ON CERTAIN MEAN VALUES OF LOGARITHMIC DERIVATIVES OF L-FUNCTIONS AND THE RELATED DENSITY FUNCTIONS

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Abstract. We study some "density function" related to the value-distribution of L-functions. The first example of such a density function was given by Bohr and Jessen in 1930s for the Riemann zeta-function. In this paper, we construct the density function in a wide class of L-functions. We prove that certain mean values of L-functions in the class are represented as integrals involving the related density functions.

1. Introduction

We begin with recalling a classical result on the value-distribution of the Riemann zeta-function $\zeta(s)$ obtained by Bohr and Jessen. For any $\sigma > 1/2$, let

$$G = \{s = \sigma + it \mid \sigma > 1/2\} \setminus \bigcup_{\rho=\beta+i\gamma} \{s = \sigma + i\gamma \mid 1/2 < \sigma \leq \beta\},$$

where ρ runs through all zeros of $\zeta(s)$ with $\beta > 1/2$. Then we define $\log \zeta(s)$ for $s \in G$ by analytic continuation along the horizontal line. Fix a rectangle R in the complex plane whose edges are parallel to the coordinate axes, and denote by $\mathcal{V}_\sigma(T, R)$ the Lebesgue measure of the set

$$\{t \in [-T, T] \mid \sigma + it \in G,\ \log \zeta(\sigma + it) \in R\}.$$

Bohr and Jessen [1, 2] proved that there exists the limit value

$$\mathcal{W}_\sigma(R) = \lim_{T\to\infty} \frac{1}{2T} \mathcal{V}_\sigma(T, R) \tag{1.1}$$

for any fixed $\sigma > 1/2$. They also showed that there exists a non-negative real valued continuous function $\mathcal{M}_\sigma(z)$ such that the formula

$$\mathcal{W}_\sigma(R) = \int_R \mathcal{M}_\sigma(z)\, |dz| \tag{1.2}$$

holds with $|dz| = (2\pi)^{-1} dxdy$. Their study was developed in various ways, for example, Jessen–Wintner [14], Borchsenius–Jessen [3], Laurinčikas [17], and Matsumoto [21].

Matsumoto [22] generalized limit formula (1.1) in a quite wide class of zeta-functions, which is now called the *Matsumoto zeta-functions*. On the other hand, an analogue of integral formula (1.2) was obtained only in some restricted cases, for example, the case of Dedekind zeta-functions of finite Galois extensions of $\mathbb{Q}$ [23], and automorphic L-functions of normalized holomorphic Hecke-eigen cusp forms of

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level N [25]. Thus it is worth studying "density functions" such as $\mathcal{M}_\sigma(z)$ for more general zeta- or L-functions.

Kershner and Wintner [16] proved analogues of formulas (1.1) and (1.2) for $(\zeta'/\zeta)(s)$. In this paper, we construct the density functions $M_\sigma(z;F)$ for functions $F(s)$ in a subclass of the Matsumoto zeta-functions and generalize Kershner–Wintner's result.

2. L-functions and the related density functions

2.1. **Class of L-functions.** We introduce the class $\mathcal{S}_{\mathrm{I}}$ as the set of all functions $F(s)$ represented as Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$$

in some half plane that satisfy the following axioms:

(1) *Ramanujan hypothesis.* Dirichlet coefficients $a_F(n)$ satisfy $a_F(n) \ll_\epsilon n^\epsilon$ for every $\epsilon > 0$.

(2) *Analytic continuation.* There exists a non-negative integer m such that $(s-1)^m F(s)$ is an entire function of finite order.

(3) *Functional equation.* $F(s)$ satisfies a functional equation of the form

$$\Lambda_F(s) = \omega\overline{\Lambda_F(1-\overline{s})},$$

where

$$\Lambda_F(s) = F(s) Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j),$$

with some $|\omega| = 1$, $Q > 0$, $\lambda_j > 0$, $\mathrm{Re}(\mu_j) \geq 0$.

(4) *Polynomial Euler product.* For $\sigma > 1$, $F(s)$ is expressed as the infinite product

$$F(s) = \prod_p \prod_{j=1}^{g} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1},$$

where g is a positive constant and $\alpha_j(p) \in \mathbb{C}$.

(5) *Prime mean square.* There exists a positive constant κ such that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} |a_F(p)|^2 = \kappa,$$

where $\pi(x)$ stands for the number of prime numbers less than or equal to x.

The above axioms come from two classes of L-functions introduced by Selberg [29] and Steuding [30]. We see that the class $\mathcal{S}_{\mathrm{I}}$ is just equal to the intersection of these classes, and it is also a subclass of the Matsumoto zeta-functions, see Section 2 of [30].

Let $N_F(\sigma, T)$ be the number of zeros $\rho = \beta + i\gamma$ of $F(s)$ with $\beta > \sigma$ and $0 < \gamma < T$. Then for the function $F(s)$ satisfying axioms (1)–(4), there exists a positive constant b such that for any $\epsilon > 0$,

$$N_F(T,\sigma) \ll_\epsilon T^{b(1-\sigma)+\epsilon} \tag{2.1}$$

as $T \to \infty$, uniformly for $\sigma \geq 1/2$ [15, Lemma 3]. From the proof of [15], estimate (2.1) generally holds with $b = 4(d_F + 3)$, where d_F is the degree of F defined by

$$d_F = 2\sum_{j=1}^{r} \lambda_j.$$

The constant b is taken smaller in some special cases, for example, Heath-Brown [8] showed that the Dedekind zeta-functions attached to algebraic number fields of degree $d \geq 3$ satisfy (2.1) with $b = d$, and Perelli [26] obtained it with $b = d_F$ in a subclass of the Selberg class.

Next, we define the subclass $\mathcal{S}_{\mathrm{II}}$ as the set of all $F(s)$ satisfying axioms (1)–(5) and the following (6):

(6) *Zero density estimate.* There exist positive constants c and A such that

$$N_F(T,\sigma) \ll T^{1-c(\sigma-\frac{1}{2})}(\log T)^A \tag{2.2}$$

as $T \to \infty$, uniformly for $\sigma \geq 1/2$.

There are many zeta- or L-functions that belong to the class $\mathcal{S}_{\mathrm{I}}$, for instance, the Riemann zeta-functions $\zeta(s)$, Dirichlet L-functions $L(s,\chi)$ of primitive characters χ, Dedekind zeta-functions $\zeta_K(s)$, automorphic L-functions $L(s,f)$ of normalized holomorphic Hecke-eigen cusp forms f with respect to $SL_2(\mathbb{Z})$. Furthermore, estimate (2.2) is proved for $\zeta(s)$ by Selberg [28], for $L(s,\chi)$ by Fujii [4], and for $L(s,f)$ by Luo [19], and hence they belong to the subclass $\mathcal{S}_{\mathrm{II}}$.

2.2. **Statements of results.** For an integrable function $f(z)$, we denote its Fourier transform and Fourier inverse transform by

$$\widehat{f}(z) = f^{\wedge}(z) = \int_{\mathbb{C}} f(w)\psi_z(w)\,|dw| \quad \text{and} \quad f^{\vee}(z) = \int_{\mathbb{C}} f(w)\psi_{-z}(w)\,|dw|,$$

respectively, where $\psi_w(z) = \exp(i\,\mathrm{Re}(z\overline{w}))$ is an additive character of $\mathbb{C}$ and $|dw|$ is the measure $(2\pi)^{-1}dudv$ for $w = u+iv$. According to [11, Section 9] or [12, Section 5], we then define the class Λ as

$$\Lambda = \{f \in L^1 \mid f, \widehat{f} \in L^1 \cap L^\infty \text{ and } (f^\wedge)^\vee = f \text{ holds}\}.$$

We see that any Schwartz function belongs to the class Λ, and especially, any compactly supported C^∞-function does.

The first main result of this paper is related to the mean values of L-functions.

Theorem 2.1. *Let $F \in \mathcal{S}_{\mathrm{I}}$. Let σ_1 be a large fixed positive real number. Let $\theta, \delta > 0$ be real numbers with $\delta + 3\theta < 1/2$. Let $\epsilon > 0$ be a small fixed real number. Let $\Phi \in \Lambda$. Then there exists a constant $T_{\mathrm{I}} = T_{\mathrm{I}}(F,\sigma_1,\theta,\delta,\epsilon) > 0$ such that the following formula*

$$\frac{1}{T}\int_0^T \Phi\left(\frac{F'}{F}(\sigma+it)\right)dt = \int_{\mathbb{C}} \Phi(z)M_\sigma(z;F)\,|dz| + E \tag{2.3}$$

holds for all $T \geq T_{\mathrm{I}}$ and for all $\sigma \in [1-b^{-1}+\epsilon, \sigma_1]$, where $M_\sigma(z;F)$ is a non-negative real valued continuous function uniquely determined from $F(s)$, and the constant b is that in (2.1). *The error term E is estimated as*

$$E \ll \exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right)\int_{\Omega} |\widehat{\Phi}(z)|\,|dz| + \int_{\mathbb{C}\setminus\Omega} |\widehat{\Phi}(z)|\,|dz|, \tag{2.4}$$

where the implied constant depends only on F, σ_1, ϵ, and

$$\Omega = \{z = x + iy \in \mathbb{C} \mid -(\log T)^{\delta} \leq x, y \leq (\log T)^{\delta}\}.$$

Moreover, if $F \in \mathcal{S}_{\mathrm{II}}$, then there exists a constant $T_{\mathrm{II}} = T_{\mathrm{II}}(F, \sigma_1, \theta, \delta) > 0$ such that (2.3) *and* (2.4) *hold together with $T \geq T_{\mathrm{II}}$ and $\sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1]$, where the implied constant depends only on F and σ_1.*

Then, let again R be a rectangle in the complex plane whose edges are parallel to the axes, and define $V_\sigma(T, R; F)$ as the Lebesgue measure of the set of all $t \in [0, T]$ for which $(F'/F)(\sigma + it)$ belongs to R. Denote by ν_k the usual k-dimensional Lebesgue measure. The second result is an analogue of Bohr–Jessen's limit theorem for $(F'/F)(s)$.

Theorem 2.2. *Let $F \in \mathcal{S}_{\mathrm{I}}$. Let σ be fixed with $\sigma > 1 - b^{-1}$, where the constant b is that in* (2.1)*. Let $\epsilon > 0$ be an arbitrarily small real number. Then we have*

$$\frac{1}{T} V_\sigma(T, R; F) = \int_R M_\sigma(z; F)\, |dz| + O\left((\nu_2(R) + 1)(\log T)^{-\frac{1}{2}+\epsilon}\right) \tag{2.5}$$

as $T \to \infty$, where the implied constant depends only on F, σ, and ϵ. Moreover, if $F \in \mathcal{S}_{\mathrm{II}}$, then (2.5) *holds with any fixed $\sigma > 1/2$.*

2.3. **Remarks on the related works.** The Riemann zeta-function $\zeta(s)$ is a typical example of the member of the subclass $\mathcal{S}_{\mathrm{II}}$. In this case, Theorem 2.1 is essentially Theorem 1.1.1 of [5], and the density function $M_\sigma(z; \zeta)$ was used to study of the distribution of zeros of $\zeta'(s)$ in [6].

Theorem 2.2 is related to the study on the *discrepancy estimates* for zeta-functions. Let

$$\mathcal{D}_\sigma(T, R) = \frac{1}{2T} \mathcal{V}_\sigma(T, R) - \mathcal{W}_\sigma(R).$$

We know that $\mathcal{D}_\sigma(T, R) = o(1)$ as $T \to \infty$ by (1.1). Matsumoto [20] gave a better upper bound for $\mathcal{D}_\sigma(T, R)$, which was improved by Harman and Matsumoto [7]. They proved

$$\mathcal{D}_\sigma(T, R) \ll (\nu_2(R) + 1)(\log T)^{-A(\sigma)+\epsilon}$$

for an arbitrarily small $\epsilon > 0$, where

$$A(x) = \begin{cases} (x-1)/(3+2x) & \text{if } x > 1, \\ (4x-2)/(21+8x) & \text{if } 1/2 < x \leq 1. \end{cases}$$

Matsumoto [24] also generalized this result for Dedekind zeta-functions even in the case of non-Galois extensions. We note that $A(x) \leq 1/2$ for any $x > 1/2$. Though the difference of logarithms and logarithmic derivatives exists, Theorem 2.2 gives a better estimate on the discrepancy for $(F'/F)(s)$.

Recently, Ihara and Matsumoto studied density functions such as $\mathcal{M}_\sigma(z)$ more precisely, and named them "M-functions" for L-functions, see [10–13].

3. Proof of Theorem 2.1

We begin with considering the case of $\Phi = \psi_z$ in Theorem 2.1. The following proposition is a key for the proof of the theorem:

Proposition 3.1. *Let $F(s)$ be a function satisfying axioms* (1)–(4)*. Let σ_1 be a large fixed positive real number. Let $\theta, \delta > 0$ be real numbers with $\delta + 3\theta < 1/2$. Let $\epsilon > 0$ be a small fixed real number. Then there exists a constant $T_{\mathrm{I}} = T_{\mathrm{I}}(F, \sigma_1, \theta, \delta, \epsilon) > 0$ such that we have*

$$\frac{1}{T}\int_0^T \psi_z\left(\frac{F'}{F}(\sigma+it)\right)dt = \widetilde{M_\sigma}(z;F) + O\left(\exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right)\right) \tag{3.1}$$

for all $T \geq T_{\mathrm{I}}$, for all $\sigma \in [1-b^{-1}+\epsilon, \sigma_1]$, and for all $z \in \Omega$, where $\widetilde{M_\sigma}(z;F)$ is a function uniquely determined from $F(s)$. The implied constant depends only on F, σ_1 and ϵ. If $F(s)$ further satisfies axiom (6)*, there exists a constant $T_{\mathrm{II}} = T_{\mathrm{II}}(F, \sigma_1, \theta, \delta) > 0$ such that* (3.1) *holds together with $T \geq T_{\mathrm{II}}$ and $\sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1]$, where the implied constant depends only on F and σ_1.*

We first prove Proposition 3.1 in Section 3.1. We sometimes omit details of the proofs there since they strongly follow Guo's method in [5]. Towards the proof of Theorem 2.1, we next consider in Section 3.2 the growth of the function $\widetilde{M_\sigma}(z;F)$ of (3.1). We finally complete the proof of Theorem 2.1 in Section 3.3.

3.1. **Proof of Proposition 3.1.** Let $F(s)$ be a function satisfying axiom (4). Then we see that

$$\frac{F'}{F}(s) = -\sum_{n=1}^{\infty}\frac{\Lambda_F(n)}{n^s}, \qquad \sigma > 1,$$

where $\Lambda_F(n)$ is given by $\Lambda_F(n) = (\alpha_1(p)^m + \cdots + \alpha_g(p)^m)\log p$ if $n = p^m$ and $\Lambda_F(n) = 0$ otherwise. In this section, we approximate $(F'/F)(\sigma+it)$ by some Dirichlet polynomials. First, we define

$$w_X(n) = \begin{cases} 1 & \text{if } 1 \leq n \leq X, \\ \dfrac{\log(X^2/n)}{\log X} & \text{if } X \leq n \leq X^2 \end{cases}$$

for $X > 1$. We approximate $(F'/F)(\sigma+it)$ by the following function $f_X(t,\sigma;F)$:

$$f_X(t,\sigma;F) = -\sum_{n\leq X^2}\frac{\Lambda_F(n)}{n^{\sigma+it}}w_X(n).$$

Lemma 3.2. *Let $F(s)$ be a function satisfying axioms* (1)–(4)*. Let σ_1 be a large fixed positive real number. Let $\epsilon > 0$ be a small fixed real number. Then there exists an absolute constant $T_0 > 0$ such that we have*

$$\frac{1}{T}\int_0^T \psi_z\left(\frac{F'}{F}(\sigma+it)\right)dt = \frac{1}{T}\int_0^T \psi_z(f_X(t,\sigma;F))\,dt + E_1 \tag{3.2}$$

for all $T \geq T_0$, for all $\sigma \in [1-b^{-1}+\epsilon, \sigma_1]$, and for all $z \in \mathbb{C}$. The error term E_1 is estimated as for any $X, Y > 1$

(3.3)

$$\begin{aligned} E_1 \ll & \frac{1}{T} + YT^{-\frac{b}{2}\{\sigma-(1-b^{-1}+\frac{\epsilon}{2})\}} \\ & + \frac{|z|}{\log X}\left(\frac{X\log Y\log T}{Y} + \frac{X^{-\frac{1}{2}\{\sigma-(1-b^{-1}+\frac{\epsilon}{2})\}}\log T}{\{\sigma-(1-b^{-1}+\frac{\epsilon}{2})\}^2} + \frac{X}{T} + X^{-\sigma}\log^2 T\right), \end{aligned}$$

where the implied constant depends only on F. If $F(s)$ further satisfies axiom (6), then (3.2) holds with $\sigma \in [1/2+(\log T)^{-\theta}, \sigma_1]$, and we have

$$E_1 \ll \frac{1}{T} + YT^{-\frac{c}{2}(\sigma-\frac{1}{2})}(\log T)^A + \frac{|z|}{\log X}\left(\frac{X\log Y\log T}{Y} + \frac{X^{-\frac{1}{2}(\sigma-\frac{1}{2})}\log T}{(\sigma-\frac{1}{2})^2} + \frac{X}{T} + X^{-\sigma}\log^2 T\right), \tag{3.4}$$

where the implied constant depends only on F.

Proof. This lemma is an analogue of Lemma 2.1.4 of [5]. Let $\mathscr{B}_Y(\sigma,T;F)$ be the set of all $t\in[0,T]$ for which $|\gamma-t|\le Y$ holds with some zeros $\rho=\beta+i\gamma$ of $F(s)$ satisfying $\beta\ge\frac{1}{2}(\sigma+1-b^{-1}+\frac{\epsilon}{2})$. Then we see that E_1 is

$$\ll \frac{1}{T} + \frac{\nu_1(\mathscr{B}_Y(\sigma,T;F))}{T} + \frac{|z|}{T}\int_{[1,T]\cap\mathscr{B}_Y(\sigma,T;F)^c}\left|\frac{F'}{F}(\sigma+it)-f_X(t,\sigma;F)\right|dt, \tag{3.5}$$

since $|\psi_z(w)-\psi_z(w')|\le|z||w-w'|$. By the definition of $\mathscr{B}_Y(\sigma,T;F)$, we have

$$\nu_1(\mathscr{B}_Y(\sigma,T;F))\le 2YN_F\left(\frac{1}{2}\left(\sigma+1-b^{-1}+\frac{\epsilon}{2}\right),T\right).$$

Furthermore, estimate (2.1) implies that the second term of (3.5) is

$$\ll YT^{1-\frac{b}{2}\{\sigma-(1-b^{-1}+\frac{\epsilon}{2})\}}$$

for $\sigma>1-b^{-1}+\epsilon/2$. Then we estimate the third term. For this, Guo used the formula of [27, Lemma 2], and we need a similar formula for general $F(s)$. We first recall that the following estimate

$$\frac{F'}{F}(s)\ll\log^2(|t|+2) \tag{3.6}$$

holds if $s=\sigma+it$ satisfies $-1\le\sigma\le 2$ and has distance $\gg\log(|t|+2)^{-1}$ from zeros and poles of $F(s)$. This can be easily deduced from axioms (1)–(4). Let $c=\max\{2,1+\sigma\}$ and choose $T_m\in(m,m+1]$ and $0<\delta<1$ such that the edges $[c+iT_m,-\delta+iT_m]$, $[c-iT_m,-\delta-iT_m]$, and $[-\delta-iT_m,-\delta+iT_m]$ have distance $\gg\log(|t|+2)^{-1}$ from zeros and poles of $F(s)$. Then, we consider the integral

$$\frac{1}{2\pi i}\int_{c-iT_m}^{c+iT_m}\frac{F'}{F}(z)\frac{X^{z-s}-X^{2(z-s)}}{(z-s)^2}\,dz.$$

We see that

$$\lim_{m\to\infty}\frac{1}{2\pi i}\int_{c-iT_m}^{c+iT_m}\frac{F'}{F}(z)\frac{X^{z-s}-X^{2(z-s)}}{(z-s)^2}\,dz=-f_X(t,\sigma;F)\log X$$

and change the contour by the edges $[c+iT_m,-\delta+iT_m]$, $[c-iT_m,-\delta-iT_m]$, and $[-\delta-iT_m,-\delta+iT_m]$. The integrals on the horizontal edges tend to 0 as $m\to\infty$ due to estimate (3.6), and we have also by (3.6),

$$\frac{1}{2\pi i}\int_{-\delta-iT_m}^{-\delta+iT_m}\frac{F'}{F}(z)\frac{X^{z-s}-X^{2(z-s)}}{(z-s)^2}\,dz\ll_{\sigma_0}X^{-\sigma}\log^2 T$$

for any $\sigma \geq \sigma_0 > 0$ and $t \in [1, T]$. Calculating the residues, we obtain the following formula:

$$\frac{F'}{F}(s) = f_X(t, \sigma; F) - \frac{m_1}{\log X} \frac{X^{1-s} - X^{2(1-s)}}{(1-s)^2} + \frac{m_0}{\log X} \frac{X^{-s} - X^{-2s}}{s^2} + \frac{1}{\log X} \sum_{\rho} \frac{X^{\rho-s} - X^{2(\rho-s)}}{(\rho-s)^2} + O_{\sigma_0}\left(\frac{1}{\log X} X^{-\sigma} \log^2 T\right), \tag{3.7}$$

where $m_1, m_0 \geq 0$ are orders of the possible pole of $F(s)$ at $s = 1$ and the possible zero of $F(s)$ at $s = 0$, respectively, and ρ runs through nontrivial zeros of $F(s)$. In order to complete the proof of Lemma 3.2, we must consider the contributions of the second, third, and fourth terms of (3.7). They are estimated by an argument similar to the proof of Lemma 2.1.4 of [5]. Thus we find the first part of Lemma 3.2.

All changes that we need for the proof of the second part are just replacing the definition of $\mathscr{B}_Y(\sigma, T; F)$ with the set of all $t \in [0, T]$ for which $|\gamma - t| \leq Y$ holds with some zeros $\rho = \beta + i\gamma$ of $F(s)$ satisfying $\beta \geq \frac{1}{2}(\sigma + \frac{1}{2})$. By the axiom (6), we have

$$\nu_1(\mathscr{B}_Y(\sigma, T; F)) \leq 2Y N_F\left(\frac{1}{2}\left(\sigma + \frac{1}{2}\right), T\right) \ll Y T^{1-\frac{c}{2}(\sigma-\frac{1}{2})} (\log T)^A.$$

The remaining estimates are given in a similar way. □

Towards the next step, we define

$$g_X(t, \sigma; F) = -\sum_{n \leq X^2} \frac{\Lambda_F(n)}{n^{\sigma+it}} \quad \text{and} \quad h_X(t, \sigma; F) = -\sum_{p \leq X^2} \sum_{m=1}^{\infty} \frac{\Lambda_F(p^m)}{p^{m(\sigma+it)}}$$

for $X > 1$. Then we have the following three lemmas:

Lemma 3.3. *Let $F(s)$ be a function satisfying axioms* (1) *and* (4)*. Then there exists an absolute constant $T_0 > 0$ such that we have*

$$\frac{1}{T} \int_0^T \psi_z(f_X(t, \sigma; F))\, dt = \frac{1}{T} \int_0^T \psi_z(g_X(t, \sigma; F))\, dt + E_2$$

for all $T \geq T_0$, for all $\sigma > 1/2$, and for all $z \in \mathbb{C}$. The error term E_2 is estimated as

$$E_2 \ll \frac{g|z| \log X}{(2\sigma - 1)^{\frac{1}{2}}} \left(1 + \frac{X^2}{T}\right)^{\frac{1}{2}} X^{\frac{1}{2}-\sigma} \tag{3.8}$$

for any $X > 1$. The implied constant is absolute.

Lemma 3.4. *Let $F(s)$ be a function satisfying axioms* (1) *and* (4)*. Then there exists an absolute constant $T_0 > 0$ such that we have*

$$\frac{1}{T} \int_0^T \psi_z(g_X(t, \sigma; F))\, dt = \frac{1}{R} \int_0^R \psi_z(g_X(r, \sigma; F))\, dr + E_3$$

for all $R \geq T \geq T_0$, for all $\sigma > 1/2$, and for all $z \in \mathbb{C}$. The error term E_3 is estimated as

$$E_3 \ll \frac{g^N X^{5N}}{T}(1+|z|^2)^{\frac{N}{2}} + \frac{(8g|z|)^N}{N!}\left(1+\frac{X^N}{T}\right)\left\{(\zeta(2\sigma)^{\frac{1}{2}}\log X)^N \left(\frac{N}{2}\right)! + \zeta'(2\sigma)^N\right\} \tag{3.9}$$

for any $X > 1$ and any large even integer N. The implied constant is absolute.

Lemma 3.5. *Let $F(s)$ be a function satisfying axioms* (1) *and* (4)*. Then there exists an absolute constant $T_0 > 0$ such that we have*

$$\frac{1}{R}\int_0^R \psi_z(g_X(r,\sigma;F))\,dr = \frac{1}{R}\int_0^R \psi_z(h_X(r,\sigma;F))\,dr + E_4$$

for all $R \geq T \geq T_0$, for all $\sigma > 1/2$, and for all $z \in \mathbb{C}$. The error term E_4 is estimated as

$$E_4 \ll \frac{g|z|\log X}{2\sigma - 1}X^{1-2\sigma} \tag{3.10}$$

for any $X > 1$. The implied constant is absolute.

These lemmas are analogues of Lemmas 2.2.5, 2.1.6, and 2.1.10 in [5]. Note that we have $|\Lambda_F(n)| \leq g\Lambda(n)$ due to axioms (1) and (4), where $\Lambda(n) = \Lambda_\zeta(s)$ is the usual von Mangolt function. In fact, by axiom (4) we have $\Lambda_F(p^m) = (\alpha_1(p)^m + \ldots + \alpha_g(p)^m)\log p$, and by axiom (1) the absolute values of $\alpha_j(p)$ are less than or equal to 1; see Lemma 2.2 of [30]. Therefore we obtain these lemmas by replacing $\Lambda(n)$ with $\Lambda_F(n)$ in the proofs of the corresponding lemmas in [5].

Let $F(s)$ be a function satisfying axioms (1)–(4). Let σ_1 be a large fixed positive real number. Let $\epsilon > 0$ be a small fixed real number. By the above lemmas, we have for all $R \geq T \geq T_0$ and for all $\sigma \in [1 - b^{-1} + \epsilon, \sigma_1]$,

$$\frac{1}{T}\int_0^T \psi_z\left(\frac{F'}{F}(\sigma+it)\right)dt = \frac{1}{R}\int_0^R \psi_z(h_X(r,\sigma;F))\,dr + E_1 + E_2 + E_3 + E_4, \tag{3.11}$$

where the error terms E_j are estimated as in (3.3), (3.8), (3.9), and (3.10). Let $\theta, \delta > 0$ with $\delta + 3\theta < 1/2$. We take X, Y, and N as the following functions in T:

$$X = \exp((\log T)^{\theta_1}), \quad Y = \exp((\log T)^{\theta_2}), \quad \text{and} \quad N = 2\lfloor(\log T)^{\theta_3}\rfloor,$$

where $\theta_1 = (5/3)\theta$, $\theta_2 = (\theta_1 + 1 - \theta)/2$, $\theta_3 = ((2\delta + \theta + 2\theta_1) + (1 - \theta_1))/2$. Moreover, let $T_0' = T_0'(\theta, \epsilon) \geq T_0$ with

$$(\log T_0')^{-\theta} \leq \epsilon/2.$$

Then we have $\sigma \geq 1 - b^{-1} + \epsilon/2 + (\log T)^{-\theta}$ for $T \geq T_0'$. Hence, there exists a positive real number $T_{\mathrm{I}} = T_{\mathrm{I}}(F, \theta, \delta, \epsilon) \geq T_0'$ such that we have

$$E_1 + E_2 + E_3 + E_4 \ll \exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right) \tag{3.12}$$

for all $T \geq T_{\mathrm{I}}$ and for all $z \in \Omega$ with the implied constant depending only on F and ϵ.

Then, let $F(s)$ further satisfy axiom (6). In this case, we obtain that the formula (3.11) holds for all $R \geq T \geq T_0$ and for all $\sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1]$, where the error terms E_j are estimated as in (3.4), (3.8), (3.9), and (3.10). Therefore there exists a positive real number $T_{\mathrm{II}} = T_{\mathrm{II}}(F, \theta, \delta) > T_0$ such that we have the same estimate as (3.12) for all $T \geq T_{\mathrm{II}}$ and for all $z \in \Omega$.

Next, applying Lemma 2 of [9], we see that

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \psi_z(h_X(r,\sigma;F))\,dr = \prod_{p\leq X^2}\int_0^1 \psi_z\left(\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{p^{m\sigma}}e^{2\pi i m\theta}\right)d\theta \tag{3.13}$$

since the system

$$\left\{\frac{\log p}{2\pi}\;\middle|\; p \text{ is a prime number}\right\}$$

is linearly independent over $\mathbb{Q}$. We define

$$\widetilde{M}_{\sigma,p}(z;F) = \int_0^1 \psi_z\left(\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{p^{m\sigma}}e^{2\pi i m\theta}\right)d\theta. \tag{3.14}$$

Then we obtain the following lemma on $\widetilde{M}_{\sigma,p}(z;F)$, which is proved in Section 3.2.

Lemma 3.6. *Let $F(s)$ be a function satisfying axioms* (1) *and* (4)*. Let σ_1 be a large fixed positive real number. Let $\theta, \delta > 0$ be real numbers with $\delta + 3\theta < 1/2$. Then there exists a positive real number $T_0 = T_0(F, \sigma_1, \theta, \delta)$ such that we have*

$$\prod_{p>X^2}\widetilde{M}_{\sigma,p}(z;F) = 1 + O\left(\exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right)\right)$$

for all $T \geq T_0$, for all $\sigma \in [1/2 + (\log T)^{-\theta}, \sigma_1]$, and for all $z \in \Omega$. Here we denote $X = \exp((\log T)^{\frac{5}{3}\theta})$, and the implied constant depends only on F and σ_1.

We prove Proposition 3.1 with the above preliminary lemmas.

Proof of Proposition 3.1. By (3.11), (3.12), and (3.13), we have

$$\frac{1}{T}\int_0^T \psi_z\left(\frac{F'}{F}(\sigma+it)\right)dt = \prod_{p\leq X^2}\widetilde{M}_{\sigma,p}(z;F) + O\left(\exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right)\right).$$

We consider the replacement of the product $\prod_{p\leq X^2}\widetilde{M}_{\sigma,p}(z;F)$ with $\prod_p \widetilde{M}_{\sigma,p}(z;F)$, where the error is estimated as

$$\left|\prod_p \widetilde{M}_{\sigma,p}(z;F) - \prod_{p\leq X^2}\widetilde{M}_{\sigma,p}(z;F)\right| \leq \left|\prod_{p>X^2}\widetilde{M}_{\sigma,p}(z;F) - 1\right|,$$

since $|\widetilde{M}_{\sigma,p}(z;F)| \leq 1$ by definition. Hence we have

$$\prod_{p\leq X^2}\widetilde{M}_{\sigma,p}(z;F) = \prod_p \widetilde{M}_{\sigma,p}(z;F) + O\left(\exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right)\right)$$

by Lemma 3.6. Therefore Proposition 3.1 follows if we define

$$\widetilde{M}_\sigma(z;F) = \prod_p \widetilde{M}_{\sigma,p}(z;F).$$

□

3.2. **Estimates on $\widetilde{M}_\sigma(z;F)$.** In this section, we examine some analytic properties of the function $\widetilde{M}_\sigma(z;F)$. By definition (3.14) and $\psi_z(w)=\exp(i\operatorname{Re}(z\overline{w}))$, we have

$$\widetilde{M}_{\sigma,p}(z;F)=\int_0^1\exp(ixa_p(\theta,\sigma;F)+iyb_p(\theta,\sigma;F))\,d\theta,$$

where $z=x+iy$ and $a_p(\theta,\sigma;F),b_p(\theta,\sigma;F)$ are functions such that

$$a_p(\theta,\sigma;F)=\sum_{m=1}^\infty\frac{1}{p^{m\sigma}}\{\operatorname{Re}\Lambda_F(p^m)\cos(2\pi m\theta)-\operatorname{Im}\Lambda_F(p^m)\sin(2\pi m\theta)\},$$
$$b_p(\theta,\sigma;F)=\sum_{m=1}^\infty\frac{1}{p^{m\sigma}}\{\operatorname{Re}\Lambda_F(p^m)\sin(2\pi m\theta)+\operatorname{Im}\Lambda_F(p^m)\cos(2\pi m\theta)\}.$$

Then we define

$$\widetilde{M}_p(s,z_1,z_2;F)=\int_0^1\exp(iz_1a_p(\theta,s;F)+iz_2b_p(\theta,s;F))\,d\theta \tag{3.15}$$

for $\operatorname{Re}s>0$ and $z_1,z_2\in\mathbb{C}$. We have $\widetilde{M}_{\sigma,p}(x+iy;F)=\widetilde{M}_p(\sigma,x,y;F)$ if $\sigma>0$ and $x,y\in\mathbb{R}$. For the study on the function $\widetilde{M}_p(s,z_1,z_2;F)$, the following lemma is fundamental, which is easily deduced from the expansion of $\exp(z)$ and the calculations of integrals.

Lemma 3.7. *Let $F(s)$ be a function that satisfies axiom* (4). *Then we have*

$$\widetilde{M}_p(s,z_1,z_2;F)=1-\mu_p+R_p \tag{3.16}$$

for $\sigma=\operatorname{Re}s>0$ and $z_1,z_2\in\mathbb{C}$, where

$$\mu_p=\mu_p(s,z_1,z_2;F)=\frac{z_1^2+z_2^2}{4}\sum_{m=1}^\infty\frac{|\Lambda_F(p^m)|^2}{p^{2ms}},$$
$$R_p=R_p(s,z_1,z_2;F)=\int_0^1\sum_{k=3}^\infty\frac{i^k}{k!}\{z_1a_p(\theta,s;F)+z_2b_p(\theta,s;F)\}^k\,d\theta.$$

Therefore, if μ_p and R_p are sufficiently small, we have

$$\log\widetilde{M}_p(s,z_1,z_2;F)=-\mu_p+R_p+O(|\mu_p|^2+|R_p|^2), \tag{3.17}$$

where log is the principal blanch of logarithm. Using Lemma 3.7, we study the function

$$\widetilde{M}(s,z_1,z_2;F)=\prod_p\widetilde{M}_p(s,z_1,z_2;F). \tag{3.18}$$

Proposition 3.8. *Let $F(s)$ be a function satisfying axioms* (1) *and* (4). *Assume that (s,z_1,z_2) varies on $\{\operatorname{Re}s>1/2\}\times\mathbb{C}\times\mathbb{C}$. If we fix two of the variables, the function $\widetilde{M}(s,z_1,z_2;F)$ is holomorphic with respect to the reminder variable.*

Proof. Let K be any compact subset on the half plane $\{\operatorname{Re}s>1/2\}$, and let K_1,K_2 be any compact subsets on $\mathbb{C}$. Assume that $(s,z_1,z_2)\in K\times K_1\times K_2$, and let σ_0 be the smallest real part of $s\in K$. As in Section 3.1, we have $|\Lambda(p^m)|\le g\log p$, where g is the constant in axiom (4). Then we obtain

$$\mu_p\ll\frac{g^2(\log p)^2}{p^{2\sigma_0}}\quad\text{and}\quad R_p\ll\frac{g^3(\log p)^3}{p^{3\sigma_0}},$$

where the implied constants depend only on K, K_1, K_2. Thus, by (3.17), we have $\log \widetilde{M}_p(s, z_1, z_2; F) \ll g^2(\log p)^2 p^{-2\sigma_0}$ for all $p > M$, where $M = M(K, K_1, K_2)$ is a sufficiently large constant that depends only on K, K_1, and K_2. The series $\sum_p (\log p)^2 p^{-2\sigma_0}$ converges since $\sigma_0 > 1/2$; therefore infinite product (3.18) uniformly converges on $K \times K_1 \times K_2$. Every local parts $\widetilde{M}_p(s, z_1, z_2; F)$ are holomorphic, and hence we have the result. □

We estimate the growth of $\widetilde{M}(s, z_1, z_2; F)$ with z_1 and z_2 near the real axis.

Proposition 3.9. *Let $F(s)$ be a function satisfying axioms* (1), (4), *and* (5). *Let $\sigma > 1/2$ be an arbitrarily fixed real number. Then there exist positive constants $K = K(\sigma; F)$ and $c = c(\sigma; F)$ such that for all $x, y \in \mathbb{R}$ with $|x| + |y| \geq K$, and for all non-negative integers m and n, we have*

$$\frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} \widetilde{M}(\sigma, z_1, z_2; F) \ll \exp\left(-c(|x| + |y|)^{\frac{1}{\sigma}} (\log(|x| + |y|))^{\frac{1}{\sigma} - 1}\right)$$

for any $z_1, z_2 \in \mathbb{C}$ with $|z_1 - x| < 1/4$, $|z_2 - y| < 1/4$. The implied constant depends only on m and n.

Proof. Let $K > 1$ and $c_0 < 1$ be positive constants chosen later, and assume that $x, y \in \mathbb{R}$ with $|x| + |y| \geq K$. We define

$$P_0 = \left(\frac{g(|x| + |y|)}{c_0} \log \frac{g(|x| + |y|)}{c_0}\right)^{\frac{1}{\sigma}}$$

for any fixed $\sigma > 1/2$. Then for any $p \geq P_0$, we see that

$$\frac{(|x| + |y|) g \log p}{p^\sigma} \leq \frac{(|x| + |y|) g \log P_0}{P_0^\sigma} \leq c_0 c_1$$

with an absolute constant $c_1 > 0$. Hence, we estimate μ_p and R_p in Lemma 3.7 arbitrarily small if we let the constant c_0 suitably small. Thus formula (3.17) holds. We then replace μ_p in (3.17) with the real number

$$\mu_p' = \mu_p'(\sigma, x, y; F) = \frac{x^2 + y^2}{4} \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|^2}{p^{2m\sigma}}.$$

The error of the replacement is estimated as

$$|\mu_p - \mu_p'| \leq (|x| + |y|) \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|^2}{p^{2m\sigma}}$$

if we assume that $|z_1 - x| < 1/2$ and $|z_2 - y| < 1/2$. Moreover, we have

$$\begin{aligned}\mu_p^2 &\ll \left(\frac{(|x| + |y|) g \log p}{p^\sigma}\right)^4 \leq \frac{(|x| + |y|) g \log P_0}{P_0^\sigma} \left(\frac{(|x| + |y|) g \log p}{p^\sigma}\right)^3 \\ &\ll \frac{(|x| + |y|)^3 g^3 (\log p)^3}{p^{3\sigma}}\end{aligned}$$

and

$$R_p \ll (|x| + |y|)^3 \left(\sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|}{p^{m\sigma}}\right)^3 \ll \frac{(|x| + |y|)^3 g^3 (\log p)^3}{p^{3\sigma}},$$

where all implied constants are absolute. Therefore by (3.17) we have for any $p \geq P_0$,

$$\left|\log \widetilde{M}_p(\sigma, z_1, z_2; F) + \frac{x^2+y^2}{4} \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|^2}{p^{2m\sigma}}\right| \leq (|x|+|y|) \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|^2}{p^{2m\sigma}} + B(|x|+|y|)^3 \frac{g^3(\log p)^3}{p^{3\sigma}}$$

with some absolute constant $B > 0$. Thus for sufficiently large K, if $|x|+|y| \geq K$, then we obtain

$$\begin{aligned}&\operatorname{Re} \log \widetilde{M}_p(\sigma, z_1, z_2; F)\\ &\leq -A(|x|+|y|)^2 \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|^2}{p^{2m\sigma}} + B(|x|+|y|)^3 \frac{g^3(\log p)^3}{p^{3\sigma}}\\ &\leq -A(|x|+|y|)^2 \frac{|\Lambda_F(p)|^2}{p^{2\sigma}} + B(|x|+|y|)^3 \frac{g^3(\log p)^3}{p^{3\sigma}}\end{aligned}$$

with some absolute constant $A > 0$. Note that

$$\Lambda_F(p) = (\alpha_1(p) + \cdots \alpha_g(p)) \log p = -a_F(p) \log p$$

from axiom (4). Hence, we have

(3.19)

$$\begin{aligned}&\left|\prod_{p \geq P_0} \widetilde{M}_p(\sigma, z_1, z_2; F)\right|\\ &\leq \exp\left(-A(|x|+|y|)^2 \sum_{p \geq P_0} \frac{|a_F(p)|^2(\log p)^2}{p^{2\sigma}} + B(|x|+|y|)^3 g^3 \sum_{p \geq P_0} \frac{(\log p)^3}{p^{3\sigma}}\right)\\ &\leq \exp\left(-A(|x|+|y|)^2 \sum_{p \geq P_0} \frac{(\log p)^2}{p^{2\sigma}} |a_F(p)|^2 + Bc_0c_1g^2(|x|+|y|)^2 \sum_{p \geq P_0} \frac{(\log p)^2}{p^{2\sigma}}\right).\end{aligned}$$

Then we estimate

$$\sum_{p \geq P_0} \frac{(\log p)^2}{p^{2\sigma}} |a_F(p)|^2 \quad \text{and} \quad \sum_{p \geq P_0} \frac{(\log p)^2}{p^{2\sigma}}.$$

We see that for any $\sigma > 1/2$, there exists a constant $X_0(\sigma; F) > 0$ such that for any $X \geq X_0(\sigma; F)$,

$$\sum_{p \geq X} \frac{(\log p)^2}{p^{2\sigma}} |a_F(p)|^2 \geq \frac{\kappa}{2(2\sigma-1)} X^{1-2\sigma} \log X$$

$$\sum_{p \geq X} \frac{(\log p)^2}{p^{2\sigma}} \leq \frac{2}{2\sigma-1} X^{1-2\sigma} \log X.$$

Indeed, the first inequality is deduced by summing by parts with axiom (5), and we obtain the second inequality in a similar way. Then, we let $c_0 = c_0(F)$ smaller so that $2Bc_0c_1g^2 < A\kappa/2$. If we let $K = K(\sigma; F)$ suitably large, then we obtain for $|u| + |v| \geq K$,

$$\begin{aligned}
&- A(|x| + |y|)^2 \sum_{p \geq P_0} \frac{(\log p)^2}{p^{2\sigma}} |a_F(p)|^2 + Bc_0c_1g^2(|x| + |y|)^2 \sum_{p \geq P_0} \frac{(\log p)^2}{p^{2\sigma}} \\
&\leq -c(|x| + |y|)^{\frac{1}{\sigma}} (\log(|x| + |y|))^{\frac{1}{\sigma}-1}
\end{aligned}$$

with some positive constant $c = c(\sigma; F)$. Hence we obtain

$$\left| \prod_{p \geq P_0} \widetilde{M}_p(\sigma, z_1, z_2; F) \right| \leq \exp\left(-c(|x| + |y|)^{\frac{1}{\sigma}} (\log(|x| + |y|))^{\frac{1}{\sigma}-1} \right). \tag{3.20}$$

The estimate on the contributions of $\widetilde{M}_p(\sigma, z_1, z_2; F)$ for $p < P_0$ remains. By definition (3.15), we see that

$$\begin{aligned}
\left| \widetilde{M}_p(\sigma, z_1, z_2; F) \right| &\leq \int_0^1 \exp(-\operatorname{Im}(z_1) a_p(\theta, \sigma; F) - \operatorname{Im}(z_2) b_p(\theta, \sigma; F)) \, d\theta \\
&\leq \int_0^1 \exp(|a_p(\theta, \sigma; F)| + |b_p(\theta, \sigma; F)|) \, d\theta \\
&\leq \exp\left(C \frac{g \log p}{p^\sigma} \right)
\end{aligned}$$

with some absolute positive constant C since $|z_1 - x| < 1/2$, $|z_2 - y| < 1/2$, and $x, y \in \mathbb{R}$. Thus we have

$$\left| \prod_{p < P_0} \widetilde{M}_p(\sigma, z_1, z_2; F) \right| \leq \exp\left(C \sum_{p < P_0} \frac{g \log p}{p^\sigma} \right) \leq \exp\left(Cg \log P_0 P_0^{\frac{1}{2}} \right).$$

Then we see that for $|x| + |y| \geq K$,

$$\left| \prod_{p < P_0} \widetilde{M}_p(\sigma, z_1, z_2; F) \right| \leq \exp\left(C'(|x| + |y|)^{\frac{3}{4\sigma}} \right), \tag{3.21}$$

where $C' = C'(F)$ is some positive constant. Therefore we obtain

$$\left| \widetilde{M}(\sigma, z_1, z_2; F) \right| \leq \exp\left(-c(|x| + |y|)^{\frac{1}{\sigma}} (\log(|x| + |y|))^{\frac{1}{\sigma}-1} \right) \tag{3.22}$$

by (3.20) and (3.21), where $c = c(\sigma; F)$ is some positive constant. We finally assume that $|z_1 - x| < 1/4$ and $|z_2 - y| < 1/4$. Then, applying Cauchy's integral formula, we have

$$\frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} \widetilde{M}(\sigma, z_1, z_2; F) = \frac{m!n!}{(2\pi i)^2} \iint_{\substack{|\xi_1 - z_1| = 1/4, \\ |\xi_2 - z_2| = 1/4}} \frac{\widetilde{M}(\sigma, \xi_1, \xi_2; F)}{(\xi_1 - z_1)^{m+1} (\xi_2 - z_2)^{n+1}} \, d\xi_1 d\xi_2.$$

Therefore by estimate (3.22), the desired result follows. □

Remark 3.10. We find that $\widetilde{M}_\sigma(z; F)$ is a Schwartz function according to Proposition 3.9. Hence its Fourier inverse

$$M_\sigma(z; F) = \int_{\mathbb{C}} \widetilde{M}_\sigma(w; F) \psi_{-z}(w) \, |dw|$$

is also a Schwartz function, and belongs to the class Λ. Thus we have $\widetilde{M}_\sigma(z;F) = (M_\sigma(z;F))^\wedge$. By a simple calculation, we see that $M_\sigma(z;F)$ is real valued.

Finally, we prove Lemma 3.6 in Section 3.1.

Proof of Lemma 3.6. Assume $p \geq X^2$ with $X = \exp((\log T)^{\frac{5}{3}\theta})$. Then we see that $\mu_p = \mu_p(\sigma, x, y; F)$ and $R_p = R_p(\sigma, x, y; F)$ in Lemma 3.7 are small when T is sufficiently large. In fact, we have for $p \geq X^2$,

$$\mu_p \ll (x^2+y^2)\frac{g^2(\log p)^2}{p^{2\sigma}} \ll \{(|x|+|y|)gX^{1-2\sigma}\log X\}^2.$$

By the setting for $X, z = x+iy$, and σ, we have

$$X^{1-2\sigma}\log X \leq \exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right) \to 0$$

as $T \to \infty$. The argument for R is similar. Hence by (3.17), we obtain

$$\log \widetilde{M}_{\sigma,p}(z;F) \ll (x^2+y^2)\frac{(\log p)^2}{p^{2\sigma}},$$

where the implied constant depends only on F. Therefore we have

$$\begin{aligned}\prod_{p\geq X^2}\widetilde{M}_{\sigma,p}(z;F) &= \exp\left(\sum_{p\geq X^2}\log \widetilde{M}_{\sigma,p}(z;F)\right)\\ &= 1 + O\left((x^2+y^2)\sum_{p\geq X^2}\frac{(\log p)^2}{p^{2\sigma}}\right).\end{aligned}$$

Applying the prime number theorem, we estimate the above error term as

$$(x^2+y^2)\sum_{p\geq X^2}\frac{(\log p)^2}{p^{2\sigma}} \ll (x^2+y^2)\frac{X^{2(1-2\sigma)}\log X}{(\sigma-\frac{1}{2})^2} \leq \exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right)$$

by the assumptions on X, $z = x+iy$, and σ. Here the implied constant depends only on F and σ_1. □

3.3. **Completion of the proof.**

Proof of Theorem 2.1. We only consider the case of $F \in \mathcal{S}_{\mathrm{I}}$ since the case $F \in \mathcal{S}_{\mathrm{II}}$ follows completely in an analogous way. By the definition of the class Λ, for any $\Phi \in \Lambda$ we have

$$\Phi(w) = \int_{\mathbb{C}}\widehat{\Phi}(z)\psi_{-z}(w)\,|dw|.$$

Hence, by Proposition 3.1, we see that for all $T \geq T_I$,

$$\begin{aligned}\frac{1}{T}\int_0^T \Phi\left(\frac{F'}{F}(\sigma+it)\right)dt &= \int_\Omega \widehat{\Phi}(z)\frac{1}{T}\int_0^T\psi_{-z}\left(\frac{F'}{F}(\sigma+it)\right)dt\,|dz| + E_1\\ &= \int_\Omega \widehat{\Phi}(z)\widetilde{M}_\sigma(-z;F)\,|dz| + E_1 + E_2\\ &= \int_{\mathbb{C}} \widehat{\Phi}(z)\widetilde{M}_\sigma(-z;F)\,|dz| + E_1 + E_2 + E_3,\end{aligned}$$

where the error terms are estimated as

$$\begin{aligned}
E_1 &= \int_{\mathbb{C}\setminus\Omega} \widehat{\Phi}(z)\frac{1}{T}\int_0^T \psi_{-z}\left(\frac{F'}{F}(\sigma+it)\right) dt\,|dz| \ll \int_{\mathbb{C}\setminus\Omega} |\widehat{\Phi}(z)|\,|dz|,\\
E_2 &\ll \exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right)\int_{\Omega} |\widehat{\Phi}(z)|\,|dz|,\\
E_3 &\ll \int_{\mathbb{C}\setminus\Omega} |\widehat{\Phi}(z)|\,|dz|.
\end{aligned}$$

Here all implied constants depend at most only on F, σ_1, and ϵ. We find that

$$\int_{\mathbb{C}} \widehat{\Phi}(w)\widetilde{M}_\sigma(-w;F)\,|dw| = \int_{\mathbb{C}} \widehat{\Phi}(w)\overline{\widetilde{M}_\sigma(w;F)}\,|dw| = \int_{\mathbb{C}} \Phi(z)M_\sigma(z;F)\,|dw|$$

due to Parseval's identity, and therefore (2.3) and (2.4) follow. The proof of the non-negativity of the function $M_\sigma(z;F)$ remains. For this, we assume $M_\sigma(z;F)<0$ for some region U. If we take $\Phi(z)$ as a non-negative function with a support included in U, then we have the contradiction. Due to the continuity of $M_\sigma(z;F)$, we see that $M_\sigma(z;F)$ is everywhere non-negative. $\square$

4. Proof of Theorem 2.2

We find that Theorem 2.1 imply Theorem 2.2 by the following lemma.

Lemma 4.1. *Let*

$$K(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2.$$

Then for any $a,b\in\mathbb{R}$ with $a<b$, there exists a continuous function $F_{a,b}:\mathbb{R}\to\mathbb{R}$ such that the following conditions hold: for any $\omega>0$,

1. $F_{a,b}(x) - 1_{[a,b]}(x) \ll K(\omega(x-a)) + K(\omega(x-b))$ *for any* $x\in\mathbb{R}$*;*
2. $\int_{\mathbb{R}} (F_{a,b}(x) - 1_{[a,b]}(x))\,dx \ll \omega^{-1}$*;*
3. *if* $|x|\geq\omega$*, then* $\widehat{F}_{a,b}(x)=0$*;*
4. $\widehat{F}_{a,b}(x) \ll (b-a)+\omega^{-1}$.

Here,

$$\widehat{F}_{a,b}(x) = \int_{\mathbb{R}} F_{a,b}(u)e^{ixu}\,|du|$$

is the Fourier transformation of $F_{a,b}(x)$ with $|du|=(2\pi)^{-\frac{1}{2}}du$.

Proof. This is Lemma 4.1 of [18] except for the difference of the definition of the Fourier transform, which does not affect the result. $\square$

Proof of Theorem 2.2. Again we consider only the case of $F\in\mathcal{S}_{\mathrm{I}}$. Assume that the rectangle R is given as

$$R = \{z = x+iy\in\mathbb{C} \mid a\leq x\leq b,\ c\leq y\leq d\}.$$

Then we define for $z=x+iy\in\mathbb{C}$

$$\Phi(z) = F_{a,b}(x)F_{c,d}(y). \tag{4.1}$$

We first find that the function $\Phi(z)$ belongs to the class Λ. The class Λ is also written as

$$\Lambda = \{f \in L^1 \mid f \text{ is continuous and } \widehat{f} \in L^1\},$$

and hence we must check that $\Phi \in L^1$, Φ is continuous, and $\widehat{\Phi} \in L^1$. Since

$$\int_{\mathbb{C}} \Phi(z)\,|dz| = \int_{\mathbb{R}} F_{a,b}(x)\,|dx| \int_{\mathbb{R}} F_{c,d}(y)\,|dy|,$$

we see that $\Phi \in L^1$ by condition (2) of Lemma 4.1. The function $\Phi(z)$ is continuous by its definition (4.1), and furthermore, we have

$$\widehat{\Phi}(z) = \widehat{F}_{a,b}(x)\widehat{F}_{c,d}(y) = 0$$

if $|x|, |y| \geq \omega$ by condition (3). Thus also we have $\widehat{\Phi} \in L^1$. Therefore $\Phi(z)$ belongs to the class Λ, and we apply Theorem 2.1 for this function. Note that

$$\Phi(z) - 1_R(z) \ll K(\omega(x-a)) + K(\omega(x-b)) + K(\omega(y-c)) + K(\omega(y-d)) \tag{4.2}$$

by condition (1) of Lemma 4.1. Then, let $\sigma > 1 - b^{-1}$ be fixed, and let $\theta, \delta > 0$ with $\delta + 3\theta > 0$. We take $\omega = (\log T)^{\delta}$. Due to inequality (4.2), Theorem 2.1 gives

$$\frac{1}{T}V_\sigma(T, R; F) = \int_R M_\sigma(z; F)\,|dz| + E_1 + E_2 + E_3 \tag{4.3}$$

for large T, where

$$E_1 \ll \exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right) \int_\Omega |\widehat{\Phi}(z)|\,|dz| + \int_{\mathbb{C}\setminus\Omega} |\widehat{\Phi}(z)|\,|dz|, \tag{4.4}$$

$$\begin{aligned} E_2 \ll{}& \frac{1}{T}\int_0^T K\left(\omega\left(\operatorname{Re}\frac{F'}{F}(\sigma+it) - a\right)\right) dt \\ &+ \frac{1}{T}\int_0^T K\left(\omega\left(\operatorname{Re}\frac{F'}{F}(\sigma+it) - b\right)\right) dt \\ &+ \frac{1}{T}\int_0^T K\left(\omega\left(\operatorname{Im}\frac{F'}{F}(\sigma+it) - c\right)\right) dt \\ &+ \frac{1}{T}\int_0^T K\left(\omega\left(\operatorname{Im}\frac{F'}{F}(\sigma+it) - d\right)\right) dt, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} E_3 \ll{}& \int_{\mathbb{C}} K(\omega(x-a))M_\sigma(z;F)\,|dz| + \int_{\mathbb{C}} K(\omega(x-b))M_\sigma(z;F)\,|dz| \\ &+ \int_{\mathbb{C}} K(\omega(y-c))M_\sigma(z;F)\,|dz| + \int_{\mathbb{C}} K(\omega(y-d))M_\sigma(z;F)|dz|. \end{aligned} \tag{4.6}$$

All implied constants depend on $F, \sigma, \theta, \delta, \epsilon$. We estimate three error terms E_1, E_2, and E_3. The first term of the right hand side of (4.4) is estimated as

$$\begin{aligned} \exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right) \int_\Omega |\widehat{\Phi}(z)|\,|dz| &\ll \exp\left(-\frac{1}{4}(\log T)^{\frac{2}{3}\theta}\right) (\log T)^{2\delta}(b-a)(d-c) \\ &\ll (\log T)^{-\delta}\nu_2(R) \end{aligned}$$

for sufficiently large T by condition (4) of Lemma 4.1. We have

$$\int_{\mathbb{C}\setminus\Omega} |\widehat{\Phi}(z)|\,|dz| = 0$$

since $\widehat{\Phi}(z) = 0$ if $|x|, |y| \geq \omega$. Therefore we obtain

$$E_1 \ll \nu_2(R)(\log T)^{-\delta}. \tag{4.7}$$

Next we estimate E_2. Since we have

$$K(\omega x) = \frac{2}{\omega^2}\int_0^\omega (\omega - u)\cos(2\pi x u)\,du = \frac{2}{\omega^2}\operatorname{Re}\int_0^\omega (\omega - u)e^{2\pi i x u}\,du,$$

the first term of the right hand side of (4.5) is estimated as

$$\begin{aligned} &\frac{1}{T}\int_0^T K\left(\omega\left(\operatorname{Re}\frac{F'}{F}(\sigma+it) - a\right)\right)dt \\ &\ll \frac{1}{\omega^2}\int_0^\omega (\omega-u)\left|\frac{1}{T}\int_0^T \exp\left(2\pi i u \operatorname{Re}\frac{F'}{F}(\sigma+it)\right)dt\right|du. \end{aligned} \tag{4.8}$$

Proposition 3.1 deduces

$$\frac{1}{T}\int_0^T \exp\left(2\pi i u \operatorname{Re}\frac{F'}{F}(\sigma+it)\right)dt \ll \left|\widetilde{M}_\sigma(2\pi u; F)\right|$$

as $T \to \infty$, hence (4.8) is

$$\ll \frac{1}{\omega^2}\int_0^\omega (\omega-u)\left|\widetilde{M}_\sigma(2\pi u; F)\right|du \ll \frac{1}{\omega} = (\log T)^{-\delta}.$$

The last inequality follows from Proposition 3.9. Since the reminder terms of (4.5) are estimated in a similar way, we have

$$E_2 \ll (\log T)^{-\delta}. \tag{4.9}$$

The work of the estimate of E_3 remains. For this, we define

$$m_\sigma(x; F) = \int_{\mathbb{R}} M_\sigma(x+iy; F)\,|dy|.$$

Then the first term of the right hand side of (4.6) is equal to

$$\int_{\mathbb{R}} K(\omega(x-a))m_\sigma(x; F)\,|dx|.$$

The function $m_\sigma(x; F)$ is bounded on $\mathbb{R}$. In fact, it is continuous, and we see that

$$\int_{\mathbb{R}} m_\sigma(x; F)\,|dx| = \int_{\mathbb{C}} M_\sigma(x; F)\,|dz| = \widetilde{M}_\sigma(0; F) = 1.$$

Therefore, we obtain

$$\int_{\mathbb{C}} K(\omega(x-a))\mathcal{M}_\sigma(z; F)\,|dz| \ll \int_{\mathbb{R}} K(\omega(x-a))\,dx \ll \frac{1}{\omega} = (\log T)^{-\delta}.$$

Estimating the remaining terms of (4.6) similarly, we have

$$E_3 \ll (\log T)^{-\delta}. \tag{4.10}$$

By estimates (4.7), (4.9), and (4.10), formula (4.3) gives

$$\frac{1}{T}V_\sigma(T,R;F) - \int_R M_\sigma(z;F)\,|dz| \ll (\nu_2(R)+1)(\log T)^{-\delta}.$$

Taking care of the assumption $\delta + 3\theta < 1/2$, we put $\theta = \epsilon/4$ and $\delta = 1/2 - \epsilon$ for arbitrarily small $\epsilon > 0$. Then we obtain

$$(\nu_2(R)+1)(\log T)^{-\delta} = (\nu_2(R)+1)(\log T)^{-\frac{1}{2}+\epsilon},$$

which gives the result. □

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