

A LOGARITHMIC $\bar{\partial}$ -EQUATION ON A COMPACT KÄHLER MANIFOLD ASSOCIATED TO A SMOOTH DIVISOR

XUEYUAN WAN

ABSTRACT. In this paper, we solve a logarithmic $\bar{\partial}$ -equation on a compact Kähler manifold associated to a smooth divisor by using the cyclic covering trick. As applications, we discuss the closedness of logarithmic forms, injectivity theorems and obtain a kind of degeneration of spectral sequence at E_1 , and we also prove that the pair (X, D) has unobstructed deformations for any smooth divisor $D \in |-2K_X|$. Lastly, we prove some logarithmic vanishing theorems for q -ample divisors on compact Kähler manifolds.

CONTENTS

Introduction	1
1. Preliminaries	4
1.1. Logarithmic connection	4
1.2. Cyclic covering	6
2. Logarithmic $\bar{\partial}$ -equation	7
2.1. Logarithmic $\bar{\partial}$ -equation	7
2.2. Solving the $\bar{\partial}$ -equation	8
3. Some applications	11
3.1. Closedness of logarithmic forms	12
3.2. An injectivity theorem	12
3.3. Degeneration of spectral sequences	13
3.4. Logarithmic deformation	14
4. Logarithmic vanishing theorems	16
5. Further discussions	18
References	19

INTRODUCTION

It is well-known that Deligne's degeneration of logarithmic Hodge to de Rham spectral sequences at E_1 -level [9] is a fundamental result and has great impact in algebraic geometry, such as vanishing and injectivity theorems [12, 13]. P. Deligne and L. Illusie [10] also proved this degeneration by using a purely algebraic positive characteristic method. In terms of Kähler geometry, K. Liu, S. Rao and the author [23] developed a method so that the degeneration of the spectral sequence can be reduced to solve a logarithmic $\bar{\partial}$ -equation. By using the harmonic integral theory [6, 20], we can solve the logarithmic $\bar{\partial}$ -equation and thus give a geometric and

simpler proof to Deligne's degeneration [23, Theorem 0.4]. In this paper, we will continue to study another kind of logarithmic $\bar{\partial}$ -equation associated to a smooth divisor by combining with the cyclic covering trick.

Let X be a compact Kähler manifold of dimension n and $D = \sum_{i=1}^r D_i$ be a smooth divisor in X , i.e., $D_i \cap D_j = \emptyset$ for $i \neq j$. Let L be a holomorphic line bundle over X with $L^N = \mathcal{O}_X(D)$ for some integer $N \geq 1$, and let $\Omega_X^p(\log D)$ denote the sheaf of differential forms with logarithmic poles along D . From Proposition 1.2, one can endow an integrable logarithmic connection $d = \partial + \bar{\partial}$ along D on L^{-1} (see Definition 4.1), namely a \mathbb{C} -linear map

$$d : L^{-1} \rightarrow \Omega_X^1(\log D) \otimes L^{-1}$$

satisfies Leibniz's rule and $d^2 = 0$. Let $A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1})$ denote the space of smooth $(0, q)$ -forms with valued in $\Omega_X^p(\log D) \otimes L^{-1}$. For any $\alpha \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1})$ with $\bar{\partial}\alpha = 0$, we will consider the following logarithmic $\bar{\partial}$ -equation:

$$(0.1) \quad \bar{\partial}x = \partial\alpha$$

such that $x \in A^{0,q-1}(X, \Omega_X^{p+1}(\log D) \otimes L^{-1})$. In order to solve (0.1), we will use the cyclic covering trick (see Subsection 1.2). More precisely, let s be a canonical section of $\mathcal{O}_X(D)$ and taking a cyclic cover obtained by taking the N -th root out of s , we will get a compact Kähler manifold Y and a finite morphism $\pi : Y \rightarrow X$. By pullback the logarithmic $\bar{\partial}$ -equation (0.1) to Y , one gets a solvable logarithmic $\bar{\partial}$ -equation. This will lead to the vanishing of the harmonic projection of the logarithmic form (see (2.9)). By these and the bundle-valued Hodge decomposition theorem, we have

Theorem 0.1 (=Theorem 2.4). *Let X be a compact Kähler manifold and $D = \sum_{i=1}^r D_i$ be a smooth divisor in X , let L be a holomorphic line bundle over X with $L^N = \mathcal{O}_X(D)$. For any $\alpha \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1})$, the following equation*

$$\bar{\partial}x = \partial\alpha$$

has a solution $x \in A^{0,q-1}(X, \Omega_X^{p+1}(\log D) \otimes L^{-1})$.

The d -closedness of logarithmic forms has been studied in [9, 29, 23]. As an immediate application of Theorem 0.1, we obtain the following type of closedness of logarithmic forms.

Corollary 0.2 (=Corollary 3.1). *If $\alpha \in A^{0,0}(X, \Omega_X^p(\log D) \otimes L^{-1})$ with $\bar{\partial}\alpha = 0$, then $\partial\alpha = 0$.*

In [23, Corollary 3.6], K. Liu, S. Rao and the author reproved an injectivity theorem of F. Ambro [1, Theorem 2.1], the key step is to prove the following mapping

$$H^q(X, \Omega_X^n(\log D)) \rightarrow H^{q+n}(X, \Omega_X^\bullet(\log D)), \quad [\alpha] \mapsto [\alpha]_d,$$

is injective by solving a logarithmic $\bar{\partial}$ -equation. As a comparison, the second application of Theorem 0.1 is the following injectivity theorem.

Proposition 0.3 (=Proposition 3.2). *The following mapping is injective:*

$$\iota : H^q(X, \Omega_X^n(\log D) \otimes L^{-1}) \rightarrow H^{q+n}(X, \Omega_X^\bullet(\log D) \otimes L^{-1}),$$

where $\iota([\alpha]) = [\alpha]_d$ and $H^{q+n}(X, \Omega_X^\bullet(\log D) \otimes L^{-1})$ denotes the cohomology of the complex of sections $(\Gamma(X, \Omega_X^\bullet(\log D) \otimes L^{-1}), d)$ of $\Omega_X^\bullet(\log D) \otimes L^{-1}$.

By logarithmic analogue of the general description on the terms in the Frölicher spectral sequence as in [5, Theorems 1 and 3], $E_r^{p,q} \cong Z_r^{p,q}/B_r^{p,q}$ and $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$, where $Z_r^{p,q}$ and $B_r^{p,q}$ can be described as in (3.4) and (3.5). one would have $d_i = 0, \forall i \geq 1$ if (0.1) is solvable. Therefore, we get the following E_1 -degeneration of spectral sequence.

Theorem 0.4 (=Theorem 3.4). *The spectral sequence*

$$(0.2) \quad E_1^{p,q} = H^q(X, \Omega_X^p(\log D) \otimes L^{-1}) \implies \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D) \otimes L^{-1})$$

associated to the logarithmic de Rham complex

$$(\Omega_X^\bullet(\log D) \otimes L^{-1}, d)$$

degenerates in E_1 . Here $\mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D) \otimes L^{-1})$ denotes the hypercohomology.

It is well-known that a Calabi-Yau manifold has unobstructed deformations [34, 35]. More precisely, if X is a Calabi-Yau manifold, then for any $[\varphi_1] \in H^1(X, T_X)$, one can construct a holomorphic family $\varphi(t) \in A^{0,1}(X, T_X)$ on t , and it satisfies the integrable equation $\bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$, $(\partial\varphi(t)/\partial t)|_{t=0} = \varphi_1$. In the case of logarithmic deformations, the set of infinitesimal logarithmic deformations is the space $H^1(X, T_X(-\log D))$. The pair (X, D) has unobstructed deformations if one can construct a holomorphic family

$$\varphi := \varphi(t) \in A^{0,1}(X, T_X(-\log D))$$

satisfying the following integrability and initial conditions:

$$\bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi], \quad \frac{\partial\varphi}{\partial t}(0) = \varphi_1.$$

By using an iterative method originally from [34, 35, 25] and developed in [24, 37, 38, 32, 30, 31, 23, 27], we prove

Theorem 0.5 (=Theorem 3.8). *Let X be a compact Kähler manifold and D a smooth divisor such that $D \in |-2K_X|$. Then, the pair (X, D) has unobstructed deformations.*

Note that for X is projective, the above theorem was proved in [18, Proposition 6.4, Remark 6.5] and [21, Section 4.3.3] by a purely algebraic method.

By using cyclic covering trick and [26, Theorem 1.2], we obtain the following logarithmic vanishing theorem for q -ample divisors (see Definition 4.1) on compact Kähler manifolds.

Proposition 0.6 (=Proposition 4.2). *Let D be a smooth divisor and it is the support of a q -ample effective divisor. If L is a holomorphic line bundle satisfies $L^N = \mathcal{O}_X(D)$ for some $N \geq 0$, then*

$$H^j(X, \Omega_X^i(\log D) \otimes L^{-k}) = 0 \quad \text{for } i + j \geq n + q + 1, \quad 0 \leq k \leq N - 1.$$

The following logarithmic vanishing theorem is an application of [17, Theorem 3.1] and [26, Theorem 1.2]. By [17, Theorem 3.1], the cohomology $H^j(X, \Omega_X^i(\log D) \otimes L)$ is isomorphic to the L^2 -cohomology on the open manifold $U := X - D$ with respect to some metric h on $L|_U$ and the Poincaré metric on U . Moreover, the L^2 -cohomology is invariant under some small perturbations of the metric h . On the other hand, if U is simple connected, then one can take a global N -th root of the canonical section of $L^N = \mathcal{O}_X(D)$ on U . Combining these observations with [26, Theorem 1.2] shows that

Proposition 0.7 (=Proposition 4.3). *Let $D = \sum_{i=1}^r D_i$ be a simple normal crossing divisor and D is the support of an effective q -ample divisor, if $L^N = \mathcal{O}_X(\sum_{i=1}^r b_i D_i)$ with $b_i \in (-N, N) \cap \mathbb{Z}$ for some $N \in \mathbb{Z}$, and $X - D$ is simply connected, then*

$$H^j(X, \Omega_X^i(\log D) \otimes L) = 0, \quad \text{for } i + j \geq n + q + 1.$$

From the results [12, Theorem 3.2 (b)] and [21, Section 4.3.3 (iii)], there are more general E_1 -degenerations of spectral sequences and the unobstructed deformations for smooth projective varieties. Both these results would be proved for compact Kähler manifolds if one can solve a more general logarithmic $\bar{\partial}$ -equation. Base on these, it is natural to propose the following conjecture.

Conjecture 0.8. Let X be a compact Kähler manifold and $D = \sum_{i=1}^r D_i$ a simple normal crossing divisor in X . If L is a holomorphic bundle over X with $L^N = \mathcal{O}_X(\sum_{i=1}^r a_i D_i)$, $0 \leq a_i \leq N$ and $a_i \in \mathbb{Z}$, then the following logarithmic $\bar{\partial}$ -equation:

$$(0.3) \quad \bar{\partial}x = \partial\alpha$$

has a solution $x \in A^{0,q-1}(X, \Omega_X^{p+1}(\log D) \otimes L^{-1})$ for any $\alpha \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1})$ with $\bar{\partial}\alpha = 0$.

This article is organized as follows. In Section 1, we will recall some basic definitions and facts on logarithmic connection and cyclic covering, and prove that there is a canonical integrable logarithmic connection along D on L . In Section 2, by using cyclic covering trick, we will solve a logarithmic $\bar{\partial}$ -equation on a compact Kähler manifold associated to a smooth divisor, Theorem 0.1 is proved in this section. In Section 3, we will give some applications to Theorem 0.1 and prove Corollary 0.2, Proposition 0.3, Theorem 0.4 and Theorem 0.5. In Section 4, we will prove two logarithmic vanishing theorems for q -ample divisors on compact Kähler manifolds and prove Proposition 0.6, Proposition 0.7. In Section 5, we make some further discussions and propose Conjecture 0.8 on solving a more general logarithmic $\bar{\partial}$ -equation.

Acknowledgement: The author would like to express his gratitude to Professor Kefeng Liu for suggesting related problems and many insightful discussions on using cyclic covering trick to study the geometry of logarithmic forms, and thank Professors Sheng Rao and Xiaokui Yang for many helpful discussions.

1. PRELIMINARIES

In this section, we will recall some basic definitions and facts on logarithmic connection and cyclic covering. For more details one may refer to [2, 3, 11, 12, 16, 19, 23, 28].

1.1. Logarithmic connection. Let X be a compact complex manifold of dimension n and D be a simple normal crossing divisor in X , i.e., $D = \sum_{i=1}^r D_i$, where the $D_i, 1 \leq i \leq r$ are distinct smooth hypersurfaces intersecting transversely in X .

Denote by $\tau : U := X - D \rightarrow X$ the natural inclusion and

$$\Omega_X^p(*D) = \lim_{\substack{\rightarrow \\ \nu}} \Omega_X^p(\nu \cdot D) = \tau_* \Omega_U^p.$$

Then $(\Omega_X^\bullet(*D), d)$ is a complex. The sheaf of logarithmic forms

$$\Omega_X^p(\log D)$$

(introduced by Deligne in [7]) is defined as the subsheaf of $\Omega_X^p(*D)$ with logarithmic poles along D , i.e., for any open subset $V \subset X$,

$$\Gamma(V, \Omega_X^p(\log D)) = \{\alpha \in \Gamma(V, \Omega_X^p(*D)) : \alpha \text{ and } d\alpha \text{ have simple poles along } D\}.$$

From ([8, II, 3.1-3.7] or [12, Properties 2.2]), the log complex $(\Omega_X^\bullet(\log D), d)$ is a subcomplex of $(\Omega_X^\bullet(*D), d)$ and $\Omega_X^p(\log D)$ is locally free,

$$\Omega_X^p(\log D) = \wedge^p \Omega_X^1(\log D).$$

For any $z \in X$, which k of these D_i pass, we may choose local holomorphic coordinates $\{z^1, \dots, z^n\}$ in a small neighborhood U of $z = (0, \dots, 0)$ such that

$$D \cap U = \{z^1 \cdots z^k = 0\}$$

is the union of coordinates hyperplanes. Such a pair

$$(U, \{z^1, \dots, z^n\})$$

is called a *logarithmic coordinate system* [22, Definition 1]. Then $\Omega_X^p(\log D)$ is generated by the holomorphic forms and logarithmic differentials dz^i/z^i ($i = 1, \dots, k$), i.e.,

$$\Omega_X^p(\log D) = \Omega_X^p \left\{ \frac{dz^1}{z^1}, \dots, \frac{dz^k}{z^k} \right\}.$$

Denote by

$$A^{0,q}(X, \Omega_X^p(\log D))$$

the space of smooth $(0, q)$ -forms on X with values in $\Omega_X^p(\log D)$, and call an element of $A^{0,q}(X, \Omega_X^p(\log D))$ a *logarithmic (p, q) -form*.

Now we recall the definition of logarithmic connection on a locally free coherent sheaf.

Definition 1.1 ([12, Definition 2.4]). Let \mathcal{E} be a locally free coherent sheaf on X and let

$$\nabla : \mathcal{E} \rightarrow \Omega_X^1(\log D) \otimes \mathcal{E}$$

be a \mathbb{C} -linear map satisfying

$$(1.1) \quad \nabla(f \cdot e) = f \cdot \nabla e + df \otimes e.$$

One defines

$$\nabla_a : \Omega_X^a(\log D) \otimes \mathcal{E} \rightarrow \Omega_X^{a+1}(\log D) \otimes \mathcal{E}$$

by the rule

$$\nabla_a(\omega \otimes e) = d\omega \otimes e + (-1)^a \omega \wedge \nabla e.$$

We assume that $\nabla_{a+1} \circ \nabla_a = 0$ for all a . Such ∇ will be called an *integrable logarithmic connection along D* , or just a connection. The complex

$$(\Omega_X^\bullet(\log D) \otimes \mathcal{E}, \nabla_\bullet)$$

is called the *logarithmic de Rham complex* of (\mathcal{E}, ∇) .

Let L be a holomorphic line bundle over X satisfying

$$L^N = \mathcal{O}_X \left(\sum_{i=1}^r a_i D_i \right)$$

for some $a_i \in \mathbb{Z}$, $1 \leq i \leq r$. Then

Proposition 1.2. *There exists an integrable logarithmic connection along D on L .*

Proof. Let σ be the canonical meromorphic section of $\mathcal{O}_X(\sum_{i=1}^r a_i D_i)$, and let e be a local frame of L satisfying

$$(1.2) \quad e^N = \prod_{i=1}^r (z^i)^{-a_i} \sigma.$$

We define

$$(1.3) \quad \partial e := \frac{1}{N} \partial \log \frac{e^N}{\sigma} e = - \sum_{i=1}^r \frac{a_i}{N} \frac{dz^i}{z^i} e,$$

which is taken valued in $\Omega_X^1(\log D) \otimes L$. It is well-defined since for any local frame e' of L with $e'^N = e^N$, one has $\partial(e'/e) = 0$ so

$$\partial e' = - \sum_{i=1}^r \frac{a_i}{N} \frac{dz^i}{z^i} e' = \frac{\partial e}{e} e' = \frac{e'}{e} \partial e = \partial \left(\frac{e'}{e} \right) \otimes e + \frac{e'}{e} \partial e,$$

which satisfies (1.1). And we define a logarithmic connection d on L by

$$d : L \rightarrow \Omega_X^1(\log D) \otimes L \quad d(f \cdot e) := f \cdot \partial e + df \otimes e.$$

It induces a logarithmic connection on $\Omega_X^p(\log D) \otimes L$ by

$$d(\omega \otimes e) = d\omega \otimes e + (-1)^p \omega \wedge \partial e = d\omega \otimes e + (-1)^p \omega \wedge \left(- \sum_{i=1}^r \frac{a_i}{N} \frac{dz^i}{z^i} \right) \otimes e.$$

By a direct calculation, one has $d^2 = 0$. Therefore, d is an integrable logarithmic connection along D on L . \square

1.2. Cyclic covering. In this subsection, let L be a holomorphic line bundle over X satisfies

$$L^N = \mathcal{O}_X \left(\sum_{i=1}^r a_i D_i \right),$$

where $a_i \geq 0$ and $a_i \in \mathbb{Z}$. Let s be the canonical section of $\mathcal{O}_X(\sum_{i=1}^r a_i D_i)$, and denote by \mathbb{L} the total space of line bundle L . Let $\pi : \mathbb{L} \rightarrow X$ be the bundle projection. If $v \in \Gamma(\mathbb{L}, \pi^* L)$ is the tautological section, then the zero divisor of $\pi^* s - v^N$ defines an analytic subspace, say $X[\sqrt[N]{s}]$, in \mathbb{L} . We denote by $\overline{X}[\sqrt[N]{s}]$ the normalization of $X[\sqrt[N]{s}]$, and $\overline{\pi} : \overline{X}[\sqrt[N]{s}] \rightarrow X$. We will call $\overline{X}[\sqrt[N]{s}]$ the *cyclic cover obtained by taking the N -th root out of s* . The map $\overline{\pi}$ is flat and finite morphism, $\overline{X}[\sqrt[N]{s}]$ is a normal variety and

$$\overline{\pi}_* \mathcal{O}_{\overline{X}[\sqrt[N]{s}]} = \bigoplus_{i=0}^{N-1} L^{-i} \left(\left[\frac{i}{N} \sum_{j=1}^r a_j D_j \right] \right),$$

(see e.g. [19, Section 2.11] or [12, Corollary 3.11]).

Let e be a local frame of L , any element of L can be represented as the form $v \cdot e$, so as a complex manifold, the local coordinate of \mathbb{L} is given by (z, v) . In terms of local coordinates, then

$$X[\sqrt[N]{s}] := \{(z, v) \in \mathbb{L} \mid v^N - s(z) = 0\},$$

where $s = s(z)e$. Therefore, $(z, v) \in \mathbb{L}$ is a singular point of $X[\sqrt[N]{s}]$ if and only if

$$\nabla(v^N - s(z)) = (Nv^{N-1}, \nabla s(z)) = 0,$$

which is equivalent to $z \in \text{Sing}(\sum_{i=1}^r a_i D_i)$.

If $\sum_{i=1}^r a_i D_i$ is reduced and smooth, then $\text{Sing}(\sum_{i=1}^r a_i D_i) = \emptyset$ so $X[\sqrt[N]{s}]$ is smooth. In this case, $X[\sqrt[N]{s}] = \overline{X}[\sqrt[N]{s}]$.

Proposition 1.3 ([3, Lemma 17.1, 17.2], [18, Page 25]). *Let $\pi : X[\sqrt[N]{s}] \rightarrow X$ be the N -cyclic covering of X branched along a smooth divisor D and determined by L , where $L^N = \mathcal{O}_X(D)$ and let D' be the reduced divisor $\pi^{-1}(D)$ on $X[\sqrt[N]{s}]$, then*

- (i) $\mathcal{O}_{X[\sqrt[N]{s}]}(D') = \pi^* L$;
- (ii) $\pi^* D = N D'$;
- (iii) $K_{X[\sqrt[N]{s}]} = \pi^*(K_X \otimes L^{N-1})$;
- (iv) $\pi_* \mathcal{O}_X = \bigoplus_{i=0}^{N-1} L^{-i}$;
- (v) $\pi^* \Omega_X^\bullet(\log D) = \Omega_{X[\sqrt[N]{s}]}^\bullet(\log(\pi^* D))$.

2. LOGARITHMIC $\bar{\partial}$ -EQUATION

In this section, by using the cyclic covering trick, we will solve a logarithmic $\bar{\partial}$ -equation on a compact Kähler manifold associated to a smooth divisor.

2.1. Logarithmic $\bar{\partial}$ -equation. Denote by

$$A^{0,q}(X, \Omega_X^p(\log D) \otimes L)$$

the space of smooth $(0, q)$ -forms with valued in $\Omega_X^p(\log D) \otimes L$. From Proposition 1.2, if $L^N = \mathcal{O}_X(\sum_{i=1}^r a_i D_i)$, then there is an integral logarithmic connection d along D on L . So there is a logarithmic de Rham complex of (L, d) ,

$$(\Omega_X^\bullet(\log D) \otimes L, d).$$

By taking $d = \partial + \bar{\partial}$ and using Proposition 1.2, we have

$$\partial : A^{0,q}(X, \Omega_X^p(\log D) \otimes L) \rightarrow A^{0,q}(X, \Omega_X^{p+1}(\log D) \otimes L)$$

and

$$\bar{\partial} : A^{0,q}(X, \Omega_X^p(\log D) \otimes L) \rightarrow A^{0,q+1}(X, \Omega_X^p(\log D) \otimes L).$$

For any $\alpha \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L)$ with $\bar{\partial}\alpha = 0$, one may ask that whether the following logarithmic $\bar{\partial}$ -equation

$$(2.1) \quad \bar{\partial}x = \partial\alpha$$

has a solution $x \in A^{0,q-1}(X, \Omega_X^{p+1}(\log D) \otimes L)$.

Remark 2.1. For the case that L is trivial or $L = \mathcal{O}_X(-D)$, and X is a compact Kähler manifold, the logarithmic $\bar{\partial}$ -equation (2.1) is solvable [23, Theorem 0.1, Theorem 0.2].

In this section, let X be a compact Kähler manifold and $D = \sum_{i=1}^r D_i$ be a smooth divisor, i.e. $D_i \cap D_j = \emptyset$ for $i \neq j$. Let L be a holomorphic line bundle over X satisfying $L^N = \mathcal{O}_X(D)$, and we consider the following logarithmic $\bar{\partial}$ -equation:

$$(2.2) \quad \bar{\partial}x = \partial\alpha$$

for any $\alpha \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1})$ with $\bar{\partial}\partial\alpha = 0$.

2.2. Solving the $\bar{\partial}$ -equation. In this subsection, we will solve the logarithmic $\bar{\partial}$ -equation (2.2) by using the cyclic covering trick.

Since $D = \sum_{i=1}^r D_i$ is a smooth divisor in X and $L^N = \mathcal{O}_X(D)$, from the discussions in Subsection 1.2, $Y := X[\sqrt[N]{s}]$ is a compact complex submanifold of \mathbb{L} . Let $\pi : Y \rightarrow X$ be the cyclic covering. For any point $x \in D$, one can take a small neighborhood U around x , such that

$$U \cap D = \{z^1 \cdots z^k = 0\}.$$

Since D is smooth, by take U small sufficiently, so $k = 1$. Without loss of generality, for any $x \in D$, one may take a local coordinate neighborhood around x such that

$$\pi : (w^1, \dots, w^n) \mapsto (z^1 = (w^1)^N, z^2 = w^2, \dots, z^n = w^n).$$

For any $\alpha \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1})$, locally

$$\alpha = f \frac{dz^1}{z^1} \wedge dz^2 \wedge \cdots \wedge dz^p \otimes e^*$$

where f is a local smooth $(0, q)$ -form and e^* is a local frame of L^{-1} , then

$$(2.3) \quad \begin{aligned} \pi^*\alpha &= \pi^* f N \frac{dw^1}{w^1} \wedge dw^2 \wedge \cdots \wedge dw^p \otimes \pi^* e^* \in A^{0,q}(Y, \Omega_Y^p(\log(\pi^*D)) \otimes \pi^*L^{-1}) \\ &\cong A^{0,q}(Y, \Omega_Y^p(\log D') \otimes \mathcal{O}_Y(-D')), \end{aligned}$$

where the isomorphism is followed from Proposition 1.3 (i), which is given by dividing the canonical section of $\mathcal{O}_X(-D')$.

Firstly, we note that

$$(2.4) \quad A^{0,q}(Y, \Omega_Y^p(\log D') \otimes \mathcal{O}_Y(-D')) \cong A^{n,q}(Y, T_Y^{n-p}(-\log D')) \hookrightarrow A^{p,q}(Y).$$

where $T_Y^{n-p}(-\log D') := \wedge^{n-p} (\Omega_Y^1(\log D'))^*$ and the last embedding is given by [23, Page 17-18]. In fact, the first isomorphism is given by the following: for any $\beta \in A^{0,q}(Y, \Omega_Y^p(\log D') \otimes$

$\mathcal{O}_Y(-D')$), in terms of local coordinates, one has

$$\begin{aligned} \beta &= g \frac{dw^1}{w^1} \wedge dw^2 \wedge \cdots \wedge dw^p w^1 \\ &= g dw^1 \wedge \cdots \wedge dw^p \\ &\simeq g dw^1 \wedge \cdots \wedge dw^n \otimes \left(\frac{\partial}{\partial w^{p+1}} \otimes \cdots \otimes \frac{\partial}{\partial w^r} \otimes \frac{\partial}{\partial w^{r+1}} \otimes \cdots \otimes \frac{\partial}{\partial w^n} \right) \\ &\in A^{n,q}(Y, T_Y^{n-p}(-\log D')). \end{aligned}$$

Here g is a local smooth $(0, q)$ -form.

Let σ' be the canonical meromorphic section of $\mathcal{O}_Y(-D')$, then

$$\begin{aligned} (2.5) \quad \partial(\pi^* e^*) &:= \partial \log \frac{\pi^* e^*}{\sigma'} \otimes \pi^* e^* \\ &= \frac{1}{N} \partial \log \frac{\pi^* e^{*N}}{\sigma'^N} \otimes \pi^* e^* \\ &= \pi^*(\partial e^*), \end{aligned}$$

where the last equality holds by (1.3) and Proposition 1.3 (ii).

For any $\alpha \in A^{0,q}(X, \Omega_X^{p-1}(\log D) \otimes L^{-1})$ with $\bar{\partial}\alpha = 0$, by (2.5), then

$$\bar{\partial}\partial((\sigma')^{-1}\pi^*\alpha) = \bar{\partial}((\sigma')^{-1}\partial(\pi^*\alpha)) = (\sigma')^{-1}\pi^*(\bar{\partial}\alpha) = 0.$$

From (2.3) and (2.4), we have

$$(\sigma')^{-1}\pi^*\alpha \in A^{n,q}(Y, T_Y^{n-p}(-\log D')) \hookrightarrow A^{p,q}(Y).$$

Proposition 2.2. *If (X, ω) is a compact Kähler manifold, then Y is also a compact Kähler manifold.*

Proof. By the construction of cyclic covering, Y is a compact complex submanifold of the total space \mathbb{L} . Let h be a Hermitian metric on L , since Y is compact, we may assume that

$$(2.6) \quad Y \hookrightarrow Y_R := \{ve \in L, \|v\|_h^2 = h|v|^2 < R\} \subset \mathbb{L},$$

for some large $R > 0$. By a direct calculation, one has

$$\sqrt{-1}\partial\bar{\partial}\|v\|_h^2 = \sqrt{-1}(\partial\bar{\partial}\log h)\|v\|_h^2 + \sqrt{-1}h\delta v \wedge \delta\bar{v},$$

where $\delta v := dv + v\partial\log h$. When restricting to the open manifold Y_R , one has

$$\sqrt{-1}\partial\bar{\partial}\|v\|_h^2 \geq -CR\pi^*\omega + \sqrt{-1}h\delta v \wedge \delta\bar{v},$$

where $C > 0$ is a uniform constant satisfying $\sqrt{-1}\partial\bar{\partial}\log h > -C\omega$, and $\pi : Y_R(\subset \mathbb{L}) \rightarrow X$ is the natural projection. Thus, $C_1\pi^*\omega + \sqrt{-1}\partial\bar{\partial}\|v\|_h^2$ is a Kähler metric on Y_R for any $C_1 > CR$. From (2.6) and by the restriction, one obtains a Kähler metric on Y . \square

From [23, Theorem 0.2], one can solve the equation $\bar{\partial}x = \partial((\sigma')^{-1}\pi^*\alpha)$ with

$$(2.7) \quad x = \mathcal{I}^* \bar{\partial}_E^* \mathbb{G}_E''(\mathcal{I}^*)^{-1} \partial((\sigma')^{-1}\pi^*\alpha) \in A^{n,q-1}(Y, T_Y^{n-p-1}(-\log D')).$$

Here $(\mathcal{I}^*)^{-1} : A^{n,q}(Y, T_Y^{n-p-1}(-\log D')) \rightarrow A^{n,q}(Y, (E_Y^{n-p-1})^*)$ is the canonical isomorphism, E_Y^{n-p-1} is the holomorphic vector bundle associated to the local free sheaf $\Omega_Y^{n-p-1}(\log D')$.

Now we define the following isomorphism by

$$(\mathcal{I}^*)^{-1} : A^{0,q}(X, \Omega_X^{p+1}(\log D) \otimes L^{-1}) \rightarrow A^{n,q}(X, (E_X^{n-p-1})^* \otimes L^{N-1})$$

$$\alpha = f \frac{dz^1}{z^1} \wedge dz^2 \wedge \cdots \wedge dz^{p+1} \otimes e^* \mapsto f dz \otimes (e_{p+2} \otimes \cdots \otimes e_n) \otimes (e^*)^{1-N},$$

where $\{e_{i_1} \otimes \cdots \otimes e_{i_{n-p-1}}\}$ is a basis of $(E_X^{n-p-1})^*$. The definition is well-defined since

$$\begin{aligned} \alpha &= f \frac{dz^1}{z^1} \wedge dz^2 \wedge \cdots \wedge dz^{p+1} \otimes e^* \\ &\simeq f \frac{dz}{z^1} \otimes \left(\frac{\partial}{z^{p+2}} \otimes \cdots \otimes \frac{\partial}{\partial z^n} \right) \otimes e^* \\ &= f \frac{dz}{e^N/\sigma} \otimes \left(\frac{\partial}{z^{p+2}} \otimes \cdots \otimes \frac{\partial}{\partial z^n} \right) \otimes e^* \\ &= f dz \otimes \left(\frac{\partial}{z^{p+2}} \otimes \cdots \otimes \frac{\partial}{\partial z^n} \right) \otimes (e^*)^{1-N} \otimes \sigma \\ &\simeq f dz \otimes (e_{p+2} \otimes \cdots \otimes e_n) \otimes (e^*)^{1-N} \otimes \sigma \\ &\simeq f dz \otimes (e_{p+2} \otimes \cdots \otimes e_n) \otimes (e^*)^{1-N}. \end{aligned}$$

Here we denote $dz := dz^1 \wedge \cdots \wedge dz^n$. So one has the following lemma.

Lemma 2.3. *It holds the following commutative diagram:*

$$\begin{array}{ccc} A^{n,q}(Y, T_Y^{n-p-1}(-\log D')) & \xrightarrow{(\mathcal{I}^*)^{-1}} & A^{n,q}(Y, (E_Y^{n-p-1})^*) \\ \uparrow (\sigma')^{-1} \cdot \pi^* & & \uparrow (\sigma')^{N-1} \cdot \pi^* \\ A^{0,q}(X, \Omega_X^{p+1}(\log D) \otimes L^{-1}) & \xrightarrow{(\mathcal{I}^*)^{-1}} & A^{n,q}(X, (E_X^{n-p-1})^* \otimes L^{N-1}) \end{array}$$

Proof. One can check the above commutative diagram directly. For any

$$\alpha = f \frac{dz^1}{z^1} \wedge dz^2 \wedge \cdots \wedge dz^{p+1} \otimes e^* \in A^{0,q}(X, \Omega_X^{p+1}(\log D) \otimes L^{-1}),$$

one has

$$\begin{aligned} (\sigma')^{N-1} \pi^* ((\mathcal{I}^*)^{-1}(\alpha)) &= (\sigma')^{N-1} \pi^* (f dz \otimes (e_{p+2} \otimes \cdots \otimes e_n) \otimes (e^*)^{1-N}) \\ &= N \pi^* f \wedge dw \otimes (\tilde{e}_{p+2} \otimes \cdots \otimes \tilde{e}_n) (\pi^* e^* / (w_1 \sigma'))^{1-N} \\ &= N \pi^* f \wedge dw \otimes (\tilde{e}_{p+2} \otimes \cdots \otimes \tilde{e}_n) (\pi^* e^* / (w_1 \sigma')), \end{aligned}$$

where $\tilde{e}_i := \pi^* e_i$, $dw = dw^1 \wedge \cdots \wedge dw^n$ and the third equality holds since one can take the local frame e^* satisfies $(\pi^* e^* / (w_1 \sigma'))^N = 1$ as (1.2). While

$$\begin{aligned} (\mathcal{I}^*)^{-1} (\sigma')^{-1} \pi^* (\alpha) &= (\mathcal{I}^*)^{-1} (\sigma')^{-1} \left(N \pi^* f \frac{dw^1}{w^1} \wedge dw^2 \wedge \cdots \wedge dw^p \otimes \pi^* e^* \right) \\ &= N \pi^* f \wedge dw \otimes (\tilde{e}_{p+2} \otimes \cdots \otimes \tilde{e}_n) (\pi^* e^* / (w_1 \sigma')) \\ &= (\sigma')^{N-1} \pi^* ((\mathcal{I}^*)^{-1}(\alpha)). \end{aligned}$$

□

From (2.7) and Lemma 2.3, one has

$$(\mathcal{I}^*)^{-1}x = \bar{\partial}_E^* \mathbb{G}_E'' (\mathcal{I}^*)^{-1} \partial((\sigma')^{-1} \pi^* \alpha),$$

which is a solution of $\bar{\partial}y = (\mathcal{I}^*)^{-1}(\sigma')^{-1} \pi^*(\partial\alpha) = (\sigma')^{N-1} \pi^*((\mathcal{I}^*)^{-1} \partial\alpha) = (\sigma')^{N-1} \pi^* \beta$, where $\beta = (\mathcal{I}^*)^{-1} \partial\alpha \in A^{n,q}(X, (E_X^{n-p-1})^* \otimes L^{N-1})$. For any smooth Hermitian metric h on the vector space $(E_X^{n-p-1})^* \otimes L^{N-1}$ and any harmonic element $\eta \in A^{n,q}(X, (E_X^{n-p-1})^* \otimes L^{N-1})$, i.e. $\bar{\partial}\eta = 0 = \bar{\partial} *' \eta$, where $*'$ is the Hodge $*$ -operator. Then

$$\begin{aligned} \langle \beta, \eta \rangle &= \int_X \beta \wedge *' \eta \\ &= \frac{1}{N} \int_Y \pi^*(\beta \wedge *' \eta) \\ (2.8) \quad &= \frac{1}{N} \int_Y (\sigma')^{1-N} \bar{\partial}((\mathcal{I}^*)^{-1}x) \wedge \pi^*(*' \eta) \\ &= \frac{1}{N} (-1)^{n+q} \int_Y (\sigma')^{1-N} (\mathcal{I}^*)^{-1}x \wedge \pi^* \bar{\partial}(*' \eta) = 0, \end{aligned}$$

where the second equality follows from [36, Lemma 2.2]. It follows that

$$(2.9) \quad \mathbb{H}(\beta) = 0.$$

By the Hodge theorem for bundle-valued and noting $\bar{\partial}\beta = 0$, we have

$$\beta = \bar{\partial} \bar{\partial}^* \mathbb{G} \beta + \bar{\partial}^* \bar{\partial} \mathbb{G} \beta + \mathbb{H}(\beta) = \bar{\partial} \bar{\partial}^* \mathbb{G} \beta = \bar{\partial} \bar{\partial}^* \mathbb{G} (\mathcal{I}^*)^{-1} \partial\alpha,$$

which is equivalent to

$$(\mathcal{I}^*)^{-1} (\bar{\partial}(\mathcal{I}^* \bar{\partial}^* \mathbb{G} (\mathcal{I}^*)^{-1} \partial\alpha) - \partial\alpha) = 0.$$

Since $(\mathcal{I}^*)^{-1}$ is an isomorphism, so we obtain a solution

$$(2.10) \quad x = \mathcal{I}^* \bar{\partial}^* \mathbb{G} (\mathcal{I}^*)^{-1} \partial\alpha \in A^{0,q-1}(X, \Omega_X^{p+1}(\log D) \otimes L^{-1})$$

of the equation $\bar{\partial}x = \partial\alpha$. In one word, we obtain

Theorem 2.4. *Let X be a compact Kähler manifold and $D = \sum_{i=1}^r D_i$ be a smooth divisor in X , let L be a holomorphic line bundle over X with $L^N = \mathcal{O}_X(D)$. For any $\alpha \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1})$, the following equation*

$$(2.11) \quad \bar{\partial}x = \partial\alpha$$

has a solution $x \in A^{0,q-1}(X, \Omega_X^{p+1}(\log D) \otimes L^{-1})$.

3. SOME APPLICATIONS

In this section, we will give some applications to Theorem 2.4. Throughout this section, let X be a compact Kähler manifold of dimension n and $D = \sum_{i=1}^r D_i$ be a smooth divisor, let L be a holomorphic line bundle over X with $L^N = \mathcal{O}_X(D)$.

3.1. Closedness of logarithmic forms. The theory of logarithmic forms has been playing very important roles in various aspects of analytic-algebraic geometry, in which the understanding of closedness of logarithmic forms is fundamental. In 1971, Deligne [9, (3.2.14)] proved the d -closedness of logarithmic forms on a smooth complex quasi-projective variety by showing the degeneration of logarithmic Hodge to de Rham spectral sequence. In 1995, Noguchi [29] gave a short proof of this result. In [23, Corollary 0.3], K. Liu, S. Rao and the author generalized Deligne's result and obtained: if $\alpha \in A^{0,0}(X, \Omega_X^p(\log D))$ with $\bar{\partial}\alpha = 0$ then $\partial\alpha = 0$.

As the first application of Theorem 2.4, we obtain

Corollary 3.1. *If $\alpha \in A^{0,0}(X, \Omega_X^p(\log D) \otimes L^{-1})$ with $\bar{\partial}\alpha = 0$, then $\partial\alpha = 0$.*

Proof. By Theorem 2.4, there exists a solution

$$x \in A^{0,-1}(X, \Omega_X^p(\log D) \otimes L^{-1}) = \{0\}$$

such that $\bar{\partial}x = \alpha$ and then

$$\partial\alpha = \bar{\partial}x = 0$$

since $x = 0$. □

3.2. An injectivity theorem. In this subsection, we will prove an injectivity theorem by using Theorem 2.4.

From Proposition 1.2, d is an integrable logarithmic connection along D on L^{-1} , and

$$(\Omega_X^\bullet(\log D) \otimes L^{-1}, d)$$

is a logarithmic de Rham complex of (L^{-1}, d) . Let

$$H^k(X, \Omega_X^\bullet(\log D) \otimes L^{-1})$$

denote the cohomology of the complex of sections $(\Gamma(X, \Omega_X^\bullet(\log D) \otimes L^{-1}), d)$ of $\Omega_X^\bullet(\log D) \otimes L^{-1}$.

Proposition 3.2. *The following mapping is injective:*

$$\iota : H^q(X, \Omega_X^n(\log D) \otimes L^{-1}) \rightarrow H^{q+n}(X, \Omega_X^\bullet(\log D) \otimes L^{-1}), \quad [\alpha] \mapsto \iota([\alpha]) := [\alpha]_d.$$

Proof. For any $[\alpha] \in H^q(X, \Omega_X^n(\log D) \otimes L^{-1})$, then $\bar{\partial}\alpha = 0$. By considering the degree of α , so

$$d\alpha = \partial\alpha + \bar{\partial}\alpha = \bar{\partial}\alpha = 0.$$

It follows that $\iota([\alpha]) = [\alpha]_d \in H^{q+n}(X, \Omega_X^\bullet(\log D) \otimes L^{-1})$. If $\iota([\alpha]) = [\alpha]_d = 0$, then there exists a logarithmic form $\beta \in A^{q+n-1}(X, \Omega_X^\bullet(\log D) \otimes L^{-1})$ such that $\alpha = d\beta$. Therefore, the components $\beta_{n-1,q} \in A^{0,q}(X, \Omega_X^{n-1}(\log D) \otimes L^{-1})$ and $\beta_{n,q-1} \in A^{0,q-1}(X, \Omega_X^n(\log D) \otimes L^{-1})$ of β satisfying

$$(3.1) \quad \alpha = \partial\beta_{n-1,q} + \bar{\partial}\beta_{n,q-1}.$$

Notice that $\bar{\partial}\beta_{n-1,q} = \bar{\partial}\alpha = 0$, by Theorem 2.4, there exists $\gamma \in A^{0,q-1}(X, \Omega_X^n(\log D) \otimes L^{-1})$ such that

$$(3.2) \quad \partial\beta_{n-1,q} = \bar{\partial}\gamma.$$

Combining (3.1) with (3.2), one has

$$\alpha = \bar{\partial}(\gamma + \beta_{n,q-1}),$$

which implies that $[\alpha] = 0 \in H^q(X, \Omega_X^n(\log D) \otimes L^{-1})$. So we get the injectivity of ι . \square

Remark 3.3. For the case that L is trivial and $D = \sum_{i=1}^r D_i$ is a simple normal crossing divisor, Proposition 3.2 was proved in [23, Corollary 3.6]. More precisely, we proved that the restriction homomorphism

$$H^q(X, \Omega_X^n(\log D)) \rightarrow H^q(X - D, K_{X-D})$$

is injective, which was first proved in [1, Theorem 2.1] by using algebraic method.

3.3. Degeneration of spectral sequences. In this subsection, as an application of Theorem 2.4, we will prove the following E_1 -degeneration of spectral sequences.

Theorem 3.4. *The spectral sequence*

$$(3.3) \quad E_1^{p,q} = H^q(X, \Omega_X^p(\log D) \otimes L^{-1}) \implies \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D) \otimes L^{-1})$$

associated to the logarithmic de Rham complex

$$(\Omega_X^\bullet(\log D) \otimes L^{-1}, d)$$

degenerates in E_1 . Here $\mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D) \otimes L^{-1})$ denotes the hypercohomology.

Proof. The proof needs a logarithmic analogue of the general description on the terms in the Frölicher spectral sequence as in [5, Theorems 1 and 3]. By Dolbeault isomorphism theorem, one has

$$H^q(X, \Omega_X^p(\log D) \otimes L^{-1}) \cong H_{\bar{\partial}}^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1}) := \frac{\text{Ker}(\bar{\partial}) \cap A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1})}{\bar{\partial}A^{0,q-1}(X, \Omega_X^p(\log D) \otimes L^{-1})}.$$

By the definition of spectral sequences

$$E_r^{p,q} \cong Z_r^{p,q} / B_r^{p,q},$$

where $Z_r^{p,q}$ lies between the $\bar{\partial}$ -closed and d -closed logarithmic (p, q) -forms and $B_r^{p,q}$ lies between the $\bar{\partial}$ -exact and d -exact logarithmic (p, q) -forms in some senses. Actually,

$$Z_1^{p,q} = \{\alpha \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1}) \mid \bar{\partial}\alpha = 0\},$$

$$B_1^{p,q} = \{\alpha \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1}) \mid \alpha = \bar{\partial}\beta, \beta \in A^{0,q-1}(X, \Omega_X^p(\log D) \otimes L^{-1})\}.$$

For $r \geq 2$,

(3.4)

$$Z_r^{p,q} = \{\alpha_{p,q} \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1}) \mid \bar{\partial}\alpha_{p,q} = 0, \text{ and there exist}$$

$$\alpha_{p+i,q-i} \in A^{0,q-i}(X, \Omega_X^{p+i}(\log D) \otimes L^{-1})$$

$$\text{such that } \partial\alpha_{p+i-1,q-i+1} + \bar{\partial}\alpha_{p+i,q-i} = 0, 1 \leq i \leq r-1\},$$

$$B_r^{p,q} = \{\partial\beta_{p-1,q} + \bar{\partial}\beta_{p,q-1} \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1}) \mid \text{there exist}$$

(3.5)

$$\beta_{p-i,q+i-1} \in A^{0,q+i-1}(X, \Omega_X^{p-i}(\log D) \otimes L^{-1}), 2 \leq i \leq r-1,$$

$$\text{such that } \partial\beta_{p-i,q+i-1} + \bar{\partial}\beta_{p-i+1,q+i-2} = 0, \bar{\partial}\beta_{p-r+1,q+r-2} = 0\},$$

and the map $d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$ is given by

$$d_r[\alpha_{p,q}] = [\partial\alpha_{p+r-1,q-r+1}],$$

where $[\alpha_{p,q}] \in E_r^{p,q}$ and $\alpha_{p+r-1,q-r+1}$ appears in (3.4). Hence, a direct and exact application of Theorem 2.4 implies

$$d_i = 0, \quad \forall i \geq 1,$$

which is indeed the desired degeneration. \square

Remark 3.5. For the case that X is projective and D is simple normal crossing divisor, the above theorem is proved in [12, Theorem 3.2 (b)].

3.4. Logarithmic deformation. In this subsection, we will discuss the logarithmic deformation by using an iterative method originally from [34, 35, 25] and developed in [24, 37, 38, 32, 30, 31, 23, 27].

For the definition of logarithmic definition, one can refer to [22, Definition 3]. Let $T_X(-\log D)$ be the dual sheaf of $\Omega_X^1(\log D)$. Then the set of infinitesimal logarithmic deformations is the space $H^1(X, T_X(-\log D))$. Moreover, as shown in [22, Page 251], the semi-universal family [22, Definition 5] can be obtain from a subspace of

$$\Gamma_{\text{real analytic}}(X, T_X(-\log D) \otimes \Lambda^{0,1}T^*X),$$

which, usually called the space of *Beltrami differentials*, consists of sections satisfying the integrability condition:

$$(3.6) \quad \bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi].$$

Suppose that D is a smooth divisor with $D \in |-2K_X|$, and for any $[\varphi_1] \in H^{0,1}(X, T_X(-\log D))$ and any t in a small ϵ -disk Δ_ϵ of 0 in $\mathbb{C}^{\dim_{\mathbb{C}} H^{0,1}(X, T_X(-\log D))}$, we try to construct a holomorphic family

$$\varphi := \varphi(t) \in A^{0,1}(X, T_X(-\log D))$$

satisfying the following integrability and initial conditions:

$$(3.7) \quad \bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi], \quad \frac{\partial\varphi}{\partial t}(0) = \varphi_1.$$

To solve the above equation, we need the following lemma, and we will omit its proof because it is same as to [23, Lemma 4.9].

Lemma 3.6. *Let $\Omega' \in A^{0,0}(X, \Omega_X^n(\log D) \otimes K_X)$ be a logarithmic $(n,0)$ -form without zero points. Then*

$$\bullet \lrcorner \Omega' : A^{0,1}(X, T_X(-\log D)) \rightarrow A^{0,1}(X, \Omega_X^{n-1}(\log D) \otimes K_X)$$

is an isomorphism, whose inverse we denote by

$$\Omega'^* \lrcorner \bullet : A^{0,1}(X, \Omega_X^{n-1}(\log D) \otimes K_X) \rightarrow A^{0,1}(X, T_X(-\log D)).$$

By assumption, $D \in |-2K_X|$, so $\Omega_X^n(\log D) \otimes K_X \cong \mathcal{O}_X(D + 2K_X)$ is trivial, one may take an element

$$(3.8) \quad \Omega \in H^0(X, \Omega_X^n(\log D) \otimes K_X)$$

without zero points.

Proposition 3.7. *If there are two smooth families*

$$\varphi(t) \in A^{0,1}(X, T_X(-\log D))$$

and

$$\Omega(t) \in A^{0,0}(X, \Omega_X^n(\log D) \otimes K_X)$$

satisfying the system of equations

$$(3.9) \quad \begin{cases} (\bar{\partial} + \frac{1}{2}\partial \circ i_\varphi)(i_\varphi\Omega(t)) = 0, \\ (\bar{\partial} + \partial \circ i_\varphi)\Omega(t) = 0, \\ \Omega_0 = \Omega, \end{cases}$$

then $\varphi(t)$ satisfies (3.6) for sufficiently small t .

Proof. From (3.9), one has

$$(3.10) \quad \begin{aligned} \bar{\partial}(\varphi \lrcorner \Omega(t)) &= -\frac{1}{2}\partial \circ i_\varphi \circ i_\varphi\Omega(t) \\ &= \frac{1}{2}[\varphi, \varphi] \lrcorner \Omega(t) - i_\varphi \circ \partial \circ i_\varphi\Omega(t) \\ &= \frac{1}{2}[\varphi, \varphi] \lrcorner \Omega(t) + i_\varphi \circ \bar{\partial}\Omega(t). \end{aligned}$$

Therefore,

$$(\bar{\partial}\varphi) \lrcorner \Omega(t) = \bar{\partial}(\varphi \lrcorner \Omega(t)) - i_\varphi \circ \bar{\partial}\Omega(t) = \frac{1}{2}[\varphi, \varphi] \lrcorner \Omega(t).$$

Since $\Omega(t)$ is smooth and $\Omega(0) = \Omega_0 = \Omega$, $\Omega(t) \in A^{0,0}(X, \Omega_X^n(\log D) \otimes K_X)$ also has no zero point for small t . One has

$$\bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi].$$

□

By the above Proposition, our goal is to construct two smooth families $\varphi(t)$ and $\Omega(t)$ satisfy (3.9) and $(\partial\varphi(t)/\partial t)|_{t=0} = \varphi_1$. By taking $L = K_X^{-1}$ and using Theorem 2.4, and the same argument as [23, Page 34-38], one can solve the system of equations (3.9) and obtain

Theorem 3.8. *Let X be a compact Kähler manifold and D a smooth divisor such that $D \in |-2K_X|$. Then, the pair (X, D) has unobstructed deformations. More precisely, for any $[\varphi_1] \in H^{0,1}(X, T_X(-\log D))$, there is a holomorphic family*

$$\varphi(t) \in A^{0,1}(X, T_X(-\log D)),$$

satisfying (3.7).

Remark 3.9. For the case that X is projective and D is a smooth divisor with $D \in |-NK_X|$ for some positive integer N , (X, D) was also proved to have unobstructed deformations [18]. More general, if X is projective, $D = \sum_{i=1}^r D_i$ is a simple normal crossing divisor, and there is a collection of weights $\{a_i\}_{i \in I} \subset [0, 1] \cap \mathbb{Q}$ so that

$$\sum_{i \in I} a_i [D_i] = -K_X \in \text{Pic}(X) \otimes \mathbb{Q},$$

then the pair (X, D) has also unobstructed deformations [21, Section 4.3.3 (iii)].

4. LOGARITHMIC VANISHING THEOREMS

In this section, we will discuss some logarithmic vanishing theorems on q -ample divisors by using cyclic covering trick. Let us first recall the definitions of q -ample line bundles and q -ample divisors.

Definition 4.1. Let X be a compact complex manifold and L be a holomorphic line bundle over X . L is called q -ample if for any coherent sheaf \mathcal{F} on X there exists a positive integer $m_0 = m_0(X, L, \mathcal{F}) > 0$ such that

$$H^j(X, \mathcal{F} \otimes L^m) = 0, \quad \text{for } j > q, \quad m \geq m_0.$$

A divisor D is called q -ample divisor if $\mathcal{O}_X(D)$ is a q -ample line bundle.

Let X be a compact Kähler manifold of dimension n .

Proposition 4.2. *Let D be a smooth divisor and it is the support of a q -ample effective divisor. If L is a holomorphic line bundle satisfies $L^N = \mathcal{O}_X(D)$ for some $N \geq 0$, then*

$$H^j(X, \Omega_X^i(\log D) \otimes L^{-k}) = 0 \quad \text{for } i + j \geq n + q + 1, \quad 0 \leq k \leq N - 1.$$

Proof. Let $\pi : Y \rightarrow X$ be the N -cyclic covering of X branched along D . Since π is a flat and finite morphism, for any $0 \leq k \leq N - 1$, by Proposition 1.2 (iv) and [18, (6)], one has

$$\begin{aligned} (4.1) \quad H^j(X, \Omega_X^i(\log D) \otimes L^{-k}) &\subset H^j(X, \Omega_X^i(\log D) \otimes \bigoplus_{k=0}^{N-1} L^{-k}) \\ &= H^j(X, \pi_*(\Omega_Y^i(\log D'))) \\ &\cong H^j(Y, \Omega_Y^i(\log D')). \end{aligned}$$

Suppose that \tilde{D} is the effective q -ample divisor with $\text{Supp}(\tilde{D}) = D$. Then $\pi^*\tilde{D}$ is also an effective q -ample divisor on Y with $\text{Supp}(\pi^*(\tilde{D})) = D'$. Indeed,

$$\text{Supp}(\pi^*\tilde{D}) = \pi^{-1}(\tilde{D}) = \pi^{-1}(D) = D'.$$

Using the projection formula and the Leray spectral sequence to π [33, Lemma 5.28], we have

$$H^j(Y, (\mathcal{O}_Y(\pi^*\tilde{D}))^{\otimes l} \otimes \mathcal{F}) \cong H^j(X, (\mathcal{O}_X(\tilde{D}))^{\otimes l} \otimes \pi_*\mathcal{F})$$

for any coherent sheaf \mathcal{F} . By Definition 4.1, $\pi^*\tilde{D}$ is also a q -ample divisor on Y . By Proposition 3.7 and [26, Theorem 1.2], we have

$$H^j(Y, \Omega_Y^i(\log D')) = 0, \quad i + j \geq n + q + 1.$$

Combining with (4.1) shows that

$$H^j(X, \Omega_X^i(\log D) \otimes L^{-k}) = 0, \quad i + j \geq n + q + 1, \quad 0 \leq k \leq N - 1.$$

□

The following proposition is an application of [17, Theorem 3.1] and [26, Theorem 1.2]. Let L be a holomorphic line bundle over X .

Proposition 4.3. *Let $D = \sum_{i=1}^r D_i$ be a simple normal crossing divisor and D is the support of an effective q -ample divisor, if $L^N = \mathcal{O}_X(\sum_{i=1}^r b_i D_i)$ with $b_i \in (-N, N) \cap \mathbb{Z}$ for some $N \in \mathbb{Z}$, and $X - D$ is simply connected, then*

$$H^j(X, \Omega_X^i(\log D) \otimes L) = 0, \quad \text{for } i + j \geq n + q + 1.$$

Proof. Let σ_i be the canonical section of $\mathcal{O}_X(D_i)$. By conditions, one has

$$\bigotimes_{i=1}^r \sigma_i^{\otimes b_i} \in H^0(X, L^N),$$

which does not vanishing on $U := X - D$. Moreover, since U is simply connected, so one can select a branch section of $L|_U$, which we denote it by

$$\tilde{\sigma} := \left(\bigotimes_{i=1}^r \sigma_i^{\otimes b_i} \right)^{1/N} \in H^0(U, L|_U).$$

In fact, the total space \mathbb{L}^N is an open complex manifold of dimension $n + 1$ since L^N is holomorphic line bundle over X , so

$$\bigotimes_{i=1}^r \sigma_i^{\otimes b_i} : X \rightarrow L^N$$

is a holomorphic mapping. Note that

$$p : L - \{0\} \rightarrow L^N - \{0\} \quad e \mapsto e^N$$

is a covering, where $\{0\}$ denotes the zero section of L . Since U is simply connected, by [14, Proposition 13.5], there is a lifting

$$\tilde{\sigma} : U \rightarrow L - \{0\}$$

of the mapping $\bigotimes_{i=1}^r \sigma_i^{\otimes b_i} : U \rightarrow L - \{0\}$, such that

$$\tilde{\sigma}^N = p \circ \tilde{\sigma} = \bigotimes_{i=1}^r \sigma_i^{\otimes b_i}.$$

Since p is local biholomorphic, so $\tilde{\sigma}$ is a holomorphic section of $L|_U$.

Denote by $H_{(2)}^{i,j}(U, L|_U, \omega_U, h_U^L)$ the L^2 Dolbeault cohomology (see [17, Section 2.3]). Then we have the following isomorphism,

$$(4.2) \quad \otimes \tilde{\sigma} : H_{(2)}^{i,j}(U, \mathcal{O}_U, \omega_P, h_1) \rightarrow H_{(2)}^{i,j}(U, L|_U, \omega_P, h_2),$$

where ω_P is a Poincaré type metric on U (see [17, Definition 2.3]), h_1, h_2 are smooth metrics on $\mathcal{O}_U, L|_U$ respectively. More precisely,

$$h_1 = \prod_{i=1}^r \|\sigma_i\|_{D_i}^{2\tau_i} (\log(\|\sigma_i\|_{D_i}^2))^\alpha$$

and

$$h_2 = \prod_{i=1}^r \|\sigma_i\|_{D_i}^{2(\tau_i - \frac{b_i}{N})} (\log(\|\sigma_i\|_{D_i}^2))^\alpha \cdot \left(\prod_{i=1}^r \|\cdot\|_{D_i}^{2b_i} \right)^{\frac{1}{N}},$$

where $\alpha > 0$ is a large even integer and $\tau_i \in (0, 1]$, $\|\cdot\|_{D_i}^2$ is a smooth metric on $\mathcal{O}_X(D_i)$. Indeed, (4.2) is an isomorphism since

$$\|\tilde{\sigma}\|_{h_2}^2 = \prod_{i=1}^r \|\sigma_i\|_{D_i}^{2(\tau_i - \frac{b_i}{N})} (\log(\|\sigma_i\|_{D_i}^2))^\alpha \cdot \left(\prod_{i=1}^r \|\sigma_i\|_{D_i}^{2b_i} \right)^{\frac{1}{N}} = h_1.$$

Since $b_i \in (-N, N) \cap \mathbb{Z}$, so there exists $\tau_i \in (0, 1]$ such that

$$\tau_i - \frac{b_i}{N} \in (0, 1].$$

From [17, Theorem 3.1], one has

$$(4.3) \quad H^j(X, \Omega_X^i(\log D)) \cong H_{(2)}^{i,j}(U, \mathcal{O}_U, \omega_P, h_1)$$

and

$$(4.4) \quad H^j(X, \Omega_X^i(\log D) \otimes L) \cong H_{(2)}^{i,j}(U, L|_U, \omega_P, h_2).$$

From (4.3), (4.4) and the isomorphism (4.2), one has

$$H^j(X, \Omega_X^i(\log D) \otimes L) \cong H^j(X, \Omega_X^i(\log D)).$$

Combining with [26, Theorem 1.2] shows that

$$H^j(X, \Omega_X^i(\log D) \otimes L) = 0, \quad \text{for } i + j \geq n + q + 1.$$

□

Remark 4.4. For the case that X is projective, there are various logarithmic vanishing theorems for q -ample divisors have been studied in [4, 15, 11].

5. FURTHER DISCUSSIONS

Let X be a smooth projective variety, the injective theorems, vanishing theorems and unobstructed deformations have been studied widely by using the E_1 -degeneration of some spectral sequences [12, 13, 18]. In particular, by [12, Theorem 3.2 (b)], the following spectral sequence

$$(5.1) \quad E_1^{ab} = H^b(X, \Omega_X^a(\log D) \otimes L^{-1}) \implies \mathbb{H}^{a+b}(X, \Omega_X^\bullet(\log D) \otimes L^{-1})$$

associated to the logarithmic de Rham complex

$$(\Omega_X^\bullet(\log D) \otimes L^{-1}, \nabla_\bullet^{(i)})$$

degenerates in E_1 . Here L is a holomorphic line bundle over X satisfying $L^N = \mathcal{O}_X(\sum_{i=1}^r a_i D_i)$, $0 < a_j < N$. For the proof of (5.1), one can first take a cyclic cover obtained by taking the N -th root out of a canonical section of L^N and get a normal variety Y , then by using a Kawamata's result [12, Lemma 3.19] (which needs an ample invertible sheaf), one can get a projective manifold T and a finite morphism $\delta : T \rightarrow Y$, so that the degeneration (5.1) can be reduced to the more familiar degeneration of the Hodge spectral sequence

$$E_1^{ab} = H^b(T, \Omega_T^a) \implies \mathbb{H}^{a+b}(T, \Omega_T^\bullet).$$

On the other hand, if X is a smooth projective complex variety, $D = \sum_{i=1}^r D_i$ is a simple normal crossing divisor, and there is a collection of weights $\{a_i\}_{i \in I} \subset [0, 1] \cap \mathbb{Q}$ so that

$$(5.2) \quad \sum_{i \in I} a_i [D_i] = -K_X \in \text{Pic}(X) \otimes \mathbb{Q},$$

then the pair (X, D) has also unobstructed deformations by using a purely algebraic method [21, Section 4.3.3 (iii)]. More precisely, they used Dolbeault type complexes to construct a differential Batalin-Vilkovisky algebra such that the associated differential graded Lie algebra (DGLA) controls the deformation problem. If the differential Batalin-Vilkovisky algebra has a degeneration property then the associated DGLA is homotopy abelian.

In terms of Kähler geometry, K. Liu, S. Rao and the author [23] developed a method so that the E_1 -degeneration of spectral sequences and the unobstructed deformations can be reduced to a logarithmic $\bar{\partial}$ -equation. From the discussions in Section 3, if one can solve (2.11) in the case of $L^N = \mathcal{O}_X(\sum_{i=1}^r a_i D_i)$, $0 \leq a_i \leq N$, then by the same argument as in Section 3, one can immediately obtain the degeneration (5.1) and give an analytic proof for the unobstructed deformations in the case of (5.2). Inspired by the above discussions, one may naturally conjecture that:

Conjecture 5.1. Let X be a compact Kähler manifold and $D = \sum_{i=1}^r D_i$ a simple normal crossing divisor in X . If L is a holomorphic bundle over X with $L^N = \mathcal{O}_X(\sum_{i=1}^r a_i D_i)$, $0 \leq a_i \leq N$ and $a_i \in \mathbb{Z}$, then the following logarithmic $\bar{\partial}$ -equation:

$$(5.3) \quad \bar{\partial}x = \partial\alpha$$

has a solution $x \in A^{0,q-1}(X, \Omega_X^{p+1}(\log D) \otimes L^{-1})$ for any $\alpha \in A^{0,q}(X, \Omega_X^p(\log D) \otimes L^{-1})$ with $\bar{\partial}\partial\alpha = 0$.

Note that (5.3) can be solved for the case that L is trivial or $L = \mathcal{O}_X(D)$ [23, Theorem 0.1, 0.2], and Theorem 2.4 is a special case that $L^N = \mathcal{O}_X(D)$ for a smooth divisor D .

REFERENCES

- [1] F. Ambro, *An injectivity theorem*, *Compositio Math.* **150** (2014), 999-1023.
- [2] V. Ancona, B. Gaveau, *Differential Forms on Singular Varieties, De Rham and Hodge Theory Simplified*, Chapman Hall/CRC, 2005.
- [3] W. Barth, K. Hulek, C. Peters, A. Van de Ven, *Compact complex surfaces*, Second edition. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, 4. Springer-Verlag, Berlin, 2004.
- [4] N. Broomhead, J.-C. Ottem, A. Prendergast-Smith, *Partially ample line bundles on toric varieties*, *Glasgow Math. J.* **58** (2016) 587-598.
- [5] L. A. Cordero, M. Fernandez, A. Gray, L. Ugarte, *A general description of the terms in the Frölicher spectral sequence*, *Diff. Geom. Applic.* **7** (1997), 75-84.
- [6] G. de Rham, K. Kodaira, *Harmonic integrals*, (Mimeographed notes), Institute for Advanced Study, Princeton (1950).
- [7] P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, *Publ. Math. Inst. Hautes Études Sci.* **35** (1969), 107-126.
- [8] P. Deligne, *Equations différentielles à points singuliers réguliers*, *Springer Lect. Notes Math.* **163** (1970).
- [9] P. Deligne, *Théorie de Hodge, II*, *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 5-57.

- [10] P. Deligne, L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Inventiones math. **89** (1987), 247-270.
- [11] H. Esnault, E. Viehweg, *Logarithmic de Rham complexes and vanishing theorems*, Invent. Math. **86** (1986), 161-194.
- [12] H. Esnault, E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, **20**, Birkhäuser, Verlag, Basel, 1992.
- [13] O. Fujino, *Introduction to the log minimal model program for log canonical pairs*, preprint, arXiv:0907.1506 [math.AG]. Published as *Foundations of the minimal model program*, MSJ Memoirs, 35. Mathematical Society of Japan, Tokyo, 2017.
- [14] W. Fulton, *Algebraic Topology, A First Course*, Springer-Verlag, New York, 1995.
- [15] D. Greb, A. Küronya, *Partial positivity: geometry and cohomology of q -ample line bundles*, Recent advances in algebraic geometry, 207–239, London Math. Soc. Lecture Note Ser., 417, Cambridge Univ. Press, Cambridge, 2015.
- [16] P. Griffith, J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.
- [17] C. Huang, K. Liu, X. Wan, X. Yang, *Logarithmic vanishing theorems on compact Kähler manifolds I*, arXiv: 1611. 07671v1, 2016.
- [18] D. Iacono, *Deformations and obstructions of pairs (X, D)* , International Mathematics Research Notices **19** (2015), 9660-9695.
- [19] J. Kollár, *Singularities of pairs*. Algebraic Geom.-Santa Cruz. In: Proceedings of Symposia Pure Mathematics 62, Part 1, pp. 221-287. Amer. Math. Soc., Providence, RI (1995).
- [20] K. Kodaira, *The theorem of Riemann-Roch on compact analytic surfaces*, Amer. J. Math. **73** (1951), 1-46.
- [21] L. Katzarkov, M. Kontsevich, T. Pantev, *Hodge theoretic aspects of mirror symmetry*, In From Hodge theory to integrability and TQFT tt*-geometry, volume 78 of Proc. Sympos. Pure Math., pages 87-174. Amer. Math. Soc., Providence, RI, 2008.
- [22] Y. Kawamata, *On deformations of compactifiable complex manifolds*, Math. Ann. **235** (1978), 247-265.
- [23] K. Liu, S. Rao, X. Wan, *Geometry of logarithmic forms and deformations of complex structures*, arXiv:1708.00097v2, has been accepted for publication in Journal of Algebraic Geometry.
- [24] K. Liu, S. Rao, X. Yang, *Quasi-isometry and deformations of Calabi-Yau manifolds*, Invent. Math. **199** (2015), no. 2, 423-453.
- [25] K. Liu, X. Sun, S.-T. Yau, *Recent development on the geometry of the Teichmüller and moduli spaces of Riemann surfaces*, Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces, 221-259, (2009).
- [26] K. Liu, X. Wan, X. Yang, *Logarithmic vanishing theorems for effective q -ample divisors*, submitted.
- [27] K. Liu, S. Zhu, *Solving equations with Hodge theory*, arXiv:1803.01272v1.
- [28] R. Lazarsfeld, *Positivity in Algebraic Geometry, I, II*, A Series of Modern Surveys in Mathematics 48. Berlin: Springer, 2004.
- [29] J. Noguchi, *A short analytic proof of closedness of logarithmic forms*, Kodai Math. J. **18** (1995), No. 2, 295-299.
- [30] S. Rao, X. Wan, Q. Zhao, *Power series proofs for local stabilities of Kähler and balanced structures with mild $\partial\bar{\partial}$ -lemma*, arXiv: 1609.05637v1.
- [31] S. Rao, X. Wan, Q. Zhao, *On local stabilities of p -Kähler structures*, arXiv:1801.01031v2.
- [32] S. Rao, Q. Zhao, *Several special complex structures and their deformation properties*, J. Geom. Anal. (2017), <https://doi.org/10.1007/s12220-017-9944-7>, arXiv: 1604.05396v3.
- [33] B. Shiffman, A. Sommese, *Vanishing theorems on complex manifolds*. Progress in Mathematics, vol. 56. Birkhauser Boston Inc, Boston (1985).
- [34] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, Mathematical aspects of string theory (San Diego, Calif., 1986), 629-646, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, (1987).
- [35] A. Todorov, *The Weil-Petersson geometry of the moduli space of $\mathbb{S}\mathbb{U}(n \geq 3)$ (Calabi-Yau) manifolds I*, Comm. Math. Phys., **126** (2), (1989), 325-346.

- [36] R. Wells, *Comparison of De Rham and Dolbeault cohomology for proper surjective mappings*, Pacific J. Math. **53** (1974), no. 1, 281-300.
- [37] Q. Zhao, S. Rao, *Applications of deformation formula of holomorphic one-forms*, Pacific J. Math. Vol. **266**, No. 1, 2013, 221-255.
- [38] Q. Zhao, S. Rao, *Extension formulas and deformation invariance of Hodge numbers*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 11, 979-984.

XUEYUAN WAN: MATHEMATICAL SCIENCES, CHALMERS UNIVERSITY OF TECHNOLOGY, 41296 GOTHENBURG, SWEDEN

E-mail address: `xwan@chalmers.se`