

FROM PÓLYA FIELDS TO PÓLYA GROUPS

(I) GALOISIAN EXTENSIONS

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ABSTRACT. The Pólya group of a number field K is the subgroup of the class group of K generated by the classes of the products of the maximal ideals with same norm. A Pólya field is a number field whose Pólya group is trivial. Our purpose is to start with known assertions about Pólya fields to find results concerning Pólya groups. This first paper inspired by Zantema's results focuses on the Pólya group of the compositum of two galoisian extensions of \mathbb{Q} .

1. INTRODUCTION

1.1. Pólya fields.

The notion of Pólya fields goes back to 1919 with two papers by Pólya [22] and Ostrowski [21] where they undertook a study of the integer-valued polynomials in number fields. For a fixed number field K with ring of integers \mathcal{O}_K , they were interested in the fact that the \mathcal{O}_K -module $\text{Int}(\mathcal{O}_K) = \{f(X) \in K[X] \mid f(\mathcal{O}_K) \subseteq \mathcal{O}_K\}$ admits a regular basis, that is, a basis with one and only one polynomial of each degree. They proved that such a basis exists if and only if, for each integer q which is the norm of a maximal ideal of \mathcal{O}_K , the product of all maximal ideals of \mathcal{O}_K with norm q is a principal ideal.

Let us introduce a notation: for each positive integer q , let

$$(1) \quad \Pi_q(K) = \prod_{\mathfrak{m} \in \text{Max}(\mathcal{O}_K), |\mathcal{O}_K/\mathfrak{m}|=q} \mathfrak{m}.$$

If q is not the norm of a maximal ideal of \mathcal{O}_K , then by convention $\Pi_q(K) = \mathcal{O}_K$. A formal definition was introduced only in 1982 by Zantema [29]. Because of the characterization previously given, it could be stated in the following way:

Definition 1.1. A *Pólya field* is a number field K such that the following equivalent assertions are satisfied:

- (1) the \mathcal{O}_K -module $\text{Int}(\mathcal{O}_K)$ admits a regular basis,
- (2) all the ideals $\Pi_q(K)$ are principal.

Between 1919 and 1982, nothing was done about Pólya fields. Zantema's paper is a great contribution to the subject which from our point of view is not sufficiently known. After 1982, one can find several papers on the subject, most often they are studies of particular cases: Spickermann [24] in 1986, Leriche ([16], [17], [18]) in 2011–2014, Heidaryan and Rajaei ([10], [11], [12]) in 2014–2017.

Examples 1.2. The following number fields are Pólya fields:

- the fields K with class number $h_K = 1$,

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- the cyclotomic fields [29, Proposition 2.6],
- the abelian extensions of \mathbb{Q} with only one ramified prime [29, Proposition 2.5],
- the Hilbert class fields [18, Corollary 3.2].

The notion of Pólya field introduced for number fields has been extended to functions fields by Van Der Linden [28] in 1988. His results were then generalized by Adam [1] in 2008. But we will not speak here about the case of positive characteristic.

1.2. Pólya groups.

The notion of Pólya-Ostrowski group was introduced later, in 1997, in a monography about integer-valued polynomials ‘as a measure of the obstruction’ for K to be a Pólya field [4, §II.4]. For simplicity, we will call it just Pólya group.

Definition 1.3. The *Pólya group* of a number field K is the subgroup $\mathcal{P}o(K)$ of the class group $\mathcal{C}l(K)$ generated by the classes of the ideals $\Pi_q(K)$.

The Pólya group of K is also the subgroup of $\mathcal{C}l(K)$ generated by the classes of the factorial ideals of K as defined by Bhargava [3], but here we will not use this property.

Of course, the field K is a Pólya field if and only if its Pólya group is trivial. So that, every result about Pólya groups has a corollary which is a result about Pólya fields. For instance, the first important result about Pólya groups is due to Hilbert [13, Prop. 105–106] who studied the ambiguous ideals since, for a galoisian number field K , the Pólya group of K is nothing else than the group formed by the classes of ambiguous ideals:

Proposition 1.4 (Hilbert, 1897). *Let K be a quadratic number field. If s_K denotes the number of ramified prime numbers in the extension K/\mathbb{Q} , then*

$$(2) \quad |\mathcal{P}o(K)| = \begin{cases} 2^{s_K-2} & \text{if } K \text{ is real and } N_{K/\mathbb{Q}}(\mathcal{O}_K^\times) = \{1\} \\ 2^{s_K-1} & \text{else.} \end{cases}$$

It is then easy to describe the quadratic number fields which are Pólya fields (cf. [29, Example 3.3] ou [4, Cor. II.4.5]).

Since, as previously said, if K is a galoisian number field, the Pólya group of K is the group of strongly ambiguous classes, we find in the literature some results about Pólya groups before their introduction. For instance, if K is a quadratic number field, Mollin [19, 1993] gave a sufficient condition for $\mathcal{C}l(K)$ to be generated by the classes of ramified primes, that is, such that $\mathcal{P}o(K) = \mathcal{C}l(K)$. In Section 4 we recall some classical results about the order of this group of strongly ambiguous classes. About Pólya groups we may also cite more recent papers: [2],[5], [9], [26], [27], [30].

1.3. The aim of this paper.

Here we are going to make a reverse lecture of this fact that an assertion about Pólya groups implies, as a particular case, a result about Pólya fields. We make the conjecture that an assertion about Pólya fields is the evidence of a statement about Pólya groups, statement that we have to formulate as a conjecture, and then, to try to prove. For instance, consider the following proposition due to Ostrowski:

Proposition 1.5. [21] *If the extension K/\mathbb{Q} is galoisian, then K is a Pólya field if and only if, for every non ramified prime number p , the ideal $\Pi_{p^{f_p(K/\mathbb{Q})}}(K)$ is principal.*

This is an easy consequence of the fact that, if the extension K/\mathbb{Q} is galoisian, if p is not ramified, and if $q = p^{f_p(K/\mathbb{Q})}$, then $\Pi_q(K) = p\mathcal{O}_K$ is principal. Thus, clearly:

Corollary 1.6. *If the extension K/\mathbb{Q} is galoisian, then the Pólya group of K is generated by the classes of the ideals $\Pi_{p^{f_p(K/\mathbb{Q})}}(K)$ where p is ramified in the extension K/\mathbb{Q} .*

We are going to take advantage of the results of Zantema's paper on Pólya fields to obtain results on Pólya groups. In this first study we restrict ourselves to the easier case, that is the galoisian case. Here is Zantema's theorem that we want to transpose. It concerns the compositum of two galoisian extensions of \mathbb{Q} .

Theorem 1.7. [29, Thm 3.4] *Let K_1 and K_2 be finite Galois extensions of \mathbb{Q} , $L = K_1 \cdot K_2$, $K = K_1 \cap K_2$. For a prime p , let $e_i(p)$ be the ramification index of p in K_i/K , $i = 1, 2$. Then*

- (a) *If $\gcd(e_1(p), e_2(p)) = 1$ for all primes p and if K_1 and K_2 are Pólya fields, then L is a Pólya field.*
- (b) *If either $[K_1 : K], [K_2 : K], [K : \mathbb{Q}]$ are pairwise relatively prime, or K is a Pólya field and $\gcd([K_1 : K], [K_2 : K]) = 1$, then K_1 and K_2 are Pólya fields if L is a Pólya field.*

The transposition of this theorem in terms of Pólya groups will be done with Proposition 3.5 and Theorems 3.7 and 3.11.

2. GALOISIAN EXTENSIONS OF \mathbb{Q}

Notation. For every number field K , we denote by \mathcal{O}_K its ring of integers, \mathcal{O}_K^\times its group of units, \mathcal{I}_K the group of nonzero fractional ideals, \mathcal{P}_K the subgroup of nonzero principal fractional ideals, $\mathcal{Cl}(K) = \mathcal{I}_K/\mathcal{P}_K$ its class group, and $h_K = |\mathcal{Cl}(K)|$ its class number.

Recall that, for every $q \in \mathbb{N}^*$, $\Pi_q(K)$ denotes the product of the maximal ideals of \mathcal{O}_K with norm q and that $\mathcal{Po}(K)$ is the subgroup of $\mathcal{Cl}(K)$ generated by the classes of these $\Pi_q(K)$'s.

If K is a galoisian extension of \mathbb{Q} , for simplicity we will write $\Pi_{p^*}(K)$ instead of $\Pi_{p^{f_p(K/\mathbb{Q})}}(K)$, and if G denotes the Galois group $\text{Gal}(K/\mathbb{Q})$, \mathcal{I}_K^G will denote the subgroup of \mathcal{I}_K formed by the ambiguous ideals of K .

Note that, in the galoisian case, the group of ambiguous ideals \mathcal{I}_K^G is generated by the ideals $\Pi_{p^*}(K)$, in fact, this is the free abelian group with basis formed by the $\Pi_{p^*}(K)$ that are distinct from \mathcal{O}_K . As a consequence, the image of \mathcal{I}_K^G in the class group $\mathcal{Cl}(K)$ is the Pólya group of K :

$$(3) \quad \mathcal{Po}(K) = \mathcal{I}_K^G/\mathcal{P}_K^G.$$

In order to establish links between different Pólya groups, we have to recall several homomorphisms of groups (cf. [23, I, §5]). If L/K is a finite extension, there is a natural injective morphism

$$(4) \quad j_K^L : \mathfrak{J} \in \mathcal{I}_K \mapsto \mathfrak{J}\mathcal{O}_L \in \mathcal{I}_L$$

which induces a (non necessarily injective) morphism:

$$(5) \quad \varepsilon_K^L : \overline{\mathfrak{J}} \in \mathcal{Cl}(K) \mapsto \overline{\mathfrak{J}\overline{\mathcal{O}_L}} \in \mathcal{Cl}(L).$$

On the other hand, the norm morphism

$$N_L^K : \mathcal{I}_L \rightarrow \mathcal{I}_K,$$

which is determined by its values on the maximal ideals \mathfrak{n} of \mathcal{O}_L :

$$(6) \quad N_L^K(\mathfrak{n}) = \mathfrak{m}^{[\mathcal{O}_L/\mathfrak{n} : \mathcal{O}_K/\mathfrak{m}]} \quad \text{where } \mathfrak{m} = \mathfrak{n} \cap \mathcal{O}_K,$$

induces a morphism:

$$(7) \quad \nu_L^K : \overline{\mathfrak{J}} \in \mathcal{Cl}(L) \mapsto \overline{N_L^K(\mathfrak{J})} \in \mathcal{Cl}(K).$$

And, for every $\mathfrak{J} \in \mathcal{I}_K$, one has:

$$(8) \quad (N_L^K \circ j_K^L)(\mathfrak{J}) = \mathfrak{J}^{[L:K]}.$$

As noticed in [5], if K and L are galoisian extensions of \mathbb{Q} with Galois groups H and G respectively, then the subgroups \mathcal{I}_K^H and \mathcal{I}_L^G on the one hand, and the subgroups $\mathcal{P}o(K)$ and $\mathcal{P}o(L)$ on the other hand, behave well with respect to the previous morphisms:

Lemma 2.1. *If $K \subseteq L$ are two galoisian extensions of \mathbb{Q} with respective Galois groups H and G , then:*

$$j_K^L(\mathcal{I}_K^H) \subseteq \mathcal{I}_L^G \quad \text{and} \quad \varepsilon_K^L(\mathcal{P}o(K)) \subseteq \mathcal{P}o(L),$$

$$N_L^K(\mathcal{I}_L^G) \subseteq \mathcal{I}_K^H \quad \text{and} \quad \nu_L^K(\mathcal{P}o(L)) \subseteq \mathcal{P}o(K).$$

This is no more true if one of the fields is not a galoisian extension of \mathbb{Q} (cf. [16, Ex. 2]). We begin with a lemma which is a ‘local version’ of the behaviour of the Pólya groups.

Lemma 2.2. *Let $K_0 \subseteq K_1 \subseteq L$ be three galoisian extensions of \mathbb{Q} and let $p \in \mathbb{P}$ be such that:*

- (1) $\Pi_{p^*}(K_0)$ is principal,
- (2) $e_p(K_1/K_0)$ et $[L : K_1]$ are relatively prime.

Then, the morphism $\nu_L^{K_1} \circ \varepsilon_{K_1}^L : \mathcal{P}o(K_1) \rightarrow \mathcal{P}o(L) \rightarrow \mathcal{P}o(K_1)$ induces an automorphism of the group $\langle \overline{\Pi_{p^}(K_1)} \rangle$. In particular, $\langle \overline{\Pi_{p^*}(L)} \rangle$ contains a subgroup isomorphic to $\langle \overline{\Pi_{p^*}(K_1)} \rangle$.*

Proof. Let $\Pi = \Pi_{p^*}(K_1)$. The equality $\Pi_{p^*}(K_0)\mathcal{O}_K = \Pi^{e_p(K_1/K_0)}$ implies with hypothesis (1) that the order of the class $\overline{\Pi}$ of Π divides $e_p(K_1/K_0)$. It follows from hypothesis (2) that this order is prime to $[L : K_1]$. The formula $\nu_L^K(\varepsilon_K^L(\overline{\Pi})) = \overline{\Pi}^{[L:K_1]}$ shows then that the morphism $\nu_L^{K_1} \circ \varepsilon_{K_1}^L$ induces an automorphism of the group generated by $\overline{\Pi_{p^*}(K_1)}$. The last assertion is obvious. \square

By ‘globalization’ Lemma 2.2 leads to:

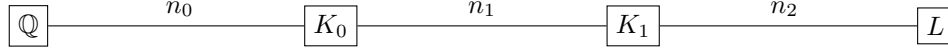
Proposition 2.3. *Let $K_0 \subseteq K_1 \subseteq L$ be three galoisian extensions of \mathbb{Q} such that:*

- (1) K_0 is a Pólya field,
- (2) $([L : K_1], [K_1 : K_0]) = 1$.

Then, $\mathcal{P}o(L)$ contains a subgroup which is isomorphic to $\mathcal{P}o(K_1)$. More precisely,

$$\mathcal{P}o(L) \simeq \mathcal{P}o(K_1) \oplus (\mathcal{P}o(L) / \varepsilon_{K_1}^L(\mathcal{P}o(K_1)))$$

Proof. Let $n_0 = [K_0 : \mathbb{Q}]$, $n_1 = [K_1 : K_0]$ et $n_2 = [L : K_1]$.



The group $\mathcal{P}o(K_1)$ is generated by the classes of the $\Pi_{p^*}(K_1)$'s whose orders divide $e_p(K_1/K_0)$, which itself divides n_1 . Thus, the order of every element of the abelian group $\mathcal{P}o(K_1)$ divides n_1 . Consequently the restriction of the morphism $\nu_L^{K_1} \circ \varepsilon_{K_1}^L$ to $\mathcal{P}o(K_1)$, which corresponds to the elevation to the power n_2 , is an automorphism of $\mathcal{P}o(K_1)$ since $(n_1, n_2) = 1$. Thus, the subgroup $\varepsilon_{K_1}^L(\mathcal{P}o(K_1))$ of $\mathcal{P}o(L)$ is isomorphic to $\mathcal{P}o(K_1)$.

On the other hand, the group $\mathcal{P}o(L)$ is generated by the classes of the $\Pi_{p^*}(L)$'s. But, the subgroup $\varepsilon_{K_1}^L(\mathcal{P}o(K_1))$ contains $\Pi_{p^*}(K_1)\mathcal{O}_L = \Pi_{p^*}(L)^{e_p(L/K_1)}$. Thus, mod $\varepsilon_{K_1}^L(\mathcal{P}o(K_1))$, the order of $\overline{\Pi_{p^*}(L)}$ divides $e_p(L/K_1)$ which itself divides n_2 . Consequently, the order of every element of $\mathcal{P}o(L)$ mod $\varepsilon_{K_1}^L(\mathcal{P}o(K_1))$ divides n_2 .

As all the orders of the elements of $\mathcal{P}o(L)$ divide $n_1 n_2$, one has:

$$\mathcal{P}o(L) = G_{n_1} \oplus G_{n_2} \quad \text{where} \quad G_{n_i} = \{\alpha \in \mathcal{P}o(L) \mid \alpha^{n_i} = 1\}.$$

Clearly,

$$G_{n_1} = \varepsilon_{K_1}^L(\mathcal{P}o(K_1)) \simeq \mathcal{P}o(K_1) \quad \text{and} \quad G_2 \simeq \mathcal{P}o(L) / \varepsilon_{K_1}^L(\mathcal{P}o(K_1)). \quad \square$$

In particular, if $K_0 = \mathbb{Q}$, one has:

Corollary 2.4. *Let $K_1 \subseteq L$ be two galoisian extensions of \mathbb{Q} such that one has $([L : K_1], [K_1 : \mathbb{Q}]) = 1$. Then $\mathcal{P}o(L)$ contains a subgroup isomorphic to $\mathcal{P}o(K_1)$. In particular, if L is a Pólya field, K_1 is also a Pólya field.*

Example 2.5. Let K be a non-galoisian cubic number field. Denote by L the galoisian closure of K and by K' the unique quadratic number field such that $L = KK'$. Then, $\mathcal{P}o(K') \subseteq \mathcal{P}o(L)$.

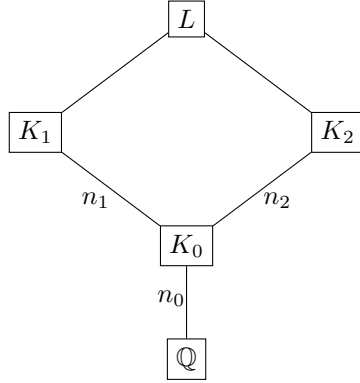
3. COMPOSITUM OF GALOISIAN EXTENSIONS

First let us recall a strong version of Abhyankar's lemma (see for instance [25, Prop. III.8.9] for the function field case):

Proposition 3.1. [8] *Let A be a Dedekind domain with fraction field K . Let \mathfrak{p} be a maximal ideal of A and assume that the residue field A/\mathfrak{p} is perfect. Let K_1 and K_2 be two finite extensions of K . Denote by L the compositum of K_1 and K_2 and assume that the extension L/K is separable. Let \mathfrak{q} be a prime ideal of L such that $\mathfrak{q} \cap K = \mathfrak{p}$. Let $\mathfrak{p}_1 = \mathfrak{q} \cap K_1$, $\mathfrak{p}_2 = \mathfrak{q} \cap K_2$, and $p\mathbb{Z} = \mathfrak{q} \cap \mathbb{Z}$. If p does not divide one of the ramification indices $e(\mathfrak{p}_i/\mathfrak{p})$ ($i = 1, 2$), then*

$$e(\mathfrak{q}/\mathfrak{p}) = \text{lcm}\{e(\mathfrak{p}_1/\mathfrak{p}), e(\mathfrak{p}_2/\mathfrak{p})\}.$$

Hypotheses and notation. Let K_1 and K_2 be two galoisian extensions of \mathbb{Q} .



$$L = K_1 K_2$$

$$K_0 = K_1 \cap K_2$$

Lemma 3.2. *Let p be a prime number such that p does not divide at least one of the ramification indices $e_p(K_i/K_0)$. Then, in the group $I_{\mathcal{O}_L}$, one has the following equality of subgroups:*

$$(9) \quad \langle \Pi_{p^*}(K_1)\mathcal{O}_L, \Pi_{p^*}(K_2)\mathcal{O}_L \rangle = \langle \Pi_{p^*}(L) \rangle,$$

and hence, in the group $Cl(L)$:

$$(10) \quad \varepsilon_{K_1}^L(\langle \overline{\Pi_{p^*}(K_1)} \rangle) + \varepsilon_{K_2}^L(\langle \overline{\Pi_{p^*}(K_2)} \rangle) = \langle \overline{\Pi_{p^*}(L)} \rangle.$$

In particular, the ideal $\Pi_{p^*}(L)$ is principal if and only if the ideals $\Pi_{p^*}(K_1)\mathcal{O}_L$ and $\Pi_{p^*}(K_2)\mathcal{O}_L$ are themselves principal.

Proof. Clearly, L and K_0 are galoisian extensions of \mathbb{Q} . Note that K_0 could be replaced by any field contained in $K_1 \cap K_2$ provided that K_0 is a galoisian extension of \mathbb{Q} . Let $e_i = e_p(K_i/K_0)$ ($i = 1, 2$) and $m = \text{lcm}(e_1, e_2)$. Following Abhyankar's lemma, $e_p(L/K) = m$. Let

$$\pi = \Pi_{p^*}(K_0), \quad \Pi_i = \Pi_{p^*}(K_i) \quad (i = 1, 2) \quad \text{and} \quad \Pi = \Pi_{p^*}(L).$$

Then,

$$(11) \quad \pi\mathcal{O}_{K_i} = \Pi_i^{e_i}, \quad \pi\mathcal{O}_L = \Pi^m, \quad \Pi_i\mathcal{O}_L = \Pi^{\frac{m}{e_i}} \quad (i = 1, 2).$$

Let u_1 and $u_2 \in \mathbb{Z}$ be such that $u_1 \times \frac{m}{e_1} + u_2 \times \frac{m}{e_2} = 1$. Then, one has:

$$(12) \quad \Pi_1^{u_1} \Pi_2^{u_2} \mathcal{O}_L = \Pi^{u_1 \frac{m}{e_1} + u_2 \frac{m}{e_2}} = \Pi.$$

Clearly, equality (9) follows from equalities (11) and (12). \square

Lemma 3.3. *Under the hypothesis of lemma 3.2, the equality*

$$j_{K_1}^L(\langle \Pi_{p^*}(K_1) \rangle) \cap j_{K_2}^L(\langle \Pi_{p^*}(K_2) \rangle) = j_K^L(\langle \Pi_{p^*}(K_0) \rangle)$$

holds if and only if $e(\mathfrak{p}_1/\mathfrak{p})$ and $e(\mathfrak{p}_2/\mathfrak{p})$ are coprime

Proof. Clearly $j_{K_0}^L(\Pi_{p^*}(K_0)) \subseteq j_{K_1}^L(\Pi_{p^*}(K_1)) \cap j_{K_2}^L(\Pi_{p^*}(K_2))$. Do we have an equality? The ideals I belonging to this intersection are of the form $I = \Pi_i^{k_i} \mathcal{O}_L$ where $k_i \in \mathbb{Z}$ ($i = 1, 2$), thus of the form $I = \Pi^{k_i \frac{m}{e_i}}$ ($i = 1, 2$). Consequently, $I = \Pi^\alpha$ where α is divisible by $\frac{m}{e_1}$ and $\frac{m}{e_2}$, that is, multiple of $\text{lcm}(\frac{m}{e_1}, \frac{m}{e_2})$, namely, multiple of $\frac{m}{d}$ where $d = \text{gcd}(e_1, e_2)$. Letting $k_i = \frac{e_i}{d}$, one has $I = \Pi^{\frac{m}{d}}$, while $\pi\mathcal{O}_L = \Pi^m$. \square

By 'globalization', Lemmas 3.2 and 3.3 lead to both following propositions.

Proposition 3.4. *Let K_1 and K_2 be two finite galoisian extensions of \mathbb{Q} . Let $K_0 = K_1 \cap K_2$, $L = K_1 K_2$, $G = \text{Gal}(L/\mathbb{Q})$ and $G_i = \text{Gal}(K_i/\mathbb{Q})$ for $0 \leq i \leq 2$. If $([K_1 : K_0], [K_2 : K_0]) = 1$, then,*

$$j_{K_1}^L(\mathcal{I}_{K_1}^{G_1}) \cdot j_{K_2}^L(\mathcal{I}_{K_2}^{G_2}) = \mathcal{I}_L^G \quad \text{and} \quad j_{K_1}^L(\mathcal{I}_{K_1}^{G_1}) \cap j_{K_2}^L(\mathcal{I}_{K_2}^{G_2}) = j_{K_0}^L(\mathcal{I}_{K_0}^{G_0}).$$

Thus, \mathcal{I}_L^G is isomorphic to the amalgamated sum of $\mathcal{I}_{K_1}^{G_1}$ and $\mathcal{I}_{K_2}^{G_2}$ above $\mathcal{I}_{K_0}^{G_0}$:

$$\mathcal{I}_L^G \simeq \mathcal{I}_{K_1}^{G_1} *_{\mathcal{I}_{K_0}^{G_0}} \mathcal{I}_{K_2}^{G_2}$$

with respect to the morphisms j .

Proof. This is a consequence of the fact that the groups $\mathcal{I}_{K_i}^{G_i}$ are free abelian groups with bases formed by the $\Pi_{p^*}(K_i)$'s. \square

Proposition 3.5. *Let K_1 and K_2 be two finite galoisian extensions of \mathbb{Q} . Let $K_0 = K_1 \cap K_2$ and $L = K_1 K_2$. If, for each prime number p , at least one extension K_i/K_0 ($i = 1, 2$) is tamely ramified with respect to p , then*

$$\mathcal{P}o(L) = \varepsilon_{K_1}^L(\mathcal{P}o(K_1)) + \varepsilon_{K_2}^L(\mathcal{P}o(K_2)).$$

Example 3.6. Let $K_1 = \mathbb{Q}(\sqrt{d})$ and $K_2 = \mathbb{Q}(\sqrt{d'})$. If $d \equiv 1 \pmod{4}$, then

$$\mathcal{P}o(\mathbb{Q}(\sqrt{d}, \sqrt{d'})) = \varepsilon_{K_1}^L(\mathcal{P}o(K_1)) + \varepsilon_{K_2}^L(\mathcal{P}o(K_2)).$$

Proposition 3.5 implies as a particular case assertion (a) of Zantema's Theorem 1.7. Both following theorems describe the group $\mathcal{P}o(L)$ by generalizing assertion (b) of Theorem 1.7.

Theorem 3.7. *Let K_1 and K_2 be two finite galoisian extensions of \mathbb{Q} . Let $K_0 = K_1 \cap K_2$ and $L = K_1 K_2$. Assume that:*

- (a) K_0 is a Pólya field,
- (b) $([K_1 : K_0], [K_2 : K_0]) = 1$.

Then,

$$\mathcal{P}o(L) \simeq \mathcal{P}o(K_1) \oplus \mathcal{P}o(K_2).$$

More precisely,

$$\mathcal{P}o(L) = \varepsilon_{K_1}^L(\mathcal{P}o(K_1)) \oplus \varepsilon_{K_2}^L(\mathcal{P}o(K_2)).$$

Proof. This is a consequence of Proposition 3.5 and of the fact that the intersection of the two subgroups $\varepsilon_{K_1}^L(\mathcal{P}o(K_1))$ and $\varepsilon_{K_2}^L(\mathcal{P}o(K_2))$ in $\mathcal{P}o(L)$ is $\{1\}$. Indeed, the order of every element of $\mathcal{P}o(K_i)$ divides $[K_i : K_0]$ ($i = 1, 2$) and, by hypothesis, these degrees are coprime. \square

Example 3.8. Let $K_1 = \mathbb{Q}(i, j)$ and $K_2 = \mathbb{Q}(j, \sqrt[3]{7})$. Then, $\mathcal{P}o(K_1) = \{1\}$ [17, Thm. 5.1] and $\mathcal{P}o(K_2) = \text{Cl}(K_2) \simeq \mathbb{Z}/3\mathbb{Z}$ [17, Ex. 6.8]. As $K_0 = \mathbb{Q}(j)$ is a Pólya field, we obtain: $\mathcal{P}o(\mathbb{Q}(i, j, \sqrt[3]{7})) \simeq \mathbb{Z}/3\mathbb{Z}$.

Corollary 3.9. *Assume K_0 is a Pólya field and that L/K_0 is a galoisian extension whose Galois group is of the form $\prod_{i \in I} G_i$ where, for $i \neq j$, $(|G_i|, |G_j|) = 1$. Then,*

$$\mathcal{P}o(L) \simeq \bigoplus_{i \in I} \mathcal{P}o(K_i)$$

where K_i denotes the subfield of L fixed by all the G_j where $j \neq i$.

As a particular case, we have:

Corollary 3.10. *Assume that K_0 is a Pólya field and that L/K_0 is an abelian extension of degree $n = \prod_{p \in \mathbb{P}} p^{\nu(p)}$. Then,*

$$\mathcal{P}o(L) \simeq \bigoplus_{p|n} \mathcal{P}o(K_p)$$

where, for each p dividing n , K_p denotes the subfield of L of degree $p^{\nu(p)}$ on K_0 .

As a particular case, we obtain [29, Zantema, Cor. 3.5].

Theorem 3.11. *Let K_1 and K_2 be two finite galoisian extensions of \mathbb{Q} . Let $K_0 = K_1 \cap K_2$ and $L = K_1 K_2$. If the degrees $[K_1 : K_0]$, $[K_2 : K_0]$ and $[K_0 : \mathbb{Q}]$ are pairwise relatively prime, then*

$$\mathcal{P}o(L) \simeq \mathcal{P}o(K_0) \oplus \left(\mathcal{P}o(K_1) / \varepsilon_{K_0}^{K_1}(\mathcal{P}o(K_0)) \right) \oplus \left(\mathcal{P}o(K_2) / \varepsilon_{K_0}^{K_2}(\mathcal{P}o(K_0)) \right).$$

Proof. This is a consequence of Propositions 2.3 and 3.5. Indeed, for $i = 1, 2$, one has the canonical injective morphisms :

$$\varepsilon_{K_0}^{K_i} : \mathcal{P}o(K_0) \rightarrow \mathcal{P}o(K_i) = \varepsilon_{K_0}^{K_i}(\mathcal{P}o(K_0)) \oplus W_i \text{ where } W_i \simeq \mathcal{P}o(K_i) / \varepsilon_{K_0}^{K_i}(\mathcal{P}o(K_0)).$$

Moreover,

$$\mathcal{P}o(L) = \varepsilon_{K_1}^L(\mathcal{P}o(K_1)) + \varepsilon_{K_2}^L(\mathcal{P}o(K_2)).$$

Thus,

$$\mathcal{P}o(L) = \varepsilon_{K_0}^L(\mathcal{P}o(K_0)) + \varepsilon_{K_1}^L(W_1) + \varepsilon_{K_2}^L(W_2)$$

and this sum is direct since the orders of the elements of the three subgroups are pairwise relatively prime. \square

4. THE RELATIVE PÓLYA GROUP OF AN EXTENSION

We introduce the following notation for every finite extension L/K of number fields:

$$(13) \quad \forall \mathfrak{p} \in \text{Max}(\mathcal{O}_K) \quad \forall f \in \mathbb{N}^* \quad \Pi_{\mathfrak{p}^f}(L/K) = \prod_{\mathfrak{P} \in \text{Max}(\mathcal{O}_L), N_{L/K}(\mathfrak{P}) = \mathfrak{p}^f} \mathfrak{P}.$$

This is clearly a generalization of the ideals $\Pi_q(K)$'s since $\Pi_{\mathfrak{p}^f}(K) = \Pi_{(p\mathbb{Z})^f}(K/\mathbb{Q})$. Analogously, we generalize the notion of Pólya group:

Definition 4.1. The *relative Pólya group* of an extension L/K is the subgroup $\mathcal{P}o(L/K)$ of the class group of L generated by the classes of the ideals $\Pi_{\mathfrak{p}^f}(L/K)$.

Clearly, $\mathcal{P}o(K) = \mathcal{P}o(K/\mathbb{Q})$. Here, we are interested in galoisian extensions.

Notation. In the remaining of this section, *the extension L/K is assumed to be galoisian with Galois group H* . We may forget the exponent f and write:

$$\Pi_{\mathfrak{p}^*}(L/K) = \Pi_{\mathfrak{p}^{f_{\mathfrak{p}}}(L/K)}(L/K) = \prod_{\mathfrak{P} \in \text{Max}(\mathcal{O}_L), \mathfrak{P}|\mathfrak{p}} \mathfrak{P}.$$

For every $\mathfrak{p} \in \text{Max}(\mathcal{O}_K)$, one has $\mathfrak{p}\mathcal{O}_L = \Pi_{\mathfrak{p}^*}(L/K)^{e_{\mathfrak{p}}(L/K)}$. Clearly, the subgroup \mathcal{I}_L^H of \mathcal{I}_L formed by the invariant ideals is generated by the $\Pi_{\mathfrak{p}^*}(L/K)$'s:

$$\mathcal{I}_L^H = \langle \Pi_{\mathfrak{p}^*}(L/K) \mid \mathfrak{p} \in \text{Max}(\mathcal{O}_K) \rangle.$$

In fact, \mathcal{I}_L^H is the free abelian group with basis formed by the $\Pi_{\mathfrak{p}^*}(L/K)$'s. Note that the relative Pólya group $\mathcal{P}o(L/K)$ of a galoisian extension L/K is nothing else than the subgroup $\mathcal{I}_L^H / \mathcal{P}_L^H$ of $\mathcal{C}l(L)$ formed by the strongly ambiguous classes with respect to the extension L/K .

Recall that Spickermann generalized the notion of Pólya field in the following way:

Definition 4.2. [24] A galoisian extension of number fields L/K with Galois group H is said to be a *galoisian Pólya extension* if $\mathcal{I}_L^H \subseteq \mathcal{I}_K \cdot \mathcal{P}_L$

This is a generalization of the definition of a Pólya field L only if L is a galoisian extension of \mathbb{Q} . Moreover, with respect to the notion of relative Pólya group, to say that the extension L/K is a galoisian Pólya extension means that:

$$\mathcal{P}o(L/K) = \varepsilon_K^L(\mathcal{C}l(K)).$$

If L/\mathbb{Q} is galoisian, then $\Pi_{p^*}(L) = \prod_{\mathfrak{p}|p} \Pi_{\mathfrak{p}^*}(L/K)$, and hence,

$$\mathcal{P}o(L/\mathbb{Q}) = \mathcal{P}o(L) \subseteq \mathcal{P}o(L/K).$$

Remark 4.3. There is another notion of Pólya extension introduced in [6, § II.9] and [16], that is more directly linked to the integer-valued polynomials and that is completely distinct from the previous definition: in these papers, an extension of number fields L/K is said to be a Pólya extension if $\varepsilon_K^L(\mathcal{P}o(K)) = \{1\}$.

We recall now some known results with respect to the group formed by the strongly ambiguous classes.

Proposition 4.4. *Let L/K be a galoisian extension with Galois group H . Then,*

$$(14) \quad |\mathcal{P}o(L/K)| = h_K \times \frac{\prod_{\mathfrak{p} \in \text{Max}(\mathcal{O}_K)} e_{\mathfrak{p}}(L/K)}{|H^1(H, \mathcal{O}_L^\times)|}$$

Proof. Consider the surjective morphism:

$$\varphi : \prod_{\mathfrak{p} \in \text{Max}(\mathcal{O}_K)} \Pi_{\mathfrak{p}^*}(L/K)^{\alpha_{\mathfrak{p}}} \in \mathcal{I}_L^H \mapsto (\alpha_{\mathfrak{p}} \bmod e_{\mathfrak{p}}(L/K)) \in \bigoplus_{\mathfrak{p} \in \text{Max}(\mathcal{O}_K)} \mathbb{Z}/e_{\mathfrak{p}}(L/K)\mathbb{Z}.$$

Clearly, $I = \prod_{\mathfrak{p} \in \text{Max}(\mathcal{O}_K)} \Pi_{\mathfrak{p}^*}(L/K)^{\alpha_{\mathfrak{p}}}$ belongs to $\text{Ker}(\varphi)$ if and only if $e_{\mathfrak{p}}(L/K) \mid \alpha_{\mathfrak{p}}$ for each \mathfrak{p} , that is, $I = \prod_{\mathfrak{p} \in \text{Max}(\mathcal{O}_K)} (\mathfrak{p}\mathcal{O}_L)^{\beta_{\mathfrak{p}}}$. Thus, $\text{Ker}(\varphi) = j_K^L(\mathcal{I}_K)$ and we have the following exact sequence:

$$(15) \quad 1 \rightarrow \mathcal{I}_K \xrightarrow{j_K^L} \mathcal{I}_L^H \xrightarrow{\varphi} \bigoplus_{\mathfrak{p} \in \text{Max}(\mathcal{O}_K)} \mathbb{Z}/e_{\mathfrak{p}}(L/K)\mathbb{Z} \rightarrow 0$$

On the other hand, applying the functor $U \mapsto U^H$ to the exact sequence

$$1 \rightarrow \mathcal{O}_L^\times \rightarrow L^* \rightarrow \mathcal{P}_L \rightarrow 1,$$

leads with Hibert 90 to the following exact sequence:

$$(16) \quad 1 \rightarrow \mathcal{P}_K \xrightarrow{j_K^L} \mathcal{P}_L^H \xrightarrow{\delta} H^1(H, \mathcal{O}_L^\times) \rightarrow 1.$$

Putting together the exact sequences (15) and (16), we obtain the following commutative diagram:

$$(17) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{P}_K & \xrightarrow{j_K^L} & \mathcal{P}_L^H & \xrightarrow{\delta} & H^1(H, \mathcal{O}_L^\times) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \tau_{L/K} & & \downarrow \\ 1 & \longrightarrow & \mathcal{I}_K & \xrightarrow{j_K^L} & \mathcal{I}_L^H & \xrightarrow{\varphi} & \bigoplus_{\mathfrak{p}} \mathbb{Z}/e_{\mathfrak{p}}(L/K)\mathbb{Z} & \longrightarrow & 0 \end{array}$$

Note that the morphisme $\tau_{L/K}$ is naturally defined by the commutativity of the diagram. The snake's lemma leads to the following exact sequence:

$$(18) \quad 1 \rightarrow \text{Ker}(\tau_{L/K}) \rightarrow \mathcal{C}l(K) \xrightarrow{\varepsilon_K^L} \mathcal{P}o(L/K) \rightarrow \text{Coker}(\tau_{L/K}) \rightarrow 1.$$

As

$$|\text{Coker}(\tau_{L/K})| = \prod_{\mathfrak{p}} e_{\mathfrak{p}}(L/K) \times |\text{Ker}(\tau_{L/K})| / |H^1(H, \mathcal{O}_L^\times)|,$$

Formula (14) follows from sequence (18). \square

Corollary 4.5. *Let L/K be a galoisian extension with Galois group H and assume that $h_K = 1$. Then, the following sequence of abelian groups is exact:*

$$(19) \quad 1 \rightarrow H^1(H, \mathcal{O}_L^\times) \xrightarrow{\tau_{L/K}} \bigoplus_{\mathfrak{p} \in \text{Max}(\mathcal{O}_K)} \mathbb{Z}/e_{\mathfrak{p}}(L/K)\mathbb{Z} \rightarrow \mathcal{P}o(L/K) \rightarrow 0.$$

Moreover, L/K is a galoisian Pólya extension if and only if

$$|H^1(H, \mathcal{O}_L^\times)| = \prod_{\mathfrak{p} \in \text{Max}(\mathcal{O}_K)} e_{\mathfrak{p}}(L/K).$$

Recall that the morphisme $\tau_{L/K}$ was defined by the commutativity of diagram (17).

Proof. If $h_K = 1$, sequence (18) shows that $\text{Ker}(\tau_{L/K}) = \{1\}$ and $\text{Coker}(\tau_{L/K}) = \mathcal{I}_L^H / \mathcal{P}_L^H = \mathcal{P}o(L/K)$ which leads to sequence (19). Moreover, since $h_K = 1$, to say that L/K is a galoisian Pólya extension means that $\mathcal{P}o(L/K) = \{1\}$. \square

As a particular case, sequence (19) leads, when $K = \mathbb{Q}$, to Zantema's sequence:

Corollary 4.6. [29, p. 163] *If L/\mathbb{Q} is a galoisian extension with Galois group G , then the following sequence is exact:*

$$(20) \quad 1 \rightarrow H^1(G, \mathcal{O}_L^\times) \rightarrow \bigoplus_{p \in \mathbb{P}} \mathbb{Z}/e_p(L/\mathbb{Q})\mathbb{Z} \rightarrow \mathcal{P}o(L) \rightarrow 0.$$

Corollary 4.7. *If L/K is a cyclic extension, then*

$$|\mathcal{P}o(L/K)| = \frac{h_K}{[L : K]} \times \frac{\prod_{\mathfrak{p} \in \text{Max}(\mathcal{O}_K)} e_{\mathfrak{p}}(L/K) \times \prod_{\mathfrak{p}|\infty} e_{\mathfrak{p}}(L/K)}{(\mathcal{O}_K^\times : N_{L/K}(\mathcal{O}_L^\times))}.$$

Proof. Assume that the cyclic group $H = \text{Gal}(L/K)$ is generated by σ . One knows that [20, IV.3.7]:

$$H^1(H, \mathcal{O}_L^\times) \simeq H^{-1}(H, \mathcal{O}_L^\times) = \frac{\{a \in \mathcal{O}_L^\times \mid N_{L/K}(a) = 1\}}{\{\sigma(a)/a \mid a \in \mathcal{O}_L^\times\}}.$$

Following [14, p. 192, Cor. 2] or [15, Thm 2], the index

$$(\{a \in \mathcal{O}_L^\times \mid N_{L/K}(a) = 1\} : \{\sigma(a)/a \mid a \in \mathcal{O}_L^\times\})$$

is equal to the product:

$$(\mathcal{O}_K^\times : N_{L/K}(\mathcal{O}_L^\times)) \times [L : K] \times \frac{1}{\prod_{\mathfrak{p}|\infty} e_{\mathfrak{p}}(L/K)}.$$

\square

Corollary 4.8. [15, Thm 1] *If L/K is cyclic of prime degree p , then*

$$|\mathcal{P}o(L/K)| = \frac{h_K \times p^{s_{L/K}-1}}{(\mathcal{O}_K^\times : N_{L/K}(\mathcal{O}_L^\times))}$$

where $s_{L/K}$ denotes the number of ramified primes of K including those at infinity.

If L/\mathbb{Q} is cyclic of prime degree, we obtain [5, Cor. 3.11] and [29, Prop. 3.2]. In particular, if L/\mathbb{Q} is cyclic with odd prime degree:

$$|\mathcal{P}o(L)| = p^{s_L-1}$$

where s_L denotes the number of prime numbers which are ramified in L .

Remark 4.9. Assume that $K \subseteq L$ and that K and L are both galoisian extensions of \mathbb{Q} . Let $G = \text{Gal}(L/\mathbb{Q})$ and $H = \text{Gal}(L/K)$. Then, putting together the exact sequence (20) written for K and for L leads to the following commutative diagram:

$$(21) \quad \begin{array}{ccccccc} 1 & \longrightarrow & H^1(G/H, \mathcal{O}_K^\times) & \xrightarrow{\tau_K} & \oplus_p \mathbb{Z}/e_p(K/\mathbb{Q})\mathbb{Z} & \longrightarrow & \mathcal{P}o(K) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \varepsilon_K^L \downarrow \\ & & \text{Inf} & & & & \\ 1 & \longrightarrow & H^1(G, \mathcal{O}_L^\times) & \xrightarrow{\tau_L} & \oplus_p \mathbb{Z}/e_p(L/\mathbb{Q})\mathbb{Z} & \longrightarrow & \mathcal{P}o(L) \longrightarrow 0 \end{array}$$

The morphisms τ_K and τ_L were defined in diagramm (17) while Inf denotes the inflation morphism. The snake's lemma leads to the following exact sequence:

$$1 \rightarrow \text{Ker}(\varepsilon_K^L) \rightarrow \text{Coker}(\text{Inf}) \rightarrow \oplus_p \mathbb{Z}/e_p(L/K)\mathbb{Z} \rightarrow \text{Coker}(\varepsilon_K^L) \rightarrow 0.$$

If $([L : K], [K : \mathbb{Q}]) = 1$, then $\text{Ker}(\varepsilon_K^L) = \{1\}$, and hence, we have the following exact sequence:

$$1 \rightarrow \text{Coker}(\text{Inf}) \rightarrow \oplus_p \mathbb{Z}/e_p(L/K)\mathbb{Z} \rightarrow \mathcal{P}o(L)/\varepsilon_K^L(\mathcal{P}o(K)) \rightarrow 0.$$

5. NON-GALOISIAN EXTENSIONS

In the non-galoisian case, the characterization of the Pólya fields is more difficult and it is probably the same for the description of the Pólya groups. Let us look at the case of a cubic number field K . If K is a galoisian cubic number field, it follows from Corollary 4.8 that $|\mathcal{P}o(K)| = 3^{s_K-1}$, and hence, K is a Pólya field if and only if there is exactly one ramified prime in the extension K/\mathbb{Q} . If K is a non-galoisian cubic number field, it follows from [29, Thm 1.1] that K is a Pólya field if and only if $h_K = 1$, that is, the Pólya group of K is trivial if and only if the class group of K is trivial. What could be the generalized assertion about Pólya groups? Perhaps, the Pólya group of a non-galoisian cubic number field is equal to its class group? On this line, let us recall Zantema's main result in the non-galoisian case:

Theorem 5.1. [29, Thm 6.9] *Let K be a number field of degree $n \geq 3$ and let G be the Galois group of the normal closure of K over \mathbb{Q} . If G is either isomorphic to the symmetric group \mathcal{S}_n where $n \neq 4$, or to the alternating group \mathcal{A}_n where $n \neq 3, 5$, or is a Frobenius group, then the following assertions are equivalent:*

- (1) every ideal $\Pi_{p^f}(K)$ where p is not ramified is principal,
- (2) K is a Pólya field,
- (3) $h_K = 1$.

In other words, under the previous hypotheses, denoting by $\mathcal{P}o(K)_{nr}$ the subgroup of $\mathcal{P}o(K)$ generated by the classes of the ideals $\Pi_{p^f}(K)$ where p is not ramified, one has the equivalences: $\mathcal{P}o(K)_{nr} = \{1\} \Leftrightarrow \mathcal{P}o(K) = \{1\} \Leftrightarrow \mathcal{C}l(K) = \{1\}$. Following the spirit if the translation of Zantema's results in Section 3, we formulate the following unreasonable conjecture:

Conjecture. *Under the hypotheses of Theorem 5.1, the following equalities hold:*

$$\mathcal{P}o(K)_{nr} = \mathcal{P}o(K) = Cl(K).$$

This conjecture will be proved in a forthcoming paper [7].

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