ON THE DISTRIBUTION OF PRIMES IN THE ALTERNATING SUMS OF CONCECUTIVE PRIMES

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ABSTRACT. Quite recently, in [8] the authoor of this paper considered the distribution of primes in the sequence (S_n) whose nth term is defined as $S_n = \sum_{k=1}^{2n} p_k$, where p_k is the kth prime. Some heuristic arguments and the numerical evidence lead to the conjecture that the primes are distributed among sequence (S_n) in the same way that they are distributed among positive integers. More precisely, Conjecture 3.3 in [8] asserts that $\pi_n \sim \frac{n}{\log n}$ as $n \to \infty$, where π_n denotes the number of primes in the set $\{S_1, S_2, \ldots, S_n\}$. Motivated by this, here we consider the distribution of primes in aletrnating sums of first $2n$ primes, i.e., in the sequences (A_n) and (T_n) defined by $A_n := \sum_{i=1}^{2n}(-1)^i p_i$ and $T_n := A_n - 2 = \sum_{i=2}^{2n}(-1)^i p_i$ $(n = 1, 2, \ldots)$.

Heuristic arguments and computational results suggest the conjecture that (Conjecture 2.5)

$$\pi_{(A_k)}(A_n) \sim \pi_{(T_k)}(T_n) \sim \frac{2n}{\log n} \quad \text{as } n \to \infty,$$

where $\pi_{(A_k)}(A_n)$ (respectively, $\pi_{(T_k)}(T_n)$) denotes the number of primes in the set $\{A_1, A_2, \ldots, A_n\}$ (respectively, $\{T_1, T_2, \ldots, T_n\}$). Under Conjecture 2.5 and Pillai's conjecture, we establish two results concerning the expressions for the kth prime in the sequences (A_n) and (T_n). Furthermore, we propose some other related conjectures and we deduce some their consequences.

1. INTRODUCTION, MOTIVATION AND PRELIMINARIES

Motivated by the notion of generalized prime system or g-*prime system*) $\mathcal{G}$ introduced by A. Beurling in [2] which generalizes the notion of primes and positive integers, in [8, Section 1] it was considered a system described as follows.

Let $\mathcal{P} := \{p_1, p_2, p_3, \ldots\}$ be the set of all primes $2 = p_1 < p_2 < p_3 < \cdots$ and let $\mathcal{N}$ be an increasing integer sequence $(a_k)_{k=1}^{\infty}$.

Let $(\mathcal{P}, \mathcal{N} := (a_k)_{k=1}^{\infty})$ be a pair defined above. Then we define its *counting function* [8, p. 3] $N_{(a_k)}(x)$ $x \in [1, \infty))$ as

$$N_{(a_k)}(x) = \#\{i : i \in \mathbb{N} \text{ and } a_i \leq x\}.$$

Furthermore, the *prime counting function* for $(\mathcal{P}, \mathcal{N})$ is the function $x \mapsto \pi_{(a_k)}(x)$ defined on $[1, \infty)$ as

$$\pi_{(a_k)}(x) = \#\{q : q \in \mathcal{P} \text{ and } q = a_i \text{ for some } i \text{ with } a_i \leq x\}. \tag{1}$$

Some heuristic and computational results show that for many "natural pairs" $(\mathcal{P}, \mathcal{N} := (a_k)_{k=1}^{\infty})$ the associated counting function $N_{(a_k)}(x)$ satisfies certain asymptotic growth as $x \to \infty$ (see [8, Section 2]). Notice that for each positive integer n we define [8, the equality (2) with $\mathcal{G} = \mathcal{P}$]

$$\pi_{(a_k)}(a_n) = \#\{q : q \in \mathcal{P} \text{ and } q = a_i \text{ for some } i \text{ with } 1 \leq i \leq n\}. \tag{2}$$

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Accordingly, we give the following definition [8, Definition 1.1].

Definition 1.1. Let Ω be a set of all nonnegative continuous real functions defined on $[1, +\infty)$ and let $(a_k)_{k=1}^{\infty} := (a_k)$ be an increasing sequence of positive integers. We say that (a_k) satisfies ω-*Restricted Prime Number Theorem* if there exists the function $\omega_{(a_k)} = \omega \in \Omega$ such that the function $n \mapsto \pi_{(a_k)}(a_n)$ defined by (2) is asymptotically equivalent to $\omega(n)$ as $n \to \infty$.

In particular, if $\omega(x) \sim x/\log x$ as $x \to \infty$, then we say that a sequence (a_k) satisfies the *Restricted Prime Number Theorem* (RPNT).

Notice that by the *Prime Number Theorem*,

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1,$$

where $\pi(x)$ is the prime counting function, i.e., $\pi(x)$ denotes the number of primes less than x. For history, see [1] and [7, p. 21].

Motivated by our recent paper [8] concerning the distribution of primes in the sequence (S_n) with $S_n = \sum_{i=1}^{2n} p_i$ $(n = 1, 2, \ldots)$, computations (Table 1), Pillai's conjecture (Conjecture 2.5) and some heuristic arguments, in the following section we propose the conjecture (Conjecture 2.5) on the distribution of primes in the sequences (A_n) and $(A_n - 2)$ with $A_n = \sum_{i=1}^{2n}(-1)^i p_i$. Namely, Conjecture 2.2 asserts that the number of primes in in the set $\{A_1, A_2, \ldots, A_n\}$ is $\sim \frac{2n}{\log n}$ as $n \to \infty$. Under Pillai's conjecture and Conjecture 2.5, we deduce two consequences concerning the asymptotic expressions for the kth prime in the sequences (A_n) and $(A_n - 2)$. Some related conjectures and their corollaries are also presented. Finally, by using computational results up to $n = 5 \cdot 10^8$, we propose some conjectures on the estimates of differences $A_n - p_n$.

2. THE DISTRIBUTION OF PRIMES IN ALTERNATING SUMS (A_n) AND $(A_n - 2)$ WITH $A_n := \sum_{i=1}^{2n}(-1)^i p_i$

In this section we consider the distribution of primes in alternating sums of consecutive primes; namely, in the sequences (A_n) and (T_n) respectively defined by

$$A_n = \sum_{i=1}^{2n}(-1)^i p_i, \quad n = 1, 2, \ldots,$$

and

$$T_n = \sum_{i=2}^{2n}(-1)^i p_i, \quad n = 1, 2, \ldots.$$

Notice that $T_n = A_n - 2$ $(n = 1, 2, \ldots)$ and here we present conjectures and related results based on computational results concerning the sequence (A_n). Notice that computational results and heuristic arguments related to the sequence (T_n) suggest the same conjectures and their consequences as these for the sequence (A_n).

For computational investigations of distribution of primes presented in Table 1, we proceed similarly as in [8, Section 6], where the analogous study is considered for the sequence (S_n) with $S_n = \sum_{i=1}^{2n} p_i$ $(n = 1, 2, \ldots)$.

Remark **2.1.** Note that (A_n) is the sequence consisting of terms of Sloane's sequence A008347 [12] (firstly introduced by N.J.A. Sloane and J.H. Conway) with even indices defined as $a_n = \sum_{i=0}^{n-1}(-1)^i p_{n-i}$ $(a_0 = 0, 2, 1, 4, 3, 8, 5, \ldots)$; namely, $A_n = a_{2n}$ for all $n = 1, 2, \ldots$. Notice also that the "complement" (with respect to $\mathbb{N}$) of Sloane's sequence A008347 is the sequence A226913 $(6, 9, 10, 11, 14, 15, 17, \ldots)$. Furthermore, the sequence (A_n) is also closely related to Sloane's sequence A131694-numbers n such that $b_n := \sum_{i=1}^{n}(-1)^i p_i$ is a prime $(1, 4, 6, 8, 10, 12, 18, \ldots)$.

Recall also that Sloane's sequence A066033 is defined as $a_n = 2 + \sum_{i=2}^{n}(-1)^i p_i$ with $a_1 = 2$ $(2, 5, 0, 7, -4, 9, -8, 11, -12, 17, \ldots)$; the sequence A136288 defined as primes which are the absolute value of the alternating sum and the difference of the first n primes $(2, 3, 5, 7, 13, 19, 29, 53, 61, \ldots)$ (cf. the sequences A163057-an alternating sum from the nth odd number up to the nth odd prime $(2, 4, 6, 9, 11, 14, 16, \ldots)$, and the related sequences A163058-primes in A163057 $(2, 11, 19, 23, \ldots)$. Notice also that the sequences A264834, A242188, A240860, A233809, A226743, A131196 and A131197 are closely related to the sequence A008347.

In 1982 D.A. Goldston [6] has proved assuming the Riemann Hypothesis that

$$\sum_{\substack{p_i < x \\ p_i - p_{i-1} \geq d}} (p_i - p_{i-1}) = O\left(\frac{x \log x}{d}\right)$$

uniformly for $d \geq 2$, which for $d = 2$ putting $x = p_{2n}$ and $p_{2n} \sim 2n \log n$ immediately yields

$$A_n := \sum_{i=1}^{2n}(p_i - p_{i-1}) = O(n \log^2 n).$$

Assuming the Riemann Hypothesis, as a consequence of a conjecture posed in 2011 by M. Wolf [13, Conjecture 1], Wolf [13, the asymptotic relation (41) of Section 4] noticed that

$$\sum_{\substack{p_i < x \\ p_i - p_{i-1} \geq d}} (p_i - p_{i-1}) \sim x + \frac{d(d-1)}{2} \cdot \frac{x}{\log^2 x} + O\left(\frac{1}{\log^3 x}\right)$$

which for x so large that $\log x > d$ is indeed smaller than the above upper bound of Goldston. In particular, for $d = 2$, $k = p_{2n}$ and $p_{2n} \sim 2n \log n$ the previous estimate gives

$$A_n := \sum_{i=1}^{2n}(p_i - p_{i-1}) \sim p_{2n} \sim 2n \log n.$$

However, a computation shows that the above asymptotic relation is probably false, i.e., it is probably true with $n \log n$ instead of $2n \log n$ (i.e., with p_n instead of p_{2n}) on the right hand side. This is in fact the following conjecture due to Pillai [10, p. 84, Conjecture 34] (also cf. [12, Comments of Joseph L.Pe in Sloane's sequence A008347]).

Conjecture 2.2. *If $k \in \mathbb{N}$, then*

$$\Big|\sum_{i \leq k}(-1)^{i-1} p_i\Big| \sim \frac{p_k}{2} \quad \text{as } k \to \infty. \tag{3}$$

Corollary 2.3. *Under Pillai's Conjecture 2.2 we have*

$$A_n \sim T_n \sim n \log n \quad \text{as } n \to \infty. \tag{4}$$

More precisely,

$$A_n \sim T_n \sim n \log n + n \log\log n - n + o(n) \quad \text{as} \quad n \to \infty. \tag{5}$$

Proof. Taking $k = 2n$ and the well known asymptotic relation $p_{2n} \sim 2n\log 2n \sim 2n\log n$ into (3) (see, e.g., [9]), we immediately obtain (4).

Furthermore, by Cipolla's formula [3] for the approximation to the kth prime,

$$p_k = k\log k + k\log\log k - k + o(k).$$

Taking the above expression with $k = 2n$ into (3), we immediately obtain (5). □

Remark **2.4.** Since

$$A_n = p_{2n} - (p_{2n-1} - p_{2n-2}) - \cdots - (p_3 - p_2) - p_1,$$

we see that $A_n < p_{2n}$ for all $n = 1, 2, \ldots$.

Using the asymptotic relation (4) and the fact that A_n is an odd integer for all $n \in \mathbb{N}$, some heuristic arguments together with the Prime Number Theorem suggest that the "probability" of A_n being a prime is $2/\log n$. Consequently, there are $\sim 2n/\log n$ primes that belong to the set $\{A_1, A_2, \ldots, A_n\}$ (of course, the same assertion holds for the sequence (T_n)). This together with computational results given in Table 1 leads to the following conjecture.

Conjecture 2.5. *Let (A_n) and (T_n) be the sequences for which $A_n = \sum_{i=1}^{2n}(-1)^i p_i$ and $T_n = A_n - 2 = \sum_{i=2}^{2n}(-1)^i p_i$. Then in accordance to the notion of Definition* 1.1*,*

$$\omega_{(A_n)}(x) = \omega_{(T_n)}(x) \sim \frac{2x}{\log x} \quad \text{as } x \to \infty, \tag{6}$$

or equivalently,

$$\begin{aligned}\pi_{(A_k)}(A_n) &= \#\{p : p \text{ is a prime and } p = A_i \text{ for some } i \text{ with } 1 \le i \le n\}\\ &\sim \frac{2n}{\log n} \quad \text{as } n \to \infty\end{aligned} \tag{7}$$

and

$$\begin{aligned}\pi_{(T_k)}(T_n) &= \#\{p : p \text{ is a prime and } p = T_i \text{ for some } i \text{ with } 1 \le i \le n\}\\ &\sim \frac{2n}{\log n} \quad \text{as } n \to \infty.\end{aligned} \tag{8}$$

Notice that in all our results of this section (Theorem 2.8 and Corollaries) we assume the truth of Conjectures 2.2 and 2.5.

Observe that $|\sum_{i=1}^{n}(-1)^i p_i|$ is equal to $A_{n/2}$ for even n, while $|\sum_{i=1}^{n}(-1)^i p_i|$ is even for odd n. This fact shows that Conjecture 2.5 is equivalent with the following one.

Conjecture 2.5.' *Let (a_n) be the sequence defined as $a_n = |\sum_{i=1}^{n}(-1)^i p_i|$. Then*

$$\omega_{(a_n)}(x) = \frac{x}{\log x}. \tag{9}$$

In other words, the sequence (a_n) satisfies the Restricted Prime Number Theorem.

As a direct application of Conjectures 2.2 and 2.5, we obtain the following A_n (T_n)-analogue of Corollary 3.6 in [8] concerning the sequence (S_n) with $S_n = \sum_{i=1}^{2n} p_i$.

Corollary 2.6 (The asymptotic expression for the kth prime in the sequences (A_n) and (T_n))**.** *Let r_k $(k = 1, 2, \ldots)$ be the kth prime in the sequence (A_n) (or (T_n)). Then*

$$r_k \sim \frac{k\log^2 k}{2} \quad \text{as } k \to \infty. \tag{10}$$

Proof. If a pair (k, m) satisfies $r_k = A_m$, then by (6) of Conjecture 2.5, $2m \sim k \log m$ as $k \to \infty$, and hence, $\log m \sim \log k$ as $k \to \infty$. The previous two asymptotic relations immediately give

$$m \log m \sim \frac{k \log^2 k}{2} \quad \text{as } k \to \infty. \tag{11}$$

Since by (4) of Corollary 2.3, $r_k \sim m \log m$ as $k \to \infty$, substituting this into (11), we immediately obtain (10). □

Notice that the Prime Number Theorem and Corollary 2.3 immediately yield the following result.

Corollary 2.7. *Under Conjecture* 2.2 *there holds*

$$\pi(A_n) \sim \pi(T_n) \sim n \quad as\ n \to \infty. \tag{12}$$

where $\pi(x)$ *is the prime counting function.*

Furthermore, we have the following A_n-analogue of Theorem 4.4 of [8] concerning the sequence (S_n) with $S_n = \sum_{i=1}^{2n} p_i$.

Theorem 2.8 (The asymptotic expression for the kth prime in the sequence (A_n))**.** *Let* r_k *be the* k*th prime in the sequence* (A_n) $(k = 2, 3, \ldots)$. *Then under Conjectures* 2.2 *and* 2.5, *there exists a positive sequence* (R_k) *such that* $\lim_{k\to\infty} R_k = 1$ *and*

$$r_k = \frac{1}{2} R_k^3 k \log k(\log k + \log\log k + 2 \log R_k). \tag{13}$$

Proof of Theorem 2.8 is based on the following result.

Lemma 2.9. *Let* $r_k = A_m$ *be the* k*th prime in the sequence* (A_n). *Then under Conjectures* 2.2 *and* 2.5,

$$r_k \sim \frac{m\sqrt{2m} \log m}{\sqrt{k \log k}} \quad as\ k \to \infty. \tag{14}$$

Proof of Lemma 2.9. From the proof of Corollary 2.6 we see that $2m \sim k \log k$ and $r_k \sim m \log m$ as $k \to \infty$. The previous two asymptotic relations immediately imply (14). □

Proof of Theorem 2.8. Proof of Theorem 2.8 is based on Lemma 2.9. Since this proof is completely similar to those of Theorem 4.4 of [8], it can be omitted. □

Computational results (see seventh column in Table 1) suggest the following conjecture (cf. Conjecture 4.6 of [8]).

Conjecture 2.10. *For each pair* (k, m) *with* $k \geq 1$ *and* $r_k = A_m$ *we have*

$$\lfloor k \log k \rfloor + 1 \leq 2m, \tag{15}$$

or equivalently,

$$r_k \geq A_{\lfloor (k+1)/2 \rfloor}. \tag{16}$$

Consequently, we can obtain the following two corollaries (cf. Corollaries 4.7, 4.8 and their proofs from [8]).

Corollary 2.11. *If the inequality* (15) *of Conjecture* 2.10 *is true, then for each* $k \geq 1$ *there holds*

$$r_k > \frac{k}{2}\log k(\log k + \log\log k). \tag{17}$$

Corollary 2.12. *If the inequality* (15) *of Conjecture* 2.10 *is true, then* $R_k > 1$ *for each* $k \geq 1$*, where* (R_k) *is the sequence defined by* (13) *in Theorem* 2.8.

Remark **2.13.** As observed above, $A_n = a_{2n}$ for all $n = 1, 2, \ldots$, where (a_n) is Sloane's sequence A008347 [12] defined as $a_n = \sum_{i=0}^{n-1}(-1)^i p_{n-i}$. Z.-W. Sun [12, Conjectures (i)–(iv) in Comments of Sloane's sequence A008347] proposed certain conjectures involving the sequence A008347. In particular, Sun conjectured that for each $n > 9$,

$$a_{n+1} < (a_{n-1})^{(1+2/(n+2))}. \tag{18}$$

The conjecture has been verified by Sun for n up to 10^8. Notice that (18) with $2n-1$ instead of n can be written as

$$A_n < (A_{n-1})^{(1+2/(2n+1))}, \quad n = 6, 7, \ldots.$$

Sun also conjectured that (a_n) contains infinitely many *Sophie Germain primes* (given as Sloane's sequence A005384 in [12]), and that there are infinitely many positive integers n such that $a_n - 1$ and $a_n + 1$ are twin primes.

Table 1 obtained via `Mathematica 9` presents our computational results concerning the number of "alternating prime sums" r_k (under Conjecture 2.5) and related expression (the equality (13) of Theorem 2.8). The value k in the second column of Table 1 presents the number of primes in set $\mathcal{A}_n := \{A_1, A_2, \ldots, A_n\}$, where n is a corresponding value given in the first column of this table. Hence, under notations of Section 1 and Conjecture 2.5,

$$k := \pi_{(A_k)}(A_n) = \#\{p : p \text{ is a prime and } p = A_i \text{ for some } i \text{ with } 1 \leq i \leq n\}.$$

The appropriate value of the greatest prime r_k in $\mathcal{A}_n$ is given in the third column, while after the value of r_k in the bracket it is written the value $n - m$, where m is the index such that $q_k = A_m$. In the fourth column we present the corresponding values of R_k obtained as solutions of the equation (13) in Theorem 2.8. The fifth column of Table 1 presents the values $R_k^{(u)} := A_n/(n \log n)$ which are upper bounds of R_k. Notice that weakly but for computational purposes more suitable upper bounds of R_k than $R_k^{(u)}$, are given as $R_k^{(u')} := A_k/(k \log k)$.

The values of seventh column suggest that Conjecture 2.2 is probably true, but we believe that the values of these column are close to 1 for large values $n \gg 5 \cdot 10^8$. Notice also that the values in the last column of Table 1 suggest the truth of Lemma 2.9 and Conjecture 2.14.

For example, from Table 1 we see that $r_{33} = A_{96} = 563$, $r_{15234} = A_{9992} = 1379813$, $r_{129447} = A_{999994} = 16230881$ and $r_{9833766} = A_{99999972} = 2111199529$.

Table 1. Distribution of primes in the sequence (A_n) in the range $1 \le n \le 5 \cdot 10^8$

n	k	r_k with $(n-m)$	R_k	$R_k^{(u)} := \frac{A_n}{n \log n}$	$\frac{A_n - p_n}{n \log \log n}$	$\frac{k \log m}{2m}$	$\frac{r_k \sqrt{k \log k}}{m\sqrt{2m} \log m}$
10	6	29(1)	1.24290	1.43317	0.47960	0.73241	1.03381
10^2	33	563(4)	1.23573	1.29637	0.36670	0.78450	1.00799
10^3	254	8807(1)	1.23573	1.27523	0.46051	0.87804	0.95632
10^4	1982	113557(4)	1.15094	1.23334	0.39931	0.91307	0.92720
10^5	15234	1379813(8)	1.15567	1.19862	0.32845	0.87700	1.02665
10^6	129447	16230881(6)	1.13701	1.17484	0.28379	0.89412	1.02546
10^7	1116732	186806173(11)	1.12667	1.15899	0.26554	0.89998	1.02000
$2 \cdot 10^7$	2144771	388274699(4)	1.12395	1.11113	0.26022	0.90141	1.02100
$5 \cdot 10^7$	5097220	1019145103(2)	1.12042	1.10839	0.25525	0.90361	1.02016
$7 \cdot 10^7$	7007444	1451570059(19)	1.12926	1.10742	0.25307	0.90416	1.01966
10^8	9822766	2111199529(28)	1.11807	1.14610	0.25099	0.90471	1.01916
$10^8 + 5 \cdot 10^7$	14431395	3230666071(2)	1.11662	1.14404	0.24854	0.90563	1.01877
$2 \cdot 10^8$	18966586	4368109771(8)	1.11561	1.14266	0.24721	0.90631	1.01858
$3 \cdot 10^8$	27883839	6680071639(1)	1.11427	1.14076	0.24538	0.90712	1.01823
$4 \cdot 10^8$	36664392	9027893009(0)	1.11332	1.13948	0.24436	0.90776	1.01807
$5 \cdot 10^8$	45345672	11401770283(28)	1.11126	1.13846	0.24322	0.90828	1.01791

In view of the data of the last column in Table 1, we propose the following two conjectures (cf. Conjecture 4.9 of [8] concerning the kth prime in the sequence (S_n) with $S_n = \sum_{i=1}^{2n} p_i$).

Conjecture 2.14. *For every $k \ge 15234$ with $r_k = A_m$ there holds*

$$r_k > \frac{m\sqrt{2m} \log m}{\sqrt{k \log k}}.$$

Furthermore, heuristic arguments, some computational results and Conjecture 2.5 lead to the following its two generalizations (cf. [8, Conjectures 3.9 and 3.18]).

Conjecture 2.15. *For any fixed nonnegative integer d the sequence $(A_n^{(d)})_{n=1}^{\infty}$ defined as*

$$A_n^{(d)} = 2d + A_n = 2d + \sum_{i=1}^{2n} (-1)^i p_i, \quad n = 1, 2, \ldots$$

satisfies the Restricted Prime Number Theorem. In other words, as $n \to \infty$,

$$\begin{aligned} \pi_{(2d+A_k)}(2d + A_n) &:= \#\{p : p \text{ is a prime and } p = 2d + S_i \\ &\text{for some } i \text{ with } 1 \le i \le n\} \sim \frac{2n}{\log n}. \end{aligned} \tag{19}$$

Notice that For $d = -1$, Conjecture 2.15 is in fact the part of Conjecture 2.5 concerning the sequence (T_n) with $T_n = A_n - 2$ $(n = 1, 2, \ldots)$.

Conjecture 2.16. *For any fixed positive integer k, let $(A_n^{(k)}) := (A_n^{(k)})_{n=1}^{\infty}$ be the sequence whose nth term is defined as*

$$A_n^{(k)} = \sum_{i=1}^{2n+1} (-1)^{i-1} p_{i+k}, \quad n = 1, 2, \ldots.$$

Then the sequence $(A_n^{(k)})$ satisfies the Restricted Prime Number Theorem.

Suppose that a and d are relatively prime positive integers. Then Dirichlet's theorem [4] asserts that that there are infinitely many primes of the form $kd + a$ with $k \in \mathbb{N} \cup \{0\}$. Dirichlet's theorem, Conjecture 2.5 and some computational results lead to the following conjecture.

Conjecture 2.17 (Dirichlet's theorem for the sequences A_n and T_n)**.** *Suppose that a and d are relatively prime positive integers. Then in the sequence A_n (T_n) there are infinitely many primes of the form $kd + a$ with $k \in \mathbb{N} \cup \{0\}$.*

Finally, sixth column of Table 1 suggests the following conjecture.

Conjecture 2.18. *For each $n \geq 50$,*

$$A_n - p_n > \frac{n \log\log n}{5}, \tag{20}$$

and for each $n \geq 10^8 + 5 \cdot 10^7$,

$$A_n - p_n < \frac{n \log\log n}{4}. \tag{21}$$

Notice that the inequalitiy $p_n < n\log n + n \log\log n$ with $n \geq 6$ (see [5] and [11, (3.13) of Corollary, p.69]) together with some additional computations immediately yields the consequence of the first part of Conjecture 2.18 given as follows.

Corollary 2.19. *Under the inequality* (20) *of the first part of Conjecture* 2.18*, we have*

$$\frac{A_n}{p_n} - 1 > \frac{\log\log n}{5\log n} \quad \text{for each} \quad n \geq 50. \tag{22}$$

Similarly, the inequalitiy $p_n > n\log n$ with $n \geq 1$ (see, e.g., [11, (3.12) of Corollary, p.69]) immediately yields the consequence of the second part of Conjecture 2.18 given as follows.

Corollary 2.20. *Under the inequality* (21) *of the second part of Conjecture* 2.18*, we have*

$$\frac{A_n}{p_n} - 1 < \frac{\log\log n}{4\log n} \quad \text{for each} \quad n \geq 10^8 + 5 \cdot 10^7. \tag{23}$$

Remark **2.21.** From (22) and (23) it follows that for every $n \geq 10^8 + 5 \cdot 10^7$ there exists a real number C_n with $1/5 < C_n < 1/4$ such that

$$A_n - p_n = C_n n \log\log n. \tag{24}$$

Observe that the equality (24) is a refined version of "even case" of Pillai's conjecture (i.e., Conjecture 2.2 for even positive integers k such that $k \geq 2(10^8 + 5 \cdot 10^7)$).

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