gromov-Witten invariants in all genera, that is, a sequence of operators are conjectured to annihilate the generating functions of Gromov-Witten invariants of smooth projective varieties. It is equivalent to Witten conjecture [19] proved by Kontsevich [11] when the underlying mainfold is one point. For manifolds with semisimple quantum product, this conjecture was proved in lower genera [3, 4, 6, 12, 13, 16, 17], and completely solved by Teleman [18]. The genus-0 part of the Virasoro conjecture has been proved firstly in [12] without assumption of semisimplicity, and later by other authors [3, 6]. It is still open for the general case of the genus-1 Virasoro conjecture.

In [13], it is proved that the genus-1 Virasoro conjecture can be reduced to some simple equation derived by restricting the genus-1 L_1-constraint on the small phase space. This equation is motivated by using Getzler's genus one relation [7] as follows

$$\mathbf{G}(E^{k_1}, E^{k_2}, E^{k_3}, E^{k_4}) = 0, \tag{1}$$

where E^k is k^{th} quantum power of Euler vector field, and the definition of $\mathbf{G}$ is presented in section 2. Furtherly, the author in [15] shows that to prove the genus-1 Virasoro conjecture, it is enough to prove the property that the derivative of that simple equation along the direction of any vector field vanishes. Besides the identity of the ordinary cohomology ring satisfying this property, he also find one more vector field called quantum volume element by the computation of

$$\sum_\alpha \mathbf{G}(E, E, \gamma^\alpha, \gamma_\alpha) = 0, \tag{2}$$

where γ^α and γ_α belong to the space of cohomology classes.

In this paper, we will consider the following form

$$\mathbf{G}(E^k, \gamma_\alpha, \gamma_\beta, \gamma_\sigma) = 0. \tag{3}$$

The explicit formula of equation (3) is computed in section 3 (cf. Theorem 3.8), which implies the explicit expressions (cf. Theorems 3.13 and 3.16) for the following two equations

$$\mathbf{G}(E^{k_1}, E^{k_2}, \gamma_\alpha, \gamma_\beta) = 0, \tag{4}$$

$$\mathbf{G}(E^{k_1}, E^{k_2}, E^{k_3}, \gamma_\alpha) = 0. \tag{5}$$

The explicit formula (5) generalizes equation (1), and its equivalent result, i.e., Theorem 4.1 implies some generalized version (cf. Corollaries 4.3 and 4.4) of the Virasoro type relation for $\{\Phi_k\}$ in [13]. And formula (4) implies equation (2), and provides more evidences (cf. Theorem 4.6) for the genus-1 Virasoro conjecture by computing

$$\sum_\mu \mathbf{G}(E^{k_1}, E^{k_2}, \gamma_\mu, \gamma^\mu \circ \gamma_\alpha) = 0.$$

These evidences give an alternative proof of the genus-1 Virasoro conjecture for any manifold with semisimple quantum cohomology (cf. Corollary 4.7). We also derive some new relation (cf. Theorem 4.8) from the general equation (3).

An outline of this paper is as follows. In section 2, we recall some basic definitions, present some known facts for the genus-1 Virasoro conjecture and recollect universal equations for primary Gromov-Witten invariants which will be used later. In section 3, we firstly obtain an explicit formula for the first derivative of Φ_k, and then derive explicit universal equations from Getzler's genus one relation involving some quantum powers of Euler vector field. In section 4, we consider applications to the genus-1 Virasoro conjecture.

Acknowledgements. The author would like to thank Professor Xiaobo Liu for many helpful suggestions, and Professor Jian Zhou for his encouragement.

2. Preliminaries

In this section, we recall Gromov-Witten invariants, quantum product, and some known facts for the genus-1 Virasoro conjecture. We will also recollect some universal equations for primary Gromov-Witten invariants, and fix some notation.

2.1. **Gromov-Witten invariants, quantum product and Virasoro conjecture.** Let X be a smooth projective variety of dimension d, and denote by N the dimension of the space of cohomology classes $H^*(X, \mathbb{C})$. We assume $H^{odd}(X; \mathbb{C}) = 0$ for simplicity, and fix a basis $\{\gamma_1, \cdots, \gamma_N\}$ of $H^*(X, \mathbb{C})$. Let γ_1 be the identity of the cohomology ring of X and $\gamma_\alpha \in H^{p_\alpha, q_\alpha}(X, \mathbb{C})$ for each α. Let $\eta_{\alpha\beta} = \int_X \gamma_\alpha \cup \gamma_\beta$ be the intersection form on $H^*(X, \mathbb{C})$, and $\mathcal{C} = (\mathcal{C}_\alpha^\beta)$ be the matrix satisfying

$$c_1(X) \cup \gamma_\alpha = \sum_\beta \mathcal{C}_\alpha^\beta \gamma_\beta.$$

The symmetric matrices $\eta = (\eta_{\alpha\beta})$ and $\eta^{-1} = (\eta^{\alpha\beta})$ are used to lower and raise indices. For example, $\gamma^\alpha = \sum_\mu \gamma_\mu \eta^{\mu\alpha}$. Let $b_\alpha = p_\alpha - \frac{1}{2}(d-1)$. It is easy to verify that if $\eta^{\alpha\beta} \neq 0$ or $\eta_{\alpha\beta} \neq 0$, then $b_\alpha = 1 - b_\beta$. For $\Lambda \in H_2(X, \mathbb{Z})$, let $\overline{\mathcal{M}}_{g,k}(X, \Lambda)$ be the moduli space of stable map with $[\overline{\mathcal{M}}_{g,k}(X, \Lambda)]^{vir}$ as its virtual fundamental class, and $\mathbb{L}_i$ the tautological line bundle over $\overline{\mathcal{M}}_{g,k}(X, \Lambda)$. Let $\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_k \in H^*(X, \mathbb{C})$, the genus-$g$ descendant Gromov-Witten invariants are defined by

$$\langle \tau_{n_1}(\widetilde{\gamma}_1) \cdots \tau_{n_k}(\widetilde{\gamma}_k) \rangle_g$$

$$= \sum_{\Lambda \in H_2(X, \mathbb{Z})} q^\Lambda \int_{[\overline{\mathcal{M}}_{g,k}(X, \Lambda)]^{vir}} \prod_{i=1}^k c_1(\mathbb{L}_1)^{n_i} \cup \mathbf{ev}_i^*(\widetilde{\gamma}_i), \tag{6}$$

where q^Λ is in Novikov ring with the product defined by $q^{\Lambda_1} q^{\Lambda_2} = q^{\Lambda_1 + \Lambda_2}$, and $\mathbf{ev}_i : \overline{\mathcal{M}}_{g,k}(X, \Lambda) \longrightarrow X$ is the i-th evaluation map $(C; x_1, \cdots, x_k; f) \mapsto f(x_i)$. One

can refer [2] for more details. As in [13], for any $\tau_n(\gamma_\alpha)$, one associates a parameter t_n^α, and call the space of all $T = \{t_n^\alpha : n \in \mathbb{Z}^{\geq 0}, \alpha = 1, \cdots, N\}$ the big phase space and its subspace spanned by $\{T | t_n^\alpha = 0 \text{ if } n > 0\}$ the small phase space. For simplicity, we identify $\tau_n(\gamma_\alpha)$ with tangent vector field $\frac{\partial}{\partial t_n^\alpha}$ on the big phase space, and on the small phase space, we write t_0^α as t^α and identify γ_α with $\frac{\partial}{\partial t^\alpha}$. If we restrict the invariant (6) on the small phase space, i.e., setting $n_i = 0$ for all $1 \leq i \leq k$, the resulting invariants are called primary Gromov-Witten invariants.

The generating function of genus-g Gromov-Witten invariants is defined as

$$F_g(T) := \sum_{k \geq 0} \frac{1}{k!} \sum_{n_1, \alpha_1, \cdots, n_k, \alpha_k} t_{n_1}^{\alpha_1} \cdots t_{n_k}^{\alpha_k} \langle \tau_{n_1}(\gamma_{\alpha_1}) \cdots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_g.$$

And the generating function for Gromov-Witten invariants involving all genera is defined to be

$$Z(T; \lambda) := \exp \sum_{g \geq 0} \lambda^{2g-2} F_g(T).$$

As in [17], we define k-point (correlation) function

$$\langle\langle \mathcal{W}_1 \mathcal{W}_2 \cdots \mathcal{W}_k \rangle\rangle_g := \sum_{n_1, \alpha_1, \cdots, n_k, \alpha_k} f_{n_1, \alpha_1}^1 \cdots f_{n_k, \alpha_k}^k \frac{\partial^k}{\partial t_{n_1}^{\alpha_1} \partial t_{n_2}^{\alpha_2} \cdots \partial t_{n_k}^{\alpha_k}} F_g$$

for vector fields $\mathcal{W}_i = \sum_{n, \alpha} f_{n, \alpha}^i \frac{\partial}{\partial t_n^\alpha}$. Let ∇ be the covariant derivative defined by

$$\nabla_\mathcal{W} \mathcal{V} = \sum_{n, \alpha} (\mathcal{W} f_{n, \alpha}) \tau_n(\gamma_\alpha) \tag{7}$$

for any vector fields $\mathcal{W}$ and $\mathcal{V} = \sum_{n, \alpha} f_{n, \alpha} \tau_n(\gamma_\alpha)$. It is simple to show that

$$[\mathcal{V}, \mathcal{W}] = \nabla_\mathcal{V} \mathcal{W} - \nabla_\mathcal{W} \mathcal{V} \tag{8}$$

and

$$\mathcal{W} \langle\langle \mathcal{V}_1 \cdots \mathcal{V}_k \rangle\rangle_g = \langle\langle \mathcal{W} \mathcal{V}_1 \cdots \mathcal{V}_k \rangle\rangle_g + \sum_{i=1}^k \langle\langle \mathcal{V}_1 \cdots (\nabla_\mathcal{W} \mathcal{V}_i) \cdots \mathcal{V}_k \rangle\rangle_g \tag{9}$$

for any vector fields $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{V}_i$.

Next, as in [17], we define the quantum product of any two vector fields $\mathcal{V}$ and $\mathcal{W}$ by

$$\mathcal{V} \circ \mathcal{W} = \sum_\alpha \langle\langle \mathcal{V} \mathcal{W} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha.$$

Obviously, this product is commutative by definition and associative by the following well known generalized WDVV equation

$$\langle\langle \{\mathcal{V}_1 \circ \mathcal{V}_2\} \mathcal{V}_3 \mathcal{V}_4 \rangle\rangle_0 = \langle\langle \{\mathcal{V}_1 \circ \mathcal{V}_3\} \mathcal{V}_2 \mathcal{V}_4 \rangle\rangle_0. \tag{10}$$

It follows easily from formula (9) that

$$\nabla_\mathcal{U} (\mathcal{V} \circ \mathcal{W}) = (\nabla_\mathcal{U} \mathcal{V}) \circ \mathcal{W} + \mathcal{V} \circ (\nabla_\mathcal{U} \mathcal{W}) + \sum_\alpha \langle\langle \mathcal{U} \mathcal{V} \mathcal{W} \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \tag{11}$$

for any vector fields $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$. Define

$$G(\mathcal{V}) := \sum_{n, \alpha} (n + b_\alpha) f_{n, \alpha} \tau_n(\gamma_\alpha) \tag{12}$$

for any vector field $\mathcal{V} = \sum\limits_{n,\alpha} f_{n,\alpha}\tau_n(\gamma_\alpha)$. We have the following easy observations

$$
\sum_\mu \langle\langle\{\mathcal{V}_1 \circ \gamma^\mu\}\{\mathcal{V}_2 \circ \mathcal{V}_3 \circ \gamma_\mu\}\mathcal{V}_4\cdots\mathcal{V}_k\rangle\rangle_g
$$
$$
= \sum_\mu \langle\langle\{\mathcal{V}_1 \circ \gamma_\mu\}\{\mathcal{V}_2 \circ \mathcal{V}_3 \circ \gamma^\mu\}\mathcal{V}_4\cdots\mathcal{V}_k\rangle\rangle_g
$$
$$
= \sum_\mu \langle\langle\{\mathcal{V}_1 \circ \mathcal{V}_2 \circ \gamma^\mu\}\{\mathcal{V}_3 \circ \gamma_\mu\}\mathcal{V}_4\cdots\mathcal{V}_k\rangle\rangle_g \tag{13}
$$

and

$$
\sum_\mu \langle\langle G(\mathcal{V}_1 \circ \mathcal{V}_2 \circ \gamma^\mu)\{\mathcal{V}_3 \circ \mathcal{V}_4 \circ \gamma_\mu\}\mathcal{V}_5\cdots\mathcal{V}_k\rangle\rangle_g
$$
$$
= \sum_\mu \langle\langle G(\mathcal{V}_1 \circ \mathcal{V}_2 \circ \gamma_\mu)\{\mathcal{V}_3 \circ \mathcal{V}_4 \circ \gamma^\mu\}\mathcal{V}_5\cdots\mathcal{V}_k\rangle\rangle_g
$$
$$
= \sum_\mu \langle\langle G(\mathcal{V}_1 \circ \mathcal{V}_2 \circ \mathcal{V}_3 \circ \gamma^\mu)\{\mathcal{V}_4 \circ \gamma_\mu\}\mathcal{V}_5\cdots\mathcal{V}_k\rangle\rangle_g
$$
$$
= \sum_\mu \langle\langle G(\mathcal{V}_1 \circ \gamma^\mu)\{\mathcal{V}_2 \circ \mathcal{V}_3 \circ \mathcal{V}_4 \circ \gamma_\mu\}\mathcal{V}_5\cdots\mathcal{V}_k\rangle\rangle_g \tag{14}
$$

for any vector fields $\mathcal{V}_i$ ($1 \le i \le k$).

We recollect the following Virasoro operators defined in [12]

$$
L_{-1}: \quad = \quad \sum_{m,\alpha} \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m-1}^\alpha} + \frac{1}{2\lambda^2}\sum_{\alpha,\beta}\eta_{\alpha\beta}t_0^\alpha t_0^\beta,
$$

$$
L_0: \quad = \quad \sum_{m,\alpha}(m+b_\alpha)\tilde{t}_m^\alpha \frac{\partial}{\partial t_m^\alpha} + \sum_{m,\alpha,\beta}\mathcal{C}_\alpha^\beta \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m-1}^\beta} + \frac{1}{2\lambda^2}\sum_{\alpha,\beta}\mathcal{C}_{\alpha\beta}t_0^\alpha t_0^\beta
$$
$$
+\frac{1}{24}\left(\frac{3-d}{2}\chi(X) - \int_X c_1(X)\cup c_{d-1}(X)\right),
$$

and for $n \ge 1$,

$$
L_n: \quad = \quad \sum_{m,\alpha,\beta}\sum_{j=0}^{m+n} A_\alpha^{(j)}(m,n)(\mathcal{C}^j)_\alpha^\beta \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m+n-j}^\beta}
$$
$$
+\frac{\lambda^2}{2}\sum_{\alpha,\beta,\gamma}\sum_{j=0}^{n-1}\sum_{k=0}^{n-j-1} B_\alpha^{(j)}(k,n)(\mathcal{C}^j)_\alpha^\beta \eta^{\alpha\gamma} \frac{\partial}{\partial t_k^\gamma}\frac{\partial}{\partial t_{n-k-1-j}^\beta}
$$
$$
+\frac{1}{2\lambda^2}\sum_{\alpha,\beta}(\mathcal{C}^{n+1})_{\alpha\beta}t_0^\alpha t_0^\beta,
$$

where

$$
A_\alpha^{(j)}(m,n) := \frac{\Gamma(b_\alpha+m+n+1)}{\Gamma(b_\alpha+m)}\sum_{m\le l_1<l_2<\cdots<l_j\le m+n}\left(\prod_{i=1}^{j}\frac{1}{b_\alpha+l_i}\right),
$$

$$
B_\alpha^{(j)}(m,n) := \frac{\Gamma(m+2-b_\alpha)\Gamma(n-m+b_\alpha)}{\Gamma(1-b_\alpha)\Gamma(b_\alpha)}\sum_{-m-1\le l_1<l_2<\cdots<l_j\le n-m-1}\left(\prod_{i=1}^{j}\frac{1}{b_\alpha+l_i}\right).
$$

These operators satisfy the following bracket relation

$$[L_m, L_n] = (m - n)L_{m+n}, \tag{15}$$

for $m, n \geq -1$. The Virasoro conjecture is

Conjecture 2.1. $L_n Z \equiv 0$ *(called the L_n-constraint) for all $n \geq -1$.*

It is well known that the L_{-1}-constraint and the L_0-constraint hold for all manifolds. In fact, the L_{-1}-constraint is equivalent to the string equation, and $L_0 Z = 0$ is true by Hori [9]. Assume that

$$L_n Z(T; \lambda) = \left\{ \sum_{g \geq 0} \Omega_{g,n} \lambda^{2g-2} \right\} Z(T; \lambda).$$

Then the L_n-constraint is equivalent to $\Omega_{g,n} = 0$ for all $g \geq 0$. As in [12], equation $\Omega_{g,n} = 0$ is called the genus-g L_n-constraint, and the so called genus-g Virasoro conjecture predicts that the genus-g L_n-constraint is true for all $n \geq -1$.

We will concentrate on the study of the genus-1 Virasoro conjecture. As mentioned in Introduction, it suffices to consider some simple equation on the small phase space. Therefore, in the rest of this paper, everything will be considered on the small phase space. Notice that all the above definitions and results hold when restricted on the small phase space. The notation $\langle\langle \cdots \rangle\rangle_g$ is used again to denote some correlation function on the small phase space which is written as $\langle\langle \cdots \rangle\rangle_{g,s}$ in [13]. We start with Euler vector field on the small phase space, which is defined by

$$E := c_1(X) + \sum_\alpha (b_1 + 1 - b_\alpha) t^\alpha \gamma_\alpha.$$

It is showed in [12] that the following quasi-homogeneity equation holds

$$\langle\langle E \rangle\rangle_g = (3 - d)(1 - g) F_g + \frac{1}{2} \delta_{g,0} \sum_{\alpha, \beta} \mathcal{C}_{\alpha\beta} t^\alpha t^\beta - \frac{1}{24} \delta_{g,1} \int_X c_1(X) \cup c_{d-1}(X).$$

And its derivatives are

$$\langle\langle E v_1 \cdots v_k \rangle\rangle_g$$

$$= \sum_{i=1}^k \langle\langle v_1 \cdots G(v_i) \cdots v_k \rangle\rangle_g - (2g + k - 2)(b_1 + 1) \langle\langle v_1 \cdots v_k \rangle\rangle_g$$

$$+ \delta_{g,0} \nabla^k_{v_1, \cdots, v_k} \left(\frac{1}{2} \sum_{\alpha, \beta} \mathcal{C}_{\alpha\beta} t^\alpha t^\beta \right) \tag{16}$$

for any vector fields v_i $(1 \leq i \leq k)$ on the small phase space. In particular, we have

$$\sum_\alpha \langle\langle E v_1 v_2 \gamma^\alpha \rangle\rangle_0 \gamma_\alpha = G(v_1) \circ v_2 + v_1 \circ G(v_2) - G(v_1 \circ v_2) - b_1 v_1 \circ v_2. \tag{17}$$

It is shown in [15] that

$$\nabla_v E = -G(v) + (b_1 + 1)v. \tag{18}$$

More generally, we have

Lemma 2.2. *For all $k \geq 0$ and any vector field v on the small phase space, let E^k be the kth quantum power of E, we have*

$$\nabla_v E^k = \sum_{i=1}^k G(E^{i-1}) \circ v \circ E^{k-i} - \sum_{i=1}^k G(v \circ E^{k-i}) \circ E^{i-1} + k v \circ E^{k-1}. \tag{19}$$

Proof. We will prove this lemma by induction on k. By the definition (7) and $E^0 = \gamma_1$, it holds for $k = 0$ and also for $k = 1$ due to formula (18). Assume equation (19) holds for $k \leq n$. For $k = n+1$, by equations (11) and (17), we have

$$
\begin{aligned}
&\nabla_v E^{n+1} \\
={} &\nabla_v(E \circ E^n) \\
={} &(\nabla_v E) \circ E^n + E \circ (\nabla_v E^n) + \sum_\alpha \langle\langle EE^n v \gamma^\alpha \rangle\rangle_0 \gamma_\alpha \\
={} &-G(v) \circ E^n + (b_1 + 1)v \circ E^n + \sum_{i=1}^n G(E^{i-1}) \circ v \circ E^{n-i+1} \\
&-\sum_{i=1}^n G(v \circ E^{n-i}) \circ E^i + nv \circ E^n + G(E^n) \circ v + G(v) \circ E^n \\
&-G(E^n \circ v) - b_1 E^n \circ v \\
={} &\sum_{i=1}^{n+1} G(E^{i-1}) \circ v \circ E^{n+1-i} - \sum_{i=1}^{n+1} G(v \circ E^{n+1-i}) \circ E^{i-1} + (n+1)v \circ E^n.
\end{aligned}
$$

The prooof is completed. $\qquad\square$

Then by equations (8) and (19), we have

Corollary 2.3 ([3, 8, 13]). *For $k, m \geq 0$,*

$$[E^k, E^m] = (m - k)E^{m+k-1}.$$

Secondly, functions Φ_k are defined in terms of genus-0 data in [13]. They are

$$
\begin{aligned}
\Phi_0 &= 0, \\
\Phi_1 &= -\frac{1}{24} \int_X c_1(X) \cup c_{d-1}(X), \\
\Phi_k &= -\frac{1}{24} \sum_{m=0}^{k-1} \sum_{\alpha,\beta,\sigma} b_\alpha \langle\langle \gamma_1 E^m \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha E^{k-1-m} \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta \gamma_\sigma \gamma^\sigma \rangle\rangle_0 \\
&\quad -\frac{1}{4} \sum_{m=0}^{k-1} \sum_{\alpha,\beta} b_\alpha b_\beta \langle\langle \gamma_\alpha E^m \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta E^{k-1-m} \gamma^\alpha \rangle\rangle_0 \\
&\quad +\frac{k}{12} \sum_\sigma \langle\langle \gamma_\sigma E^{k-1} \gamma^\sigma \rangle\rangle_0,
\end{aligned}
\tag{20}
$$

for $k \geq 2$. It follows from the definitions of quantum product and G that

$$
\begin{aligned}
24\Phi_k &= -\sum_{i=0}^{k-1} \langle\langle G(E^i)\Delta E^{k-i-1} \rangle\rangle_0 - k \sum_\mu \langle\langle E^{k-1} \gamma_\mu \gamma^\mu \rangle\rangle_0 \\
&\quad +6\sum_{i=0}^{k-1} \sum_\mu \langle\langle G(E^i \circ \gamma^\mu)G(\gamma_\mu)E^{k-i-1} \rangle\rangle_0,
\end{aligned}
\tag{21}
$$

where $\Delta = \sum_\alpha \gamma^\alpha \circ \gamma_\alpha$ and $k \geq 2$.

Remark 2.4. *It is easy to check that the expression* (20) *holds for $k = 0$. It also holds for $k = 1$ by the string equation* (22) *and the following equality* [1] *derived by*

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$$\sum_{\alpha} b_{\alpha}(1 - b_{\alpha}) - \frac{b_1 + 1}{6}\chi(X) = -\frac{1}{6}\int_X c_1(X) \cup c_{d-1}(X).$$

The same argument is applied in the proof of $\gamma_1\Phi_2 = 2\Phi_1$ in [13]. Therefore, we adopt equation (20) or (21) as the definition of Φ_k for all $k \geq 0$. And notice that it is shown in [13] that $\langle\langle E^k \rangle\rangle_1 = \Phi_k$ for $k = 0, 1$.

One important result is

Theorem 2.5 ([13]). *For any manifold X, the genus-1 Virasoro conjecture holds if and only if $\langle\langle E^2 \rangle\rangle_1 = \Phi_2$.*

It is shown in [14] that for any given $k \geq 2$, the genus-1 Virasoro conjecture holds if and only if $\langle\langle E^k \rangle\rangle_1 = \Phi_k$. Due to the equation $E(\langle\langle E^2 \rangle\rangle_1 - \Phi_2) = \langle\langle E^2 \rangle\rangle_1 - \Phi_2$ proved in [13], this conjecture is reduced to prove that $\upsilon(\langle\langle E^2 \rangle\rangle_1 - \Phi_2) = 0$ for any $\upsilon \in H^*(X, \mathbb{C})$. It is verified in [13] that $\gamma_1(\langle\langle E^2 \rangle\rangle_1 - \Phi_2) = 0$. And one more evidence was discovered in the following

Theorem 2.6 ([15]). *For all smooth projective varieties, we have*

$$\Delta(\langle\langle E^2 \rangle\rangle_1 - \Phi_2) = 0.$$

We will find more evidences in Section 4. By the fact $\gamma_1(\langle\langle E^k \rangle\rangle_1 - \Phi_k) = k(\langle\langle E^{k-1} \rangle\rangle_1 - \Phi_{k-1})$ proved in Lemma 6.3 of [13], it is easy to show that for any given $k \geq 2$, the genus-1 Virasoro conjecture holds if and only if $\upsilon(\langle\langle E^k \rangle\rangle_1 - \Phi_k) = 0$ for any $\upsilon \in H^*(X, \mathbb{C})$.

Remark 2.7. *By the above argument, to prove the genus-1 Virasoro conjecture, it is also reduced to verify that for any given $k, l \geq 2$, $\upsilon_l \upsilon_{l-1} \cdots \upsilon_1(\langle\langle E^k \rangle\rangle_1 - \Phi_k) = 0$ holds for any $\upsilon_1, \cdots, \upsilon_l \in H^*(X, \mathbb{C})$. Notice that the computation in Section 3 only yields the first derivative of $\{\langle\langle E^k \rangle\rangle_1 - \Phi_k\}$, hence to derive its l-th derivative, one may take derivatives of our results (cf. Remark 3.9) in section 3 or by applying the pull-back of Getzler's genus one relation to quantum powers of Euler vector field.*

2.2. **Universal equations for Gromov-Witten invariants.** In this subsection, we breifly recall some universal equations for primary Gromov-Witten invariants. Firstly, the string equation on the small phase space shows that

Lemma 2.8 ([13]).

$$\langle\langle \gamma_1 \gamma_{\alpha} \gamma_{\beta} \rangle\rangle_0 = \eta_{\alpha\beta}, \tag{22}$$

$$\langle\langle \gamma_1 \gamma_{\alpha_1} \cdots \gamma_{\alpha_k} \rangle\rangle_0 = 0 \text{ if } k \geq 3, \tag{23}$$

$$\langle\langle \gamma_1 \gamma_{\alpha_1} \cdots \gamma_{\alpha_k} \rangle\rangle_g = 0 \text{ if } g \geq 1 \text{ and } k \geq 0. \tag{24}$$

Next, the important universal equation for genus-0 primary Gromov-Witten invariants is the WDVV equation (10), which is implicitly used for computation throughout the paper. Its first derivative has the form

$$\langle\langle \{v_1 \circ v_2\}v_3 v_4 v_5 \rangle\rangle_0$$
$$= \langle\langle \{v_1 \circ v_3\}v_2 v_4 v_5 \rangle\rangle_0 + \langle\langle \{v_2 \circ v_5\}v_1 v_3 v_4 \rangle\rangle_0 - \langle\langle \{v_3 \circ v_5\}v_1 v_2 v_4 \rangle\rangle_0. \tag{25}$$

It is well known that equation (25) is the topological recursion relation derived from Keel relation [10] on the moduli space of stable curves $\overline{\mathcal{M}}_{0,5}$. And it implies

Lemma 2.9 ([13]). *For any vector field v on the small phase space, let v^k be the kth quantum power of v. Then for any α, β and μ, we have*

$$
\begin{aligned}
\langle\langle v^k \gamma_\alpha \gamma_\beta \gamma_\mu \rangle\rangle_0 \quad = \quad & -\sum_{i=1}^{k-1} \langle\langle v^{k-i}(\gamma_\alpha \circ \gamma_\beta \circ v^{i-1}) v \gamma_\mu \rangle\rangle_0 \\
& + \sum_{i=1}^{k} \langle\langle (v^{k-i} \circ \gamma_\alpha)(\gamma_\beta \circ v^{i-1}) v \gamma_\mu \rangle\rangle_0.
\end{aligned}
$$

It follows easily from Keel relation on $\overline{\mathcal{M}}_{0,6}$ that

$$
\begin{aligned}
& \langle\langle \{v_1 \circ v_2\} v_3 v_4 v_5 v_6 \rangle\rangle_0 \\
= \quad & \sum_\rho \langle\langle v_1 v_3 v_6 \gamma^\rho \rangle\rangle_0 \langle\langle \gamma_\rho v_2 v_4 v_5 \rangle\rangle_0 + \sum_\rho \langle\langle v_1 v_3 v_5 \gamma^\rho \rangle\rangle_0 \langle\langle \gamma_\rho v_2 v_4 v_6 \rangle\rangle_0 \\
& + \langle\langle \{v_2 \circ v_4\} v_1 v_3 v_5 v_6 \rangle\rangle_0 + \langle\langle \{v_1 \circ v_3\} v_2 v_4 v_5 v_6 \rangle\rangle_0 \\
& - \sum_\rho \langle\langle v_1 v_2 v_6 \gamma^\rho \rangle\rangle_0 \langle\langle \gamma_\rho v_3 v_4 v_5 \rangle\rangle_0 - \sum_\rho \langle\langle v_1 v_2 v_5 \gamma^\rho \rangle\rangle_0 \langle\langle \gamma_\rho v_3 v_4 v_6 \rangle\rangle_0 \\
& - \langle\langle \{v_3 \circ v_4\} v_1 v_2 v_5 v_6 \rangle\rangle_0, \tag{26}
\end{aligned}
$$

which is also derived from the second derivative of the WDVV equation (10). We will use the following lemma for the computation in Section 3.

Lemma 2.10. *For any vector field v on the small phase space, let v^k be the kth quantum power of v. Then for any α, β, μ, and σ, we have*

$$
\begin{aligned}
& \langle\langle v^k \gamma_\alpha \gamma_\beta \gamma_\mu \gamma_\sigma \rangle\rangle_0 \\
= \quad & -\sum_{i=1}^{k-1} \sum_\rho \langle\langle v^{k-i} \gamma_\sigma v \gamma^\rho \rangle\rangle_0 \langle\langle \gamma_\rho \gamma_\alpha \{\gamma_\beta \circ v^{i-1}\} \gamma_\mu \rangle\rangle_0 \\
& - \sum_{i=1}^{k-1} \langle\langle v^{k-i} \{\gamma_\alpha \circ \gamma_\beta \circ v^{i-1}\} v \gamma_\mu \gamma_\sigma \rangle\rangle_0 \\
& - \sum_{i=1}^{k-1} \sum_\rho \langle\langle v^{k-i} \gamma_\mu v \gamma^\rho \rangle\rangle_0 \langle\langle \gamma_\rho \gamma_\alpha \{\gamma_\beta \circ v^{i-1}\} \gamma_\sigma \rangle\rangle_0 \\
& + \sum_{i=1}^{k-1} \sum_\rho \langle\langle v^{k-i} \gamma_\alpha \gamma_\sigma \gamma^\rho \rangle\rangle_0 \langle\langle \gamma_\rho \{\gamma_\beta \circ v^{i-1}\} v \gamma_\mu \rangle\rangle_0 \\
& + \sum_{i=1}^{k-1} \sum_\rho \langle\langle v^{k-i} \gamma_\alpha \gamma_\mu \gamma^\rho \rangle\rangle_0 \langle\langle \gamma_\rho \{\gamma_\beta \circ v^{i-1}\} v \gamma_\sigma \rangle\rangle_0 \\
& + \sum_{i=1}^{k} \langle\langle \{v^{k-i} \circ \gamma_\alpha\} \{\gamma_\beta \circ v^{i-1}\} v \gamma_\mu \gamma_\sigma \rangle\rangle_0
\end{aligned}
$$

Proof. As in the proof of Lemma 4.2 in [13], the proof follows by choosing $v_1 = v^{k-1}$, $v_2 = v$, $v_3 = \gamma_\alpha$, $v_4 = \gamma_\beta$, $v_5 = \gamma_\mu$, and $v_6 = \gamma_\sigma$ in equation (26) and repeatedly using the resulting formula. $\qquad\square$

The essential universal equation for genus-1 primary Gromov-Witten invariants is derived from Getzler's genus one relation [7]. Adopting the presentation in [13],

it has the following form (we call it Getzler's genus one relation again)

$$\mathbf{G}(v_1, v_2, v_3, v_4) = G_0(v_1, v_2, v_3, v_4) + G_1(v_1, v_2, v_3, v_4) = 0, \qquad (27)$$

where

$$G_0(v_1, v_2, v_3, v_4)$$
$$= \sum_{h \in S_4} \sum_{\alpha, \beta} \left\{ \frac{1}{6} \langle\langle v_{h(1)} v_{h(2)} v_{h(3)} \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha v_{h(4)} \gamma_\beta \gamma^\beta \rangle\rangle_0 \right.$$
$$+ \frac{1}{24} \langle\langle v_{h(1)} v_{h(2)} v_{h(3)} v_{h(4)} \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \gamma^\beta \rangle\rangle_0$$
$$\left. - \frac{1}{4} \langle\langle v_{h(1)} v_{h(2)} \gamma^\alpha \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta v_{h(3)} v_{h(4)} \rangle\rangle_0 \right\},$$

and

$$G_1(v_1, v_2, v_3, v_4)$$
$$= 3 \sum_{h \in S_4} \langle\langle \{v_{h(1)} \circ v_{h(2)}\} \{v_{h(3)} \circ v_{h(4)}\} \rangle\rangle_1$$
$$- 4 \sum_{h \in S_4} \langle\langle \{v_{h(1)} \circ v_{h(2)} \circ v_{h(3)}\} v_{h(4)} \rangle\rangle_1$$
$$- \sum_{h \in S_4} \sum_{\alpha} \langle\langle \{v_{h(1)} \circ v_{h(2)}\} v_{h(3)} v_{h(4)} \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_1$$
$$+ 2 \sum_{h \in S_4} \sum_{\alpha} \langle\langle v_{h(1)} v_{h(2)} v_{h(3)} \gamma^\alpha \rangle\rangle_0 \langle\langle \{\gamma_\alpha \circ v_{h(4)}\} \rangle\rangle_1,$$

for any vector fields v_1, v_2, v_3, v_4 on the small phase space.

Remark 2.11. *By string equations* (23) *and* (24), *we have* $G_1(\gamma_1, v_1, v_2, v_3) \equiv 0$ *and* $G_0(\gamma_1, v_1, v_2, v_3) \equiv 0$ *for any vector fields* v_1, v_2, v_3 *on the small phase space.*

For simplicity, we will use the following notational conventions: Repeated indices are summed over their entire meaningful range unless otherwise indicated.

3. Universal equations from Getzler's genus one relation

In this section, we will derive an explicit formula for the first derivative of Φ_k, and then obtain an explicit universal equation, i.e., Theorem 3.8, by computing Getzler's genus one relation (27) when one of vector fields is substituted by quantum power of Euler vector field. With this theorem, we can derive other explicit universal equations for the cases involving more quantum powers of Euler vector field. These explicit universal equations from Getzler's genus one relation will be called Getzler equations. To see how much information of Getzler equations can be applied in the genus-1 Virasoro conjecture, we always reduce higher k-point correlation functions involving some quantum power of Euler vector field to lower l-point correlation functions, and use the first derivative of Φ_k for possible simplification.

3.1. An explicit formula for the first derivative of Φ_k. The following lemma is useful for reducing 4-point functions to 3-point functions.

Lemma 3.1. *For all* $m \geq 0$,

$$\langle\langle E^m \gamma_\alpha \gamma_\beta \gamma_\mu \rangle\rangle_0$$

$$= -\sum_{i=1}^{m} \langle\langle G(E^{m-i})\{E^{i-1} \circ \gamma_\alpha \circ \gamma_\beta\}\gamma_\mu\rangle\rangle_0 - \sum_{i=1}^{m} \langle\langle G(E^{i-1} \circ \gamma_\alpha \circ \gamma_\beta)E^{m-i}\gamma_\mu\rangle\rangle_0$$

$$+ \sum_{i=1}^{m} \langle\langle G(E^{m-i} \circ \gamma_\alpha)\{E^{i-1} \circ \gamma_\beta\}\gamma_\mu\rangle\rangle_0 + \sum_{i=1}^{m} \langle\langle G(E^{i-1} \circ \gamma_\beta)\{E^{m-i} \circ \gamma_\alpha\}\gamma_\mu\rangle\rangle_0.$$

Proof. Using Lemma 2.9, equations (16) and (10), we have

$$\langle\langle E^m \gamma_\alpha \gamma_\beta \gamma_\mu \rangle\rangle_0$$

$$= -\sum_{i=1}^{m-1} \langle\langle G(E^{m-i})\{E^{i-1} \circ \gamma_\alpha \circ \gamma_\beta\}\gamma_\mu\rangle\rangle_0 - \sum_{i=1}^{m-1} \langle\langle G(E^{i-1} \circ \gamma_\alpha \circ \gamma_\beta)E^{m-i}\gamma_\mu\rangle\rangle_0$$

$$+ \sum_{i=1}^{m} \langle\langle G(E^{m-i} \circ \gamma_\alpha)\{E^{i-1} \circ \gamma_\beta\}\gamma_\mu\rangle\rangle_0 + \sum_{i=1}^{m} \langle\langle G(E^{i-1} \circ \gamma_\beta)\{E^{m-i} \circ \gamma_\alpha\}\gamma_\mu\rangle\rangle_0$$

$$- \sum_{i=1}^{m-1} \langle\langle E^{m-1}\{\gamma_\alpha \circ \gamma_\beta\}G(\gamma_\mu)\rangle\rangle_0 + \sum_{i=1}^{m} \langle\langle E^{m-1}\{\gamma_\alpha \circ \gamma_\beta\}G(\gamma_\mu)\rangle\rangle_0$$

$$+ (b_1 + 1)\left\{ \sum_{i=1}^{m-1} \langle\langle E^{m-1}\{\gamma_\alpha \circ \gamma_\beta\}\gamma_\mu\rangle\rangle_0 - \sum_{i=1}^{m} \langle\langle E^{m-1}\{\gamma_\alpha \circ \gamma_\beta\}\gamma_\mu\rangle\rangle_0 \right\}.$$

The proof follows by using the following equation (29). □

In particular, it follows from Lemma 3.1 that

$$\langle\langle E^m \gamma_\alpha \gamma_\beta \gamma^\mu \rangle\rangle_0 \gamma_\mu$$

$$= -\sum_{i=1}^{m} G(E^{m-i}) \circ E^{i-1} \circ \gamma_\alpha \circ \gamma_\beta - \sum_{i=1}^{m} G(E^{i-1} \circ \gamma_\alpha \circ \gamma_\beta) \circ E^{m-i}$$

$$+ \sum_{i=1}^{m} G(E^{m-i} \circ \gamma_\alpha) \circ E^{i-1} \circ \gamma_\beta + \sum_{i=1}^{m} G(E^{i-1} \circ \gamma_\beta) \circ E^{m-i} \circ \gamma_\alpha. \quad (28)$$

The following lemma is useful for simplification.

Lemma 3.2. *For any vector field v_i ($1 \le i \le 4$) on the small phase space, we have*

$$\langle\langle G(v_1 \circ v_2)v_3 v_4 \rangle\rangle_0 = \langle\langle v_1 v_2\{v_3 \circ v_4\}\rangle\rangle_0 - \langle\langle v_1 v_2 G(v_3 \circ v_4)\rangle\rangle_0, \quad (29)$$

$$G(v_1 \circ \gamma^\mu) \circ \gamma_\mu = G(v_1 \circ \gamma_\mu) \circ \gamma^\mu = \frac{1}{2}\Delta \circ v_1, \quad (30)$$

$$\langle\langle G(v_1 \circ v_2 \circ v_3 \circ \gamma^\mu)G(\gamma_\mu)v_4 \rangle\rangle_0 = \langle\langle\{G(v_4 \circ \gamma^\mu) \circ G(\gamma_\mu) \circ v_1\}v_2 v_3 \rangle\rangle_0, \quad (31)$$

$$\langle\langle\{G(v_1 \circ \gamma^\mu) \circ G(v_2 \circ \gamma_\mu)\}v_3 v_4 \rangle\rangle_0 = \langle\langle\{G(v_1 \circ v_2 \circ \gamma^\mu) \circ G(\gamma_\mu)\}v_3 v_4 \rangle\rangle_0. \quad (32)$$

Proof. As $\eta_{\alpha\beta} \ne 0$ requires that $b_\alpha = 1 - b_\beta$,

$$\langle\langle G(v_1 \circ v_2)v_3 v_4 \rangle\rangle_0 = b_\alpha \langle\langle v_1 v_2 \gamma^\alpha \rangle\rangle \langle\langle \gamma_\alpha v_3 v_4 \rangle\rangle_0$$

$$= b_\alpha \eta_{\alpha\beta} \langle\langle v_1 v_2 \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma^\beta v_3 v_4 \rangle\rangle_0$$

$$= (1 - b_\beta)\eta_{\alpha\beta} \langle\langle v_1 v_2 \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma^\beta v_3 v_4 \rangle\rangle_0$$

$$= (1 - b_\beta)\langle\langle v_1 v_2 \gamma_\beta \rangle\rangle_0 \langle\langle \gamma^\beta v_3 v_4 \rangle\rangle_0.$$

The equation (29) follows. By the same argument, the equation (30) also holds. The equation (31) follows from equations (29) and (30). The last equation can be proved as follows

$$\langle\langle\{G(v_1 \circ \gamma^\mu) \circ G(v_2 \circ \gamma_\mu)\}v_3 v_4 \rangle\rangle_0$$

$$
\begin{aligned}
&= b_\alpha b_\beta \langle\!\langle v_1 \gamma^\mu \gamma^\alpha \rangle\!\rangle_0 \langle\!\langle v_2 \gamma_\mu \gamma^\beta \rangle\!\rangle_0 \langle\!\langle \{\gamma_\alpha \circ \gamma_\beta\} v_3 v_4 \rangle\!\rangle_0 \\
&= b_\alpha b_\beta \langle\!\langle v_2 \{v_1 \circ \gamma^\alpha\} \gamma^\beta \rangle\!\rangle_0 \langle\!\langle \{\gamma_\alpha \circ \gamma_\beta\} v_3 v_4 \rangle\!\rangle_0 \\
&= \langle\!\langle \{G(v_1 \circ v_2 \circ \gamma^\mu) \circ G(\gamma_\mu)\} v_3 v_4 \rangle\!\rangle_0.
\end{aligned}
$$

$\square$

Now, an explicit formula for the first derivative of Φ_k is presented below.

Theorem 3.3. *For all $k \geq 0$, and any α,*

$$
\begin{aligned}
24 \gamma_\alpha \Phi_k \;=\; & -\sum_{i=1}^{k-1} \langle\!\langle G(E^i) \{E^{k-i-1} \circ \gamma_\alpha\} \gamma_\mu \gamma^\mu \rangle\!\rangle_0 \\
& -\sum_{i=0}^{k-2} (k-i-1) \langle\!\langle \{G(E^i) \circ E^{k-i-2}\} \Delta \gamma_\alpha \rangle\!\rangle_0 \\
& +\sum_{i=0}^{k-2} (-k+2i+2) \langle\!\langle G(\Delta \circ E^{k-i-2}) E^i \gamma_\alpha \rangle\!\rangle_0 \\
& -\sum_{i=1}^{k-1} \sum_{j=1}^{i} \langle\!\langle G(E^{k-i-1} \circ \Delta) G(E^{i-j} \circ \gamma_\alpha) E^{j-1} \rangle\!\rangle_0 \\
& -\sum_{i=1}^{k-1} \sum_{j=1}^{i} \langle\!\langle \{G(E^{k-i-1}) \circ \Delta\} G(E^{i-j} \circ \gamma_\alpha) E^{j-1} \rangle\!\rangle_0 \\
& +12 \sum_{i=1}^{k-1} i \langle\!\langle \{G(E^{k-i-1} \circ \gamma^\mu) \circ G(\gamma_\mu)\} E^{i-1} \gamma_\alpha \rangle\!\rangle_0 \\
& -k(k-1) \langle\!\langle \Delta E^{k-2} \gamma_\alpha \rangle\!\rangle_0.
\end{aligned}
$$

Proof. It is trivial for $k = 0, 1$. It remains to prove this theorem when $k \geq 2$. According to equation (20), using equations (9) and (23), we have

$$
\begin{aligned}
& 24 \gamma_\alpha \Phi_k \\
=\; & -\sum_{i=0}^{k-1} \langle\!\langle G(\nabla_{\gamma_\alpha} E^i) E^{k-i-1} \Delta \rangle\!\rangle_0 - \sum_{i=0}^{k-1} \langle\!\langle E^{k-i-1} G(E^i) \Delta \gamma_\alpha \rangle\!\rangle_0 \\
& -\sum_{i=0}^{k-1} \langle\!\langle G(E^i) \{\nabla_{\gamma_\alpha} E^{k-i-1}\} \Delta \rangle\!\rangle_0 - \sum_{i=0}^{k-1} \langle\!\langle \{G(E^i) \circ E^{k-i-1}\} \gamma_\alpha \gamma_\sigma \gamma^\sigma \rangle\!\rangle_0 \\
& -12 \sum_{i=0}^{k-1} b_\mu b_\beta \langle\!\langle E^i \gamma_\alpha \gamma_\mu \gamma^\beta \rangle\!\rangle_0 \langle\!\langle \gamma_\beta E^{k-i-1} \gamma^\mu \rangle\!\rangle_0 \\
& -12 \sum_{i=0}^{k-1} b_\mu b_\beta \langle\!\langle \{\nabla_{\gamma_\alpha} E^i\} \gamma_\mu \gamma^\beta \rangle\!\rangle_0 \langle\!\langle \gamma_\beta E^{k-i-1} \gamma^\mu \rangle\!\rangle_0 \\
& +2k \langle\!\langle E^{k-1} \gamma_\sigma \gamma^\sigma \gamma_\alpha \rangle\!\rangle_0 + 2k \langle\!\langle \{\nabla_{\gamma_\alpha} E^{k-1}\} \gamma_\sigma \gamma^\sigma \rangle\!\rangle_0. \qquad (33)
\end{aligned}
$$

By equation (25), we have

$$
\begin{aligned}
& \langle\!\langle \{G(E^i) \circ E^{k-i-1}\} \gamma_\alpha \gamma_\sigma \gamma^\sigma \rangle\!\rangle_0 \\
=\; & \langle\!\langle \{E^{k-i-1} \circ \gamma_\alpha\} \gamma_\sigma \gamma^\sigma G(E^i) \rangle\!\rangle_0 + \langle\!\langle E^{k-i-1} \gamma_\sigma \gamma_\alpha \{G(E^i) \circ \gamma^\sigma\} \rangle\!\rangle_0
\end{aligned}
$$

$$-\langle\langle E^{k-i-1}\{\gamma_\alpha \circ \gamma^\sigma\}\gamma_\sigma G(E^i)\rangle\rangle_0. \tag{34}$$

Notice that

$$b_\mu b_\beta \langle\langle E^i \gamma_\alpha \gamma_\mu \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta E^{k-i-1}\gamma^\mu \rangle\rangle_0$$
$$= b_\mu(1-b_\beta)\langle\langle E^i \gamma_\alpha \gamma_\mu \gamma_\beta \rangle\rangle_0 \langle\langle \gamma^\beta E^{k-i-1}\gamma^\mu \rangle\rangle_0$$
$$= \langle\langle E^i G(\gamma_\mu)\{E^{k-i-1}\circ\gamma^\mu\}\gamma_\alpha\rangle\rangle_0 - \langle\langle E^i G(\gamma_\mu)G(E^{k-i-1}\circ\gamma^\mu)\gamma_\alpha\rangle\rangle_0 \tag{35}$$

and

$$b_\mu b_\beta \langle\langle \{\nabla_{\gamma_\alpha} E^i\}\gamma_\mu \gamma^\beta \rangle\rangle_0 \langle\langle \gamma_\beta E^{k-i-1}\gamma^\mu \rangle\rangle_0$$
$$= [1-(1-b_\mu)]\langle\langle G((\nabla_{\gamma_\alpha}E^i)\circ\gamma_\mu)E^{k-i-1}\gamma^\mu\rangle\rangle_0$$
$$= \langle\langle G((\nabla_{\gamma_\alpha}E^i)\circ\gamma_\mu)E^{k-i-1}\gamma^\mu\rangle\rangle_0 - \langle\langle G((\nabla_{\gamma_\alpha}E^i)\circ\gamma_\mu)E^{k-i-1}G(\gamma^\mu)\rangle\rangle_0 \tag{36}$$

The proof follows by firstly substituting equations (34), (35) and (36) into the equality (33), and secondly applying Lemma 3.1 and formula (19) to the resulting equality, and then using Lemma 3.2 for simplification. $\qquad\square$

Therefore, we have an alternative direct proof of

Corollary 3.4 ([13]). *For any smooth projective variety X, we have*

$$E^k\Phi_m - E^m\Phi_k = (m-k)\Phi_{k+m-1}.$$

Proof. We only consider the cases for $k+m \geq 2$, since other cases are trivial. Using Theorem 3.3, Lemma 3.1 and Lemma 3.2, we have

$$24E^m\Phi_k$$
$$= \sum_{i=1}^{k}\sum_{j=1}^{m}\langle\langle G(E^{i-1})G(E^{j-1})\{\Delta\circ E^{k+m-i-j}\}\rangle\rangle_0$$
$$-b_1\sum_{i=1}^{k+m-1}\langle\langle G(E^{i-1})\Delta E^{k+m-i-1}\rangle\rangle_0$$
$$+\sum_{i=1}^{k-1}\sum_{j=1}^{k+m-i-1}\langle\langle G(E^i)G(\Delta\circ E^{j-1})E^{k+m-i-j-1}\rangle\rangle_0$$
$$-\sum_{i=m}^{k+m-2}\sum_{j=1}^{k+m-i-1}\langle\langle G(E^i)G(\Delta\circ E^{j-1})E^{k+m-i-j-1}\rangle\rangle_0$$
$$-\sum_{i=1}^{k-1}(2k+m-2i-2)\langle\langle G(E^i)\Delta E^{k+m-i-2}\rangle\rangle_0$$
$$+\sum_{i=m}^{k+m-2}(2m+k-2i-2)\langle\langle G(E^i)\Delta E^{k+m-i-2}\rangle\rangle_0$$
$$+6\sum_{i=1}^{k-1}(k-i)\langle\langle G(E^{i-1}\circ\gamma^\mu)G(\gamma_\mu)E^{k+m-i-1}\rangle\rangle_0$$
$$-6\sum_{i=m+1}^{k+m-1}(m-i)\langle\langle G(E^{i-1}\circ\gamma^\mu)G(\gamma_\mu)E^{k+m-i-1}\rangle\rangle_0$$
$$-(k-1)(k+b_1\cdot\delta_{k\geq2})\langle\langle E^{m+k-2}\gamma^\mu\gamma_\mu\rangle\rangle_0.$$

where if $k \geq 2$, then $\delta_{k \geq 2} = 1$, otherwise $\delta_{k \geq 2} = 0$. Then it follows that

$$24E^m \Phi_k - 24E^k \Phi_m$$
$$= \left\{ -\sum_{i=0}^{k+m-2} \langle\langle G(E^i) \Delta E^{k+m-i-2} \rangle\rangle_0 - (k+m-1)\langle\langle E^{m+k-2} \gamma^\mu \gamma_\mu \rangle\rangle_0 \right.$$
$$\left. +6 \sum_{i=1}^{k+m-1} \langle\langle G(E^{i-1} \circ \gamma^\mu) G(\gamma_\mu) E^{k+m-i-1} \rangle\rangle_0 \right\} \times (k-m).$$

The proof is completed by equation (21). $\qquad\square$

3.2. Getzler equation $\mathbf{G}(E^k, \gamma_\alpha, \gamma_\beta, \gamma_\sigma) = 0$.

We start with the computation of the function G_1.

Lemma 3.5. *For any* α, β, σ,

$$G_1(E^k, \gamma_\alpha, \gamma_\beta, \gamma_\sigma)$$
$$= -24\{\gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}\langle\langle E^k \rangle\rangle_1 + 24k\langle\langle E^{k-1} \circ \gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma \rangle\rangle_1$$
$$+12 \sum_{g \in S_3} \langle\langle \{E^k \circ \gamma_{\varsigma_{g(1)}}\}\{\gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_1 - 12 \sum_{g \in S_3} \langle\langle \{E^k \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\rangle\rangle_1$$
$$+12 \sum_{g \in S_3} \sum_{i=1}^{k} \langle\langle (G(E^{i-1} \circ \gamma_{\varsigma_{g(1)}}) \circ E^{k-i} \circ \gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}} \rangle\rangle_1$$
$$-12 \sum_{g \in S_3} \sum_{i=1}^{k} \langle\langle (G(E^{i-1} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}) \circ E^{k-i} \circ \gamma_{\varsigma_{g(3)}} \rangle\rangle_1$$

where $k \geq 0$ *and* $\{\varsigma_1, \varsigma_2, \varsigma_3\} = \{\alpha, \beta, \sigma\}$.

Proof. By the definition of G_1, we have

$$G_1(E^k, \gamma_\alpha, \gamma_\beta, \gamma_\sigma)$$
$$= 12 \sum_{g \in S_3} \langle\langle \{E^k \circ \gamma_{\varsigma_{g(1)}}\}\{\gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_1 - 12 \sum_{g \in S_3} \langle\langle \{E^k \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\rangle\rangle_1$$
$$-24\langle\langle \{\gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}E^k \rangle\rangle_1 - 2 \sum_{g \in S_3} \langle\langle \{E^k \circ \gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(2)}} \gamma_{\varsigma_{g(3)}} \gamma^\mu \rangle\rangle_0 \langle\langle \gamma_\mu \rangle\rangle_1$$
$$-2 \sum_{g \in S_3} \langle\langle E^k \{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}} \gamma^\mu \rangle\rangle_0 \langle\langle \gamma_\mu \rangle\rangle_1$$
$$+6 \sum_{g \in S_3} \langle\langle E^k \gamma_{\varsigma_{g(1)}} \gamma_{\varsigma_{g(2)}} \gamma^\mu \rangle\rangle_0 \langle\langle \gamma_\mu \circ \gamma_{\varsigma_{g(3)}} \rangle\rangle_1$$
$$+12\langle\langle \gamma_\alpha \gamma_\beta \gamma_\sigma \gamma^\mu \rangle\rangle_0 \langle\langle \gamma_\mu \circ E^k \rangle\rangle_1. \tag{37}$$

Firstly, it follows from equations (9) and (19) that

$$\langle\langle \{\gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}E^k \rangle\rangle_1$$
$$= \{\gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}\langle\langle E^k \rangle\rangle_1 - \sum_{i=0}^{k-1} \langle\langle G(E^i) \circ E^{k-i-1} \circ \gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma \rangle\rangle_1$$
$$+\sum_{i=1}^{k} \langle\langle (G(E^{k-i} \circ \gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma) \circ E^{i-1} \rangle\rangle_1 - k\langle\langle E^{k-1} \circ \gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma \rangle\rangle_1. \tag{38}$$

Using equation (25), we have

$$\langle\langle\gamma_\alpha\gamma_\beta\gamma_\sigma\gamma^\mu\rangle\rangle_0\langle\langle\gamma_\mu\circ E^k\rangle\rangle_1$$
$$= \langle\langle\{E^k\circ\gamma^\mu\}\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0\langle\langle\gamma_\mu\rangle\rangle_1$$
$$= \langle\langle E^k\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\{\gamma^\mu\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0\langle\langle\gamma_\mu\rangle\rangle_1 + \langle\langle\{E^k\circ\gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma^\mu\rangle\rangle_0\langle\langle\gamma_\mu\rangle\rangle_1$$
$$-\langle\langle\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}E^k\gamma_{\varsigma_{g(2)}}\gamma^\mu\rangle\rangle_0\langle\langle\gamma_\mu\rangle\rangle_1$$

for any $g\in S_3$. Hence it is easy to show that

$$12\langle\langle\gamma_\alpha\gamma_\beta\gamma_\sigma\gamma^\mu\rangle\rangle_0\langle\langle\gamma_\mu\circ E^k\rangle\rangle_1 - 2\sum_{g\in S_3}\langle\langle\{E^k\circ\gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma^\mu\rangle\rangle_0\langle\langle\gamma_\mu\rangle\rangle_1$$
$$= 2\sum_{g\in S_3}\langle\langle E^k\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\gamma^\mu\rangle\rangle_0\langle\langle\gamma_\mu\circ\gamma_{\varsigma_{g(3)}}\rangle\rangle_1$$
$$-2\sum_{g\in S_3}\langle\langle E^k\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\gamma^\mu\rangle\rangle_0\langle\langle\gamma_\mu\rangle\rangle_1. \tag{39}$$

By substituting equations (38) and (39) into equation (37), the proof is completed by using Lemma 3.1. $\square$

Remark 3.6. *Although it is simple to show that there are other equivalent expressions for $G_1(E^k,\gamma_\alpha,\gamma_\beta,\gamma_\sigma)$ by equations (9) and (19), for example,*

$$G_1(E^k,\gamma_\alpha,\gamma_\beta,\gamma_\sigma)$$
$$= -24\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\langle\langle E^k\rangle\rangle_1 + 24k\langle\langle E^{k-1}\circ\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\rangle\rangle_1$$
$$+12\sum_{g\in S_3}\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\langle\langle E^k\circ\gamma_{\varsigma_{g(3)}}\rangle\rangle_1 - 12\sum_{g\in S_3}\gamma_{\varsigma_{g(1)}}\langle\langle E^k\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\rangle\rangle_1$$
$$+72\langle\langle\gamma_\alpha\gamma_\beta\gamma_\sigma\gamma^\mu\rangle\rangle_0\langle\langle\gamma_\mu\circ E^k\rangle\rangle_1, \tag{40}$$

we adopt the expression of G_1 in Lemma 3.5 which is useful for later applications in Section 3 and 4. Notice that equation (40) can not be used to derive $G_1(E^k,\gamma_\alpha,\gamma_\beta,E^m)$ by substituting E^m into the position of γ_σ directly.

By the definition of G_0, we have for any α,β,σ,

$$G_0(E^k,\gamma_\alpha,\gamma_\beta,\gamma_\sigma)$$
$$= \langle\langle E^k\gamma_\alpha\gamma_\beta\gamma_\sigma\Delta\rangle\rangle_0 + \frac{1}{2}\sum_{g\in S_3}\langle\langle E^k\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\gamma^\mu\rangle\rangle_0\langle\langle\gamma_\mu\gamma_{\varsigma_{g(3)}}\gamma_\rho\gamma^\rho\rangle\rangle_0$$
$$-\sum_{g\in S_3}\langle\langle E^k\gamma_{\varsigma_{g(1)}}\gamma^\mu\gamma^\rho\rangle\rangle_0\langle\langle\gamma_\mu\gamma_\rho\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$
$$+\langle\langle E^k\gamma_\rho\gamma^\rho\gamma^\mu\rangle\rangle_0\langle\langle\gamma_\mu\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0, \tag{41}$$

where $k\geq 0$ and $\{\varsigma_1,\varsigma_2,\varsigma_3\}=\{\alpha,\beta,\sigma\}$. Since the computation is quite involved, we put some intermediate results in Appendix A. With these preparation, we have the following explicit formula.

Lemma 3.7. *For any α,β,σ,*

$$G_0(E^k,\gamma_\alpha,\gamma_\beta,\gamma_\sigma)$$
$$= 24\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\Phi_k - 4k\langle\langle\{\Delta\circ E^{k-1}\}\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0$$

$$+\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\gamma_\mu\gamma^\mu\{E^{i-1}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\gamma_\mu\gamma^\mu\{E^{i-1}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$+2k\sum_{g\in S_3}\langle\langle G(E^{k-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^\mu)\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_\mu\rangle\rangle_0$$

$$-\frac{1}{2}k\sum_{g\in S_3}\langle\langle\{E^{k-1}\circ\gamma_{\varsigma_{g(1)}}\}\gamma_\mu\gamma^\mu\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$+2\sum_{g\in S_3}\sum_{i=1}^{k-1}(k-3i)\langle\langle\{G(E^{k-i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^\mu)\circ G(\gamma_\mu)\}E^{i-1}\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k-1}i\langle\langle\{\Delta\circ E^{i-1}\}G(E^{k-i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$

$$-\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k-1}(k-i)\langle\langle\{\Delta\circ E^{i-1}\}G(E^{k-i-1}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-4k\sum_{i=1}^{k-1}\langle\langle G(\Delta\circ E^{k-i-1})E^{i-1}\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\rangle_0$$

$$+5k(k-1)\langle\langle\Delta E^{k-2}\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\rangle_0,$$

where $k\geq 0$ and $\{\varsigma_1,\varsigma_2,\varsigma_3\}=\{\alpha,\beta,\sigma\}$.

Proof. Firstly, by substituting Lemmas A.5, A.6, A.7, A.8, A.9 into Lemma A.1, and substituting Lemma A.11 into Lemma A.2, and combining with Lemmas A.3 and A.4, we obtain an expression for the function G_0 by equation (41). Then subsituting Lemma A.10 to the resulting expression, and using Theorem 3.3 and Lemma 3.2 for simplification, a tedious computation shows that

$$G_0(E^k,\gamma_\alpha,\gamma_\beta,\gamma_\sigma)$$
$$=\quad 24\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\Phi_k-(b_1+k)\langle\langle\{\Delta\circ E^{k-1}\}\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0$$

$$-2\sum_{i=1}^{k}\langle\langle\{G(E^{k-i})\circ E^{i-1}\}\gamma_\mu\gamma^\mu\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\rangle_0$$

$$-\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\gamma_\mu\gamma^\mu\{E^{i-1}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$+\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ\gamma_{\varsigma_{g(2)}}\}\gamma_\mu\gamma^\mu\{E^{i-1}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma^\mu\}\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\gamma_\mu\rangle\rangle_0$$

$$+\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle G(E^{k-i})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\{\gamma_{\varsigma_{g(3)}}\circ\gamma^\mu\}\{E^{i-1}\circ\gamma_\mu\}\rangle\rangle_0$$

$$+\frac{1}{3}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^{\mu}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_{\mu}\rangle\rangle_0$$

$$-\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle\{\Delta\circ E^{i-1}\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\{G(E^{k-i})\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle\{\Delta\circ E^{i-1}\}\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}G(E^{k-i})\rangle\rangle_0$$

$$+\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{\Delta\circ E^{i-1}\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}G(E^{k-i}\circ\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+k\sum_{g\in S_3}\langle\langle G(E^{k-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^{\mu})\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_{\mu}\rangle\rangle_0$$

$$-\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ E^{i-1}\circ\gamma^{\mu}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_{\mu}\rangle\rangle_0$$

$$-k\sum_{g\in S_3}\langle\langle\{G(E^{k-1}\circ\gamma^{\mu})\circ\gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_{\mu}\rangle\rangle_0$$

$$-6k\sum_{i=1}^{k-1}\langle\langle\{G(E^{k-i-1}\circ\gamma^{\mu})\circ G(\gamma_{\mu})\}E^{i-1}\{\gamma_{\alpha}\circ\gamma_{\beta}\circ\gamma_{\sigma}\}\rangle\rangle_0$$

$$+3\sum_{g\in S_3}\sum_{i=1}^{k-1}(k-2i)\langle\langle\{G(E^{k-i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^{\mu})\circ G(\gamma_{\mu})\}E^{i-1}\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-2\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{G(\Delta\circ E^{k-i})\circ G(E^{j-1})\}E^{i-j-1}\{\gamma_{\alpha}\circ\gamma_{\beta}\circ\gamma_{\sigma}\}\rangle\rangle_0$$

$$+2\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle G(E^{k-i})E^{i-j-1}G(\Delta\circ E^{j-1}\circ\gamma_{\alpha}\circ\gamma_{\beta}\circ\gamma_{\sigma})\rangle\rangle_0$$

$$+\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{G(\Delta\circ E^{k-i})\circ E^{j-1}\}G(E^{i-j-1}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k-1}i\langle\langle\{\Delta\circ E^{i-1}\}G(E^{k-i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$

$$-\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{\Delta\circ E^{k-i}\}G(E^{j-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})G(E^{i-j-1}\circ\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{\Delta\circ E^{k-i}\circ\gamma_{\varsigma_{g(1)}}\}G(E^{j-1}\circ\gamma_{\varsigma_{g(2)}})G(E^{i-j-1}\circ\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$-k\sum_{i=0}^{k-2}\langle\langle G(\Delta\circ E^{k-i-2})E^{i}\{\gamma_{\alpha}\circ\gamma_{\beta}\circ\gamma_{\sigma}\}\rangle\rangle_0$$

$$+2k(k-1)\langle\langle\Delta E^{k-2}\{\gamma_{\alpha}\circ\gamma_{\beta}\circ\gamma_{\sigma}\}\rangle\rangle_0.$$

The proof follows from some tedious manipulation again, that is, substituting Lemmas A.12, A.13, A.15 into the above expression, and using Lemma A.14 to cancel the redundant terms in the resulting expression, and then using Lemma 3.2 for possible simplification. $\qquad\square$

The main result of this subsection is the following

Theorem 3.8. *For any α, β, σ,*

$$24\{\gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}(\langle\langle E^k\rangle\rangle_1 - \Phi_k)$$

$$= 12 \sum_{g\in S_3} \langle\langle\{E^k \circ \gamma_{\varsigma_{g(1)}}\}\{\gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_1 - 12 \sum_{g\in S_3} \langle\langle\{E^k \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_1$$

$$+12 \sum_{g\in S_3} \sum_{i=1}^{k} \langle\langle G(E^{i-1} \circ \gamma_{\varsigma_{g(1)}}) \circ E^{k-i} \circ \gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}\rangle\rangle_1$$

$$-12 \sum_{g\in S_3} \sum_{i=1}^{k} \langle\langle G(E^{i-1} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}) \circ E^{k-i} \circ \gamma_{\varsigma_{g(3)}}\rangle\rangle_1$$

$$+24k\langle\langle E^{k-1} \circ \gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\rangle\rangle_1 - 4k\langle\langle\{\Delta \circ E^{k-1}\}\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0$$

$$+\frac{1}{2} \sum_{g\in S_3} \sum_{i=1}^{k} \langle\langle G(E^{k-i} \circ \gamma_{\varsigma_{g(1)}})\gamma_\mu\gamma^\mu\{E^{i-1} \circ \gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{2} \sum_{g\in S_3} \sum_{i=1}^{k} \langle\langle G(E^{k-i} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}})\gamma_\mu\gamma^\mu\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$+2k \sum_{g\in S_3} \langle\langle G(E^{k-1} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma^\mu)\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_\mu\rangle\rangle_0$$

$$-\frac{1}{2}k \sum_{g\in S_3} \langle\langle\{E^{k-1} \circ \gamma_{\varsigma_{g(1)}}\}\gamma_\mu\gamma^\mu\{\gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$+2 \sum_{g\in S_3} \sum_{i=1}^{k-1} (k-3i)\langle\langle\{G(E^{k-i-1} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma^\mu) \circ G(\gamma_\mu)\}E^{i-1}\{\gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{2} \sum_{g\in S_3} \sum_{i=1}^{k-1} i\langle\langle\{\Delta \circ E^{i-1}\}G(E^{k-i-1} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}})\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$

$$-\frac{1}{2} \sum_{g\in S_3} \sum_{i=1}^{k-1} (k-i)\langle\langle\{\Delta \circ E^{i-1}\}G(E^{k-i-1} \circ \gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-4k \sum_{i=1}^{k-1} \langle\langle G(\Delta \circ E^{k-i-1})E^{i-1}\{\gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}\rangle\rangle_0$$

$$+5k(k-1)\langle\langle\Delta E^{k-2}\{\gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}\rangle\rangle_0,$$

where $k \geq 0$ and $\{\varsigma_1, \varsigma_2, \varsigma_3\} = \{\alpha, \beta, \sigma\}$.

Proof. It follows easily from Lemma 3.5, Lemma 3.7 and equation (27). $\qquad\square$

Remark 3.9. *In Lemma 3.5, Lemma 3.7, and Theorem 3.8, one can replace $\gamma_\alpha, \gamma_\beta$, and γ_σ by υ_1, υ_2, and υ_3 respectively for any vector fields υ_1, υ_2, and υ_3 on the small*

phase space. The same holds for the following Lemmas 3.10, 3.12, 3.14, 3.15, and Theorems 3.13, 3.16.

3.3. Getzler equation $\mathbf{G}(E^{k_1}, E^{k_2}, \gamma_\alpha, \gamma_\beta) = 0$.
In this subsection, the following explicit formulas for functions G_1, G_0 and $\mathbf{G}$ are derived from the above subsection.

Lemma 3.10. *For $k_1, k_2 \geq 0$, and any α, β,*

$$G_1(E^{k_1}, E^{k_2}, \gamma_\alpha, \gamma_\beta)$$

$$= 24\{\gamma_\alpha \circ \gamma_\beta\}\langle\langle E^{\widehat{K}}\rangle\rangle_1 - 24 \sum_{m=1}^{2}\{E^{\widehat{K}-k_m} \circ \gamma_\alpha \circ \gamma_\beta\}\langle\langle E^{k_m}\rangle\rangle_1$$

$$+24 \sum_{h\in S_2}\langle\langle\{E^{k_{h(1)}} \circ \gamma_\alpha\}\{E^{k_{h(2)}} \circ \gamma_\beta\}\rangle\rangle_1 - 24\sum_{g\in S_2}\langle\langle\{E^{\widehat{K}} \circ \gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(2)}}\rangle\rangle_1$$

$$+24 \sum_{g\in S_2}\sum_{m=1}^{2}\sum_{i=1}^{k_m}\langle\langle G(E^{i-1} \circ \gamma_{\varsigma_{g(1)}}) \circ E^{\widehat{K}-i} \circ \gamma_{\varsigma_{g(2)}}\rangle\rangle_1$$

$$-24 \sum_{g\in S_2}\sum_{i=1}^{\widehat{K}}\langle\langle G(E^{i-1} \circ \gamma_{\varsigma_{g(1)}}) \circ E^{\widehat{K}-i} \circ \gamma_{\varsigma_{g(2)}}\rangle\rangle_1,$$

where $\widehat{K} = k_1 + k_2$ and $\{\varsigma_1, \varsigma_2\} = \{\alpha, \beta\}$.

Proof. The proof follows easily from Lemma 3.5 and equations (9) and (19). □

Remark 3.11. *As in Remark 3.6, we also have the following equivalent form*

$$G_1(E^{k_1}, E^{k_2}, \gamma_\alpha, \gamma_\beta)$$

$$= 24\{\gamma_\alpha \circ \gamma_\beta\}\langle\langle E^{\widehat{K}}\rangle\rangle_1 - 24 \sum_{m=1}^{2}\{E^{\widehat{K}-k_m} \circ \gamma_\alpha \circ \gamma_\beta\}\langle\langle E^{k_m}\rangle\rangle_1$$

$$+12 \sum_{g\in S_2}\sum_{h\in S_2}\{E^{k_{h(1)}} \circ \gamma_{\varsigma_{g(1)}}\}\langle\langle E^{k_{h(2)}} \circ \gamma_{\varsigma_{g(2)}}\rangle\rangle_1 - 24\sum_{g\in S_2}\gamma_{\varsigma_{g(1)}}\langle\langle E^{\widehat{K}} \circ \gamma_{\varsigma_{g(2)}}\rangle\rangle_1$$

$$+12 \sum_{g\in S_2}\sum_{m=1}^{2}\sum_{i=1}^{k_m}\langle\langle G(E^{i-1} \circ \gamma_{\varsigma_{g(1)}}) \circ E^{\widehat{K}-i} \circ \gamma_{\varsigma_{g(2)}}\rangle\rangle_1$$

$$-24 \sum_{m=1}^{2}\sum_{i=1}^{k_m}\langle\langle G(E^{i-1} \circ \gamma_\alpha \circ \gamma_\beta) \circ E^{\widehat{K}-i}\rangle\rangle_1 + 24\widehat{K}\langle\langle E^{\widehat{K}-1} \circ \gamma_\alpha \circ \gamma_\beta\rangle\rangle_1,$$

by equations (9) and (19).

Lemma 3.12. *For $k_1, k_2 \geq 0$, and any α, β,*

$$G_0(E^{k_1}, E^{k_2}, \gamma_\alpha, \gamma_\beta)$$

$$= -24\{\gamma_\alpha \circ \gamma_\beta\}\Phi_{\widehat{K}} + 24 \sum_{m=1}^{2}\{E^{\widehat{K}-k_m} \circ \gamma_\alpha \circ \gamma_\beta\}\Phi_{k_m}$$

$$- \sum_{g\in S_2}\sum_{i=1}^{\widehat{K}}\langle\langle G(E^{i-1} \circ \gamma_{\varsigma_{g(1)}})\gamma^\mu\gamma_\mu\{E^{\widehat{K}-i} \circ \gamma_{\varsigma_{g(2)}}\}\rangle\rangle_0$$

$$+ \sum_{g\in S_2}\sum_{m=1}^{2}\sum_{i=1}^{k_m}\langle\langle G(E^{i-1} \circ \gamma_{\varsigma_{g(1)}})\gamma^\mu\gamma_\mu\{E^{\widehat{K}-i} \circ \gamma_{\varsigma_{g(2)}}\}\rangle\rangle_0$$

$$-\sum_{g\in S_2}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\langle\langle\{\Delta\circ E^{i+j-2}\}G(E^{\widehat{K}-i-j}\circ\gamma_{\varsigma_{g(1)}})\gamma_{\varsigma_{g(2)}}\rangle\rangle_0$$

$$+6\sum_{g\in S_2}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\langle\langle\{G(E^{i+j-2}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^{\mu})\circ G(\gamma_{\mu})\}E^{\widehat{K}-i-j}\gamma_{\varsigma_{g(2)}}\rangle\rangle_0$$

$$-2k_1k_2\langle\langle\Delta E^{\widehat{K}-2}\{\gamma_\alpha\circ\gamma_\beta\}\rangle\rangle_0,$$

where $\widehat{K}=k_1+k_2$ and $\{\varsigma_1,\varsigma_2\}=\{\alpha,\beta\}$.

Proof. By Lemma 3.7, we get the expression of $G_0(E^{k_1},E^{k_2},\gamma_\alpha,\gamma_\beta)$ which contains three terms: $\langle\langle G(E^{\widehat{K}-1}\circ\gamma^{\mu})\gamma_\alpha\gamma_\beta\gamma_\mu\rangle\rangle_0$, $\langle\langle\{E^{k_1-1}\circ\gamma_\alpha\}\gamma_\mu\gamma^{\mu}\{\gamma_\beta\circ E^{k_2}\}\rangle\rangle_0$, and $\langle\langle\{E^{k_1-1}\circ\gamma_\beta\}\gamma_\mu\gamma^{\mu}\{\gamma_\alpha\circ E^{k_2}\}\rangle\rangle_0$. They are computed as follows.

$$\langle\langle G(E^{\widehat{K}-1}\circ\gamma^{\mu})\gamma_\alpha\gamma_\beta\gamma_\mu\rangle\rangle_0$$
$$=\langle\langle G(\gamma_\mu)\gamma_\alpha\gamma_\beta\{E^{\widehat{K}-1}\circ\gamma^{\mu}\}\rangle\rangle_0$$
$$=\langle\langle E^{\widehat{K}-1}\{\gamma_\beta\circ\gamma^{\mu}\}\gamma_\alpha G(\gamma_\mu)\rangle\rangle_0+\langle\langle\{E^{\widehat{K}-1}\circ\gamma_\alpha\}\gamma^{\mu}G(\gamma_\mu)\gamma_\beta\rangle\rangle_0$$
$$-\langle\langle E^{\widehat{K}-1}\{\gamma_\alpha\circ\gamma_\beta\}\gamma^{\mu}G(\gamma_\mu)\rangle\rangle_0,$$

$$\langle\langle\{E^{k_1-1}\circ\gamma_\alpha\}\gamma_\mu\gamma^{\mu}\{\gamma_\beta\circ E^{k_2}\}\rangle\rangle_0$$
$$=\langle\langle E^{k_2}\{\gamma_\beta\circ\gamma^{\mu}\}\gamma_\mu\{E^{k_1-1}\circ\gamma_\alpha\}\rangle\rangle_0+\langle\langle\{E^{\widehat{K}-1}\circ\gamma_\alpha\}\gamma_\mu\gamma^{\mu}\gamma_\beta\rangle\rangle_0$$
$$-\langle\langle E^{k_2}\gamma_\beta\gamma_\mu\{E^{k_1-1}\circ\gamma_\alpha\circ\gamma^{\mu}\}\rangle\rangle_0,$$

$$\langle\langle\{E^{k_1-1}\circ\gamma_\beta\}\gamma_\mu\gamma^{\mu}\{\gamma_\alpha\circ E^{k_2}\}\rangle\rangle_0$$
$$=\langle\langle E^{k_1-1}\{\gamma_\beta\circ\gamma^{\mu}\}\gamma_\mu\{E^{k_2}\circ\gamma_\alpha\}\rangle\rangle_0+\langle\langle\{E^{\widehat{K}-1}\circ\gamma_\alpha\}\gamma_\mu\gamma^{\mu}\gamma_\beta\rangle\rangle_0$$
$$-\langle\langle E^{k_1-1}\gamma_\beta\gamma_\mu\{E^{k_2}\circ\gamma_\alpha\circ\gamma^{\mu}\}\rangle\rangle_0.$$

The proof is completed by Lemmas 3.1, 3.2 and Theorem 3.3. $\square$

Theorem 3.13. *For $k_1,k_2\geq 0$, and any α,β,*

$$24\{\gamma_\alpha\circ\gamma_\beta\}(\langle\langle E^{\widehat{K}}\rangle\rangle_1-\Phi_{\widehat{K}})-24\sum_{m=1}^{2}\{E^{\widehat{K}-k_m}\circ\gamma_\alpha\circ\gamma_\beta\}(\langle\langle E^{k_m}\rangle\rangle_1-\Phi_{k_m})$$

$$=-24\sum_{h\in S_2}\langle\langle\{E^{k_{h(1)}}\circ\gamma_\alpha\}\{E^{k_{h(2)}}\circ\gamma_\beta\}\rangle\rangle_1+24\sum_{g\in S_2}\langle\langle\{E^{\widehat{K}}\circ\gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(2)}}\rangle\rangle_1$$

$$-24\sum_{g\in S_2}\sum_{m=1}^{2}\sum_{i=1}^{k_m}\langle\langle G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}})\circ E^{\widehat{K}-i}\circ\gamma_{\varsigma_{g(2)}}\rangle\rangle_1$$

$$+24\sum_{g\in S_2}\sum_{i=1}^{\widehat{K}}\langle\langle G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}})\circ E^{\widehat{K}-i}\circ\gamma_{\varsigma_{g(2)}}\rangle\rangle_1$$

$$+\sum_{g\in S_2}\sum_{i=1}^{\widehat{K}}\langle\langle G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}})\gamma^{\mu}\gamma_\mu\{E^{\widehat{K}-i}\circ\gamma_{\varsigma_{g(2)}}\}\rangle\rangle_0$$

$$-\sum_{g\in S_2}\sum_{m=1}^{2}\sum_{i=1}^{k_m}\langle\langle G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}})\gamma^{\mu}\gamma_\mu\{E^{\widehat{K}-i}\circ\gamma_{\varsigma_{g(2)}}\}\rangle\rangle_0$$

$$+ \sum_{g \in S_2} \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \langle\langle \{ \Delta \circ E^{i+j-2} \} G(E^{\widehat{K}-i-j} \circ \gamma_{\varsigma_{g(1)}}) \gamma_{\varsigma_{g(2)}} \rangle\rangle_0$$

$$-6 \sum_{g \in S_2} \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \langle\langle \{ G(E^{i+j-2} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma^\mu) \circ G(\gamma_\mu) \} E^{\widehat{K}-i-j} \gamma_{\varsigma_{g(2)}} \rangle\rangle_0$$

$$+2 k_1 k_2 \langle\langle \Delta E^{\widehat{K}-2} \{ \gamma_\alpha \circ \gamma_\beta \} \rangle\rangle_0,$$

where $\widehat{K} = k_1 + k_2$ *and* $\{\varsigma_1, \varsigma_2\} = \{\alpha, \beta\}$.

Proof. It follows easily from Lemma 3.10, Lemma 3.12 and equation (27). $\qquad\square$

3.4. Getzler equation $\mathbf{G}(E^{k_1}, E^{k_2}, E^{k_3}, \gamma_\alpha) = 0$. In this subsection, we will derive the following results from subsection 3.3.

Lemma 3.14. *For* $k_1, k_2, k_3 \geq 0$, *and any* α,

$$G_1(E^{k_1}, E^{k_2}, E^{k_3}, \gamma_\alpha)$$

$$= -24 \gamma_\alpha \langle\langle E^{\widetilde{K}} \rangle\rangle_1 + 24 \sum_{i=1}^{3} \{ E^{k_i} \circ \gamma_\alpha \} \langle\langle E^{\widetilde{K}-k_i} \rangle\rangle_1 - 24 \sum_{i=1}^{3} \{ E^{\widetilde{K}-k_i} \circ \gamma_\alpha \} \langle\langle E^{k_i} \rangle\rangle_1,$$

where $\widetilde{K} = k_1 + k_2 + k_3$.

Proof. It follows easily from Lemma 3.10 and equations (9) and (19). $\qquad\square$

Lemma 3.15. *For* $k_1, k_2, k_3 \geq 0$, *and any* α,

$$G_0(E^{k_1}, E^{k_2}, E^{k_3}, \gamma_\alpha)$$

$$= 24 \gamma_\alpha \Phi_{\widetilde{K}} - 24 \sum_{i=1}^{3} \{ E^{k_i} \circ \gamma_\alpha \} \Phi_{\widetilde{K}-k_i} + 24 \sum_{i=1}^{3} \{ E^{\widetilde{K}-k_i} \circ \gamma_\alpha \} \Phi_{k_i},$$

where $\widetilde{K} = k_1 + k_2 + k_3$.

Proof. It follows from Lemmas 3.12, 3.1, 3.2 and Theorem 3.3. $\qquad\square$

Theorem 3.16. *For* $k_1, k_2, k_3 \geq 0$, *and any* α,

$$\gamma_\alpha (\langle\langle E^{\widetilde{K}} \rangle\rangle_1 - \Phi_{\widetilde{K}}) = \sum_{i=1}^{3} \{ E^{k_i} \circ \gamma_\alpha \} (\langle\langle E^{\widetilde{K}-k_i} \rangle\rangle_1 - \Phi_{\widetilde{K}-k_i})$$

$$- \sum_{i=1}^{3} \{ E^{\widetilde{K}-k_i} \circ \gamma_\alpha \} (\langle\langle E^{k_i} \rangle\rangle_1 - \Phi_{k_i}),$$

where $\widetilde{K} = k_1 + k_2 + k_3$.

Proof. It follows easily from Lemma 3.14, Lemma 3.15 and equation (27). $\qquad\square$

3.5. Getzler equation $\mathbf{G}(E^{k_1}, E^{k_2}, E^{k_3}, E^{k_4}) = 0$. The case in this subsection has been studied in [13]. We present alternative symmetric expressions as follows.

Lemma 3.17. *For* $k_1, k_2, k_3, k_4 \geq 0$,

$$G_1(E^{k_1}, E^{k_2}, E^{k_3}, E^{k_4})$$

$$= 36K \langle\langle E^{K-1} \rangle\rangle_1 - 24 \sum_{i=1}^{4} E^{k_i} \langle\langle E^{K-k_i} \rangle\rangle_1 + 3 \sum_{g \in S_4} E^{k_{g(1)}+k_{g(2)}} \langle\langle E^{k_{g(3)}+k_{g(4)}} \rangle\rangle_1,$$

where $K = k_1 + k_2 + k_3 + k_4$.

Proof. It follows from Lemma 3.14 and Corollary 2.3. $\square$

Remark 3.18. *By Corollary 2.3, Lemma 3.17 is equivalent to Corollary 3.2 of [13].*

Lemma 3.19. *For $k_1, k_2, k_3, k_4 \geq 0$,*

$$G_0(E^{k_1}, E^{k_2}, E^{k_3}, E^{k_4})$$

$$= -36K\Phi_{K-1} + 24\sum_{i=1}^{4} E^{k_i}\Phi_{K-k_i} - 3\sum_{g\in S_4} E^{k_{g(1)}+k_{g(2)}}\Phi_{k_{g(3)}+k_{g(4)}},$$

where $K = k_1 + k_2 + k_3 + k_4$.

Proof. It follows from Lemma 3.15 and Corollary 3.4. $\square$

Remark 3.20. *It is easy to show that Lemma 3.19 implies the definition of Φ_k ($k \geq 3$) given in [13].*

Theorem 3.21. *For $k_1, k_2, k_3, k_4 \geq 0$,*

$$12K(\langle\langle E^{K-1}\rangle\rangle_1 - \Phi_{K-1})$$

$$= 8\sum_{i=1}^{4} E^{k_i}(\langle\langle E^{K-k_i}\rangle\rangle_1 - \Phi_{K-k_i}) - \sum_{g\in S_4} E^{k_{g(1)}+k_{g(2)}}(\langle\langle E^{k_{g(3)}+k_{g(4)}}\rangle\rangle_1 - \Phi_{k_{g(3)}+k_{g(4)}}),$$

where $K = k_1 + k_2 + k_3 + k_4$.

Proof. It follows from Lemma 3.17, Lemma 3.19 and equation (27). $\square$

4. Application to the genus-1 Virasoro Conjecture

In this section, we apply Getzler equations derived in Section 3 to study the genus-1 Virasoro conjecture. Since Getzler equation $\mathbf{G}(E^{k_1}, E^{k_2}, E^{k_3}, E^{k_4}) = 0$ has been analyzed in great detail in [13], we begin with the application of Getzler equation $\mathbf{G}(E^{k_1}, E^{k_2}, E^{k_3}, \gamma_\alpha) = 0$. It provides the following relations among $\{\langle\langle E^k\rangle\rangle_1 - \Phi_k\}_{k\geq 0}$.

Theorem 4.1. *For $k \geq 0$, and any α,*

$$\gamma_\alpha(\langle\langle E^k\rangle\rangle_1 - \Phi_k) = \frac{1}{2}k(k-1)\{E^{k-2}\circ\gamma_\alpha\}(\langle\langle E^2\rangle\rangle_1 - \Phi_2).$$

Proof. It is trivial for $k = 0, 1, 2$. For $k \geq 3$, let $k_1 = k - 2$, $k_2 = k_3 = 1$ in Theorem 3.16, we have

$$\gamma_\alpha\left(\langle\langle E^k\rangle\rangle_1 - \Phi_k\right)$$

$$= 2\{E\circ\gamma_\alpha\}\left(\langle\langle E^{k-1}\rangle\rangle_1 - \Phi_{k-1}\right) - \{E^2\circ\gamma_\alpha\}\left(\langle\langle E^{k-2}\rangle\rangle_1 - \Phi_{k-2}\right)$$

$$+ \{E^{k-2}\circ\gamma_\alpha\}\left(\langle\langle E^2\rangle\rangle_1 - \Phi_2\right). \tag{42}$$

Suppose that the theorem holds for $k \leq n$ ($n \geq 2$). By equation (42), we have

$$\gamma_\alpha\left(\langle\langle E^{n+1}\rangle\rangle_1 - \Phi_{n+1}\right)$$

$$= 2\{E\circ\gamma_\alpha\}\left(\langle\langle E^n\rangle\rangle_1 - \Phi_n\right) - \{E^2\circ\gamma_\alpha\}\left(\langle\langle E^{n-1}\rangle\rangle_1 - \Phi_{n-1}\right)$$

$$+ \{E^{n-1}\circ\gamma_\alpha\}\left(\langle\langle E^2\rangle\rangle_1 - \Phi_2\right)$$

$$= n(n-1)\{E^{n-1}\circ\gamma_\alpha\}\left(\langle\langle E^2\rangle\rangle_1 - \Phi_2\right) + \{E^{n-1}\circ\gamma_\alpha\}\left(\langle\langle E^2\rangle\rangle_1 - \Phi_2\right)$$

$$-\frac{1}{2}(n-1)(n-2)\{E^{n-1}\circ\gamma_\alpha\}\left((\langle\langle E^2\rangle\rangle_1-\Phi_2\right)$$

$$=\quad\frac{1}{2}n(n+1)\{E^{n-1}\circ\gamma_\alpha\}\left((\langle\langle E^2\rangle\rangle_1-\Phi_2\right).$$

The proof is completed by induction on k. □

Remark 4.2. *By Theorem 4.1, we have*

$$\{\gamma_\alpha\circ\gamma_\beta\}((\langle\langle E^{\widehat{K}}\rangle\rangle_1-\Phi_{\widehat{K}})-\sum_{m=1}^{2}\{E^{\widehat{K}-k_m}\circ\gamma_\alpha\circ\gamma_\beta\}((\langle\langle E^{k_m}\rangle\rangle_1-\Phi_{k_m})$$

$$=\quad\frac{2k_1k_2}{\widehat{K}(\widehat{K}-1)}\{\gamma_\alpha\circ\gamma_\beta\}((\langle\langle E^{\widehat{K}}\rangle\rangle_1-\Phi_{\widehat{K}})$$

which gives an equivalent equation in Theorem 3.13.

It follows easily from Theorem 4.1 that

Corollary 4.3. *For any α,*

$$\frac{m-k}{m+k}\gamma_\alpha((\langle\langle E^{m+k}\rangle\rangle_1-\Phi_{m+k})$$

$$=\quad\{E^k\circ\gamma_\alpha\}((\langle\langle E^m\rangle\rangle_1-\Phi_m)-\{E^m\circ\gamma_\alpha\}((\langle\langle E^k\rangle\rangle_1-\Phi_k).$$

where $k+m>0$.

Corollary 4.4. *For any $k\geq 0$ and $m>0$ satisfying $m+k\geq 2$, and any α,*

$$\{E^k\circ\gamma_\alpha\}\frac{\langle\langle E^m\rangle\rangle_1-\Phi_m}{m}=\frac{m-1}{(m+k)(m+k-1)}\gamma_\alpha((\langle\langle E^{m+k}\rangle\rangle_1-\Phi_{m+k}).$$

It is easy to show that Theorem 4.1 is equivalent to Theorem 3.16. Due to the fact $\gamma_1((\langle\langle E^k\rangle\rangle_1-\Phi_k)=k((\langle\langle E^{k-1}\rangle\rangle_1-\Phi_{k-1})$, Corollary 4.3 implies Virasoro type relation for $\{\Phi_k\}$ in Theorem 6.1 of [13] by setting $\gamma_\alpha=\gamma_1$ and Corollary 2.3, while Corollary 4.4 implies Lemma 6.3 in [13] if $\gamma_\alpha=\gamma_1$.

Next, we deal with more general case, i.e., Getzler equation $\mathbf{G}(E^{k_1},E^{k_2},\gamma_\alpha,\gamma_\beta)=0$. It involves more complicated genus-1 data and genus-0 4-point functions. But thanks to formulas (13) and (14), we may consider the following universal equation

$$\mathbf{G}(E^{k_1},E^{k_2},\gamma_\mu,\gamma^\mu\circ\gamma_\alpha)=0.$$

Actually, we have

Theorem 4.5. *For $k_1,k_2\geq 0$, and any α,*

$$\{\Delta\circ\gamma_\alpha\}((\langle\langle E^{k_1+k_2}\rangle\rangle_1-\Phi_{k_1+k_2})$$

$$=\quad\{\Delta\circ E^{k_1}\circ\gamma_\alpha\}((\langle\langle E^{k_2}\rangle\rangle_1-\Phi_{k_2})+\{\Delta\circ E^{k_2}\circ\gamma_\alpha\}((\langle\langle E^{k_1}\rangle\rangle_1-\Phi_{k_1}).$$

Proof. By Lemmas 3.10, 3.12 and equations (13), (14), (30), it is easy to show that

$$G_1(E^{k_1},E^{k_2},\gamma_\mu,\gamma^\mu\circ\gamma_\alpha)$$

$$=\quad 24\{\Delta\circ\gamma_\alpha\}\langle\langle E^{k_1+k_2}\rangle\rangle_1-24\{\Delta\circ E^{k_1}\circ\gamma_\alpha\}\langle\langle E^{k_2}\rangle\rangle_1-24\{\Delta\circ E^{k_2}\circ\gamma_\alpha\}\langle\langle E^{k_1}\rangle\rangle_1$$

and

$$G_0(E^{k_1},E^{k_2},\gamma_\mu,\gamma^\mu\circ\gamma_\alpha)$$

$$=\quad -24\{\Delta\circ\gamma_\alpha\}\Phi_{k_1+k_2}+24\{\Delta\circ E^{k_1}\circ\gamma_\alpha\}\Phi_{k_2}+24\{\Delta\circ E^{k_2}\circ\gamma_\alpha\}\Phi_{k_1}.$$

The proof is completed by equation (27). □

It is equivalent to the following evidences for the genus-1 Virasoro conjecture.

Theorem 4.6. *For any smooth projective variety X, we have*

$$(\Delta \circ \gamma_\alpha)(\langle\langle E^k \rangle\rangle_1 - \Phi_k) = 0,$$

for all $k \geq 0$ and $\alpha \in \{1, 2, \ldots, N\}$.

Proof. Let $k_1 = k_2 = 1$ in Theorem 4.5, we have

$$(\Delta \circ \gamma_\alpha)(\langle\langle E^2 \rangle\rangle_1 - \Phi_2) = 0.$$

Set $k_1 = k$ ($k \geq 2$) and $k_2 = 1$ in Theorem 4.5, we have

$$(\Delta \circ \gamma_\alpha)(\langle\langle E^{k+1} \rangle\rangle_1 - \Phi_{k+1}) = (\Delta \circ E \circ \gamma_\alpha)(\langle\langle E^k \rangle\rangle_1 - \Phi_k).$$

Since α is arbitrary, the proof is completed. $\qquad\qquad\square$

By setting $k = 2$ and $\gamma_\alpha = \gamma_1$, Theorem 4.6 implies Theorem 1.1 of [15]. Hence it provides more evidences for the genus-1 Virasoro conjecture. Since by the WDVV equation (10), we have

$$
\begin{aligned}
\Delta \circ \gamma_\alpha = \sum_\mu \gamma_\mu \circ \gamma^\mu \circ \gamma_\alpha &= \sum_{\sigma,\mu,\beta} \langle\langle \gamma_\alpha \gamma_\beta \gamma^\sigma \rangle\rangle_0 \langle\langle \gamma_\sigma \gamma_\mu \gamma^\mu \rangle\rangle_0 \gamma^\beta \\
&= \sum_{\sigma,\mu,\beta} \langle\langle \gamma_\alpha \gamma^\sigma \gamma^\mu \rangle\rangle_0 \langle\langle \gamma_\sigma \gamma_\mu \gamma_\beta \rangle\rangle_0 \gamma^\beta.
\end{aligned}
$$

Hence by Theorem 4.6, we have for any $k \geq 2$

$$
\begin{pmatrix}
A_{11} & A_{12} & \ldots & A_{1N} \\
A_{21} & A_{22} & \ldots & A_{2N} \\
\vdots & \vdots & & \vdots \\
A_{N1} & A_{N2} & \ldots & A_{NN}
\end{pmatrix}
\begin{pmatrix}
\gamma^1(\langle\langle E^k \rangle\rangle_1 - \Phi_k) \\
\gamma^2(\langle\langle E^k \rangle\rangle_1 - \Phi_k) \\
\vdots \\
\gamma^N(\langle\langle E^k \rangle\rangle_1 - \Phi_k)
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
$$

where

$$A_{\alpha,\beta} = \sum_{\sigma,\mu} \langle\langle \gamma_\alpha \gamma^\sigma \gamma^\mu \rangle\rangle_0 \langle\langle \gamma_\sigma \gamma_\mu \gamma_\beta \rangle\rangle_0.$$

It is obviously that $A_{\alpha,\beta} = A_{\beta,\alpha}$. If the following symmetric matrix

$$
\begin{pmatrix}
A_{11} & A_{12} & \ldots & A_{1N} \\
A_{21} & A_{22} & \ldots & A_{2N} \\
\vdots & \vdots & & \vdots \\
A_{N1} & A_{N2} & \ldots & A_{NN}
\end{pmatrix}
= \sum_{\sigma,\mu}
\begin{pmatrix}
\langle\langle \gamma_1 \gamma^\sigma \gamma^\mu \rangle\rangle_0 \\
\langle\langle \gamma_2 \gamma^\sigma \gamma^\mu \rangle\rangle_0 \\
\vdots \\
\langle\langle \gamma_N \gamma^\sigma \gamma^\mu \rangle\rangle_0
\end{pmatrix}
\begin{pmatrix}
\langle\langle \gamma_\sigma \gamma_\mu \gamma_1 \rangle\rangle_0 \\
\langle\langle \gamma_\sigma \gamma_\mu \gamma_2 \rangle\rangle_0 \\
\vdots \\
\langle\langle \gamma_\sigma \gamma_\mu \gamma_N \rangle\rangle_0
\end{pmatrix}^T
$$

is invertible, then $\gamma^\alpha(\langle\langle E^k \rangle\rangle_1 - \Phi_k) = 0$ for any $\alpha \in \{1, 2, \cdots, N\}$, which is equivalent to $\gamma_\alpha(\langle\langle E^k \rangle\rangle_1 - \Phi_k) = 0$ for any $\alpha \in \{1, 2, \cdots, N\}$, and then the genus-1 Virasoro conjecture holds.

The above argument gives an alternative proof of

Corollary 4.7 ([4, 13, 16, 18]). *For any compact symplectic manifold with semisimple quantum cohomology, the genus-1 Virasoro conjecture holds.*

Proof. Follow the notation in [16] and restrict everything on the small phase space. Let $\{\mathcal{E}_i\}$ be the idempotents which span the space of primary vector fields on the small phase space. Recall the following formulas derived in [16]:

$$\mathcal{E}_i \circ \mathcal{E}_j = \delta_{ij}\mathcal{E}_i; \ \gamma_\alpha = \sum_{i=1}^{N} \psi_{i\alpha}\frac{\mathcal{E}_i}{\sqrt{g_i}}; \ \Delta = \sum_{i=1}^{N} \frac{1}{g_i}\mathcal{E}_i,$$

where $g_\alpha = \|\mathcal{E}_\alpha\|^2$ and $(\psi_{\alpha\beta})$ is a invertible matrix (see [16] for more detail). Hence, we have

$$\Delta \circ \gamma_\alpha = \sum_{i=1}^{N} g_i^{-\frac{3}{2}}\psi_{i\alpha}\mathcal{E}_i.$$

By Theorem 4.6, we have for any $k \geq 2$

$$\begin{pmatrix} g_1^{-\frac{3}{2}}\psi_{11} & g_2^{-\frac{3}{2}}\psi_{21} & \cdots & g_N^{-\frac{3}{2}}\psi_{N1} \\ g_1^{-\frac{3}{2}}\psi_{12} & g_2^{-\frac{3}{2}}\psi_{22} & \cdots & g_N^{-\frac{3}{2}}\psi_{N2} \\ \vdots & \vdots & & \vdots \\ g_1^{-\frac{3}{2}}\psi_{1N} & g_2^{-\frac{3}{2}}\psi_{2N} & \cdots & g_N^{-\frac{3}{2}}\psi_{NN} \end{pmatrix} \begin{pmatrix} \mathcal{E}_1(\langle\langle E^k\rangle\rangle_1 - \Phi_k) \\ \mathcal{E}_2(\langle\langle E^k\rangle\rangle_1 - \Phi_k) \\ \vdots \\ \mathcal{E}_N(\langle\langle E^k\rangle\rangle_1 - \Phi_k) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The proof is completed due to the fact that the matrix $(\psi_{\alpha\beta})$ is invertible. $\qquad\square$

Finally, we obtain one new relation from Getzler equation $\mathbf{G}(E^k, \gamma_\alpha, \gamma_\beta, \gamma_\sigma) = 0$. The equation is $\mathbf{G}(E^k, \gamma_\alpha, \gamma^\alpha \circ \gamma^\beta \circ \gamma_\mu, \gamma_\beta) = 0$ since in this case, using equations (13), (14) and (30), we have

$$G_1(E^k, \gamma_\alpha, \gamma^\alpha \circ \gamma^\beta \circ \gamma_\mu, \gamma_\beta) = -24\{\Delta^2 \circ \gamma_\mu\}\langle\langle E^k\rangle\rangle_1 + 24k\langle\langle\Delta^2 \circ E^{k-1} \circ \gamma_\mu\rangle\rangle_1 \quad (43)$$

by Lemma 3.5. Actually, we have

Theorem 4.8. *For all $k \geq 1$ and any μ,*

$$\begin{aligned}
&24\langle\langle\Delta^2 \circ E^{k-1} \circ \gamma_\mu\rangle\rangle_1 \\
=\ &5\langle\langle\{\Delta \circ E^{k-1}\}\gamma_\alpha\gamma^\alpha\{\Delta \circ \gamma_\mu\}\rangle\rangle_0 \\
&+2\langle\langle\{\Delta \circ E^{k-1}\}\gamma_\alpha\{\gamma^\alpha \circ \gamma_\mu\}\Delta\rangle\rangle_0 \\
&-6\langle\langle\{\Delta \circ E^{k-1}\}G(\gamma^\alpha \circ \gamma^\beta \circ \gamma_\mu)\gamma_\alpha\gamma_\beta\rangle\rangle_0 \\
&-6\langle\langle\{\Delta \circ E^{k-1} \circ \gamma_\mu\}G(\gamma^\alpha \circ \gamma^\beta)\gamma_\alpha\gamma_\beta\rangle\rangle_0 \\
&-6\sum_{i=1}^{k-1}\langle\langle\{G(\Delta \circ E^{i-1} \circ \gamma^\alpha) \circ G(\gamma_\alpha)\}E^{k-i-1}\{\Delta \circ \gamma_\mu\}\rangle\rangle_0 \\
&+4\sum_{i=1}^{k-1}\langle\langle G(\Delta \circ E^{i-1})E^{k-i-1}\{\Delta^2 \circ \gamma_\mu\}\rangle\rangle_0 \\
&-3\sum_{i=1}^{k-1}\langle\langle G(\Delta \circ E^{i-1} \circ \gamma_\mu)E^{k-i-1}\Delta^2\rangle\rangle_0 \\
&+(k-1)\langle\langle\Delta^3 E^{k-2}\gamma_\mu\rangle\rangle_0.
\end{aligned}$$

Proof. By Lemma 3.7, and using formulas (13), (14), (25) and Lemmas 3.1, 3.2, it can be verified that

$$G_0(E^k, \gamma_\alpha, \gamma^\alpha \circ \gamma^\beta \circ \gamma_\mu, \gamma_\beta)$$

$$
\begin{aligned}
= \quad & 24\{\Delta^2 \circ \gamma_\mu\}\Phi_k - 5k\langle\langle\{\Delta \circ E^{k-1}\}\gamma_\alpha\gamma^\alpha\{\Delta \circ \gamma_\mu\}\rangle\rangle_0 \\
& -2k\langle\langle\{\Delta \circ E^{k-1}\}\gamma_\alpha\{\gamma^\alpha \circ \gamma_\mu\}\Delta\rangle\rangle_0 \\
& +6k\langle\langle\{\Delta \circ E^{k-1}\}G(\gamma^\alpha \circ \gamma^\beta \circ \gamma_\mu)\gamma_\alpha\gamma_\beta\rangle\rangle_0 \\
& +6k\langle\langle\{\Delta \circ E^{k-1} \circ \gamma_\mu\}G(\gamma^\alpha \circ \gamma^\beta)\gamma_\alpha\gamma_\beta\rangle\rangle_0 \\
& +6k\sum_{i=1}^{k-1}\langle\langle\{G(\Delta \circ E^{i-1} \circ \gamma^\alpha) \circ G(\gamma_\alpha)\}E^{k-i-1}\{\Delta \circ \gamma_\mu\}\rangle\rangle_0 \\
& -4k\sum_{i=1}^{k-1}\langle\langle G(\Delta \circ E^{i-1})E^{k-i-1}\{\Delta^2 \circ \gamma_\mu\}\rangle\rangle_0 \\
& +3k\sum_{i=1}^{k-1}\langle\langle G(\Delta \circ E^{i-1} \circ \gamma_\mu)E^{k-i-1}\Delta^2\rangle\rangle_0 \\
& -k(k-1)\langle\langle\Delta^3 E^{k-2}\gamma_\mu\rangle\rangle_0.
\end{aligned}
\tag{44}
$$

By Theorem 4.6, we have

$$
\{\Delta^2 \circ \gamma_\mu\}(\langle\langle E^k\rangle\rangle_1 - \Phi_k) = 0.
$$

Together with equations (43), (44) and (27), the proof is completed. $\qquad\square$

In particular, we have

Corollary 4.9.

$$
\langle\langle\Delta^2\rangle\rangle_1 = \frac{7}{24}\langle\langle\Delta\gamma_\mu\gamma^\mu\Delta\rangle\rangle_0 - \frac{1}{2}\langle\langle\Delta G(\gamma^\alpha \circ \gamma^\beta)\gamma_\alpha\gamma_\beta\rangle\rangle_0.
$$

Remark 4.10. *Theorem 4.8 can be also obtained from*

$$
\boldsymbol{G}(E^k, \gamma_\alpha, \gamma^\alpha, \Delta \circ \gamma_\mu) + \boldsymbol{G}(E^k, \gamma_\alpha, \gamma^\alpha \circ \gamma_\mu, \Delta) = 0.
$$

Appendix A.

In this appendix, we present the following results which are used in Lemma 3.7. Notice that $\{\alpha, \beta, \sigma\} = \{\varsigma_1, \varsigma_2, \varsigma_3\}$ is implicit below. We start with the computation of each term on the right-hand side of equation (41) as follows.

Lemma A.1. *For any* α, β, σ,

$$
\begin{aligned}
& \langle\langle E^k \gamma_\alpha\gamma_\beta\gamma_\sigma\Delta\rangle\rangle_0 \\
= \quad & -\frac{1}{6}\sum_{g \in S_3}\sum_{i=1}^{k-1}\langle\langle\{G(E^{k-i}) \circ \Delta\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0 \\
& +\frac{1}{6}\sum_{g \in S_3}\sum_{i=1}^{k}\langle\langle\{G(\Delta \circ E^{k-i})\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0 \\
& -\frac{1}{3}b_1\sum_{g \in S_3}\langle\langle\Delta\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\{E^{k-1} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0 \\
& -\frac{1}{3}\sum_{g \in S_3}\sum_{i=1}^{k}\langle\langle\{\Delta \circ E^{k-i}\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0
\end{aligned}
$$

$$-\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle G(E^{k-i})\Delta\gamma_{\varsigma_{g(1)}}\{E^{i-1}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$+\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle\{\Delta\circ E^{i-1}\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}G(E^{k-i}\circ\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+\frac{1}{3}\sum_{g\in S_3}\langle\langle\{\Delta\circ E^{k-1}\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}G(\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+\frac{1}{6}\sum_{g\in S_3}\langle\langle\Delta\gamma_{\varsigma_{g(1)}}\{E^{k-1}\circ\gamma_{\varsigma_{g(2)}}\}G(\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$-\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle\{\Delta\circ E^{i-1}\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\{G(E^{k-i})\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$+\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{G(E^{k-i})\circ\Delta\circ E^{j-1}\}G(E^{i-j-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$

$$+\frac{1}{6}b_1\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle\{\Delta\circ E^{k-i-1}\}G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$

$$+\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle E^{k-i}G(E^{j-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})G(\Delta\circ E^{i-j-1}\circ\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$-\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle\{\Delta\circ E^{i-1}\}G(E^{k-i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})G(\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$-\frac{1}{3}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{G(\Delta\circ E^{k-i})\circ E^{j-1}\}G(E^{i-j-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$

$$-2\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\langle\langle\{G(E^{k-i-j})\circ E^{i+j-2}\}\Delta\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\rangle_0$$

$$-\frac{1}{6}(b_1+2)\sum_{g\in S_3}\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\langle\langle\{\Delta\circ E^{k-j-1}\}G(E^{j-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$

$$+\frac{1}{6}(b_1+4)\sum_{g\in S_3}\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\langle\langle\{\Delta\circ E^{k-j-1}\}G(E^{j-1}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{G(E^{k-i})\circ\Delta\circ E^{j-1}\}G(E^{i-j-1}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{6}b_1\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle\{\Delta\circ E^{i-1}\}G(E^{k-i-1}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{E^{k-i}\circ\gamma_{\varsigma_{g(1)}}\}G(\Delta\circ E^{j-1}\circ\gamma_{\varsigma_{g(2)}})G(E^{i-j-1}\circ\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle\{E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\}G(\Delta\circ E^{k-i-1}\circ\gamma_{\varsigma_{g(2)}})G(\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+\frac{2}{3}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{G(\Delta\circ E^{k-i})\circ E^{j-1}\}G(E^{i-j-1}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle\{G(\Delta\circ E^{i-1})\circ E^{k-i-1}\}\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}G(\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle\{\Delta\circ E^{k-i-1}\circ\gamma_{\varsigma_{g(1)}}\}G(E^{i-1}\circ\gamma_{\varsigma_{g(2)}})G(\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$-\frac{1}{6}b_1\sum_{g\in S_3}\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\langle\langle E^{k-i-j}G(\Delta\circ E^{i+j-2}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$+\frac{1}{3}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{G(E^{k-i})\circ E^{j-1}\}G(\Delta\circ E^{i-j-1}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{6}b_1\sum_{g\in S_3}\sum_{i=1}^{k-1}\langle\langle E^{k-i-1}G(\Delta\circ E^{i-1}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\sum_{i=1}^{k-1}\sum_{j=1}^{i-1}\langle\langle\{G(\Delta\circ E^{i-j-1})\circ G(E^{k-i})\}E^{j-1}\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\rangle_0$$

$$+\frac{1}{6}b_1\sum_{g\in S_3}\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\langle\langle E^{k-i-j}G(\Delta\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\circ E^{i+j-2})\gamma_{\varsigma_{g(3)}}\rangle\rangle_0,$$

where $k\geq 0$ and $\{\varsigma_1,\varsigma_2,\varsigma_3\}=\{\alpha,\beta,\sigma\}$.

Proof. By Lemma 2.10, we have

$$\langle\langle E^k\gamma_\alpha\gamma_\beta\gamma_\sigma\Delta\rangle\rangle_0$$

$$= -\sum_{i=1}^{k-1}\langle\langle EE^{k-i}\Delta\gamma^\rho\rangle\rangle_0\langle\langle\gamma_\rho\gamma_\alpha\{\gamma_\beta\circ E^{i-1}\}\gamma_\sigma\rangle\rangle_0$$

$$-\sum_{i=1}^{k-1}\langle\langle EE^{k-i}\{\gamma_\alpha\circ\gamma_\beta\circ E^{i-1}\}\gamma_\sigma\Delta\rangle\rangle_0$$

$$-\sum_{i=1}^{k-1}\langle\langle EE^{k-i}\gamma_\sigma\gamma^\rho\rangle\rangle_0\langle\langle\gamma_\rho\gamma_\alpha\{\gamma_\beta\circ E^{i-1}\}\Delta\rangle\rangle_0$$

$$+\sum_{i=1}^{k-1}\langle\langle E^{k-i}\gamma_\alpha\Delta\gamma^\rho\rangle\rangle_0\langle\langle\gamma_\rho\{\gamma_\beta\circ E^{i-1}\}E\gamma_\sigma\rangle\rangle_0$$

$$+\sum_{i=1}^{k-1}\langle\langle E^{k-i}\gamma_\alpha\gamma_\sigma\gamma^\rho\rangle\rangle_0\langle\langle\gamma_\rho\{\gamma_\beta\circ E^{i-1}\}E\Delta\rangle\rangle_0$$

$$+\sum_{i=1}^{k}\langle\langle\{E^{k-i}\circ\gamma_\alpha\}\{\gamma_\beta\circ E^{i-1}\}E\gamma_\sigma\Delta\rangle\rangle_0.$$

We compute each term of right-hand side of the above equation as follows. By equation (17), we have

$$\langle\langle EE^{k-i}\Delta\gamma^{\rho}\rangle\rangle_0\langle\langle\gamma_{\rho}\gamma_{\alpha}\{\gamma_{\beta}\circ E^{i-1}\}\gamma_{\sigma}\rangle\rangle_0$$
$$= \langle\langle\{G(E^{k-i})\circ\Delta\}\gamma_{\alpha}\gamma_{\sigma}\{E^{i-1}\circ\gamma_{\beta}\}\rangle\rangle_0 + \langle\langle\{G(\Delta)\circ E^{k-i}\}\gamma_{\alpha}\gamma_{\sigma}\{E^{i-1}\circ\gamma_{\beta}\}\rangle\rangle_0$$
$$-\langle\langle G(\Delta\circ E^{k-i})\gamma_{\alpha}\gamma_{\sigma}\{E^{i-1}\circ\gamma_{\beta}\}\rangle\rangle_0 - b_1\langle\langle\{\Delta\circ E^{k-i}\}\gamma_{\alpha}\gamma_{\sigma}\{E^{i-1}\circ\gamma_{\beta}\}\rangle\rangle_0,$$

$$\langle\langle EE^{k-i}\gamma_{\sigma}\gamma^{\rho}\rangle\rangle_0\langle\langle\gamma_{\rho}\gamma_{\alpha}\{\gamma_{\beta}\circ E^{i-1}\}\Delta\rangle\rangle_0$$
$$= \langle\langle\{G(E^{k-i})\circ\gamma_{\sigma}\}\gamma_{\alpha}\{E^{i-1}\circ\gamma_{\beta}\}\Delta\rangle\rangle_0 + \langle\langle\{E^{k-i}\circ G(\gamma_{\sigma})\}\gamma_{\alpha}\{E^{i-1}\circ\gamma_{\beta}\}\Delta\rangle\rangle_0$$
$$-\langle\langle G(E^{k-i}\circ\gamma_{\sigma})\gamma_{\alpha}\{E^{i-1}\circ\gamma_{\beta}\}\Delta\rangle\rangle_0 - b_1\langle\langle\{E^{k-i}\circ\gamma_{\sigma}\}\gamma_{\alpha}\{E^{i-1}\circ\gamma_{\beta}\}\Delta\rangle\rangle_0,$$

$$\langle\langle E^{k-i}\gamma_{\alpha}\Delta\gamma^{\rho}\rangle\rangle_0\langle\langle\gamma_{\rho}\{\gamma_{\beta}\circ E^{i-1}\}E\gamma_{\sigma}\rangle\rangle_0$$
$$= \langle\langle E^{k-i}\Delta\gamma_{\alpha}\{G(E^{i-1}\circ\gamma_{\beta})\circ\gamma_{\sigma}\}\rangle\rangle_0 + \langle\langle E^{k-i}\Delta\gamma_{\alpha}\{G(\gamma_{\sigma})\circ E^{i-1}\circ\gamma_{\beta}\}\rangle\rangle_0$$
$$-\langle\langle E^{k-i}\Delta\gamma_{\alpha}G(E^{i-1}\circ\gamma_{\beta}\circ\gamma_{\sigma})\rangle\rangle_0 - b_1\langle\langle E^{k-i}\Delta\gamma_{\alpha}\{E^{i-1}\circ\gamma_{\beta}\circ\gamma_{\sigma}\}\rangle\rangle_0,$$

$$\langle\langle E^{k-i}\gamma_{\alpha}\gamma_{\sigma}\gamma^{\rho}\rangle\rangle_0\langle\langle\gamma_{\rho}\{\gamma_{\beta}\circ E^{i-1}\}E\Delta\rangle\rangle_0$$
$$= \langle\langle E^{k-i}\gamma_{\alpha}\gamma_{\sigma}\{G(E^{i-1}\circ\gamma_{\beta})\circ\Delta\}\rangle\rangle_0 + \langle\langle E^{k-i}\gamma_{\alpha}\gamma_{\sigma}\{G(\Delta)\circ E^{i-1}\circ\gamma_{\beta}\}\rangle\rangle_0$$
$$-\langle\langle E^{k-i}\gamma_{\alpha}\gamma_{\sigma}G(E^{i-1}\circ\gamma_{\beta}\circ\Delta)\rangle\rangle_0 - b_1\langle\langle E^{k-i}\gamma_{\alpha}\gamma_{\sigma}\{E^{i-1}\circ\gamma_{\beta}\circ\Delta\}\rangle\rangle_0.$$

By equation (16), we have

$$\langle\langle EE^{k-i}\{\gamma_{\alpha}\circ\gamma_{\beta}\circ E^{i-1}\}\gamma_{\sigma}\Delta\rangle\rangle_0$$
$$= \langle\langle G(E^{k-i})\Delta\gamma_{\sigma}\{E^{i-1}\circ\gamma_{\alpha}\circ\gamma_{\beta}\}\rangle\rangle_0 + \langle\langle E^{k-i}\Delta\gamma_{\sigma}G(E^{i-1}\circ\gamma_{\alpha}\circ\gamma_{\beta})\rangle\rangle_0$$
$$+\langle\langle E^{k-i}\Delta\{E^{i-1}\circ\gamma_{\alpha}\circ\gamma_{\beta}\}G(\gamma_{\sigma})\rangle\rangle_0 + \langle\langle E^{k-i}\gamma_{\sigma}\{E^{i-1}\circ\gamma_{\alpha}\circ\gamma_{\beta}\}G(\Delta)\rangle\rangle_0$$
$$-2(b_1+1)\langle\langle E^{k-i}\gamma_{\sigma}\{E^{i-1}\circ\gamma_{\alpha}\circ\gamma_{\beta}\}\Delta\rangle\rangle_0$$

and

$$\langle\langle\{E^{k-i}\circ\gamma_{\alpha}\}\{\gamma_{\beta}\circ E^{i-1}\}E\gamma_{\sigma}\Delta\rangle\rangle_0$$
$$= \langle\langle G(E^{k-i}\circ\gamma_{\alpha})\{\gamma_{\beta}\circ E^{i-1}\}\gamma_{\sigma}\Delta\rangle\rangle_0 + \langle\langle\{E^{k-i}\circ\gamma_{\alpha}\}G(\gamma_{\beta}\circ E^{i-1})\gamma_{\sigma}\Delta\rangle\rangle_0$$
$$+\langle\langle\{E^{k-i}\circ\gamma_{\alpha}\}\{\gamma_{\beta}\circ E^{i-1}\}G(\gamma_{\sigma})\Delta\rangle\rangle_0 + \langle\langle\{E^{k-i}\circ\gamma_{\alpha}\}\{\gamma_{\beta}\circ E^{i-1}\}\gamma_{\sigma}G(\Delta)\rangle\rangle_0$$
$$-2(b_1+1)\langle\langle\{E^{k-i}\circ\gamma_{\alpha}\}\{\gamma_{\beta}\circ E^{i-1}\}\gamma_{\sigma}\Delta\rangle\rangle_0.$$

Next, using equation (25) to change the type of the following 4-point functions

$$\langle\langle\{G(E^{k-i})\circ\gamma_{\sigma}\}\gamma_{\alpha}\{E^{i-1}\circ\gamma_{\beta}\}\Delta\rangle\rangle_0$$
$$= \langle\langle\{\Delta\circ E^{i-1}\}\gamma_{\alpha}\gamma_{\beta}\{G(E^{k-i})\circ\gamma_{\sigma}\}\rangle\rangle_0 + \langle\langle E^{i-1}\Delta\{\gamma_{\alpha}\circ\gamma_{\beta}\}\{G(E^{k-i})\circ\gamma_{\sigma}\}\rangle\rangle_0$$
$$-\langle\langle E^{i-1}\gamma_{\beta}\{\Delta\circ\gamma_{\alpha}\}\{G(E^{k-i})\circ\gamma_{\sigma}\}\rangle\rangle_0,$$

$$\langle\langle\{G(\gamma_{\sigma})\circ E^{k-i}\}\gamma_{\alpha}\{E^{i-1}\circ\gamma_{\beta}\}\Delta\rangle\rangle_0$$
$$= \langle\langle\{\Delta\circ E^{k-i}\}\{E^{i-1}\circ\gamma_{\beta}\}\gamma_{\alpha}G(\gamma_{\sigma})\rangle\rangle_0 + \langle\langle E^{k-i}\Delta\{E^{i-1}\circ\gamma_{\beta}\}\{\gamma_{\alpha}\circ G(\gamma_{\sigma})\}\rangle\rangle_0$$
$$-\langle\langle E^{k-i}\{E^{i-1}\circ\gamma_{\beta}\}\{\Delta\circ\gamma_{\alpha}\}G(\gamma_{\sigma})\rangle\rangle_0,$$

$$\langle\langle G(E^{k-i}\circ\gamma_{\sigma})\gamma_{\alpha}\{E^{i-1}\circ\gamma_{\beta}\}\Delta\rangle\rangle_0$$
$$= \langle\langle\{\Delta\circ E^{i-1}\}\gamma_{\alpha}\gamma_{\beta}G(E^{k-i}\circ\gamma_{\sigma})\rangle\rangle_0 + \langle\langle E^{i-1}\Delta\{\gamma_{\alpha}\circ\gamma_{\beta}\}G(E^{k-i}\circ\gamma_{\sigma})\rangle\rangle_0$$
$$-\langle\langle E^{i-1}\gamma_{\beta}\{\Delta\circ\gamma_{\alpha}\}G(E^{k-i}\circ\gamma_{\sigma})\rangle\rangle_0,$$

$$\langle\langle\{E^{k-i}\circ\gamma_{\sigma}\}\gamma_{\alpha}\{E^{i-1}\circ\gamma_{\beta}\}\Delta\rangle\rangle_0$$

$$
\begin{aligned}
= \ & \langle\!\langle\{\Delta \circ E^{k-i}\}\{E^{i-1} \circ \gamma_\beta\}\gamma_\alpha\gamma_\sigma\rangle\!\rangle_0 + \langle\!\langle E^{k-i}\Delta\{E^{i-1} \circ \gamma_\beta\}\{\gamma_\alpha \circ \gamma_\sigma\}\rangle\!\rangle_0 \\
& -\langle\!\langle E^{k-i}\{E^{i-1} \circ \gamma_\beta\}\{\Delta \circ \gamma_\alpha\}\gamma_\sigma\rangle\!\rangle_0,
\end{aligned}
$$

$$
\begin{aligned}
& \langle\!\langle G(E^{k-i} \circ \gamma_\alpha)\{\gamma_\beta \circ E^{i-1}\}\gamma_\sigma\Delta\rangle\!\rangle_0 \\
= \ & \langle\!\langle\{\Delta \circ E^{i-1}\}\gamma_\beta\gamma_\sigma G(E^{k-i} \circ \gamma_\alpha)\rangle\!\rangle_0 + \langle\!\langle E^{i-1}\Delta\{\gamma_\beta \circ \gamma_\sigma\}G(E^{k-i} \circ \gamma_\alpha)\rangle\!\rangle_0 \\
& -\langle\!\langle E^{i-1}\{\Delta \circ \gamma_\sigma\}\gamma_\beta G(E^{k-i} \circ \gamma_\alpha)\rangle\!\rangle_0,
\end{aligned}
$$

$$
\begin{aligned}
& \langle\!\langle\{E^{k-i} \circ \gamma_\alpha\}G(\gamma_\beta \circ E^{i-1})\gamma_\sigma\Delta\rangle\!\rangle_0 \\
= \ & \langle\!\langle\{\Delta \circ E^{k-i}\}\gamma_\alpha\gamma_\sigma G(E^{i-1} \circ \gamma_\beta)\rangle\!\rangle_0 + \langle\!\langle E^{k-i}\Delta\{\gamma_\alpha \circ \gamma_\sigma\}G(E^{i-1} \circ \gamma_\beta)\rangle\!\rangle_0 \\
& -\langle\!\langle E^{k-i}\{\Delta \circ \gamma_\sigma\}\gamma_\alpha G(E^{k-i} \circ \gamma_\beta)\rangle\!\rangle_0,
\end{aligned}
$$

$$
\begin{aligned}
& \langle\!\langle\{E^{k-i} \circ \gamma_\alpha\}\{\gamma_\beta \circ E^{i-1}\}G(\gamma_\sigma)\Delta\rangle\!\rangle_0 \\
= \ & \langle\!\langle\{\Delta \circ E^{k-i}\}\gamma_\alpha\{\gamma_\beta \circ E^{i-1}\}G(\gamma_\sigma)\rangle\!\rangle_0 + \langle\!\langle E^{k-i}\Delta\{\gamma_\alpha \circ \gamma_\beta \circ E^{i-1}\}G(\gamma_\sigma)\rangle\!\rangle_0 \\
& -\langle\!\langle E^{k-i}\gamma_\alpha\{\Delta \circ \gamma_\beta \circ E^{i-1}\}G(\gamma_\sigma)\rangle\!\rangle_0,
\end{aligned}
$$

$$
\begin{aligned}
& \langle\!\langle\{E^{k-i} \circ \gamma_\alpha\}\{\gamma_\beta \circ E^{i-1}\}\gamma_\sigma G(\Delta)\rangle\!\rangle_0 \\
= \ & \langle\!\langle\{G(\Delta) \circ E^{k-i}\}\gamma_\alpha\{\gamma_\beta \circ E^{i-1}\}\gamma_\sigma\rangle\!\rangle_0 + \langle\!\langle E^{k-i}\gamma_\sigma\{\gamma_\alpha \circ \gamma_\beta \circ E^{i-1}\}G(\Delta)\rangle\!\rangle_0 \\
& -\langle\!\langle E^{k-i}\gamma_\alpha\gamma_\sigma\{G(\Delta) \circ \gamma_\beta \circ E^{i-1}\}\rangle\!\rangle_0,
\end{aligned}
$$

$$
\begin{aligned}
& \langle\!\langle\{E^{k-i} \circ \gamma_\alpha\}\{\gamma_\beta \circ E^{i-1}\}\gamma_\sigma\Delta\rangle\!\rangle_0 \\
= \ & \langle\!\langle\{\Delta \circ E^{k-i}\}\gamma_\alpha\{\gamma_\beta \circ E^{i-1}\}\gamma_\sigma\rangle\!\rangle_0 + \langle\!\langle E^{k-i}\gamma_\sigma\{\gamma_\alpha \circ \gamma_\beta \circ E^{i-1}\}\Delta\rangle\!\rangle_0 \\
& -\langle\!\langle E^{k-i}\gamma_\alpha\gamma_\sigma\{\Delta \circ \gamma_\beta \circ E^{i-1}\}\rangle\!\rangle_0.
\end{aligned}
$$

The proof is completed by a tedious computation, i.e., by firstly collecting all the above results and using Lemma 3.1 to reduce 4-point functions to 3-point functions, and secondly using Lemma 3.2 to simplify the resulting expression, and then symmetrizing the result. $\qquad\square$

Lemma A.2. *For any* α, β, σ,

$$
\frac{1}{2}\sum_{g \in S_3}\langle\!\langle E^k\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\gamma^\mu\rangle\!\rangle_0\langle\!\langle\gamma_\mu\gamma_{\varsigma_{g(3)}}\gamma_\rho\gamma^\rho\rangle\!\rangle_0
$$

$$
\begin{aligned}
= \ & -\frac{1}{2}\sum_{g \in S_3}\sum_{i=1}^{k}\langle\!\langle\{G(E^{k-i}) \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_\mu\gamma^\mu\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\!\rangle_0 \\
& -\frac{1}{2}\sum_{g \in S_3}\sum_{i=1}^{k}\langle\!\langle G(E^{k-i} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}})\gamma_\mu\gamma^\mu\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\!\rangle_0 \\
& +\sum_{g \in S_3}\sum_{i=1}^{k}\langle\!\langle\{G(E^{k-i} \circ \gamma_{\varsigma_{g(1)}}) \circ \gamma_{\varsigma_{g(2)}}\}\gamma_\mu\gamma^\mu\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\!\rangle_0 \\
& -\frac{1}{2}\sum_{g \in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\!\langle\{G(E^{k-i}) \circ \Delta \circ E^{j-1}\}G(E^{i-j-1} \circ \gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\!\rangle_0 \\
& -\frac{1}{2}\sum_{g \in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\!\langle\{G(E^{k-i}) \circ E^{j-1}\}G(\Delta \circ E^{i-j-1} \circ \gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\!\rangle_0
\end{aligned}
$$

$$+3\sum_{i=1}^{k}(i-1)\langle\langle\{G(E^{k-i})\circ E^{i-2}\}\Delta\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\rangle_0$$

$$-\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{k-i}\langle\langle\{\Delta\circ E^{j-1}\}G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})G(E^{k-i-j}\circ\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$-\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{k-i}\langle\langle E^{k-i-j}G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})G(\Delta\circ E^{j-1}\circ\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}(k-i)\langle\langle\{\Delta\circ E^{k-i-1}\}G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$

$$+\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{\Delta\circ E^{j-1}\circ\gamma_{\varsigma_{g(1)}}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(2)}})G(E^{i-j-1}\circ\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{E^{i-j-1}\circ\gamma_{\varsigma_{g(1)}}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(2)}})G(\Delta\circ E^{j-1}\circ\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$-\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{\Delta\circ E^{i-2}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0,$$

where $k\geq 0$ *and* $\{\varsigma_1,\varsigma_2,\varsigma_3\}=\{\alpha,\beta,\sigma\}$.

Proof. It follows from equation (28) that

$$\langle\langle E^k\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\gamma^\mu\rangle\rangle_0\langle\langle\gamma_\mu\gamma_{\varsigma_{g(3)}}\gamma_\rho\gamma^\rho\rangle\rangle_0$$

$$=\quad -\sum_{i=1}^{k}\langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\gamma_\rho\gamma^\rho\rangle\rangle_0$$

$$-\sum_{i=1}^{k}\langle\langle\{G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\circ E^{k-i}\}\gamma_{\varsigma_{g(3)}}\gamma_\rho\gamma^\rho\rangle\rangle_0$$

$$+\sum_{i=1}^{k}\langle\langle\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ E^{i-1}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\gamma_\rho\gamma^\rho\rangle\rangle_0$$

$$+\sum_{i=1}^{k}\langle\langle\{G(E^{i-1}\circ\gamma_{\varsigma_{g(2)}})\circ E^{k-i}\circ\gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(3)}}\gamma_\rho\gamma^\rho\rangle\rangle_0.$$

Using equation (25), we have

$$\langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\gamma_\rho\gamma^\rho\rangle\rangle_0$$

$$=\quad \langle\langle\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_\rho\gamma^\rho\{E^{i-1}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$+\langle\langle E^{i-1}\gamma_{\varsigma_{g(3)}}\gamma_\rho\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma^\rho\}\rangle\rangle_0$$

$$-\langle\langle E^{i-1}\gamma_\rho\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\{\gamma_{\varsigma_{g(3)}}\circ\gamma^\rho\}\rangle\rangle_0,$$

$$\langle\langle\{G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\circ E^{k-i}\}\gamma_{\varsigma_{g(3)}}\gamma_\rho\gamma^\rho\rangle\rangle_0$$

$$=\quad \langle\langle G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\gamma_\rho\gamma^\rho\{E^{k-i}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$+\langle\langle E^{k-i}\gamma_{\varsigma_{g(3)}}\gamma_\rho\{G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\circ\gamma^\rho\}\rangle\rangle_0$$

$$-\langle\langle E^{k-i}\gamma_\rho\{\gamma_{\varsigma_{g(3)}}\circ\gamma^\rho\}G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\rangle\rangle_0,$$

$$\langle\langle\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ E^{i-1}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\gamma_\rho\gamma^\rho\rangle\rangle_0$$
$$=\quad \langle\langle E^{i-1}\gamma_{\varsigma_{g(3)}}\gamma_\rho\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ\gamma_{\varsigma_{g(2)}}\circ\gamma^\rho\}\rangle\rangle_0$$
$$+\langle\langle\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ\gamma_{\varsigma_{g(2)}}\}\gamma_\rho\gamma^\rho\{E^{i-1}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$
$$-\langle\langle E^{i-1}\gamma_\rho\{\gamma^\rho\circ\gamma_{\varsigma_{g(3)}}\}\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ\gamma_{\varsigma_{g(2)}}\}\rangle\rangle_0,$$

and

$$\langle\langle\{G(E^{i-1}\circ\gamma_{\varsigma_{g(2)}})\circ E^{k-i}\circ\gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(3)}}\gamma_\rho\gamma^\rho\rangle\rangle_0$$
$$=\quad \langle\langle E^{k-i}\gamma_{\varsigma_{g(3)}}\gamma_\rho\{G(E^{i-1}\circ\gamma_{\varsigma_{g(2)}})\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^\rho\}\rangle\rangle_0$$
$$+\langle\langle\{G(E^{i-1}\circ\gamma_{\varsigma_{g(2)}})\circ\gamma_{\varsigma_{g(1)}}\}\gamma_\rho\gamma^\rho\{E^{k-i}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$
$$-\langle\langle E^{k-i}\gamma_\rho\{\gamma^\rho\circ\gamma_{\varsigma_{g(3)}}\}\{G(E^{i-1}\circ\gamma_{\varsigma_{g(2)}})\circ\gamma_{\varsigma_{g(1)}}\}\rangle\rangle_0.$$

The remainder of the argument is analogous to that in Lemma A.1. □

Lemma A.3. *For any α,β,σ,*

$$\sum_{g\in S_3}\langle\langle E^k\gamma_{\varsigma_{g(1)}}\gamma^\mu\gamma^\rho\rangle\rangle_0\langle\langle\gamma_\mu\gamma_\rho\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$

$$=\quad -\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^\mu\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\{E^{i-1}\circ\gamma_\mu\}\rangle\rangle_0$$

$$-\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle G(E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^\mu)\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\{E^{k-i}\circ\gamma_\mu\}\rangle\rangle_0$$

$$+\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ\gamma^\mu\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\{E^{i-1}\circ\gamma_\mu\}\rangle\rangle_0$$

$$+\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{G(E^{i-1}\circ\gamma^\mu)\circ\gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\{E^{k-i}\circ\gamma_\mu\}\rangle\rangle_0$$

$$-6\sum_{i=1}^{k}\sum_{j=1}^{k-i}\langle\langle G(E^{i+j-2}\circ\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\circ\gamma^\mu)G(\gamma_\mu)E^{k-i-j}\rangle\rangle_0$$

$$+3\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{k-i}\langle\langle\{G(E^{k-j-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma^\mu)\circ G(\gamma_\mu)\}E^{j-1}\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$

$$-2\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{k-i}\langle\langle\{G(E^{k-j-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^\mu)\circ G(\gamma_\mu)\}E^{j-1}\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{k-i}\langle\langle\{G(E^{i+j-2}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^\mu)\circ G(\gamma_\mu)\}E^{k-i-j}\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$+6\sum_{i=1}^{k}\sum_{j=1}^{k-i}\langle\langle\{G(E^{k-j-1}\circ\gamma^\mu)\circ G(\gamma_\mu)\}E^{j-1}\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\rangle_0,$$

where $k\geq 0$ and $\{\varsigma_1,\varsigma_2,\varsigma_3\}=\{\alpha,\beta,\sigma\}$.

Proof. The proof is completed by the same argument in that of Lemma A.2. $\qquad \square$

Lemma A.4. *For $k \geq 0$, and any α, β, σ,*

$$\langle\langle E^k \gamma_\rho \gamma^\rho \gamma^\mu \rangle\rangle_0 \langle\langle \gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\sigma \rangle\rangle_0$$

$$= -\sum_{i=1}^{k} \langle\langle \{G(E^{k-i}) \circ \Delta \circ E^{i-1}\} \gamma_\alpha \gamma_\beta \gamma_\sigma \rangle\rangle_0 - \sum_{i=1}^{k} \langle\langle \{G(\Delta \circ E^{i-1}) \circ E^{k-i}\} \gamma_\alpha \gamma_\beta \gamma_\sigma \rangle\rangle_0$$

$$+ \sum_{i=1}^{k} \langle\langle \{\Delta \circ E^{k-1}\} \gamma_\alpha \gamma_\beta \gamma_\sigma \rangle\rangle_0.$$

Proof. It follows from equations (28) and (30). $\qquad \square$

By equation (41) and Lemmas A.1, A.2, A.3, A.4, one can obtain an expression for $G_0(E^k, \gamma_\alpha, \gamma_\beta, \gamma_\sigma)$. To simplify this expression, we have to reduce the number of 4-point functions. In order to cancel redundant 4-point functions, the strategy is to repeatedly use equation (25) to obtain some common 4-point functions, and then use Lemma 3.1 and Lemma 3.2 for possible reduction. The following results are used to maximally reduce the number of 4-point functions.

Lemma A.5.

$$\langle\langle \{G(E^{k-i}) \circ \Delta\} \gamma_{\varsigma_{g(1)}} \gamma_{\varsigma_{g(2)}} \{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\} \rangle\rangle_0$$

$$= \langle\langle \{G(E^{k-i}) \circ \Delta \circ E^{i-1}\} \gamma_\alpha \gamma_\beta \gamma_\sigma \rangle\rangle_0$$

$$- \sum_{j=1}^{i-1} \langle\langle \{G(E^{k-i}) \circ E^{i-j-1}\} \Delta G(E^{j-1} \circ \gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma) \rangle\rangle_0$$

$$+ \sum_{j=1}^{i-1} \langle\langle \{G(E^{k-i}) \circ \Delta \circ E^{j-1}\} G(E^{i-j-1} \circ \gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}) \gamma_{\varsigma_{g(1)}} \rangle\rangle_0$$

$$+ \sum_{j=1}^{i-1} \langle\langle \{G(E^{k-i}) \circ \Delta \circ E^{i-j-1}\} G(E^{j-1} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(3)}}) \gamma_{\varsigma_{g(2)}} \rangle\rangle_0$$

$$- \sum_{j=1}^{i-1} \langle\langle \{G(E^{k-i}) \circ \Delta \circ E^{j-1}\} G(E^{i-j-1} \circ \gamma_{\varsigma_{g(3)}}) \{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\} \rangle\rangle_0.$$

Proof. By equation (25), we have

$$\langle\langle \{G(E^{k-i}) \circ \Delta\} \gamma_{\varsigma_{g(1)}} \gamma_{\varsigma_{g(2)}} \{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\} \rangle\rangle_0$$

$$= \langle\langle \{G(E^{k-i}) \circ \Delta \circ E^{i-1}\} \gamma_\alpha \gamma_\beta \gamma_\sigma \rangle\rangle_0 + \langle\langle E^{i-1} \{\gamma_{\varsigma_{g(2)}} \circ \gamma_{\varsigma_{g(3)}}\} \gamma_{\varsigma_{g(1)}} \{G(E^{k-i}) \circ \Delta\} \rangle\rangle_0$$

$$- \langle\langle E^{i-1} \gamma_{\varsigma_{g(1)}} \gamma_{\varsigma_{g(3)}} \{G(E^{k-i}) \circ \Delta \circ \gamma_{\varsigma_{g(2)}}\} \rangle\rangle_0.$$

The proof is completed by using Lemma 3.1. $\qquad \square$

By the same argument as in Lemma A.5, we have the following results, i.e., Lemmas A.6, A.7, A.8, A.9.

Lemma A.6.

$$\langle\langle G(\Delta \circ E^{k-i}) \gamma_{\varsigma_{g(1)}} \gamma_{\varsigma_{g(2)}} \{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\} \rangle\rangle_0$$

$$= \langle\langle \{G(\Delta \circ E^{k-i}) \circ E^{i-1}\} \gamma_\alpha \gamma_\beta \gamma_\sigma \rangle\rangle_0$$

$$-\sum_{j=1}^{i-1}\langle\langle G(\Delta\circ E^{k-i})E^{i-j-1}G(E^{j-1}\circ\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma)\rangle\rangle_0$$

$$+\sum_{j=1}^{i-1}\langle\langle\{G(\Delta\circ E^{k-i})\circ E^{j-1}\}G(E^{i-j-1}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}})\gamma_{\varsigma_{g(1)}}\rangle\rangle_0$$

$$+\sum_{j=1}^{i-1}\langle\langle\{G(\Delta\circ E^{k-i})\circ E^{i-j-1}\}G(E^{j-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}})\gamma_{\varsigma_{g(2)}}\rangle\rangle_0$$

$$-\sum_{j=1}^{i-1}\langle\langle\{G(\Delta\circ E^{k-i})\circ E^{j-1}\}G(E^{i-j-1}\circ\gamma_{\varsigma_{g(3)}})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\rangle\rangle_0.$$

Lemma A.7.

$$\langle\langle\{\Delta\circ E^{k-i}\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\{E^{i-1}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$=\quad\langle\langle\{\Delta\circ E^{k-1}\}\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0$$

$$-\sum_{j=1}^{i-1}\langle\langle\Delta E^{k-j-1}G(E^{j-1}\circ\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma)\rangle\rangle_0$$

$$+\sum_{j=1}^{i-1}\langle\langle\{\Delta\circ E^{k-i+j-1}\}G(E^{i-j-1}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}})\gamma_{\varsigma_{g(1)}}\rangle\rangle_0$$

$$+\sum_{j=1}^{i-1}\langle\langle\{\Delta\circ E^{k-j-1}\}G(E^{j-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}})\gamma_{\varsigma_{g(2)}}\rangle\rangle_0$$

$$-\sum_{j=1}^{i-1}\langle\langle\{\Delta\circ E^{k-i+j-1}\}G(E^{i-j-1}\circ\gamma_{\varsigma_{g(3)}})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\rangle\rangle_0.$$

Lemma A.8.

$$\langle\langle G(E^{k-i})\Delta\gamma_{\varsigma_{g(1)}}\{E^{i-1}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$=\quad\langle\langle\{\Delta\circ E^{i-1}\}\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\gamma_{\varsigma_{g(1)}}G(E^{k-i})\rangle\rangle_0$$

$$+\sum_{j=1}^{i-1}\langle\langle\{G(\Delta\circ E^{i-j-1})\circ G(E^{k-i})\}E^{j-1}\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\rangle_0$$

$$+\sum_{j=1}^{i-1}\langle\langle\{G(E^{k-i})\circ E^{i-j-1}\}\Delta G(E^{j-1}\circ\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma)\rangle\rangle_0$$

$$-\sum_{j=1}^{i-1}\langle\langle\{G(E^{k-i})\circ\Delta\circ E^{j-1}\}G(E^{i-j-1}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}})\gamma_{\varsigma_{g(1)}}\rangle\rangle_0$$

$$-\sum_{j=1}^{i-1}\langle\langle\{G(E^{k-i})\circ E^{i-j-1}\}G(\Delta\circ E^{j-1}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0.$$

Lemma A.9.

$$\langle\langle\{E^{k-1}\circ\gamma_{\varsigma_{g(2)}}\}\Delta\gamma_{\varsigma_{g(1)}}G(\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$=\quad\langle\langle\{\Delta\circ E^{k-1}\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}G(\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+ \sum_{i=1}^{k-1} \langle\langle\{G(\Delta \circ E^{k-i-1}) \circ E^{i-1}\}\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}G(\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+ \sum_{i=1}^{k-1} \langle\langle\{\Delta \circ E^{k-i-1}\}G(E^{i-1} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}})G(\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$- \sum_{i=1}^{k-1} \langle\langle\{\Delta \circ E^{i-1}\}G(E^{k-i-1} \circ \gamma_{\varsigma_{g(2)}})\{\gamma_{\varsigma_{g(1)}} \circ G(\gamma_{\varsigma_{g(3)}})\}\rangle\rangle_0$$

$$- \sum_{i=1}^{k-1} \langle\langle E^{k-i-1}G(\Delta \circ E^{i-1} \circ \gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}} \circ G(\gamma_{\varsigma_{g(3)}})\}\rangle\rangle_0.$$

Next, the following two lemmas produce the 4-point function appearing in $\{\gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}\Phi_k$ which is used for simplification.

Lemma A.10.

$$\sum_{i=1}^{k} \langle\langle\{G(E^{k-i}) \circ E^{i-1} \circ \Delta\}\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0$$

$$= -\sum_{i=1}^{k-1} \langle\langle\{E^{i-1} \circ \gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}\gamma_\mu\gamma^\mu G(E^{k-i})\rangle\rangle_0$$

$$+ \frac{1}{3}\sum_{g \in S_3}\sum_{i=1}^{k} \langle\langle\{G(E^{k-i}) \circ E^{i-1} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma^\mu\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_\mu\rangle\rangle_0$$

$$+ \sum_{i=1}^{k} \langle\langle\{G(E^{k-i}) \circ E^{i-1}\}\gamma_\mu\gamma^\mu\{\gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}\rangle\rangle_0$$

$$- \sum_{i=1}^{k}\sum_{j=1}^{i-1} \langle\langle G(E^{k-i})E^{i-j-1}G(\Delta \circ E^{j-1} \circ \gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma)\rangle\rangle_0$$

$$- \frac{1}{6}\sum_{g \in S_3}\sum_{i=1}^{k} \langle\langle\{G(E^{k-i}) \circ \gamma^\mu\}\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\{E^{i-1} \circ \gamma_\mu\}\rangle\rangle_0$$

$$- \sum_{i=1}^{k}\sum_{j=1}^{i-1} \langle\langle\{G(E^{k-i}) \circ E^{i-j-1}\}\Delta G(E^{j-1} \circ \gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma)\rangle\rangle_0$$

$$+ \sum_{i=1}^{k} (i-1)\langle\langle\{G(E^{k-i}) \circ E^{i-2}\}\Delta\{\gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}\rangle\rangle_0.$$

Proof. By equation (25), we have

$$\langle\langle\{G(E^{k-i}) \circ E^{i-1} \circ \Delta\}\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0$$
$$= \langle\langle\{G(E^{k-i}) \circ E^{i-1} \circ \gamma^\mu \circ \gamma_\mu\}\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0$$
$$= \langle\langle\{G(E^{k-i}) \circ E^{i-1} \circ \gamma^\mu\}\gamma_\alpha\gamma_\beta\{\gamma_\mu \circ \gamma_\sigma\}\rangle\rangle_0$$
$$+ \langle\langle\{G(E^{k-i}) \circ E^{i-1} \circ \gamma_\alpha \circ \gamma^\mu\}\gamma_\beta\gamma_\sigma\gamma_\mu\rangle\rangle_0$$
$$- \langle\langle\{G(E^{k-i}) \circ E^{i-1} \circ \gamma^\mu\}\gamma_\beta\{\gamma_\alpha \circ \gamma_\sigma\}\gamma_\mu\rangle\rangle_0$$
$$= \langle\langle\{G(E^{k-i}) \circ E^{i-1} \circ \gamma_\sigma \circ \gamma^\mu\}\gamma_\alpha\gamma_\beta\gamma_\mu\rangle\rangle_0$$

$$+\langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma_\alpha\circ\gamma^\mu\}\gamma_\beta\gamma_\sigma\gamma_\mu\rangle\rangle_0$$
$$-\langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma^\mu\}\gamma_\beta\{\gamma_\alpha\circ\gamma_\sigma\}\gamma_\mu\rangle\rangle_0.$$

Using equation (25) again, we have

$$\langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma^\mu\}\gamma_\beta\{\gamma_\alpha\circ\gamma_\sigma\}\gamma_\mu\rangle\rangle_0$$
$$=\quad\langle\langle\{G(E^{k-i})\circ E^{i-1}\}\{\gamma_\beta\circ\gamma^\mu\}\{\gamma_\alpha\circ\gamma_\sigma\}\gamma_\mu\rangle\rangle_0$$
$$+\langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma_\alpha\circ\gamma_\sigma\}\gamma_\mu\gamma^\mu\gamma_\beta\rangle\rangle_0$$
$$-\langle\langle\{G(E^{k-i})\circ E^{i-1}\}\gamma_\mu\gamma^\mu\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\rangle_0,$$

$$\langle\langle\{G(E^{k-i})\circ E^{i-1}\}\{\gamma_\beta\circ\gamma^\mu\}\{\gamma_\alpha\circ\gamma_\sigma\}\gamma_\mu\rangle\rangle_0$$
$$=\quad\langle\langle\{E^{i-1}\circ\gamma_\mu\}\{\gamma_\beta\circ\gamma^\mu\}\{\gamma_\alpha\circ\gamma_\sigma\}G(E^{k-i})\rangle\rangle_0$$
$$+\langle\langle E^{i-1}\gamma_\mu\{\gamma_\beta\circ\gamma^\mu\}\{G(E^{k-i})\circ\gamma_\alpha\circ\gamma_\sigma\}\rangle\rangle_0$$
$$-\langle\langle E^{i-1}\{\gamma_\beta\circ\gamma^\mu\}\{\gamma_\alpha\circ\gamma_\sigma\circ\gamma_\mu\}G(E^{k-i})\rangle\rangle_0,$$

$$\langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma_\alpha\circ\gamma_\sigma\}\gamma_\mu\gamma^\mu\gamma_\beta\rangle\rangle_0$$
$$=\quad\langle\langle\{E^{i-1}\circ\gamma_\alpha\circ\gamma_\sigma\}\gamma_\beta\gamma^\mu\{G(E^{k-i})\circ\gamma_\mu\}\rangle\rangle_0$$
$$+\langle\langle\{E^{i-1}\circ\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\gamma_\mu\gamma^\mu G(E^{k-i})\rangle\rangle_0$$
$$-\langle\langle\{E^{i-1}\circ\gamma_\alpha\circ\gamma_\sigma\}G(E^{k-i})\gamma^\mu\{\gamma_\beta\circ\gamma_\mu\}\rangle\rangle_0,$$

$$\langle\langle\{E^{i-1}\circ\gamma_\alpha\circ\gamma_\sigma\}\gamma_\beta\gamma^\mu\{G(E^{k-i})\circ\gamma_\mu\}\rangle\rangle_0$$
$$=\quad\langle\langle\{E^{i-1}\circ\gamma^\mu\}\{\gamma_\alpha\circ\gamma_\sigma\}\gamma_\beta\{G(E^{k-i})\circ\gamma_\mu\}\rangle\rangle_0$$
$$+\langle\langle E^{i-1}\gamma_\beta\gamma^\mu\{G(E^{k-i})\circ\gamma_\alpha\circ\gamma_\sigma\circ\gamma_\mu\}\rangle\rangle_0$$
$$-\langle\langle E^{i-1}\{\gamma_\alpha\circ\gamma_\sigma\}\gamma_\beta\{G(E^{k-i})\circ\Delta\}\rangle\rangle_0,$$

and

$$\langle\langle\{E^{i-1}\circ\gamma_\alpha\circ\gamma_\sigma\}G(E^{k-i})\gamma^\mu\{\gamma_\beta\circ\gamma_\mu\}\rangle\rangle_0$$
$$=\quad\langle\langle\{E^{i-1}\circ\gamma^\mu\}\{\gamma_\alpha\circ\gamma_\sigma\}\{\gamma_\beta\circ\gamma_\mu\}G(E^{k-i})\rangle\rangle_0$$
$$+\langle\langle E^{i-1}\gamma^\mu\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\circ\gamma_\mu\}G(E^{k-i})\rangle\rangle_0$$
$$-\langle\langle E^{i-1}\{\gamma_\alpha\circ\gamma_\sigma\}\{\gamma_\beta\circ\Delta\}G(E^{k-i})\rangle\rangle_0.$$

Collecting all the above equations and using Lemma 3.1 and equation (30), and then symmetrizing the resulting expression, the proof follows. $\qquad\square$

Lemma A.11.

$$\langle\langle\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_\mu\gamma^\mu\{E^{i-1}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$
$$=\quad\langle\langle\{E^{i-1}\circ\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\gamma_\mu\gamma^\mu G(E^{k-i})\rangle\rangle_0$$
$$+\langle\langle\{G(E^{k-i})\circ\gamma^\mu\}\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\{E^{i-1}\circ\gamma_\mu\}\rangle\rangle_0$$
$$-\langle\langle G(E^{k-i})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\{E^{i-1}\circ\gamma_\mu\}\{\gamma_{\varsigma_{g(3)}}\circ\gamma^\mu\}\rangle\rangle_0$$
$$+\sum_{j=1}^{i-1}\langle\langle\{G(E^{k-i})\circ E^{i-j-1}\}\Delta G(E^{j-1}\circ\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma)\rangle\rangle_0$$
$$-\sum_{j=1}^{i-1}\langle\langle\{G(E^{k-i})\circ\Delta\circ E^{j-1}\}G(E^{i-j-1}\circ\gamma_{\varsigma_{g(3)}})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\rangle\rangle_0.$$

Proof. By equation (25), we have

$$
\langle\langle\{G(E^{k-i}) \circ \gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_\mu\gamma^\mu\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0
$$
$$
= \langle\langle\{E^{i-1} \circ \gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}\gamma_\mu\gamma^\mu G(E^{k-i})\rangle\rangle_0
$$
$$
+\langle\langle\{G(E^{k-i}) \circ \gamma^\mu\}\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\}\gamma_\mu\rangle\rangle_0
$$
$$
-\langle\langle G(E^{k-i})\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_\mu\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}} \circ \gamma^\mu\}\rangle\rangle_0,
$$

and

$$
\langle\langle\{G(E^{k-i}) \circ \gamma^\mu\}\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\}\gamma_\mu\rangle\rangle_0
$$
$$
= \langle\langle\{G(E^{k-i}) \circ \gamma^\mu\}\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\{E^{i-1} \circ \gamma_\mu\}\rangle\rangle_0
$$
$$
+\langle\langle E^{i-1}\gamma_\mu\{G(E^{k-i}) \circ \gamma^\mu\}\{\gamma_\alpha \circ \gamma_\beta \circ \gamma_\sigma\}\rangle\rangle_0
$$
$$
-\langle\langle E^{i-1}\gamma_{\varsigma_{g(3)}}\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}} \circ \gamma_\mu\}\{G(E^{k-i}) \circ \gamma^\mu\}\rangle\rangle_0.
$$

Notice that

$$
\langle\langle G(E^{k-i})\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_\mu\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}} \circ \gamma^\mu\}\rangle\rangle_0
$$
$$
= \langle\langle G(E^{k-i})\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\{E^{i-1} \circ \gamma_\mu\}\{\gamma_{\varsigma_{g(3)}} \circ \gamma^\mu\}\rangle\rangle_0.
$$

The proof is completed by using Lemma 3.1 and equation (30). □

The following results are useful for further simplification.

Lemma A.12.

$$
-\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{G(E^{k-i} \circ \gamma_{\varsigma_{g(1)}}) \circ E^{i-1} \circ \gamma^\mu\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_\mu\rangle\rangle_0
$$

$$
-\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{G(E^{k-i}) \circ E^{i-1} \circ \gamma^\mu\}\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\gamma_\mu\rangle\rangle_0
$$

$$
+\frac{1}{3}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{G(E^{k-i}) \circ E^{i-1} \circ \gamma_{\varsigma_{g(1)}} \circ \gamma^\mu\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_\mu\rangle\rangle_0
$$

$$
= -\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{G(E^{k-i} \circ \gamma_{\varsigma_{g(1)}}) \circ \gamma_{\varsigma_{g(2)}}\}\gamma_\mu\gamma^\mu\{E^{i-1} \circ \gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0
$$

$$
-\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{\Delta \circ E^{i-1}\}G(E^{k-i} \circ \gamma_{\varsigma_{g(1)}})\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\rangle\rangle_0
$$

$$
+\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle G(E^{k-i} \circ \gamma_{\varsigma_{g(1)}})\gamma_{\varsigma_{g(2)}}\{\gamma_{\varsigma_{g(3)}} \circ \gamma^\mu\}\{E^{i-1} \circ \gamma_\mu\}\rangle\rangle_0
$$

$$
-\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{\Delta \circ E^{i-1}\}\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}G(E^{k-i})\rangle\rangle_0
$$

$$
+\frac{1}{3}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle G(E^{k-i})\{\gamma_{\varsigma_{g(1)}} \circ \gamma_{\varsigma_{g(2)}} \circ \gamma^\mu\}\gamma_{\varsigma_{g(3)}}\{E^{i-1} \circ \gamma_\mu\}\rangle\rangle_0
$$

$$+\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle\{\Delta\circ E^{i-1}\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\{G(E^{k-i})\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\frac{1}{6}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\langle G(E^{k-i})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\{\gamma_{\varsigma_{g(3)}}\circ\gamma^{\mu}\}\{E^{i-1}\circ\gamma_{\mu}\}\rangle\rangle_0$$

$$-\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{G(\Delta\circ E^{i-j-1})\circ E^{j-1}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$

$$-\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{\Delta\circ E^{i-j-1}\circ\gamma_{\varsigma_{g(1)}}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(2)}})G(E^{j-1}\circ\gamma_{\varsigma_{g(3)}})\rangle\rangle_0$$

$$+\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\langle\{\Delta\circ E^{i-2}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0.$$

Proof. Using equation (25), we have

$$\langle\langle\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ E^{i-1}\circ\gamma^{\mu}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_{\mu}\rangle\rangle_0$$
$$=\ \langle\langle\{\Delta\circ E^{i-1}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$
$$+\langle\langle\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ\gamma_{\varsigma_{g(3)}}\}\{E^{i-1}\circ\gamma^{\mu}\}\gamma_{\mu}\gamma_{\varsigma_{g(2)}}\rangle\rangle_0$$
$$-\langle\langle\{E^{i-1}\circ\gamma^{\mu}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\gamma_{\varsigma_{g(2)}}\{\gamma_{\varsigma_{g(3)}}\circ\gamma_{\mu}\}\rangle\rangle_0,$$

$$\langle\langle\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ\gamma_{\varsigma_{g(3)}}\}\{E^{i-1}\circ\gamma^{\mu}\}\gamma_{\mu}\gamma_{\varsigma_{g(2)}}\rangle\rangle_0$$
$$=\ \langle\langle E^{i-1}\Delta\gamma_{\varsigma_{g(2)}}\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$
$$+\langle\langle\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ\gamma_{\varsigma_{g(3)}}\}\gamma_{\mu}\gamma^{\mu}\{E^{i-1}\circ\gamma_{\varsigma_{g(2)}}\}\rangle\rangle_0$$
$$-\langle\langle E^{i-1}\gamma^{\mu}\{\gamma_{\varsigma_{g(3)}}\circ\gamma_{\mu}\}\{G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0,$$

and

$$\langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma^{\mu}\}\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\gamma_{\mu}\rangle\rangle_0$$
$$=\ \langle\langle\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\gamma_{\mu}\{E^{i-1}\circ\gamma^{\mu}\}\rangle\rangle_0$$
$$+\langle\langle\{\Delta\circ E^{i-1}\}G(E^{k-i})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$
$$-\langle\langle\{E^{i-1}\circ\gamma^{\mu}\}G(E^{k-i})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_{\mu}\}\gamma_{\varsigma_{g(3)}}\rangle\rangle_0.$$

Since by equation (25) again, we have

$$\langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^{\mu}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_{\mu}\rangle\rangle_0$$
$$=\ \langle\langle\{E^{i-1}\circ\gamma^{\mu}\}\gamma_{\varsigma_{g(2)}}\gamma_{\mu}\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0$$
$$+\langle\langle\{\Delta\circ E^{i-1}\}\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\rangle\rangle_0$$
$$-\langle\langle\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\}\{E^{i-1}\circ\gamma^{\mu}\}\gamma_{\varsigma_{g(2)}}\{\gamma_{\varsigma_{g(3)}}\circ\gamma_{\mu}\}\rangle\rangle_0,$$

and

$$\langle\langle\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\}\{E^{i-1}\circ\gamma^{\mu}\}\gamma_{\varsigma_{g(2)}}\{\gamma_{\varsigma_{g(3)}}\circ\gamma_{\mu}\}\rangle\rangle_0$$
$$=\ \langle\langle\{G(E^{k-i})\circ E^{i-1}\circ\gamma^{\mu}\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\{\gamma_{\varsigma_{g(3)}}\circ\gamma_{\mu}\}\rangle\rangle_0$$
$$+\langle\langle G(E^{k-i})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\{\gamma_{\varsigma_{g(3)}}\circ\gamma_{\mu}\}\{E^{i-1}\circ\gamma^{\mu}\}\rangle\rangle_0$$

$$-\langle\!\langle G(E^{k-i})\gamma_{\varsigma_{g(1)}}\{\gamma_{\varsigma_{g(3)}}\circ\gamma_\mu\}\{E^{i-1}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma^\mu\}\rangle\!\rangle_0$$

$$= \quad \langle\!\langle\{G(E^{k-i})\circ\gamma_{\varsigma_{g(3)}}\circ E^{i-1}\circ\gamma^\mu\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\gamma_\mu\rangle\!\rangle_0$$

$$+\langle\!\langle G(E^{k-i})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\{\gamma_{\varsigma_{g(3)}}\circ\gamma_\mu\}\{E^{i-1}\circ\gamma^\mu\}\rangle\!\rangle_0$$

$$-\langle\!\langle G(E^{k-i})\gamma_{\varsigma_{g(1)}}\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\circ\gamma_\mu\}\{E^{i-1}\circ\gamma^\mu\}\rangle\!\rangle_0.$$

Hence we have

$$\langle\!\langle\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\circ E^{i-1}\circ\gamma^\mu\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_\mu\rangle\!\rangle_0$$

$$+\langle\!\langle\{G(E^{k-i})\circ\gamma_{\varsigma_{g(3)}}\circ E^{i-1}\circ\gamma^\mu\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\gamma_\mu\rangle\!\rangle_0$$

$$= \quad \langle\!\langle\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}\gamma_{\varsigma_{g(2)}}\gamma_\mu\{E^{i-1}\circ\gamma^\mu\}\rangle\!\rangle_0$$

$$+\langle\!\langle\{\Delta\circ E^{i-1}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\{G(E^{k-i})\circ\gamma_{\varsigma_{g(1)}}\}\rangle\!\rangle_0$$

$$-\langle\!\langle G(E^{k-i})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\{\gamma_{\varsigma_{g(3)}}\circ\gamma_\mu\}\{E^{i-1}\circ\gamma^\mu\}\rangle\!\rangle_0$$

$$+\langle\!\langle G(E^{k-i})\gamma_{\varsigma_{g(1)}}\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_\mu\}\{E^{i-1}\circ\gamma^\mu\}\rangle\!\rangle_0.$$

The proof follows by using Lemma 3.1 and equation (30).   □

Lemma A.13.

$$-2\sum_{i=1}^{k}\langle\!\langle\{G(E^{k-i})\circ E^{i-1}\}\gamma_\mu\gamma^\mu\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\!\rangle_0$$

$$= \quad 2\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\!\langle\{G(\Delta\circ E^{i-j-1})\circ G(E^{k-i})\}E^{j-1}\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\!\rangle_0$$

$$-2\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\!\langle G(E^{k-i})E^{i-j-1}G(\Delta\circ E^{j-1}\circ\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma)\rangle\!\rangle_0$$

$$-\frac{1}{3}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\!\langle G(E^{k-i})\gamma_{\varsigma_{g(1)}}\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\circ\gamma^\mu\}\{E^{i-1}\circ\gamma_\mu\}\rangle\!\rangle_0$$

$$-\frac{1}{3}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\!\langle G(E^{k-i})\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\{\gamma_{\varsigma_{g(3)}}\circ\gamma^\mu\}\{E^{i-1}\circ\gamma_\mu\}\rangle\!\rangle_0$$

$$+\frac{1}{3}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\!\langle G(E^{k-i})\{\Delta\circ E^{i-1}\}\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_{\varsigma_{g(3)}}\rangle\!\rangle_0.$$

Proof. By equation (25), we have

$$\langle\!\langle\{G(E^{k-i})\circ E^{i-1}\}\gamma_\mu\gamma^\mu\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\!\rangle_0$$

$$= \quad \langle\!\langle\{G(E^{k-i})\circ E^{i-1}\}\gamma_\mu\gamma_\alpha\{\gamma_\beta\circ\gamma_\sigma\circ\gamma^\mu\}\rangle\!\rangle_0$$

$$+\langle\!\langle\{G(E^{k-i})\circ E^{i-1}\}\gamma^\mu\{\gamma_\alpha\circ\gamma_\mu\}\{\gamma_\beta\circ\gamma_\sigma\}\rangle\!\rangle_0$$

$$-\langle\!\langle\{G(E^{k-i})\circ E^{i-1}\}\Delta\gamma_\alpha\{\gamma_\beta\circ\gamma_\sigma\}\rangle\!\rangle_0,$$

$$\langle\!\langle\{G(E^{k-i})\circ E^{i-1}\}\gamma_\mu\gamma_\alpha\{\gamma_\beta\circ\gamma_\sigma\circ\gamma^\mu\}\rangle\!\rangle_0$$

$$= \quad \langle\!\langle G(E^{k-i})\gamma_\alpha\{\gamma_\beta\circ\gamma_\sigma\circ\gamma^\mu\}\{E^{i-1}\circ\gamma_\mu\}\rangle\!\rangle_0$$

$$+\langle\!\langle E^{i-1}\{G(E^{k-i})\circ\gamma_\alpha\}\gamma_\mu\{\gamma_\beta\circ\gamma_\sigma\circ\gamma^\mu\}\rangle\!\rangle_0$$

$$-\langle\!\langle E^{i-1}\{\gamma_\alpha\circ\gamma_\mu\}\{\gamma_\beta\circ\gamma_\sigma\circ\gamma^\mu\}G(E^{k-i})\rangle\!\rangle_0,$$

$$\langle\!\langle\{G(E^{k-i})\circ E^{i-1}\}\gamma^{\mu}\{\gamma_{\alpha}\circ\gamma_{\mu}\}\{\gamma_{\beta}\circ\gamma_{\sigma}\}\rangle\!\rangle_0$$
$$= \langle\!\langle G(E^{k-i})\{\gamma_{\beta}\circ\gamma_{\sigma}\}\{\gamma_{\alpha}\circ\gamma_{\mu}\}\{E^{i-1}\circ\gamma^{\mu}\}\rangle\!\rangle_0$$
$$+\langle\!\langle E^{i-1}\{\gamma_{\beta}\circ\gamma_{\sigma}\}\gamma^{\mu}\{G(E^{k-i})\circ\gamma_{\alpha}\circ\gamma_{\mu}\}\rangle\!\rangle_0$$
$$-\langle\!\langle E^{i-1}\{\gamma_{\beta}\circ\gamma_{\sigma}\}\{\gamma_{\alpha}\circ\Delta\}G(E^{k-i})\rangle\!\rangle_0,$$

and

$$\langle\!\langle\{G(E^{k-i})\circ E^{i-1}\}\Delta\gamma_{\alpha}\{\gamma_{\beta}\circ\gamma_{\sigma}\}\rangle\!\rangle_0$$
$$= \langle\!\langle G(E^{k-i})\{\Delta\circ E^{i-1}\}\{\gamma_{\beta}\circ\gamma_{\sigma}\}\gamma_{\alpha}\rangle\!\rangle_0$$
$$+\langle\!\langle E^{i-1}\Delta\{\gamma_{\beta}\circ\gamma_{\sigma}\}\{G(E^{k-i})\circ\gamma_{\alpha}\}\rangle\!\rangle_0$$
$$-\langle\!\langle E^{i-1}\{\Delta\circ\gamma_{\alpha}\}\{\gamma_{\beta}\circ\gamma_{\sigma}\}G(E^{k-i})\rangle\!\rangle_0.$$

The proof is completed by using Lemma 3.1 and equation (30). □

Lemma A.14.

$$\sum_{g\in S_3}\sum_{i=1}^{k}\langle\!\langle G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\gamma_{\varsigma_{g(2)}}\{\gamma_{\varsigma_{g(3)}}\circ\gamma^{\mu}\}\{E^{i-1}\circ\gamma_{\mu}\}\rangle\!\rangle_0$$

$$= \frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\!\langle\{\Delta\circ E^{i-1}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\rangle\!\rangle_0$$

$$+\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\langle\!\langle G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\gamma_{\mu}\gamma^{\mu}\{E^{i-1}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\!\rangle_0$$

$$+\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\!\langle\{G(\Delta\circ E^{i-j-1})\circ E^{j-1}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\!\rangle_0$$

$$+\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\!\langle\{\Delta\circ E^{i-j-1}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})G(E^{j-1}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}})\rangle\!\rangle_0$$

$$-\frac{1}{2}\sum_{g\in S_3}\sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\!\langle\{\Delta\circ E^{i-2}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\!\rangle_0.$$

Proof. By equation (25), we have

$$\langle\!\langle G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\gamma_{\varsigma_{g(2)}}\{\gamma_{\varsigma_{g(3)}}\circ\gamma^{\mu}\}\{E^{i-1}\circ\gamma_{\mu}\}\rangle\!\rangle_0$$
$$+\langle\!\langle G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\gamma_{\varsigma_{g(3)}}\{\gamma_{\varsigma_{g(2)}}\circ\gamma^{\mu}\}\{E^{i-1}\circ\gamma_{\mu}\}\rangle\!\rangle_0$$
$$= \langle\!\langle\{\Delta\circ E^{i-1}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\rangle\!\rangle_0$$
$$+\langle\!\langle G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\gamma^{\mu}\{E^{i-1}\circ\gamma_{\mu}\}\rangle\!\rangle_0,$$

and

$$\langle\!\langle G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\gamma^{\mu}\{E^{i-1}\circ\gamma_{\mu}\}\rangle\!\rangle_0$$
$$= \langle\!\langle G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\gamma_{\mu}\gamma^{\mu}\{E^{i-1}\circ\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\!\rangle_0$$
$$+\langle\!\langle E^{i-1}\Delta\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\rangle\!\rangle_0$$
$$-\langle\!\langle E^{i-1}\gamma_{\mu}\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\circ\gamma^{\mu}\}G(E^{k-i}\circ\gamma_{\varsigma_{g(1)}})\rangle\!\rangle_0.$$

The proof is completed by using Lemma 3.1 and equation (30). □

Lemma A.15.

$$
-\sum_{g\in S_3}\langle\langle\{G(E^{k-1}\circ\gamma^\mu)\circ\gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_\mu\rangle\rangle_0
$$

$$
=\ -\frac{1}{2}\sum_{g\in S_3}\langle\langle\{E^{k-1}\circ\gamma_{\varsigma_{g(1)}}\}\gamma_\mu\gamma^\mu\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0
$$

$$
+\frac{1}{2}\sum_{g\in S_3}\sum_{j=1}^{k-1}\langle\langle E^{k-j-1}G(\Delta\circ E^{j-1}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0
$$

$$
-3\sum_{j=1}^{k-1}\langle\langle G(\Delta\circ E^{k-j-1})E^{j-1}\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\rangle_0
$$

$$
-\frac{1}{2}\sum_{g\in S_3}\sum_{j=1}^{k-1}\langle\langle\{\Delta\circ E^{k-j-1}\}G(E^{j-1}\circ\gamma_{\varsigma_{g(1)}})\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0
$$

$$
-\sum_{g\in S_3}\sum_{j=1}^{k-1}\langle\langle\{G(E^{k-j-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^\mu)\circ G(\gamma_\mu)\}E^{j-1}\{\gamma_{\varsigma_{g(2)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0
$$

$$
+6\sum_{j=1}^{k-1}\langle\langle\{G(E^{k-j-1}\circ\gamma^\mu)\circ G(\gamma_\mu)\}E^{j-1}\{\gamma_\alpha\circ\gamma_\beta\circ\gamma_\sigma\}\rangle\rangle_0
$$

$$
+\frac{1}{2}\sum_{g\in S_3}\sum_{j=1}^{k-1}\langle\langle\{\Delta\circ E^{k-j-1}\}G(E^{j-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(2)}})\gamma_{\varsigma_{g(3)}}\rangle\rangle_0
$$

$$
-3\langle\langle\{\Delta\circ E^{k-1}\}\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0
$$

$$
+\sum_{g\in S_3}\langle\langle G(E^{k-1}\circ\gamma_{\varsigma_{g(1)}}\circ\gamma^\mu)\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_\mu\rangle\rangle_0.
$$

Proof. Using equation (25), we have

$$
\langle\langle\{G(E^{k-1}\circ\gamma^\mu)\circ\gamma_{\varsigma_{g(1)}}\}\gamma_{\varsigma_{g(2)}}\gamma_{\varsigma_{g(3)}}\gamma_\mu\rangle\rangle_0
$$

$$
=\ \langle\langle G(E^{k-1}\circ\gamma^\mu)\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}\gamma_{\varsigma_{g(2)}}\gamma_\mu\rangle\rangle_0+\frac{1}{2}\langle\langle\{\Delta\circ E^{k-1}\}\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0
$$

$$
-\langle\langle G(E^{k-1}\circ\gamma^\mu)\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\{\gamma_\mu\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0
$$

$$
=\ \langle\langle G(\gamma_\mu)\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}\gamma_{\varsigma_{g(2)}}\{E^{k-1}\circ\gamma^\mu\}\rangle\rangle_0+\frac{1}{2}\langle\langle\{\Delta\circ E^{k-1}\}\gamma_\alpha\gamma_\beta\gamma_\sigma\rangle\rangle_0
$$

$$
-\langle\langle\{G(E^{k-1}\circ\gamma_{\varsigma_{g(3)}}\circ\gamma^\mu)\}\gamma_{\varsigma_{g(1)}}\gamma_{\varsigma_{g(2)}}\gamma_\mu\rangle\rangle_0,
$$

and

$$
\langle\langle G(\gamma_\mu)\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}\gamma_{\varsigma_{g(2)}}\{E^{k-1}\circ\gamma^\mu\}\rangle\rangle_0
$$

$$
=\ \frac{1}{2}\langle\langle E^{k-1}\Delta\gamma_{\varsigma_{g(2)}}\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0+\langle\langle\{E^{k-1}\circ\gamma_{\varsigma_{g(2)}}\}\gamma^\mu G(\gamma_\mu)\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0
$$

$$
-\langle\langle E^{k-1}\gamma^\mu\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}\{\gamma_{\varsigma_{g(2)}}\circ G(\gamma_\mu)\}\rangle\rangle_0
$$

$$
=\ \frac{1}{2}\langle\langle\{E^{k-1}\circ\gamma_{\varsigma_{g(2)}}\}\gamma_\mu\gamma^\mu\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0+\frac{1}{2}\langle\langle E^{k-1}\Delta\gamma_{\varsigma_{g(2)}}\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}\rangle\rangle_0
$$

$$
-\langle\langle E^{k-1}\gamma^\mu\{\gamma_{\varsigma_{g(1)}}\circ\gamma_{\varsigma_{g(3)}}\}\{\gamma_{\varsigma_{g(2)}}\circ G(\gamma_\mu)\}\rangle\rangle_0.
$$

The proof is completed by using Lemma 3.1. $\qquad\square$

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