

Comments on asymptotic safety in four–dimensional $\mathcal{N} = 1$ supersymmetric gauge theories

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We discuss asymptotic safety in four–dimensional $\mathcal{N} = 1$ supersymmetric gauge theories. Our main conclusion is that in a reasonable definition of this phenomenon there are currently no controlled examples of it; there are reasons to believe that it does not occur.

1. Introduction

In recent years there has been a resurgence of work on asymptotic safety. Much of the interest has been in applications to quantum gravity. There has also been work on the possible occurrence of this phenomenon in four dimensional $\mathcal{N} = 1$ supersymmetric gauge theories. Some of this work concluded that this phenomenon occurs and is exhibited in certain models, in some cases at parametrically weak coupling.

In this paper we will revisit these models and argue in these models, and more generally, in four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories that the phenomenon of asymptotic safety does not occur. The reasons that asymptotic safety does not occur in some cases involve the question what exactly is meant by asymptotic safety, therefore in the following we begin with a brief general discussion of the idea of asymptotic safety in quantum field theory (QFT).

A reasonable definition, which we will adopt here, is that an asymptotically safe QFT is one that:

- (1) approaches at short distances an interacting fixed point;
- (2) does not fall into the Wilsonian paradigm.

Item (1) is self explanatory, but (2) requires some further explanation, especially since some of the examples below do not satisfy it.

For the purpose of our discussion, the Wilsonian paradigm is the idea that a mathematically complete QFT is defined by specifying its ultraviolet limit, a conformal field theory (CFT), and a finite set of parameters that describe the RG flow to longer distances. These parameters can correspond to (marginally) relevant couplings in the theory, and/or expectation values of operators in the CFT. A theory that does not fall into this paradigm can be one that is defined in terms of its infrared CFT, and a flow up the RG.

The reason that item (2) in the definition of an asymptotically safe theory is (we believe) necessary is that without it, the phenomenon in question is empty—it is part of the standard picture of Wilsonian RG, and it is not clear why it deserves a special name.

To illustrate what we mean, consider the case of QCD, a gauge theory with gauge group $SU(N_c)$ and N_f (massless) flavors of fermions (quarks) in the fundamental representation of the gauge group. As is well known, for $N_f < 11N_c/2$ the theory is asymptotically free, thus at short distances the gauge coupling goes to zero. The long distance dynamics depends on N_f ; in particular [1], for N_f slightly below¹ $11N_c/2$, the theory flows at long

¹ More precisely, for N_f/N_c slightly below $11/2$. This is easiest to achieve when N_f, N_c are large.

distances into an interacting CFT, known as the Banks–Zaks (BZ) fixed point. As N_f decreases, this CFT becomes more strongly coupled, and below some critical value of N_f , $N_f^{(\text{cr})}$, the theory is believed to spontaneously break the chiral $SU(N_f)_L \times SU(N_f)_R$ symmetry to its diagonal subgroup, and approach in the infrared a free theory of the Goldstone bosons of the broken $SU(N_f)$. The value of $N_f^{(\text{cr})}$ is not known in general.

If we removed item (2) in the definition of asymptotic safety above, a simple way to construct asymptotically safe theories would be the following. Start with an asymptotically free theory with N_f slightly below $11N_c/2$. This theory flows in the infrared into the BZ fixed point. We can take this fixed point to be the starting point of a Wilsonian analysis, *i.e.* view it as an ultraviolet fixed point that we can perturb to flow to long distances. An example of such a flow is obtained by adding a mass term to one of the N_f flavors of quarks. This flow connects the BZ fixed point of the theory with N_c colors and N_f flavors, BZ_{N_c, N_f} , to BZ_{N_c, N_f-1} . It certainly satisfies criterion (1) for an asymptotically safe theory, as the ultraviolet fixed point, BZ_{N_c, N_f} , is interacting, but we don't need a special name to describe it.

In this particular case, the infrared fixed point, BZ_{N_c, N_f-1} , is typically also interacting (in some range of N_f , N_c), but it is easy to construct examples where the infrared theory is free. An example is turning on a mass term to multiple flavors, which describes the flow $BZ_{N_c, N_f} \rightarrow BZ_{N_c, N'_f}$. If $N_f > N_f^{(\text{cr})} > N'_f$, the infrared theory is free.

Another standard example of an RG flow that satisfies (1) above but not (2) corresponds to generic points on the Coulomb branch of $N = 4$ SYM with gauge group $SU(n)$ (say). At short distances the theory is described by a non-trivial four-dimensional CFT, while in the infrared it is a free $U(1)^{n-1}$ theory.

The main point in the above discussion is that non-trivial fixed points are ubiquitous in four-dimensional QFT, as are RG flows from such points to other (free or interacting) CFT's, so for the phenomenon of asymptotic safety to be non-trivial, we need some version of criterion (2) above.

The rest of the paper is organized as follows. In section 2 we examine the claim of [2] that a certain model of supersymmetric gauge theories exhibits asymptotic safety at weak coupling. We point out that the RG flows discussed in [2] are standard Wilsonian RG flows and thus do not satisfy criterion (2) above. These flows, however, do exhibit the phenomenon that irrelevant (relevant) operators in the ultraviolet build up, as we flow toward low energies, negative (positive) anomalous dimension, and eventually become relevant (irrelevant) in the infrared. Such operators are known as dangerously irrelevant (harmlessly

relevant), and are well known to be ubiquitous in four–dimensional supersymmetric gauge theories. More importantly for us, they are comfortably accommodated in the Wilsonian framework.

In section 3 we comment on some supersymmetric gauge theories discussed in [3] and some more recent follow–ups. These theories are free at long distances. They are constructed in such a way that (at large values of couplings) they satisfy all the (algebraic) consistency constraints that an asymptotically safe theory would satisfy. The authors’ presumption is therefore that these theories are described by interacting fixed points at short distances.

We will argue that there are actually reasons to believe that such fixed points do not exist. In such theories, there are non–perturbative techniques that allow one to study the dynamics at fixed points in a way that does not rely on a weak coupling expansion, but it often happens that these techniques break down when the coupling exceeds a certain critical value. The detailed results of [3] as well as later work makes it plausible that the breakdown happens before one reaches the regime where the strongly coupled fixed points proposed by these authors appear.

Appendix A provides a brief review of dangerously irrelevant and harmlessly relevant operators in $\mathcal{N} = 1$ four–dimensional supersymmetric gauge theories. Technical details that we will use in the main text will be found in Appendix B.

2. The model, a –function and RG flows

In this section we systematically analyze the pattern of RG flows in the model considered in [2] and show that it does not exhibit asymptotic safety, that is, it does not satisfy the criterion (2) discussed in the introduction.

In our analysis we will employ the generalized central charge a –function constructed in [4]. It offers a simple and non–perturbative approach for analyzing RG flows. A brief review of this construction with an example is provided in Appendix B. In the following we begin our discussion with a brief description of the model.

The model is a class of $\mathcal{N} = 1$ supersymmetric gauge theories with a product gauge group $SU(N_1) \times SU(N_2)$ coupled to (anti–)fundamental chiral superfields $(\tilde{\psi}_i)\psi_i(i, \tilde{i} = 1, \dots, N_f)$, $(\tilde{\chi}_i)\chi_i(i, \tilde{i} = 1, \dots, N_f)$ and $(\tilde{Q}_i)Q_i(i, \tilde{i} = 1, \dots, N_Q)$, and (an) a “(anti–)bi–fundamental” chiral superfield $(\Psi)\tilde{\Psi}$. It has $SU(N_f)_L \times SU(N_f)_R \times SU(N_Q)_L \times SU(N_Q)_R \times$

$U(1)_R$ global symmetry. The field contents of the model, and the representations under which they transform are summarized as follows.

Chiral superfields	$SU(N_1)$	$SU(N_2)$	$SU(N_f)_L$	$SU(N_f)_R$	$SU(N_Q)_L$	$SU(N_Q)_R$
$\psi_i(\tilde{\psi}_i)(i, \tilde{i} = 1, \dots, N_f)$	$\mathbf{N}_1(\overline{\mathbf{N}}_1)$	$\mathbf{1}$	$\mathbf{1}(\overline{\mathbf{N}}_f)$	$\mathbf{N}_f(\mathbf{1})$	$\mathbf{1}(\mathbf{1})$	$\mathbf{1}(\mathbf{1})$
$\chi_i(\tilde{\chi}_i)(i, \tilde{i} = 1, \dots, N_f)$	$\mathbf{1}(\mathbf{1})$	$\mathbf{N}_2(\overline{\mathbf{N}}_2)$	$\mathbf{1}(\overline{\mathbf{N}}_f)$	$\mathbf{N}_f(\mathbf{1})$	$\mathbf{1}(\mathbf{1})$	$\mathbf{1}(\mathbf{1})$
$Q_i(\tilde{Q}_i)(i, \tilde{i} = 1, \dots, N_Q)$	$\mathbf{1}(\mathbf{1})$	$\mathbf{N}_2(\overline{\mathbf{N}}_2)$	$\mathbf{1}(\mathbf{1})$	$\mathbf{1}(\mathbf{1})$	$\mathbf{1}(\overline{\mathbf{N}}_Q)$	$\mathbf{N}_Q(\mathbf{1})$
$\Psi(\tilde{\Psi})$	$\overline{\mathbf{N}}_1(\mathbf{N}_1)$	$\overline{\mathbf{N}}_2(\mathbf{N}_2)$	$\mathbf{1}(\mathbf{1})$	$\mathbf{1}(\mathbf{1})$	$\mathbf{1}(\mathbf{1})$	$\mathbf{1}(\mathbf{1})$

The authors included a Yukawa interaction of the form

$$W = y \text{Tr}[\psi \Psi \chi + \tilde{\psi} \tilde{\Psi} \tilde{\chi}]. \quad (2.1)$$

In the following we apply the construction of [4] and obtain the gradient flow equations of the generalized central charge a -function in the configuration space defined by the couplings of the model. The zeros of these equations determine RG flows fixed points. We find these equations following the general procedure outlined in Appendix B. This procedure involves constructing and extremizing an intermediate a -function.

The first step in the procedure is constructing an intermediate a -function. Adapting the expression (B.2) for the model, we find that the intermediate a -function $a(\lambda_i, R_j)$ ($i = 1, 2, y; j = \psi, \chi, Q, \Psi$) takes the following form

$$\begin{aligned}
a(\lambda_i, R_j) = & 2(N_1^2 - 1) + 2(N_2^2 - 1) \\
& + 2N_f N_1 [3(R_\psi - 1)^3 - (R_\psi - 1)] + 2N_f N_2 [3(R_\chi - 1)^3 - (R_\chi - 1)] \\
& + 2N_2 N_Q [3(R_Q - 1)^3 - (R_Q - 1)] + 2N_1 N_2 [3(R_\Psi - 1)^3 - (R_\Psi - 1)] \\
& - \lambda_1 [N_1 + N_f(R_\psi - 1) + N_2(R_\Psi - 1)] \\
& - \lambda_2 [N_2 + N_f(R_\chi - 1) + N_1(R_\Psi - 1) + N_Q(R_Q - 1)] \\
& - \lambda_y (2 - R_\psi - R_\chi - R_\Psi). \quad (2.2)
\end{aligned}$$

The three Lagrange multipliers: λ_1, λ_2 and λ_y , are related to the two gauge couplings corresponding to the gauge groups $SU(N_1)$ and $SU(N_2)$, *i.e.* $\alpha_1 := \frac{N_1 g_1^2}{(4\pi)^2}$ and $\alpha_2 := \frac{N_2 g_2^2}{(4\pi)^2}$ respectively, and the Yukawa coupling $\alpha_y := \frac{N_1 y^2}{(4\pi)^2}$. The precise relations between the multipliers and couplings, in the large N_1, N_2 limits, are given by [4-6]

$$\lambda_1 = 8N_1 \alpha_1 + \mathcal{O}(\alpha^2), \quad \lambda_2 = 8N_2 \alpha_2 + \mathcal{O}(\alpha^2), \quad \lambda_y = 8N_2 N_f \alpha_y + \mathcal{O}(\alpha^2). \quad (2.3)$$

Where α denotes any linear combination of the couplings α_i ($i = 1, 2, y$).

The next step is finding the local maximum of the intermediate a -function $a(\lambda_i, R_j)$ with respect to the unknowns R_k . That is, we evaluate $\partial_{R_i} a(\lambda_j, R_k) = 0$, which gives a set of equations that can be solved for $R_i := R_i(\lambda_j)$. In a certain finite region in the space of couplings around the origin, *i.e.* the free fixed point $\lambda_i = 0$, we find

$$\begin{aligned}
R_\psi &= 1 - \frac{1}{3} \left(1 + \frac{\lambda_1 N_f - \lambda_y}{2N_f N_1} \right)^{\frac{1}{2}}, \\
R_\chi &= 1 - \frac{1}{3} \left(1 + \frac{\lambda_2 N_f - \lambda_y}{2N_f N_2} \right)^{\frac{1}{2}}, \\
R_Q &= 1 - \frac{1}{3} \left(1 + \frac{\lambda_2}{2N_2} \right)^{\frac{1}{2}}, \\
R_\Psi &= 1 - \frac{1}{3} \left(1 + \frac{\lambda_1 N_2 + \lambda_2 N_1 - \lambda_y}{2N_1 N_2} \right)^{\frac{1}{2}}.
\end{aligned} \tag{2.4}$$

We substitute these expressions for R_i in $a(\lambda_i, R_j)$ (2.2). This finally gives the generalized central charge a -function $a(\lambda_i) := a(\lambda_i, R_j(\lambda_k))$. Since it solves the equation $\partial_{R_i} a(\lambda_i, R_j(\lambda_k)) = 0$, its gradient flow equations in the configuration space of the model are given by

$$\begin{aligned}
\frac{da(\lambda_i)}{d\lambda_1} &= - [N_1 + N_f(R_\psi - 1) + N_2(R_\Psi - 1)], \\
\frac{da(\lambda_i)}{d\lambda_2} &= - [N_2 + N_f(R_\chi - 1) + N_1(R_\Psi - 1) + N_Q(R_Q - 1)], \\
\frac{da(\lambda_i)}{d\lambda_y} &= - (2 - R_\psi - R_\chi - R_\Psi),
\end{aligned} \tag{2.5}$$

here the running R -charges R_i in (2.5) are given by their form (2.4) in terms of the running couplings λ_i —in some renormalization scheme.

These equations determine the fixed points of the model in a certain finite region in the space of couplings around $\lambda_i = 0$. In this paper, however, we work at weak coupling, that is, in some small region around $\lambda_i = 0$. For simplicity, we also work in the Veneziano large N limit², that is, we take N_1, N_2, N_f, N_Q to infinity while keeping their ratios fixed. This is compatible with the weak coupling regime; this will become more evident in a later subsection.³ The generalization to finite N is trivial.

² N is some linear combination of N_1, N_2, N_f, N_Q .

³ Basically, the idea is that, as in the Banks–Zaks analysis [1], the couplings can be taken to be arbitrarily small independently.

Therefore, we can expand the running R -charges R_j in (2.5) (given by (2.4)) to linear order in the couplings λ_i . Expanding (2.4) to leading order in λ_i yields

$$\begin{aligned}
R_\psi &= \frac{2}{3} - \frac{\lambda_1 N_f - \lambda_y}{12 N_f N_1}, \\
R_\chi &= \frac{2}{3} - \frac{\lambda_2 N_f - \lambda_y}{12 N_f N_2}, \\
R_Q &= \frac{2}{3} - \frac{\lambda_2}{12 N_2}, \\
R_\Psi &= \frac{2}{3} - \frac{\lambda_1 N_2 + \lambda_2 N_1 - \lambda_y}{12 N_1 N_2}.
\end{aligned} \tag{2.6}$$

Substituting these in (2.5) gives the linearized form of the gradient flow equations

$$12 N_1 \frac{da}{d\lambda_1} = - [4 N_1 (3 N_1 - N_f - N_2) - \lambda_1 (N_f + N_2) - \lambda_2 N_1 + 2 \lambda_y], \tag{2.7}$$

$$12 N_2 \frac{da}{d\lambda_2} = - [4 N_2 (3 N_2 - N_f - N_1 - N_Q) - \lambda_1 N_2 - \lambda_2 (N_f + N_1 + N_Q) + 2 \lambda_y], \tag{2.8}$$

$$12 N_1 N_2 N_f \frac{da}{d\lambda_y} = - [2 \lambda_1 N_2 N_f + 2 \lambda_2 N_1 N_f - \lambda_y (N_1 + N_2 + N_f)]. \tag{2.9}$$

A vanishing condition on these equations, accompanied by the positivity constraint on the couplings (2.3), specifies the fixed points of the various RG flows in the coupling space of the model. In the following subsections, we consider the cases in which we turn on only one of the gauge couplings, and the case in which we turn on both the gauge couplings; these initiate RG flows in the configuration space. We will find all the resulting RG fixed points, and we will examine the admissible RG flows that interpolate among them.

For convenience, we define the parameters ϵ_1 and ϵ_2 via

$$N_2 := 3 N_1 - N_f - \epsilon_1, \tag{2.10}$$

$$N_Q := 8 N_1 - 4 N_f - 3 \epsilon_1 - \epsilon_2. \tag{2.11}$$

Note that ϵ_1 and ϵ_2 are integers, and in the Veneziano large N limit, they are small, that is to say, $|\epsilon_1|, |\epsilon_2| \ll N$. In this limit, the couplings α_i are in the weak coupling regime, *i.e.* they are of $\mathcal{O}(1/N)$, therefore, as in the Banks–Zaks analysis [1], weak coupling (perturbative) Banks–Zaks expansion in ϵ_1, ϵ_2 is applicable; we show this as we progress. We note from (2.7) and (2.8) that near the Gaussian fixed point, *i.e.* $\lambda_i = 0$, for both gauge groups to be asymptotically free, *i.e.* $\frac{da}{d\lambda_i} < 0$ (here $i = 1, 2$), we must require both ϵ_1 and ϵ_2 to be positive.

For further convenience, we define a positive real number x as the ratio

$$x := \frac{N_f}{N_1}. \quad (2.12)$$

The condition that the number of quarks N_Q , given by (2.11), is positive reads now

$$x < 2. \quad (2.13)$$

Note that, in this regime, the degree of the gauge group $SU(N_2)$, given by (2.10), is trivially positive.

In each of the following subsections, we investigate the RG flows of the model in the cases in which near the Gaussian fixed point at least one of the gauge couplings is asymptotically free:

- (1) $\epsilon_1 > 0, \epsilon_2 > 0$;
- (2) $\epsilon_1 > 0, \epsilon_2 < 0$;
- (3) $\epsilon_1 < 0, \epsilon_2 > 0$;

and/or in the case in which neither of the gauge couplings are asymptotically free

- (4) $\epsilon_1 < 0, \epsilon_2 < 0$.

2.1. Turning the gauge coupling λ_1 on

In this subsection we assume that at the Gaussian fixed point the gauge coupling $\lambda_1 \approx \alpha_1$ is marginally relevant, that is, $\epsilon_1 > 0$. The gauge coupling $\lambda_2 \approx \alpha_2$ can be marginally relevant or irrelevant. We will investigate these possibilities here in the subsequent several cases. However, for the moment, we only focus on turning the gauge coupling λ_1 on.

Turning λ_1 on results in an RG flow toward long distances. The resulting fixed point of this flow is obtained by setting (2.7) to zero with $\lambda_2 = 0$ and $\lambda_y = 0$. Note that since we are turning only λ_1 on, we do not impose the vanishing condition on (2.8) and (2.9). One finds in the infrared the Banks–Zaks fixed point $(\lambda_1^*, 0, 0)$. Here,

$$\lambda_1^* = \frac{4}{3}\epsilon_1. \quad (2.14)$$

Note that this fixed point is the fixed point BZ_1 in [2]. To see this, we first note that the two sets of parameters, *i.e.* ϵ, P defined in [2] and ϵ_1, ϵ_2 defined here, are related as

$$N_1\epsilon = -\epsilon_1, \quad N_2P\epsilon = -\epsilon_2. \quad (2.15)$$

The desired result then follows readily using this and (2.3).

We note also using (2.14) that the gauge coupling $\alpha_1 \approx \lambda_1/N_1$ is of $\mathcal{O}(1/N)$, therefore, in the large N limit, it is in the weak coupling regime. This is consistent with our earlier assumption. One can also check that this is indeed the situation in all the other cases that we will consider below.

We note from (2.9) with $\lambda_2 = 0$ and $\lambda_y = 0$ that the Yukawa coupling $\lambda_y \approx \alpha_y$ is relevant at the fixed point (2.14) on the λ_1 axis; it is a dangerously irrelevant coupling. Therefore, one can turn it on at this fixed point. Turning on the Yukawa coupling drives the theory further down in the infrared into the fixed point $(\lambda_1^*, 0, \lambda_y^*)$. We find from (2.7) and (2.9) with $\lambda_2 = 0$ that

$$\lambda_1^* = \frac{4}{(x - \frac{3}{2})^2 + \frac{3}{4}} \epsilon_1, \quad \lambda_y^* = N_1 \cdot \frac{2x(3-x)}{(x - \frac{3}{2})^2 + \frac{3}{4}} \epsilon_1. \quad (2.16)$$

Note that this fixed point is the fixed point GY_1 in [2]. One can see this easily using (2.3) and (2.15) in combination with (2.10) which, in terms of the parameters defined in [2], is

$$x = \epsilon - R + 3. \quad (2.17)$$

In the following we investigate whether λ_2 is relevant at the fixed points (2.14) along the λ_1 axis and (2.16) on the $\lambda_2 = 0$ plane. In the case(s) in which it is relevant we can turn it on, that is to say, we can use it to probe other fixed point(s) in the infrared. There are two cases:

- (1) $\epsilon_2 > 0$

We first examine the case in which ϵ_2 is positive, that is, at the Gaussian fixed point the gauge coupling λ_2 is marginally relevant.

We first consider the RG flow that resulted the fixed point (2.14) on the λ_1 axis. It follows from (2.8) with $\lambda_2 = 0$ and $\lambda_y = 0$ that the gauge coupling λ_2 remains relevant⁴ along this flow provided the constraint

$$\lambda_1 < 4\epsilon_2, \quad (2.18)$$

⁴ To be precise, it is relevant only away from the free fixed point. At the fixed point, it is marginal.

is satisfied. Along this flow, $\lambda_1 \leq \lambda_1^*$; thus, the gauge coupling λ_2 is relevant throughout the flow (away from the free fixed point) and in particular at the fixed point (2.14) along the λ_1 axis only if the constraint

$$\frac{\epsilon_1}{\epsilon_2} < 3, \quad (2.19)$$

is satisfied.

Thus, if the constraint (2.19) is not satisfied, then what happens is that as we flow toward low energies, the gauge coupling λ_2 that was initially marginally relevant in the ultraviolet develops positive anomalous dimension, and eventually becomes irrelevant in the infrared. Therefore, in the case in which (2.19) is not satisfied the gauge coupling λ_2 is an example of a harmlessly relevant coupling (see Appendix A for a brief review and examples of this class of operators).

In the regime (2.19) we can turn on λ_2 at the fixed point (2.14) along the λ_1 axis, and depending on whether or not a condition is satisfied, the theory flows either into the fixed point (2.24) (see below) on the λ_2 axis or into the fixed point (2.32) (see below) on the $\lambda_y = 0$ plane. We will come back to this later.

We next consider the RG flow that was generated by turning the Yukawa coupling λ_y on at the fixed point (2.14) on the λ_1 axis. We ask the question that to be able to turn on λ_2 at the fixed point (2.16) on the $\lambda_2 = 0$ plane, that is, to flow further to longer distances, what are the constraints that ϵ_1 and ϵ_2 , and/or x must satisfy?

We first consider the case in which the constraint (2.19) is not satisfied, that is, λ_2 is harmlessly relevant and therefore irrelevant at the fixed point (2.14) on the λ_1 axis. There are two cases: $x < 1$ and $x \geq 1$.

First we consider the case $x < 1$. It follows from (2.8) (and the assumption that (2.19) is not satisfied) that x must satisfy the constraint

$$x_0 < x < 1, \quad (2.20)$$

here x_0 is defined via the equation

$$\frac{\epsilon_1}{\epsilon_2} = r_2(x_0) := -\frac{\left(x_0 - \frac{3}{2}\right)^2 + \frac{3}{4}}{x_0 - 1}. \quad (2.21)$$

Therefore, in the regime (2.20), λ_2 becomes once again relevant as we flow toward the fixed point (2.16) on the $\lambda_2 = 0$ plane, thus (in this flow) it is dangerously irrelevant; one can turn it on at the fixed point (2.16) on the plane $\lambda_2 = 0$, and flow further down in the infrared into the fixed point, see below, $(\lambda_1^*, \lambda_2^*, \lambda_y^*)$.

For $x \geq 1$, on the other hand, we find regardless of ϵ_1, ϵ_2 that the coupling λ_2 is relevant at the fixed point (2.16) on the $\lambda_2 = 0$ plane, and thus, in this case also one can turn it on.

In the regime (2.19), we find that the gauge coupling λ_2 is relevant at the fixed point (2.16) on the $\lambda_2 = 0$ plane with no constraints on the parameters of the model.

(2) $\epsilon_2 < 0$

In this case we take at the Gaussian fixed point the gauge coupling λ_2 to be marginally irrelevant.

We first consider the RG flow that resulted the fixed point (2.14) on the λ_1 axis. It follows from (2.8) (or equivalently from (2.18)) with $\lambda_2 = 0$ and $\lambda_y = 0$ that irrespective of the values of $|\epsilon_2|$, the gauge coupling λ_2 remains irrelevant along this flow (away from the free fixed point), and in particular it is irrelevant at the resulting fixed point (2.14) on the λ_1 axis.

We next consider the RG flow that was generated by turning the Yukawa coupling λ_y on at the fixed point (2.14) on the λ_1 axis. There are two cases: $x > 1$ and $x \leq 1$.

First we consider $x > 1$. We note using (2.8) that as we flow toward the fixed point (2.16) on the plane $\lambda_2 = 0$, the gauge coupling λ_2 can become relevant provided the constraints

$$\left| \frac{\epsilon_1}{\epsilon_2} \right| > 1, \quad x_0 < x < 2, \quad (2.22)$$

are satisfied. Here x_0 is defined via the equation

$$\left| \frac{\epsilon_1}{\epsilon_2} \right| = -r_2(x_0). \quad (2.23)$$

Therefore, in this regime, the coupling λ_2 is a dangerously irrelevant coupling. Thus, it is relevant at the fixed point (2.16) on the $\lambda_2 = 0$ plane, and it can be turned on to flow further into the fixed point, see below, $(\lambda_1^*, \lambda_2^*, \lambda_y^*)$ in the infrared.

In this case, *i.e.* $\epsilon_1 > 0$, $\epsilon_2 < 0$, and in the regime (2.22), the authors of [2] call the fixed point (2.16) on the $\lambda_2 = 0$ plane at which the dangerously irrelevant coupling λ_2 is relevant (and similarly as we will see shortly in the following subsection, in the case $\epsilon_1 < 0$, $\epsilon_2 > 0$, the authors call the fixed point on the $\lambda_1 = 0$ plane at which the dangerously irrelevant coupling λ_1 is relevant) an asymptotically safe interacting ultraviolet fixed point. As we explained in the introduction, none of these flows, however, imply asymptotic safety but

rather they merely correspond to the RG flow phenomenon in which an initially irrelevant coupling at relatively high energies develops, as we flow, negative anomalous dimension and eventually becomes relevant at low energies.

For $x \leq 1$, on the other hand, regardless of the values of $|\epsilon_2|$, the gauge coupling λ_2 is irrelevant at the fixed point (2.16) on the plane $\lambda_2 = 0$. Therefore, it switches off in the infrared.

2.2. Turning the gauge coupling λ_2 on

In this subsection we assume that at the Gaussian fixed point the gauge coupling λ_2 is marginally relevant, that is, $\epsilon_2 > 0$. The gauge coupling λ_1 can be marginally relevant or irrelevant. We will consider these possibilities shortly.

Turning λ_2 on drives the theory to flow in the infrared into the Banks–Zaks fixed point $(0, \lambda_2^*, 0)$, here λ_2^* is obtained by setting (2.8) to zero. With $\lambda_1 = 0$ and $\lambda_y = 0$ one finds

$$\lambda_2^* = \frac{4}{3}\epsilon_2. \quad (2.24)$$

This fixed point is the fixed point BZ_2 in [2].

We note from (2.9) with $\lambda_1 = 0$ and $\lambda_y = 0$ that the Yukawa coupling λ_y is relevant at the fixed point (2.24) on the λ_2 axis; it is a dangerously irrelevant coupling. If we turn on the Yukawa coupling λ_y at this fixed point, then the theory flows into the fixed point $(0, \lambda_2^*, \lambda_y^*)$ in the infrared. With $\lambda_1 = 0$, we find by setting (2.8) and (2.9) to zero that

$$\lambda_2^* = \frac{4(3-x)}{9-4x}\epsilon_2, \quad \lambda_y^* = N_1 \cdot \frac{2x(3-x)}{9-4x}\epsilon_2. \quad (2.25)$$

This is the fixed point GY_2 in [2].

In the following we investigate whether λ_1 is relevant at the fixed points (2.24) along the λ_2 axis and (2.25) on the plane $\lambda_1 = 0$. In the case(s) in which it is relevant, we can use it to access other low energy fixed point(s). Here also we have two cases:

- (1) $\epsilon_1 > 0$

We first examine the case in which ϵ_1 is positive, that is, at the Gaussian fixed point the gauge coupling λ_1 is marginally relevant.

We first consider the RG flow that resulted the fixed point (2.24) on the λ_2 axis. We find using (2.7) that for λ_1 to remain relevant along this flow (away from the free fixed point) one must demand that

$$\lambda_2 < 4\epsilon_1. \quad (2.26)$$

Since along the flow $\lambda_2 \leq \lambda_2^*$, λ_1 is relevant throughout the flow (away from the free fixed point) and in particular at the fixed point (2.24) along the λ_2 axis provided

$$\frac{\epsilon_1}{\epsilon_2} > \frac{1}{3}. \quad (2.27)$$

Thus, if the constraint (2.27) is not satisfied, then the gauge coupling λ_1 can become irrelevant in the infrared. In the case in which the constraint (2.27) is not met, it is a harmlessly relevant coupling.

In the regime (2.27) we can turn on λ_1 at the fixed point (2.24) along the λ_2 axis, and flow either into the fixed point (2.14) on the λ_1 axis or into the fixed point (2.32) (see below) on the $\lambda_y = 0$ plane depending on whether or not a condition is satisfied. We will comment on this later.

We next consider the RG flow that was generated by turning the Yukawa coupling λ_y on at the fixed point (2.24) on the λ_2 axis.

Assuming the constraint (2.27) is not satisfied, that is, assuming that λ_1 is harmlessly relevant and thus irrelevant at the fixed point (2.24) on the λ_2 axis, we now ask the question, what are the constraints that the parameters ϵ_1 and ϵ_2 , and/or x must respect so that λ_1 becomes relevant as we flow toward the fixed point $(0, \lambda_2^*, \lambda_y^*)$? There are two cases: $x < 1$ and $x \geq 1$.

We begin with the former. We find using (2.7) that λ_1 can become once again relevant provided the constraint

$$x_0 < x < 1, \quad (2.28)$$

is satisfied. Here x_0 is defined via the equation

$$\frac{\epsilon_1}{\epsilon_2} = r_1(x_0) := -\frac{(3-x_0)(x_0-1)}{9-4x_0}. \quad (2.29)$$

Therefore, in the regime (2.28), λ_1 is dangerously irrelevant, thus, it is relevant at the fixed point (2.25) on the plane $\lambda_1 = 0$, and one can turn it on at this fixed point to flow further down in the infrared into the fixed point, see below, $(\lambda_1^*, \lambda_2^*, \lambda_y^*)$.

On the other hand, we find for $x \geq 1$ that irrespective of ϵ_1 , ϵ_2 the gauge coupling λ_1 is relevant at the fixed point (2.25) on the $\lambda_1 = 0$ plane.

In the regime (2.27), the coupling λ_1 is relevant at the fixed point (2.25) on the $\lambda_1 = 0$ plane with no restrictions.

$$(2) \quad \epsilon_1 < 0$$

In this case we take at the Gaussian fixed point the gauge coupling λ_1 to be marginally irrelevant.

We first consider the RG flow that ended at the fixed point (2.24) on the λ_2 axis. It follows from (2.7)(or equivalently from (2.26)) that irrespective of the values of $|\epsilon_2|$, the gauge coupling λ_2 is irrelevant along this flow (away from the free fixed point), and in particular, it is irrelevant at the fixed point (2.24) on the λ_2 axis. Therefore, it goes to zero in the infrared.

For the gauge coupling λ_1 to become relevant in the infrared one must turn on the Yukawa coupling at the fixed point (2.24) on the λ_2 axis. There are two cases: $x > 1$ and $x \leq 1$.

In the former case, the constraints that the parameters ϵ_1 and ϵ_2 , and x must satisfy so that the gauge coupling λ_1 becomes relevant in the infrared as we flow toward the fixed point (2.25) on the plane $\lambda_1 = 0$ are

$$\left| \frac{\epsilon_1}{\epsilon_2} \right| < 1, \quad x_0 < x < 2. \quad (2.30)$$

Here x_0 is defined via the equation

$$\left| \frac{\epsilon_1}{\epsilon_2} \right| = -r_1(x_0). \quad (2.31)$$

If these constraints are satisfied, then λ_1 becomes relevant as we flow toward the infrared. Therefore, in this regime, the coupling λ_1 is a dangerously irrelevant coupling. Thus, turning on λ_1 at the fixed point (2.25) on $\lambda_1 = 0$ plane generates a flow into the infrared fixed point, see below, $(\lambda_1^*, \lambda_2^*, \lambda_y^*)$.

We also note here that in the regime (2.30) the fixed point (2.25) does not correspond to an asymptotically safe fixed point for the same reason as in the earlier case.

For $x \leq 1$, irrespective of the values of $|\epsilon_1|$, the gauge coupling λ_1 is irrelevant at the fixed point (2.25) on the plane $\lambda_1 = 0$; it flows to zero in the infrared.

2.3. Turning both the gauge couplings λ_1 and λ_2 on

There are two cases to consider: (1) $\lambda_y = 0$, and (2) $\lambda_y \neq 0$. In what follows we study these cases.

We first consider turning on only the gauge couplings, *i.e.* we take $\lambda_y = 0$. In this case the theory flows in the infrared into the fixed point $(\lambda_1^*, \lambda_2^*, 0)$. This fixed point is obtained by setting (2.7) and (2.8) to zero with $\lambda_y = 0$. One gets

$$\lambda_1^* = \frac{1}{2}(3\epsilon_1 - \epsilon_2), \quad \lambda_2^* = \frac{1}{2}(3\epsilon_2 - \epsilon_1). \quad (2.32)$$

This fixed point is the fixed point BZ_{12} in [2].

On the other hand, if we also turn on the Yukawa coupling λ_y , then the theory flows in the infrared into the fixed point $(\lambda_1^*, \lambda_2^*, \lambda_y^*)$. Setting the equations (2.7), (2.8) and (2.9) to zero we obtain

$$\lambda_1^* = \frac{\epsilon_1(9 - 4x) + \epsilon_2(3 - x)(x - 1)}{\frac{1}{4}(2 - x) \left[3\left(x - \frac{5}{3}\right)^2 + \frac{11}{3} \right]}, \quad (2.33)$$

$$\lambda_2^* = \frac{\epsilon_1(3 - x)(x - 1) + \epsilon_2(3 - x) \left[\left(x - \frac{3}{2}\right)^2 + \frac{3}{4} \right]}{\frac{1}{4}(2 - x) \left[3\left(x - \frac{5}{3}\right)^2 + \frac{11}{3} \right]}, \quad (2.34)$$

$$\lambda_y^* = N_f \cdot \frac{\epsilon_1(3 - x)(8 - 3x) + \epsilon_2(3 - x)x}{\frac{1}{2}(2 - x) \left[3\left(x - \frac{5}{3}\right)^2 + \frac{11}{3} \right]} = N_f \cdot \frac{1}{2} \left[(3 - x)\lambda_1^* + \lambda_2^* \right]. \quad (2.35)$$

This fixed point is the fixed point GY_{12} in [2].

We next check that these fixed points are in the physical regime. That is, we find the regimes in the parameter space of the model in which λ_1^* , λ_2^* (and λ_y^*) are positive. There are four cases:

- (1) $\epsilon_1 > 0$, $\epsilon_2 > 0$

In this case both gauge couplings are marginally relevant at the Gaussian fixed point. The fixed point (2.32) with $\lambda_y^* = 0$ is physical if the constraint

$$\frac{1}{3} \leq \frac{\epsilon_1}{\epsilon_2} \leq 3, \quad (2.36)$$

is satisfied. We note that this constraint consists of the intersection of the constraint regions (2.19) and (2.27). This makes sense in that one can reach the fixed point (2.32) on the plane $\lambda_y = 0$ either by first turning on λ_1 , and then λ_2 or vice versa; however, note that this is true only if the (gauge) couplings are not harmlessly relevant. We will come back to

this point shortly. On either side of the constraint region (2.36) we note from (2.19) and (2.27) that either of the gauge couplings is harmlessly relevant, and thus turning on both (at the free fixed point) drives the theory to flow either into the fixed point along the λ_1 or λ_2 axis. We will comment on this shortly. We also note that the Yukawa coupling λ_y is relevant at (2.32) on the plane $\lambda_y = 0$, and thus we can turn it on, which in turn generates a flow toward the fixed point given by (2.33), (2.34) and (2.35) in the infrared.

In subsection (2.1), in the case in which $\epsilon_2 > 0$, we saw that in the regime (2.19) the gauge coupling λ_2 is relevant at the fixed point (2.14) on the λ_1 axis. We mentioned that in this regime, therefore, one can turn on this coupling to flow in the infrared, depending on whether or not a condition is satisfied, either into the fixed point (2.24) on the λ_2 axis or into the above fixed point (2.32) on the $\lambda_y = 0$ plane. In the following we will be very specific.

Suppose that ϵ_1 and ϵ_2 satisfy the constraints (2.19) and (2.27). Thus, they also satisfy the constraint (2.36) since

$$\frac{1}{3} < \frac{\epsilon_1}{\epsilon_2} < 3. \quad (2.37)$$

In this case, it follows from our earlier discussion and (2.36) that turning λ_1 , and then λ_2 results in an RG flow that ends at the fixed point (2.32) on the plane $\lambda_y = 0$; note that in this regime none of the gauge couplings are harmlessly relevant throughout the flow.

Now suppose, to the contrary, that ϵ_1 and ϵ_2 do not satisfy (2.27) (and therefore (2.37)), that is, suppose that

$$\frac{\epsilon_1}{\epsilon_2} \leq \frac{1}{3}. \quad (2.38)$$

Note that only the equal sign satisfies the constraint (2.36). Note also that in this regime the condition (2.19) is trivially satisfied, and therefore, the coupling λ_2 is relevant at the fixed point on the λ_1 axis, and as such we can turn it on at this fixed point and flow to lower energies. In this case, turning λ_1 , and then λ_2 will take us into the fixed point (2.24) on the λ_2 axis; therefore, also in this regime, turning both the gauge couplings λ_1 and λ_2 on (at the free fixed point) leads in the infrared into the same fixed point on the λ_2 axis. This can be understood as follows.

Consider deforming the fixed point (2.14) on the λ_1 axis by turning on the relevant gauge coupling λ_2 . For a non-zero value of λ_2 , we note from (2.7) that the coupling λ_1 at this fixed point is irrelevant. Therefore, as we flow toward lower energies, the gauge coupling λ_1 shrinks toward zero, and the gauge coupling λ_2 , on the other hand, grows away from zero. In the regime (2.38), we find from (2.7) and (2.8), the rate at which the coupling

λ_1 runs to zero is much faster than the rate at which the coupling λ_2 runs away from zero. Therefore, when λ_1 (first) hit zero, we find the fixed point (2.24) at $\lambda_2 = \lambda_2^*$ on the λ_2 axis. We note from (2.27) that in the regime (2.38) the coupling λ_1 is irrelevant at this fixed point, therefore it does not grow, *i.e.* it stays zero. The change in the central charge a along the flow can be obtained using the generalized a -function. We find in accord with the weak a -theorem $\Delta a = a(\lambda_1^*, 0, 0) - a(0, \lambda_2^*, 0) > 0$.

In the regime (2.38), thus, what happens when we turn both the gauge couplings λ_1 and λ_2 on at the free fixed point is that as we flow toward infrared, λ_1 develops positive anomalous dimension, and eventually becomes irrelevant—thus it is harmlessly relevant—while λ_2 stays relevant (except at the free fixed point where it is marginally relevant). Thus, the RG flow then finally ends at the fixed point (2.24) on the λ_2 axis.

We also stated above in subsection (2.2) in the case in which $\epsilon_1 > 0$ that in the regime (2.27) one can turn on the gauge coupling λ_1 at the fixed point (2.24) on the λ_2 axis to flow in the infrared, depending on whether or not a condition is satisfied, either into the fixed point (2.14) on the λ_1 axis or into the above fixed point (2.32) on the $\lambda_y = 0$ plane. In the following we briefly discuss these two RG flows.

If ϵ_1 and ϵ_2 satisfy the constraint (2.37), then it follows from our earlier discussion and (2.36) that turning λ_1 on at the fixed point (2.24) on the λ_2 axis drives the theory into the above fixed point on the plane $\lambda_y = 0$.

On the contrary, if ϵ_1 and ϵ_2 do not satisfy (2.37), that is, if

$$\frac{\epsilon_1}{\epsilon_2} \geq 3, \tag{2.39}$$

then the theory flows into the fixed point (2.14) on the λ_1 axis. Here also the change in the central charge $\Delta a = a(0, \lambda_2^*, 0) - a(\lambda_1^*, 0, 0) > 0$.

We note from (2.19) that in the regime (2.39) the coupling λ_2 is harmlessly relevant, and for essentially the same reason as in the earlier case, therefore, in this regime turning both the gauge couplings λ_1 and λ_2 on at the free fixed point leads in the infrared into the fixed point (2.14) on the λ_1 axis.

We next examine the fixed point given by (2.33), (2.34) and (2.35). There are four cases, and we discuss them in turn in the following.

The first case is when the parameters ϵ_1 and ϵ_2 are allowed to take any values. In this case we find that this fixed point is physical if the constraint

$$1 \leq x < 2, \tag{2.40}$$

is satisfied. This is consistent with the results in subsections 2.1 and 2.2.

The next case is when the parameters ϵ_1 and ϵ_2 are such that

$$\frac{1}{3} \leq \frac{\epsilon_1}{\epsilon_2} \leq 3. \quad (2.41)$$

In this case we find that the constraint is

$$0 < x < 2. \quad (2.42)$$

We note from (2.13) that this constraint is trivially satisfied. That is, as long as we are in the regime (2.41), there is always a physical infrared fixed point that is given by (2.33), (2.34) and (2.35).

The other case is when

$$\frac{\epsilon_1}{\epsilon_2} > 3. \quad (2.43)$$

In this case the constraint on x is

$$x_0 < x < 1, \quad (2.44)$$

here x_0 is defined via the equation

$$\frac{\epsilon_1}{\epsilon_2} = r_2(x_0). \quad (2.45)$$

The final case is when

$$\frac{\epsilon_1}{\epsilon_2} < \frac{1}{3}. \quad (2.46)$$

In this case the constraint is

$$x_0 < x < 1, \quad (2.47)$$

here x_0 is defined via the equation

$$\frac{\epsilon_1}{\epsilon_2} = r_1(x_0). \quad (2.48)$$

Note that although the constraint (2.36) is the same as (2.41), the former is associated with a different fixed point that lies on the $\lambda_y = 0$ plane. We also note that the constraint (2.36) being identical to (2.41) is consistent with the fact that one can turn on the Yukawa coupling λ_y at the fixed point (2.32) on the $\lambda_y = 0$ plane, and flow into the fixed point given by (2.33), (2.34) and (2.35). Note also that the constraint (2.44) is the constraint (2.20), and the constraint (2.47) is the constraint (2.28).

(2) $\epsilon_1 > 0, \epsilon_2 < 0$

In this case at the free fixed point λ_1 is marginally relevant, and λ_2 is marginally irrelevant. Since λ_2^* in (2.32) is negative, the fixed point given by (2.32) on the $\lambda_y = 0$ plane is unphysical.

Now consider the fixed point given by (2.33), (2.34) and (2.35). Note that to have a physical fixed point one must allow x to take values only in the range $1 < x < 2$ (since for $x \leq 1, \lambda_2^* < 0$). We find that for

$$\left| \frac{\epsilon_1}{\epsilon_2} \right| > 1, \quad x_0 < x < 2, \quad (2.49)$$

this fixed point is physical. Here x_0 is defined via the equation

$$\left| \frac{\epsilon_1}{\epsilon_2} \right| = -r_2(x_0). \quad (2.50)$$

We note that these constraints are exactly the constraints (2.22). This is the case because, since λ_2 is not dangerously irrelevant, it switches off as we flow toward the infrared, and thus we are effectively turning on only λ_1 .

(3) $\epsilon_1 < 0, \epsilon_2 > 0$

Here at the free fixed point the gauge coupling λ_1 is marginally irrelevant and the gauge coupling λ_2 is marginally relevant. Since λ_1^* in (2.32) is negative, the fixed point given by (2.32) on the $\lambda_y = 0$ plane is also, as in the previous case, unphysical.

Consider the fixed point given by (2.33), (2.34) and (2.35). We note that to have a physical fixed point one must restrict x in the range $1 < x < 2$ (since for $x \leq 1, \lambda_1^* < 0$). We find that for

$$\left| \frac{\epsilon_1}{\epsilon_2} \right| < 1, \quad x_0 < x < 2, \quad (2.51)$$

this fixed point is physical. Here x_0 is defined via the equation

$$\left| \frac{\epsilon_1}{\epsilon_2} \right| = -r_1(x_0). \quad (2.52)$$

We note also here that these constraints are precisely the constraints (2.30). The reason is due to the fact that λ_1 is not dangerously irrelevant, and thus it flows to zero in the infrared.

(4) $\epsilon_1 < 0, \epsilon_2 < 0$

So far in the two preceding subsections, we discussed the cases in which at least one of the gauge groups is asymptotically free. Here, however, we consider the case where both ϵ_1 and ϵ_2 are negative. That is, at the Gaussian fixed point both gauge couplings are marginally irrelevant, thus, none of the gauge groups are asymptotically free.

From (2.32) we find that for both λ_1^* and λ_2^* to be positive we must require both the conditions $|\epsilon_1/\epsilon_2| < 1/3$ and $|\epsilon_1/\epsilon_2| > 3$. However, these conditions cannot be met simultaneously. Therefore, the fixed point (2.32) on the $\lambda_y = 0$ plane is unphysical.

We also note that λ_y^* in (2.35) is unphysical when both ϵ_1 and ϵ_2 are negative, hence the fixed point given by (2.33), (2.34) and (2.35) is unphysical as well. Therefore, there are no interacting ultraviolet fixed points of the Yukawa and gauge couplings.

We argue in the next section more generally that it is not possible to achieve asymptotic safety in four dimensional $\mathcal{N} = 1$ supersymmetric gauge theories with or without superpotentials, and in particular, at weak coupling.

3. Discussion

In this section we comment on certain supersymmetric gauge theories considered in [3] and some more recent follow-ups.

The authors argued that asymptotic safety is realizable and it exhibits certain theories. These theories satisfy all the consistency constraints that one would require an asymptotically safe theory to satisfy. These constraints are algebraic in nature. They include the constraints that at fixed points the scaling dimensions of gauge invariant operators cannot be less than one and the central charges defined via two point functions of the stress tensor and conserved currents must be positive, and that the Euler anomaly coefficient a must be positive at fixed points and satisfy the inequality $\Delta a := a_{\text{UV}} - a_{\text{IR}} > 0$, which is known as the weak a -theorem. The weak a -theorem constraint is equivalent to the constraint that at least one of the chiral superfields in the theory must have an R -charge larger than $5/3$.

An example given is an $SU(N_c)$ gauge theory with N_f pairs of fundamentals and anti-fundamentals Q, \tilde{Q} , two adjoints A_1, A_2 and a superpotential

$$W = y_1 A_1 Q \tilde{Q} + y_2 A_1^3, \tag{3.1}$$

that does not involve the adjoint A_2 .

In the regime $N_f < N_c$ this theory is free in the ultraviolet, and in the infrared it flows into a non-trivial fixed point. The R -charges of the superfields at this fixed point are

$$R_Q = R_{\tilde{Q}} = \frac{2}{3}, \quad R_{A_1} = \frac{2}{3}, \quad R_{A_2} = \frac{N_c + N_f}{3N_c}. \quad (3.2)$$

In the regime $N_f > N_c$ this theory is infrared free. The existence of an interacting ultraviolet fixed point in a certain finite region around the infrared free fixed point can be analyzed using the recent generalized central charge a -function construction [4] (see Appendix B). At weak coupling with N_f very close to N_c , *i.e.* $N_f/N_c = 1 + \epsilon$, $0 < \epsilon \ll 1$, we find that there is no fixed point in the ultraviolet. The authors took N_f very far away from N_c , *i.e.* $N_f > 4N_c$, so that at least one of the R -charges is large enough to satisfy the above $5/3$ constraint. For $N_f > 4N_c$, therefore all the constraints are satisfied.

We next argue more generally that the constraint that at least one of the R -charges must be larger than $5/3$ cannot be satisfied in four dimensional $\mathcal{N} = 1$ supersymmetric gauge theories in a regime in which the original infrared degrees of freedom, that is to say, variables, do not break down.

Without loss of generality, we consider a supersymmetric gauge theory with a gauge group G and coupling g containing chiral superfields Φ_i in irreducible representations r_i . We take a superpotential of the form

$$W = y \prod_i (\Phi_i)^{n_i}, \quad (3.3)$$

where n_i are some positive integers. The exact running R -charges R_i of Φ_i along an RG flow are given by (B.3) in Appendix B. For convenience we write it here again

$$R_i(\lambda, \lambda_y) = 1 - \frac{1}{3} \left(1 + \frac{\lambda T(r_i) - \lambda_y n_i}{|r_i|} \right)^{\frac{1}{2}}. \quad (3.4)$$

The sum in the square root is positive in some finite region in the space of running couplings containing the origin $\lambda = 0, \lambda_y = 0$ —the free fixed point. In particular, it is positive at weak coupling. The R -charges are therefore always $R_i \leq 1$. It may happen that some field develops sufficiently large, and positive anomalous dimension along a flow (at strong coupling) such that its R -charge $R_i > 1$ [7]. In this case it is argued in [7] that one has to take the other positive branch of the square root in (3.4). However, it is shown in [7-9] that even in this case it cannot surpass the upper bound $4/3$, therefore $R_i \leq 4/3$.

In the regime $R_i > 4/3$, in general, we are in the strong coupling regime and the description in terms of the original infrared degrees of freedom often breaks down [10]. A simple example in which a breakdown of degrees of freedom occurs is the electric SQCD theory with $N_f < 3N_c/2$ [10]. Thus, in a regime in which the theory is internally consistent, it is hardly possible to meet the requirement that at least one of the R -charges should be larger than $5/3$. Our discussion, therefore, strongly suggests that asymptotic safety does not occur in $\mathcal{N} = 1$ supersymmetric gauge theories with or without superpotentials, and in particular, at weak coupling.

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Appendix A. Dangerously irrelevant and harmlessly relevant operators

Consider deforming an ultraviolet conformal field theory (an ultraviolet CFT) by adding to its Lagrangian a relevant operator. This results in an RG flow away from the ultraviolet fixed point. It may happen that as we flow toward low energies, an operator that was initially irrelevant (relevant)⁵ at the ultraviolet fixed point builds up negative (positive) anomalous dimension, and eventually becomes relevant (irrelevant) in the infrared. Such an operator that becomes relevant (irrelevant) in the infrared is sometimes referred to as a dangerously irrelevant (harmlessly relevant) operator. Below, we review a few examples of this RG phenomenon in $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions.

Consider $\mathcal{N} = 1$ supersymmetric QCD (SQCD); an $SU(N_c)$ gauge theory with N_f flavors of chiral superfields Q^i and $\tilde{Q}_{\tilde{i}}$ ($i, \tilde{i} = 1, \dots, N_f$) in the fundamental and anti-fundamental representations of $SU(N_c)$ respectively. For $N_f < 3N_c$, the theory is asymptotically free in the ultraviolet, *i.e.* it describes free quarks and gluons, and it flows in the infrared into a non-trivial fixed point. Now consider adding the superpotential $W = h \left(Q\tilde{Q} \right)^2$. At zero gauge coupling g , the superpotential coupling h is irrelevant and therefore it flows to zero in the infrared. On the other hand, if we turn on the gauge coupling g , and if $N_f < 2N_c$, then h becomes relevant as we flow toward low energies [11]. Thus, in this regime of N_f the coupling h is a dangerously irrelevant coupling.

⁵ The operator can also be marginal.

Another example of dangerously irrelevant coupling is the gauge coupling g in supersymmetric QCD (SQCD) with $N_f > 3N_c$. In this regime of N_f , the gauge coupling g is marginally irrelevant; therefore, the theory is free in the infrared. Now consider giving masses to some of the quarks by adding the superpotential $W = \sum_i^n m_i Q_i^2$. In the large mass limit, the heavy quarks decouple from the dynamics. This reduces the number of fundamental quark flavors from N_f to $N_f - n$. If the number of light quarks $N_f - n < 3N_c$, then the gauge coupling g becomes marginally relevant. Therefore, it is a dangerously irrelevant coupling in this limit.

A further example of the phenomenon appears in the Seiberg dual description of $\mathcal{N} = 1$ $SU(N_c)$ SQCD. The dual description has gauge group $SU(N_f - N_c)$, N_f flavors of chiral superfields q_i and \tilde{q}^i ($i, \tilde{i} = 1, \dots, N_f$) in the fundamental and anti-fundamental representations of the gauge group respectively, and N_f^2 singlet “meson” chiral superfields M_i^j ($j, \tilde{i} = 1, \dots, N_f$). The meson fields couple to the fundamentals q, \tilde{q} through the superpotential $W = y M q \tilde{q}$. In the conformal window, *i.e.* for N_f in the range $3N_c/2 < N_f < 3N_c$, both the original and dual theories are asymptotically free in the ultraviolet, and in the infrared they both flow into non-trivial fixed points which are identified by the duality [12]. At zero gauge coupling g , the Yukawa coupling y is marginally irrelevant and therefore it switches off as we flow toward low energies. However, if we turn on gauge interactions, y becomes relevant as we flow toward the infrared fixed point [11]. Thus, the Yukawa coupling y is a dangerously irrelevant coupling. An example of a harmlessly relevant coupling is t in the meson mass matrix $W = t M^2$, which is the magnetic dual of the superpotential $W = h (Q\tilde{Q})^2$ considered earlier. At the free fixed point, that is in the ultraviolet, the mass matrix coupling t is relevant. However if we turn on the (magnetic) gauge coupling g and Yukawa coupling y , and assuming that $N_f > 2N_c$, it becomes irrelevant in the infrared [11]. Therefore, in this regime of N_f it is a harmlessly relevant coupling.

Yet another example of the phenomenon arises in adjoint $\mathcal{N} = 1$ SQCD (ASQCD), *i.e.* $\mathcal{N} = 1$ SQCD with an additional chiral superfield X in the adjoint representation of the gauge group $SU(N_c)$. ASQCD is asymptotically free in the ultraviolet for $N_f < 2N_c$. Without a superpotential, turning on the gauge coupling g drives the theory to flow into a non-trivial fixed point in the infrared [13, 14]. Consider now the operator $\text{Tr} X^{k+1}$. At the ultraviolet fixed point, that is at $g = 0$, it has, for $k > 2$, dimension larger than three, and hence it is marginal and/or irrelevant. However if we turn on g , for small N_f , it becomes

relevant as we flow toward the infrared limit of the adjoint theory with $W = 0$ [15, 16]. Thence, in this regime of N_f , it is an example of a dangerously irrelevant operator.

The adjoint SQCD considered above also admits harmlessly relevant operators [4, 10, 14]. Consider ASQCD with $W = 0$ at its infrared fixed point in a regime where the operator $\text{Tr}X^{k+1}$ is relevant. We are now interested in deforming the theory by adding the superpotential $W = g_k \text{Tr}X^{k+1} + g_{k'} \text{Tr}X^{k'+1}$. Consider first turning on only the coupling g_k . The operator $\text{Tr}X^{k+1}$ will then drive the theory into a fixed point which we will call F_k . At this fixed point, the R -charge of the superfield X is $R_X = 2/(k+1)$. Therefore with $k' < k$ the operator $\text{Tr}X^{k'+1}$ is relevant at F_k . Now if we turn on the other coupling $g_{k'}$, then the theory at F_k will evolve further into a new fixed point $F_{k'}$ at which the effective superpotential is $W \approx g_{k'} \text{Tr}X^{k'+1}$. At $F_{k'}$ the operator $\text{Tr}X^{k+1}$ is therefore irrelevant, and thus it is a harmlessly relevant operator [4, 10, 14].

Appendix B. a -function and RG flows

In this appendix we briefly review the generalized central charge a -function $a(\lambda_i)$ that is constructed in [4] (see also [5]) by extending the method of a -maximization [14] away from fixed points of an RG flow. It is an analogue of the Zamolodchikov c -function in four dimensions, and it manifestly satisfies the weak a -theorem, $a_{\text{UV}} > a_{\text{IR}}$.

In the following, we first outline the procedure for determining the form of the generalized central charge a -function $a(\lambda_i)$ of a four dimensional $\mathcal{N} = 1$ supersymmetric gauge theory. We then show how the generalized central charge a -function $a(\lambda_i)$ can be used to analyze an RG flow between an ultraviolet, and an infrared fixed points of the theory.

We consider a supersymmetric gauge theory with a gauge group G and coupling g containing chiral superfields Φ_i in irreducible representations r_i . We take a superpotential of the form

$$W = y \prod_i (\Phi_i)^{n_i}. \quad (\text{B.1})$$

We write more conveniently the gauge coupling g as $\alpha := \frac{g^2}{4\pi}$, and the coupling y as $\alpha_y := \frac{y^2}{4\pi}$. There are two steps in finding the general form of the generalized central charge a -function $a(\lambda, \lambda_y)$ of the theory along an RG flow.

The first step is constructing the following intermediate a -function; by abuse of notation we use the same letter a ,

$$\begin{aligned}
a(\lambda, \lambda_y, R_m) = & 2|G| + \sum_i |r_i| [3(R_i - 1)^3 - (R_i - 1)] \\
& - \lambda \left[T(G) + \sum_i T(r_i) (R_i - 1) \right] \\
& - \lambda_y \left(2 - \sum_i n_i R_i \right),
\end{aligned} \tag{B.2}$$

where $|r_i|$ and $T(r_i)$ are the dimension and index of the irreducible representation r_i respectively. $|G|$ is the dimension of the gauge group G , and $T(G)$ is the index for the adjoint representation of the gauge group G . For example, for $G = SU(N)$, $T(\mathbf{N}) = T(\overline{\mathbf{N}}) = 1/2$, $T(G) = N$. The Lagrange multiplier λ imposes the constraint that the superconformal $U(1)_R$ symmetry (at the non-trivial fixed points) is anomaly free (see also [17] for a review); λ_y is the Lagrange multiplier for the constraint that the general R symmetry is preserved by the superpotential W .⁶

The second step is finding the unknowns R_i and substituting them back into (B.2). The R_i are obtained by extremizing the intermediate a -function (B.2) with respect to R_i [14], which gives

$$R_i(\lambda, \lambda_y) = 1 - \frac{1}{3} \left(1 + \frac{\lambda T(r_i) - \lambda_y n_i}{|r_i|} \right)^{\frac{1}{2}}. \tag{B.3}$$

Substituting the R_i (B.3) back into (B.2) yields the generalized central charge a -function $a(\lambda, \lambda_y) := a(\lambda, \lambda_y, R_i(\lambda, \lambda_y))$ of the theory. Because $R_i(\lambda, \lambda_y)$ solves $\partial a / \partial R_i = 0$, one finds that

$$\begin{aligned}
\frac{da}{d\lambda} &= - [T(G) + T(r_i) (R_i - 1)] := \widehat{\beta}_G(\lambda, \lambda_y), \\
\frac{da}{d\lambda_y} &= - \left(2 - \sum_i n_i R_i \right) := \widehat{\beta}_y(\lambda, \lambda_y),
\end{aligned} \tag{B.4}$$

here the R_i are given by (B.3).

Since λ, λ_y were originally introduced as Lagrange multipliers, we must extremize the generalized central charge a -function $a(\lambda, \lambda_y)$ with respect to the multipliers λ, λ_y . We

⁶ This can be easily generalized for theories based on product gauge groups, and consist of more than one superpotential coupling by introducing as many Lagrange multipliers as there are couplings.

note that there are two cases to consider. We first consider the case in which both the couplings are turned on; and next, that only one of the couplings is turned on.

In the former case, we must extremize $a(\lambda, \lambda_y)$ with respect to both λ, λ_y . This gives

$$\widehat{\beta}_G(\lambda, \lambda_y) = 0, \quad \widehat{\beta}_y(\lambda, \lambda_y) = 0. \quad (\text{B.5})$$

In the latter case, if (say) only α is turned on, then $\lambda_y = 0$ and extremizing $a(\lambda, 0)$ with respect to λ gives

$$\widehat{\beta}_G(\lambda, 0) = 0. \quad (\text{B.6})$$

On the other hand, if only α_y is turned on, then $\lambda = 0$ and extremizing $a(0, \lambda_y)$ with respect to λ_y gives

$$\widehat{\beta}_y(0, \lambda_y) = 0. \quad (\text{B.7})$$

These equations fix the multipliers λ, λ_y and in turn also $R_i(\lambda, \lambda_y)$ and $a(\lambda, \lambda_y)$. We will denote the values of the multipliers which solves (B.5), (B.6), (B.7) by the same letters λ^*, λ_y^* in all these different cases; this should cause no ambiguity. We assume that λ^*, λ_y^* are such that the beta function for the coupling α is not singular in the regime $\alpha \leq \alpha^*$; the construction of [4] is valid only in a certain finite region in the space of couplings containing the origin.

The R_i (B.3) at the fixed points $\lambda = 0, \lambda_y = 0$ and $\lambda = \lambda^*, \lambda_y = \lambda_y^*$ give the R -charges of the chiral superfields Φ_i associated with the $U(1)_R$ superconformal symmetry. Similarly, the generalized central charge $a(\lambda, \lambda_y)$ at the fixed points gives the central charges a_{UV}, a_{IR} with $a_{UV} > a_{IR}$.

The Lagrange multipliers λ, λ_y are identified in the construction of [4] in some renormalization scheme with the running couplings α, α_y ; therefore, they are always positive. In the vicinity of $\alpha = 0, \alpha_y = 0$ it was shown in [4-6] that $\lambda \approx |G|\alpha + \mathcal{O}(\alpha^2), \lambda_y \approx \alpha_y + \mathcal{O}(\alpha^2)$. The positivity of the Lagrange multipliers gives non-trivial constraints on asymptotic safety.

The identification of λ, λ_y with the running couplings α, α_y suggests the interpretation of $R_i(\lambda, \lambda_y)$ and $a(\lambda, \lambda_y)$, in some renormalization scheme, as running R -charges and central charge. Thus, in (B.4) we note that, in the vicinity of $\lambda = 0, \lambda_y = 0$, $\widehat{\beta}_G(\lambda, \lambda_y)$ is proportional to the beta function for the coupling λ , and $\widehat{\beta}_y(\lambda, \lambda_y)$ is proportional to the

beta function for the coupling λ_y .⁷ We also note that this interpretation generalizes the method of a -maximization [14] away from fixed points of an RG flow.

The equations (B.4), together with the positivity constraint, determine RG fixed points in the space of running couplings λ, λ_y (in some renormalization scheme). In some finite domain in the space of couplings containing $\lambda = 0, \lambda_y = 0$, the fixed points of RG flows are found by setting (B.4) to zero, *i.e.* by solving the corresponding equation(s) among (B.5), (B.6), (B.7).

We note that the gauge coupling λ is relevant throughout an RG flow (away from the free fixed point) if $\widehat{\beta}_G(\lambda, \lambda_y) < 0$. Therefore, if λ is relevant at the fixed point (B.7), then turning it on will drive the theory further down in the infrared into the fixed point (B.5) provided λ_y is not harmlessly relevant. In the case in which λ_y is harmlessly relevant we flow instead into the fixed point (B.6).

We also note that away from the free fixed point the coupling λ_y is relevant along an RG flow if $\widehat{\beta}_y(\lambda, \lambda_y) < 0$. Similarly, if $\widehat{\beta}_y(\lambda, \lambda_y) < 0$, then one can turn on λ_y at the fixed point (B.6), and flow further down in the infrared into the fixed point (B.5). if, however, λ is harmlessly relevant, then we flow instead into the fixed point (B.7).

Turning on both couplings at the free fixed point in the case in which they are (marginally) relevant (and dangerously irrelevant) leads in general in the infrared into the fixed point (B.5). If either of the couplings is harmlessly relevant, then it leads into the fixed point (B.6) or (B.7) (see the preceding two paragraphs).

As a simple example to illustrate the procedure, we consider a class of $\mathcal{N} = 1$ supersymmetric gauge theories with a gauge group $SU(N_c)$ coupled to N_f (anti-)fundamental chiral superfields $(\widetilde{Q})Q$, and a bi-fundamental gauge singlet chiral superfield M . It has $SU(N_f)_L \times SU(N_f)_R \times U(1)_R$ global symmetry. We summarize the field contents, and the representations under which they transform as follows.

Chiral superfields	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$
$Q_i(\widetilde{Q}_{\widetilde{i}})(i, \widetilde{i} = 1, \dots, N_f)$	$\mathbf{N}_c(\overline{\mathbf{N}}_c)$	$\mathbf{1}(\overline{\mathbf{N}}_f)$	$\mathbf{N}_f(\mathbf{1})$
$M_{\widetilde{i}i}(\widetilde{i}, i = 1, \dots, N_f)$	$\mathbf{1}$	\mathbf{N}_f	$\overline{\mathbf{N}}_f$

We take the superpotential

$$W = y \text{Tr} \widetilde{Q} M Q. \tag{B.8}$$

⁷ In general one has $\frac{da}{d\lambda^i} = G_{ij} \beta^j$, where G_{ij} is the metric on the space of couplings, and β^i are the beta functions.

The first step in finding the a -function, and its gradient flow equations is constructing the intermediate a -function (B.2). In this class of theories, it follows from (B.2), the intermediate a -function takes the form

$$\begin{aligned}
a(\lambda, R_j) &= 2(N_c^2 - 1) \\
&+ 2N_f \cdot N_c \cdot [3(R_Q - 1)^3 - (R_Q - 1)] + N_f^2 \cdot 1 \cdot [3(R_M - 1)^3 - (R_M - 1)] \\
&- \lambda [N_c + N_f(R_Q - 1)] \\
&- \lambda_y (2 - 2R_Q - R_M),
\end{aligned} \tag{B.9}$$

where $R_{\tilde{Q}} = R_Q$; the $U(1)_R$ symmetry commutes with the chiral symmetry $SU(N_f)_L \times SU(N_f)_R$.

The next step is extremizing (B.9) with respect to the unknowns R_i . This gives a set of equations that can be solved for R_i in terms of the couplings λ, λ_y . We obtain the running R -charges $R_i(\lambda, \lambda_y)$

$$\begin{aligned}
R_Q(\lambda, \lambda_y) &= 1 - \frac{1}{3} \left(1 + \frac{\lambda N_f - 2\lambda_y}{2N_f N_c} \right)^{\frac{1}{2}}, \\
R_M(\lambda, \lambda_y) &= 1 - \frac{1}{3} \left(1 - \frac{\lambda_y}{N_f^2} \right)^{\frac{1}{2}}.
\end{aligned} \tag{B.10}$$

Substituting these into (B.9) finally gives the generalized a -function $a(\lambda, \lambda_y)$. The gradient flow equations of the a -function are then obtained using (B.10) in (B.4). For small couplings λ_1, λ_y , for instance, these flow equations to leading order in the couplings are

$$\begin{aligned}
\widehat{\beta}_G(\lambda, \lambda_y) &= -\frac{N_c}{3} \left[(3 - x) - \frac{1}{4}x \frac{\lambda}{N_c} + \frac{1}{2}x^2 \frac{\lambda_y}{N_f^2} \right], \\
\widehat{\beta}_y(\lambda, \lambda_y) &= \frac{1}{6} \left[(2x + 1) \frac{\lambda_y}{N_f^2} - \frac{\lambda}{N_c} \right],
\end{aligned} \tag{B.11}$$

here $x := \frac{N_f}{N_c}$. With $\alpha_g := \frac{N_c g^2}{(4\pi)^2}$, $\alpha_y := \frac{N_c y^2}{(4\pi)^2}$, to leading order in α_g, α_y we have $\lambda = 8N_c \alpha_g$, $\lambda_y = 4N_f^2 \alpha_y$ [4-6].

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