

## AN ELLIPTIC TOPOLOGICAL VERTEX

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ABSTRACT. We derive a one-parameter deformation of the refined topological vertex that, when used to compute non-periodic web diagrams, reproduces the six-dimensional topological string partition functions that are computed using the refined vertex and periodic web diagrams.

## 1. INTRODUCTION

*We motivate the study of topological vertices from a 2D conformal field theory point of view.*

**1.1. A web of relations.** The relation of 2D conformal field theories, which describe critical surface phenomena, and exact solutions in 2D statistical mechanical models, which describe off-critical surface phenomena, is well-understood since the 1970's. More recently, 4D, 5D and 6D instanton and topological string partition functions, as well as other topics in modern mathematical physics, were related to 2D conformal field theories in terms of *dualities*, and one can study any of these topics from the viewpoint of any of the others<sup>1</sup>. In the following, we motivate the present work from the viewpoint of 2D conformal field theory.

**1.2. From 2D correlation functions to plane partitions.** In 1984, Belavin, Polyakov and Zamolodchikov showed that correlation functions in 2D conformal field theories split into sums of products of structure constants, chiral conformal blocks, and anti-chiral conformal blocks. In 2009, Alday, Gaiotto and Tachikawa conjectured [3], and Alba, Fateev, Litvinov and Tarnopolskiy proved [2], that in the presence of an extra Heisenberg algebra, a 2D conformal block splits into products of 4D Nekrasov partition functions<sup>2</sup>, which are limits of 5D instanton partition functions. These 5D instanton partition functions split into products of topological vertices that are also 5D partition functions<sup>3</sup>. Since a topological vertex has a combinatorial interpretation as a generating function of weighted plane partitions that satisfy specific boundary conditions [32, 33], the (difficult) analytic problem of computing correlation functions in 2D conformal field theory is recast as a (hopefully) simpler exercise in algebraic combinatorics.

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*Key words and phrases.* Topological vertex. Refined topological vertex. Topological strings. Instanton partition function. Elliptic conformal blocks. Ding-Iohara-Miki-Saito algebra.

<sup>1</sup> For a review of recent developments, see [38, 39]

<sup>2</sup> Alday *et al.* also conjecture 4D interpretations of the structure constants as well as other aspects of 2D correlation functions, but in the present work, we focus on the conformal blocks.

<sup>3</sup> Topological vertices are 5D partition functions in the sense that they depend  $R$ , the radius of the  $M$ -theory circle. Gluing topological vertices leads to  $R$ -deformed 2D conformal blocks. In 2D terms,  $R$  is an off-critical deformation parameter, and the critical 2D conformal blocks are obtained in the  $R \rightarrow 0$  limit.

**1.3. The algebraic combinatoric point of view.** Viewing the 2D correlation functions in terms of algebraic combinatorial objects, which in this case are plane partitions with specific weights and specific boundary conditions, allows one to study more general classes of them. One way to do that is to change the weights while maintaining computability. All known topological vertices, starting from the original vertex  $\mathcal{O}_{Y_1 Y_2 Y_3}(x)$ , which depends on three Young diagrams  $Y_1, Y_2$  and  $Y_3$ , and a single parameter  $x$ <sup>4</sup>, and leads to conformal blocks in conformal field theories with integral central charges [1], to the refined vertex  $\mathcal{R}_{Y_1 Y_2 Y_3}(x, y)$ , which depends on an additional refinement parameter  $y$ , and leads to conformal blocks in conformal field theories with generic central charges [5, 6, 23], to the Macdonald vertex  $\mathcal{M}_{Y_1 Y_2 Y_3}(x, y | q, t)$  which depends on two additional Macdonald parameters  $q$  and  $t$  [16], and leads to conformal blocks in the presence of vertex-operator condensates [15], are generating functions of plane partitions that are given different weights.

**1.4. The present work.** Following Saito's construction of an elliptic version of Ding-Iohara-Miki algebra [11, 28], using *two* commuting Heisenberg algebras, one deformed by  $q$ , and the other by  $1/q$  [34–36], we construct an *elliptic* vertex  $\mathcal{E}_{Y_1 Y_2 Y_3}(x, y | \mathbf{q})$ , where  $\mathbf{Y}_1 = (Y_{1A}, Y_{1B})$  and  $\mathbf{Y}_2 = (Y_{2A}, Y_{2B})$  are pairs of Young diagrams,  $Y_3$  is a single Young diagram, and  $\mathbf{q} = (q, 1/q)$ , where  $q$  is a deformation parameter.

1.4.1. *Two components.*  $\mathcal{E}$  is a product of two components,

$$(1.1) \quad \mathcal{E}_{Y_1 Y_2 Y_3}(x, y | \mathbf{q}) = \mathcal{M}_{Y_{1A} Y_{2A} Y_3}(x, y | q) \mathcal{M}_{Y_{1B} Y_{2B} Y_3}(1/x, 1/y | 1/q),$$

where  $\mathcal{M}_{Y_{1A} Y_{2A} Y_3}(x, y | q)$  is a Macdonald vertex with Macdonald parameters  $q \neq 1$ , and  $t = 0$ , and refinement parameters  $x$  and  $y$ , and  $\mathcal{M}_{Y_{1B} Y_{2B} Y_3}(1/x, 1/y | 1/q)$  is a Macdonald vertex with Macdonald parameters  $1/q \neq 1$ , and  $t = 0$ , and refinement parameters  $1/x$  and  $1/y$ . The Young diagrams  $Y_{1A}$  and  $Y_{1B}$  that label the initial non-preferred legs of the component Macdonald vertices are independent, the Young diagrams  $Y_{2A}$  and  $Y_{2B}$  that label the final non-preferred legs are also independent, but the same Young diagram  $Y_3$  labels the (common) preferred leg of both component vertices<sup>5</sup>. The original Macdonald vertex depends on two Macdonald parameters  $q$  and  $t$  and has basically the same structure as the refined topological vertex, but the Schur functions replaced by Macdonald functions. The component Macdonald vertices depend on a single Macdonald parameter  $q$  or  $1/q$ , and the second Macdonald parameter  $t = 0$ . In this case, the Macdonald functions are one-parameter deformations of the Schur functions called  $q$ -Whittaker functions [18–20, 7–9].

1.4.2. *The refined vertex limit.* In the limit  $q \rightarrow 0$ ,

$$(1.2) \quad \mathcal{M}_{Y_{1A} Y_{2A} Y_3}(x, y | q) \rightarrow \mathcal{R}_{Y_{1A} Y_{2A} Y_3}(x, y), \quad \mathcal{M}_{Y_{1B} Y_{2B} Y_3}(1/x, 1/y | 1/q) \rightarrow 1, \\ \mathcal{E}_{Y_1 Y_2 Y_3}(x, y | \mathbf{q}) \rightarrow \mathcal{R}_{Y_{1A} Y_{2A} Y_3}(x, y),$$

<sup>4</sup> In this work, as in [16], we use  $x$  and  $y$  for the parameters of the refined vertex,  $x = \exp(-R\epsilon_1)$  and  $y = \exp(R\epsilon_2)$ , where  $\epsilon_1$  and  $\epsilon_2$  are Nekrasov's deformation parameters [30], and  $R$  is the radius of the M-theory circle. We reserve  $q$  and  $t$  for the Macdonald deformation parameters [27].

<sup>5</sup> These terms will be defined when we construct the elliptic vertex explicitly in section 9.

where  $\mathcal{R}_{Y_{1A}Y_{2A}Y_3}(x, y)$  is a refined vertex<sup>6</sup>. In the limit  $q \rightarrow \infty$ ,

$$(1.3) \quad \mathcal{M}_{Y_{1A}Y_{2A}Y_3}(x, y|q) \rightarrow 1, \quad \mathcal{M}_{Y_{1B}Y_{2B}Y_3}(1/x, 1/y|1/q) \rightarrow \mathcal{R}_{Y_{1B}Y_{2B}Y_3}(1/x, 1/y), \\ \mathcal{E}_{Y_1Y_2Y_3}(x, y|q) \rightarrow \mathcal{R}_{Y_{1B}Y_{2B}Y_3}(1/x, 1/y)$$

In this sense, the elliptic vertex  $\mathcal{E}_{Y_1Y_2Y_3}(x, y|q)$  is a one-parameter deformation of the refined vertex that interpolates  $\mathcal{R}_{Y_{1A}Y_{2A}Y_3}(x, y)$  and  $\mathcal{R}_{Y_{1B}Y_{2B}Y_3}(1/x, 1/y)$ , which are equivalent in the sense that they lead to the same 4D and 5D instanton partition functions and the same 2D conformal blocks, up an overall normalization.

1.4.3. *Twisted vertices.* We also introduce *twisted* version of  $\mathcal{E}$  that we call  $\mathcal{E}^*$ , that depend on *twisted*  $q$ -Whittaker functions that we define.

1.4.4. *6D instanton partition functions from non-periodic web diagrams.* Starting from a non-periodic web diagram, constructed by gluing refined topological vertices, such that the corresponding partition function is a 5D instanton partition function, and replacing the refined vertices by  $\mathcal{E}$  and  $\mathcal{E}^*$  alternately, we reproduce the 6D version of the 5D instanton partition function that we start with, while keeping the connectivity of the web diagram intact. These 6D instanton partition functions are the same as those obtained by taking traces, that is, by identifying opposite external legs and summing over the intermediate states to form periodic web diagrams, then computing their partition functions, as first proposed in the work of Hollowood, Iqbal and Vafa [21]<sup>7</sup>.

1.5. **Two routes.** We came to the elliptic vertex  $\mathcal{E}$  *via* two different routes.

1.5.1. *The Iqbal-Kozcaz-Vafa refined vertex route.* In [16], the first co-author, together with Jian-Feng Wu, proposed a Macdonald-type deformation of the refined vertex of Iqbal, Kozcaz and Vafa [23], in terms of the Macdonald parameters  $q$  and  $t$ , and noted its connection to Ding-Iohara-Miki algebra [11, 28, 12]. In [17], an elliptic extension of the Macdonald vertex of [16] based on Saito's elliptic extension of the Ding-Iohara-Miki algebra was obtained. This extension introduced a parameter  $p$ , and represents the initial and final states of the new vertex in terms of pairs of  $p$ -deformed Macdonald functions<sup>8</sup>. The properties of these  $p$ -deformed Macdonald functions were not completely well-understood, and to work with them, one had to conjecture that they satisfied suitable Cauchy-type identities. However, setting  $q = t$ , the  $p$ -deformed Macdonald functions reduced to  $p$ -deformed Schur functions, whose properties were still not completely well-understood, but appeared to be more amenable to analysis [13].

<sup>6</sup> In this work, we always refer to the formulation of the refined vertex in [23]

<sup>7</sup> The literature on 6D instanton partition functions and related topics has grown rapidly over the past few years. The present work was motivated by the recent works of Iqbal, Kozcaz and Yau [24] and Nieri [31].

<sup>8</sup> Saito's elliptic extension of the Ding-Iohara-Miki algebras makes use of two Heisenberg algebras, and the corresponding states are naturally represented in terms of pairs of symmetric functions [40].

1.5.2. *The Awata-Feigin-Shiraishi refined vertex route.* In [41], the second co-author proposed a Saito-type elliptic extension of the refined vertex of Awata, Feigin and Shiraishi [4], and in [42], he obtained another version in a form related to that of Iqbal, Kozcaz and Vafa [42], where the initial and final states are also labelled by pairs of  $p$ -deformed Schur functions. This elliptic vertex is exactly that of [17], restricted to the  $p$ -deformed Schur functions of [13].

1.5.3.  *$q$ -Whittaker functions.* The starting point of the present work is the observation that the  $p$ -deformed Schur functions of sections 1.5.1 and 1.5.2 are  $q$ -Whittaker functions obtained from Macdonald functions in the limit  $t \rightarrow 0, q \neq 0$ <sup>9</sup>, where the Macdonald parameter  $q$  plays the role of Saito's elliptic deformation parameter  $p$ <sup>10</sup>. This observation allows us to use the tools of Macdonald functions, in the limit  $t \rightarrow 0, q \neq 0$ , to put our derivations on a solid footing.

1.6. **Outline of contents.** In section 2, we recall basic facts related to  $q$ -Whittaker functions and their Cauchy identities, then in section 3, we introduce an involution that we use to define twisted  $q$ -Whittaker function and their Cauchy identities. In section 4, we recall Saito's pair of  $pqt$ -Heisenberg algebras and pair of  $pqt$ -vertex operators. In each pair, one component depends on  $p, q$  and  $t$ , and the other on  $1/p, 1/q$  and  $1/t$ . We show that Macdonald's parameter  $q$  and Saito's parameter  $p$  appear in all expressions on equal footing, so that setting  $q = t$ , or  $p = t$ , all dependence on the equated parameters disappears, and the remaining parameter can be identified with the parameter that deforms Schur functions to  $q$ -Whittaker functions. In section 5, we take the  $p \rightarrow t$  limit of Saito's  $pqt$ -Heisenberg algebras and  $pqt$ -vertex operators, to obtain a pair of  $q$ -deformed Heisenberg algebras, and a pair of  $q$ -deformed vertex operators, such that in each pair, one component depends on  $q$  and the other depends on  $1/q$ . In section 6, we recall the Heisenberg/power sum correspondence which allows us to derive useful operator-valued identities for one of the Heisenberg algebras of section 5, then in section 7, we do the same for the second Heisenberg algebra. In section 8, we define pairs of  $q$ -Whittaker functions and derive their Cauchy identities. In section 9, we construct the elliptic vertex, and in section 10, we show that gluing copies of this vertex produces an elliptic version of the strip partition function [22], so that gluing copies of the latter produces the 6D instanton partition functions of [21]. Section 11 includes a number of comments.

## 1.7. Notations and other conventions.

1.7.1. *Sets.*  $\iota$ , and similarly  $\mathbf{j}$ , is the set of non-zero natural numbers  $(1, 2, \dots)$ ,  $\mathbf{x} = (x_1, x_2, \dots)$  and  $\mathbf{y} = (y_1, y_2, \dots)$  are sets of (possibly infinitely-many) variables,  $\mathbf{a}_- = (a_{-1}, a_{-2}, \dots)$  and  $\mathbf{a}_+ = (a_1, a_2, \dots)$  are the free-boson creation and annihilation mode operators.

1.7.2. *Pairs of Young diagrams and pairs of variables.*  $\mathbf{Y} = (Y_1, Y_2)$  is a pair of Young diagrams,  $\emptyset = (\emptyset, \emptyset)$  is a pair of empty Young diagrams, and  $\mathbf{q}$  is the pair  $(q, 1/q)$ .

1.7.3. *Number of elements in sets and number of cells in Young diagrams.*  $|\mathbf{x}|, |\mathbf{y}|, \dots$ , are the numbers of elements in the sets  $\mathbf{x}, \mathbf{y}, \dots$ , and  $|\mathbf{Y}| = |Y_1| + |Y_2|$ , where  $|Y_i|$  is the number of cells in the Young diagram  $Y_i$ .

<sup>9</sup> The  $q$ -Whittaker functions are dual to the Hall-Littlewood functions in the sense that the former are obtained in the limit  $t \rightarrow 0, q \neq 0$  of Macdonald functions, while the latter are obtained in the limit  $q \rightarrow 0, t \neq 0$ .

<sup>10</sup> In Saito's work [34–36], and in the present work, whenever the parameters  $p$  and  $q$  are both non-zero, they appear on equal footing.

1.7.4. *Primed variables and transpose Young diagrams.* To simplify the notation, we use the primed variables  $x', y', p', q', t', \dots$ , for the inverse variables  $1/x, 1/y, 1/p, 1/q, 1/t, \dots$ , and the primed set variable  $\mathbf{x}' = (x'_1, x'_2, \dots)$  for the set of inverse variables  $(1/x_1, 1/x_2, \dots)$ . The Young diagram  $Y'$  is the transpose of the Young diagram  $Y$ . We use  $W'_q$  for the dual  $q$ -Whittaker symmetric function as in section 1.7.5.

1.7.5. *Macdonald and  $q$ -Whittaker symmetric functions.* We use  $P_Y(\mathbf{x})$  and  $Q_Y(\mathbf{x})$  for the Macdonald and dual Macdonald symmetric functions. Each of these functions is labelled by a Young diagram  $Y$ , depends on two parameters  $q$  and  $t$ , and is symmetric in a (possibly infinite) set of variables  $\mathbf{x} = (x_1, x_2, \dots)$ . We use  $W_{qY}(\mathbf{x})$  and  $W'_{qY}(\mathbf{x})$  for the  $q$ -Whittaker and dual  $q$ -Whittaker symmetric functions. Each of these functions is labelled by a Young diagram  $Y$ , a parameter  $q$ , and is symmetric in a (possibly infinite) set of variables  $\mathbf{x} = (x_1, x_2, \dots)$  <sup>11</sup>.

1.7.6. *Pairs of  $q$ -Whittaker symmetric functions.* We use  $\mathbf{W}_{qY}(\mathbf{x})$  and  $\mathbf{W}'_{qY}(\mathbf{x})$ ,  $\mathbf{q} = (q, 1/q)$ ,  $\mathbf{Y} = (Y_1, Y_2)$ , for a pair of  $q$ -Whittaker and a pair of dual  $q$ -Whittaker symmetric functions. The first symmetric function in a pair depends on a parameter  $q$ , a Young diagram  $Y_1$ , and is symmetric in a (possibly infinite) set of variables  $\mathbf{x} = (x_1, x_2, \dots)$ , while the second symmetric function in the pair depends on a parameter  $1/q$ , a Young diagram  $Y_2$ , and is symmetric in a (possibly infinite) set of variables  $\mathbf{x}' = (x'_1, x'_2, \dots)$ .

1.7.7. *Parameters.* Our refinement parameters  $(x, y)$  are the parameters  $(q, t)$  in [23] <sup>12</sup>. Our  $q$ -Whittaker deformation parameter (which will be called either  $q$  or  $1/q$ ) is the first Macdonald parameter  $q$ , while the second Macdonald parameter  $t = 0$ .

1.7.8. *Exponentiated sequences.* Given a Young diagram  $Y$  that consists of an infinite sequence of rows  $Y = (y_1, y_2, \dots)$ , such that only finitely-many rows have non-zero length, together with an infinite sequence of integers  $\iota = (1, 2, \dots)$ , and two variables  $u$  and  $v$ , we define the exponentiated sequences  $u^\iota, v^{\pm Y}, \dots$ , as,

$$(1.4) \quad u^\iota = (u, u^2, \dots), \quad u^{\iota-1} = (1, u, \dots), \quad v^{\pm Y} = (v^{\pm y_1}, v^{\pm y_2}, \dots), \quad \dots,$$

and the products of exponentiated sequences  $u^\iota v^{\pm Y}, u^{\iota-1} v^{\pm Y}, \dots$ , as,

$$(1.5) \quad u^\iota v^{\pm Y} = (u v^{\pm y_1}, u^2 v^{\pm y_2} \dots), \quad u^{\iota-1} v^{\pm Y} = (v^{\pm y_1}, u v^{\pm y_2} \dots), \quad \dots$$

1.7.9. *More on sequences.* Let  $\mathbf{x} = (x_1, \dots, x_m)$  be a set of  $m$  variables, and  $Y = (y_1, \dots, y_n)$  be a Young diagram that consists of  $n$  non-zero parts, such that  $n \leq m$ . The notation,

$$(1.6) \quad \mathbf{x}_t^Y = (x_{t_1}^{y_1}, \dots, x_{t_n}^{y_n}),$$

<sup>11</sup> We show the dependence on the variable  $q$  explicitly because this will often be a dependence on  $q' = 1/q$ .

<sup>12</sup> More precisely, our  $x$  is  $t$ , and our  $y$  is  $q$  in [23]

where  $\iota = (\iota_1, \dots, \iota_n)$ , is defined as follows. **1.** Consider the set of  $m$  integers,  $\mathbf{m} = (1, \dots, m)$ , for example,  $\mathbf{m} = (1, 2, 3, 4)$ , **2.** Choose a subset of  $n$  integers  $\mathbf{n} \subseteq \mathbf{m}$ , for example,  $\mathbf{n} = (1, 2, 4)$ , **3.** Consider a specific permutation  $\iota$  of  $\mathbf{n}$ , for example,  $\iota = (2, 4, 1)$ . **4.** The set on the right hand side of Equation 1.6 is obtained by starting with the set  $(x_{\iota_1}, \dots, x_{\iota_n})$ , and raising its elements sequentially to the powers  $(y_1, \dots, y_n)$ <sup>13</sup>.

1.7.10. *Products on sequences.* We will use the notation,

$$(1.7) \quad \frac{1}{1 - \mathbf{x} \mathbf{y} q^n} = \prod_{i,j=1}^{\infty} \left( \frac{1}{1 - x_i y_j q^n} \right), \quad 1 + \mathbf{x} \mathbf{y} q^n = \prod_{i,j=1}^{\infty} (1 + x_i y_j q^n),$$

$$\phi_{q\pm}(\mathbf{x}) = \prod_{i=1}^{\infty} \phi_{q\pm}(x_i), \quad \Gamma_{qa\pm}(\mathbf{x}) = \prod_{i=1}^{\infty} \Gamma_{qa\pm}(x_i), \quad \Gamma_{qb\pm}^{\pm}(\mathbf{x}) = \prod_{i=1}^{\infty} \Gamma_{qb\pm}^{\pm}(x_i),$$

where  $\phi_{q\pm}(x'_i)$ , etc. are two-boson vertex operators, and  $\Gamma_{qa-}(x_i)$ , and  $\Gamma_{qb-}^{\pm}(x_i)$  are one-boson vertex operators, to be defined in section 4.

1.7.11. *Products on almost theta functions.* We will also use,

$$(1.8) \quad \theta_q(\mathbf{x}, \mathbf{y}) = \prod_{i,j=1}^{\infty} \theta_q(x_i y_j), \quad \theta_q(x_i y_j) = \prod_{n=0}^{\infty} (1 - x_i y_j q^n) (1 - x'_i y'_j q^{n+1}),$$

that is,  $\theta_q(x)$  is a Jacobi theta function  $\Theta(x|q)$ , up to an  $x$ -independent factor,

$$(1.9) \quad \theta_q(x) = \Theta(x|q) / (q|q),$$

$$\Theta(x;q) = \prod_{n=0}^{\infty} (1 - q^{n+1}) (1 - x q^n) (1 - x'_i q^{n+1}), \quad (q|q) = \prod_{n=1}^{\infty} (1 - q^n)$$

## 2. THE $q$ -WHITTAKER FUNCTIONS

*Starting from the properties of the Macdonald functions, which depend on  $q$  and  $t$ , we take the limit  $t \rightarrow 0$  to obtain the corresponding properties of the  $q$ -Whittaker functions, which depend on  $q$  only.*

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<sup>13</sup> In applications of this notation, for example to the definition of the monomial symmetric functions, one sums over all permutations  $\iota$  of all possible distinct subsets  $\mathbf{n}$  of the same cardinality.

**2.1. The monomial symmetric functions.**  $m_Y(\mathbf{x})$ , where  $\mathbf{x} = (x_1, x_2, \dots)$ , indexed by a Young diagram  $Y$ , is <sup>14</sup>,

$$(2.1) \quad m_Y(\mathbf{x}) = \sum_{\iota} x_{\iota}^Y,$$

where the sum runs over all distinct permutations of the set  $\iota$ , which is defined as in section 1.7.9. For example,

$$(2.2) \quad m_{\emptyset}(\mathbf{x}) = 1, \quad m_1(\mathbf{x}) = \sum_i x_i, \quad m_2(\mathbf{x}) = \sum_i x_i^2, \quad \dots, \quad m_{443}(x_1, \dots, x_5) = \sum_{\iota} x_{\iota}^4 x_j^4 x_k^3,$$

where the sum in the last example is over all distinct permutations  $\iota$ , of all distinct subsets  $\mathbf{m} \subseteq \mathbf{n} = (1, \dots, 5)$ , such that the cardinality  $|\mathbf{m}| = 3$ , and  $i \neq j \neq k \in \mathbf{n}$ , as defined in section 1.7.9.

**2.2. Power-sum symmetric functions.**  $p_n(\mathbf{x})$ , where  $\mathbf{x} = (x_1, x_2, \dots)$ , indexed by an integer  $n \in (0, 1, \dots)$ , is <sup>15</sup>,

$$(2.3) \quad p_0(\mathbf{x}) = 1, \quad p_n(\mathbf{x}) = \sum_i x_i^n = m_n(\mathbf{x}), \quad n = 1, 2, \dots,$$

and  $p_Y(\mathbf{x})$ , indexed by a Young diagram  $Y = (y_1, y_2, \dots)$ , is <sup>16</sup>,

$$(2.4) \quad p_Y(\mathbf{x}) = p_{Y_1}(\mathbf{x}) p_{Y_2}(\mathbf{x}) \dots$$

**2.3.  $q$ -Whittaker functions as  $t \rightarrow 0$  limits of Macdonald functions.** Consider the ring of symmetric functions in the variables  $\mathbf{x} = (x_1, x_2, \dots)$ , with coefficients in the field of rational functions in two variables  $(q, t)$ . In this ring, the Macdonald functions  $P_Y(\mathbf{x})$ , each labelled by a Young diagram  $Y$ , and the dual Macdonald functions  $Q_Y(\mathbf{x})$ , each also labelled by a Young diagram  $Y$ , form two orthogonal bases. In the limit  $t \rightarrow 0$ , the coefficients of the ring of symmetric functions are in the field of rational functions in a single variable  $q$ , the Macdonald functions  $P_Y(\mathbf{x})$  reduce to the  $q$ -Whittaker functions, which we denote by  $W_{qY}(\mathbf{x})$ , and the dual Macdonald functions  $Q_Y(\mathbf{x})$  reduce to the dual  $q$ -Whittaker functions, which we denote by  $W'_{qY}(\mathbf{x})$ . These functions were introduced by Gerasimov, Lebedev and Oblazin [18–20], and further studied in [7–9]. In the rest of this section, we deduce the properties of and relations satisfied by  $W_{qY}(\mathbf{x})$  and  $W'_{qY}(\mathbf{x})$  by taking the  $t \rightarrow 0$  limit of the corresponding properties and relations satisfied by  $P_Y(\mathbf{x})$  and  $Q_Y(\mathbf{x})$ .

<sup>14</sup> Ch. I, p. 18, Equation 2.1, in [27]

<sup>15</sup> Ch. I, p. 23, in [27]

<sup>16</sup> Ch. I, p. 24, in [27]

**2.4. The  $q$ -inner product of the power-sum symmetric functions.** From the orthogonality of the power-sum symmetric functions in the ring of symmetric functions with coefficients in the field of rational functions in  $q$  and  $t$ <sup>17</sup>, the power-sum symmetric functions in the ring of symmetric functions with coefficients in the field of rational functions in  $q$  are orthogonal with respect to the  $q$ -inner product,

$$(2.5) \quad \langle p_{Y_1}(\mathbf{x}) | p_{Y_2}(\mathbf{x}) \rangle_q = z_{qY_1} \delta_{Y_1 Y_2}, \quad z_{qY} = (1^{n_1} (n_1!) 2^{n_2} (n_2!) \cdots) \prod_{i=1}^{y'_1} (1 - q^{y_i}),$$

where  $n_r$  is the number of rows of length  $r$  in  $Y$ , and  $y'_1$  is the length of the first row in  $Y'$ , that is, the number of non-zero rows in  $Y$ . This inner product can be understood as follows<sup>18</sup>. For every power-sum symmetric function  $p_Y(\mathbf{x})$ , there is a differential operator  $D_Y(\mathbf{x})$  in  $\mathbf{x} = (x_1, x_2, \dots)$ , such that acting with  $D_Y(\mathbf{x})$  on  $p_Y(\mathbf{x})$ , then setting  $x_1 = x_2 = \dots = 0$ , one obtains the right hand side of the first of Equations 2.5.

**2.5. A  $q$ -identity.** The power-sum symmetric functions  $p_n(\mathbf{x})$  satisfy the  $q$ -identity,

$$(2.6) \quad \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(1 - q^n)} p_n(\mathbf{x}) p_n(\mathbf{y}) \right) = \prod_{n=0}^{\infty} \left( \frac{1}{1 - \mathbf{x} \mathbf{y} q^n} \right),$$

which follows from expanding the exponent on the left hand side, then using,

$$(2.7) \quad \exp \left( - \sum_{n=1}^{\infty} \frac{x^n}{n} \right) = \exp(\log(1 - x)) = 1 - x,$$

to resum the result of the expansion in the form of the right hand side.

**2.6. The  $q$ -Whittaker function.** From the definition of the Macdonald function  $P_Y(\mathbf{x})$ <sup>19</sup>, we obtain the  $q$ -Whittaker function  $W_{qY}(\mathbf{x})$ ,  $Y = (y_1, y_2, \dots)$ , as the unique symmetric function in  $\mathbf{x} = (x_1, x_2, \dots)$ ,  $|\mathbf{x}| \geq y'_1$ , that satisfies two properties.

**2.6.1. The expansion in terms of monomial symmetric functions.**

$$(2.8) \quad W_{qY_1}(\mathbf{x}) = m_{Y_1}(\mathbf{x}) + \sum_{Y_1 > Y_2} u_{qY_1 Y_2} m_{Y_2}(\mathbf{x}),$$

where  $m_Y(\mathbf{x})$  is the monomial symmetric function in  $\mathbf{x}$  labelled by  $Y$ ,  $Y_1 > Y_2$  indicates that  $Y_1$  dominates  $Y_2$  in the natural partial ordering of Young diagrams<sup>20</sup>, and the coefficients  $u_{qY_1 Y_2}$  are rational functions in  $q$ .

<sup>17</sup> Ch. VI, p. 225, Equation 4.11, in [27]

<sup>18</sup> Ch. I, p. 75–76, in [27]

<sup>19</sup> Ch. VI, p. 322, in [27]

<sup>20</sup> Ch. I, p. 7, in [27]

2.6.2. *The orthogonality relation.*

$$(2.9) \quad \langle W_{qY_1}(\mathbf{x}) | W_{qY_2}(\mathbf{x}) \rangle = 0, \quad \text{for } Y_1 \neq Y_2$$

2.7. **The dual  $q$ -Whittaker function.** From the definition of the dual Macdonald function  $Q_Y(\mathbf{x})$ <sup>21</sup>, the dual  $q$ -Whittaker function  $W'_{qY}(\mathbf{x})$  is defined in terms of  $W_{qY}(\mathbf{x})$  as<sup>22</sup>,

$$(2.10) \quad W'_{qY}(\mathbf{x}) = b_Y W_{qY}(\mathbf{x}), \quad b_Y = \prod'_{\square \in Y} \left( \frac{1}{1 - q^{A_{\square Y}^+}} \right),$$

where the prime on the product indicates that the product is restricted to cells  $\square \in Y$  with leg-length  $L_{\square} = 0$ . This is obtained as follows. For  $q \neq 0$ , the product on the right hand side of Equation 2.10 is,

$$(2.11) \quad b_Y^{qt} = \prod_{\square \in Y} \left( \frac{1 - q^{A_{\square Y}^+} t^{L_{\square Y}^+}}{1 - q^{A_{\square Y}^+} t^{L_{\square Y}}} \right)$$

In the limit  $t \rightarrow 0$ , the numerator  $1 - q^{A_{\square Y}^+} t^{L_{\square Y}^+} \rightarrow 1$ , for all  $\square \in Y$ , while the denominator  $1 - q^{A_{\square Y}^+} t^{L_{\square Y}} \rightarrow 1$ , for all  $\square \in Y$  such that  $L_{\square Y} \geq 1$ , and the corresponding factors trivialize to 1. All non-trivial contributions are due to  $\square \in Y$  such that  $L_{\square Y} = 0$ .

2.8.  **$q$ -Whittaker Cauchy identities.** From the Cauchy identities for  $P_Y(\mathbf{x})$  and  $Q_Y(\mathbf{x})$ <sup>23</sup>,  $W_{qY}(\mathbf{x})$  and  $W'_{qY}(\mathbf{x})$  satisfy the Cauchy identity,

$$(2.12) \quad \sum_Y W_{qY}(\mathbf{x}) W'_{qY}(\mathbf{y}) = \prod_{n=0}^{\infty} \left( \frac{1}{1 - \mathbf{x}\mathbf{y}q^n} \right)$$

2.9. **The structure constants.** From the product of two Macdonald functions<sup>24</sup>, the product of two  $q$ -Whittaker functions can be expanded in the form,

$$(2.13) \quad W_{qY_1}(\mathbf{x}) W_{qY_2}(\mathbf{x}) = \sum_{Y_3} f_{Y_1 Y_2 Y_3} W_{qY_3}(\mathbf{x}),$$

which can be used as a definition of the  $q$ -dependent structure constants  $f_{Y_1 Y_2 Y_3}$ . Similarly, from the product of two dual Macdonald functions<sup>25</sup>, the product of two dual  $q$ -Whittaker functions can be expanded as,

$$(2.14) \quad W'_{qY_1}(\mathbf{x}) W'_{qY_2}(\mathbf{x}) = \sum_{Y_3} f_{Y_1 Y_2 Y_3} W'_{qY_3}(\mathbf{x})$$

<sup>21</sup> Ch. VI, p. 322, in [27]

<sup>22</sup> Ch. VI, p. 323, Equation 4.12, and p. 339, Equation 6.19, in [27]

<sup>23</sup> Ch. VI, p. 324, Equation 4.13, and p. 329, Equation 5.4, in [27]

<sup>24</sup> Ch. VI, p. 343, Equation 7.1', in [27]

<sup>25</sup> Ch. VI, p. 344, in [27]

From Equations 2.13 and 2.14, and the corresponding relations for Macdonald functions<sup>26</sup>,

$$(2.15) \quad f_{Y'_1 Y'_2 Y'_3} = \left( \frac{b_{Y_3}}{b_{Y_1} b_{Y_2}} \right) f_{Y_1 Y_2 Y_3},$$

where  $b_Y$  is defined in Equation 2.10. Similarly<sup>27</sup>, the structure constant  $f_{Y_1 Y_2 Y_3}$  can be written as an inner product,

$$(2.16) \quad f_{Y_1 Y_2 Y_3} = \langle W'_{qY_3}(\mathbf{x}) | W_{qY_1}(\mathbf{x}) W_{qY_2}(\mathbf{x}) \rangle$$

**2.10. Skew  $q$ -Whittaker functions.** From the definitions of the skew Macdonald function  $P_{Y_1/Y_2}(\mathbf{x})$  and the skew dual Macdonald function  $Q_{Y_1/Y_2}(\mathbf{x})$ <sup>28</sup>, the skew  $q$ -Whittaker function  $W_{qY_1/Y_2}(\mathbf{x})$  is defined in terms of the skew dual  $q$ -Whittaker function as,

$$(2.17) \quad W_{qY_1/Y_2}(\mathbf{x}) = \left( \frac{b_{Y_2}}{b_{Y_1}} \right) W'_{qY_1/Y_2}(\mathbf{x}),$$

while the skew dual  $q$ -Whittaker function  $W'_{qY_1/Y_2}(\mathbf{x})$  is defined in terms of the dual (non-skewed)  $q$ -Whittaker function as,

$$(2.18) \quad W'_{qY_1/Y_2}(\mathbf{x}) = \sum_{Y_3} f_{Y_2 Y_3 Y_1} W'_{qY_3}(\mathbf{x})$$

**2.11. Skew  $q$ -Whittaker Cauchy identities.** From the Cauchy identities for skew Macdonald functions<sup>29</sup>, the skew  $q$ -Whittaker functions satisfy the Cauchy identities,

$$(2.19) \quad \sum_Y W_{qY/Y_1}(\mathbf{x}) W'_{qY/Y_2}(\mathbf{y}) = \prod_{n=0}^{\infty} \left( \frac{1}{1 - \mathbf{x} \mathbf{y} q^n} \right) \sum_Y W_{qY_2/Y}(\mathbf{x}) W'_{qY_1/Y}(\mathbf{y}),$$

and other Cauchy identities that involve skew Hall-Littlewood functions that need not concern us here.

### 3. TWISTED $q$ -WHITTAKER FUNCTIONS

*We define an involution that we call 'a twist' that acts on the  $q$ -Whittaker functions to generate 'twisted  $q$ -Whittaker functions', then consider Cauchy identities that involve the  $q$ -Whittaker functions and their twisted versions. The reason why we need these specific Cauchy identities will be clear in the sequel.*

<sup>26</sup> Ch. VI, p. 344, Equation 7.3, in [27]

<sup>27</sup> Ch. VI, p. 343, Equation 7.1, in [27]

<sup>28</sup> Ch. VI, p. 344, Equation 7.6', and Ch. VI, p. 344, Equation 7.5, in [27]

<sup>29</sup> Ch. VI, p. 352, and p. 352, in [27]

3.1. **A twist.** Consider the twist  $\iota$ , which acts on the power sum symmetric functions as,

$$(3.1) \quad \iota \cdot p_n(\mathbf{x}) = (-1)^{n-1} p_n(\mathbf{x}), \quad n = 1, 2, \dots,$$

which, in the absence of the Macdonald  $q$  and  $t$  parameters, is identical to the involution  $\omega$  in the theory of symmetric functions<sup>30</sup>, and acts on Schur functions as,

$$(3.2) \quad \iota \cdot s_\lambda(\mathbf{x}) = \omega \cdot s_\lambda(\mathbf{x}) = s_{\lambda'}(\mathbf{x})$$

In the presence of Macdonald  $q$  and  $t$  parameters, the natural generalization of  $\omega$  acts as,

$$(3.3) \quad \omega \cdot p_n(\mathbf{x}) = (-1)^{n-1} \left( \frac{1-q^n}{1-t^n} \right) p_n(\mathbf{x}), \quad n \geq 1,$$

and acts on a  $q$ -Whittaker function with a parameter  $q$  to give a Hall-Littlewood function also with a parameter  $q$  (rather than  $t$ ) [9]. The action of our twist  $\iota$  remains the same in the presence of a Macdonald parameter (or both), and acts on  $q$ -Whittaker functions to generate *twisted  $q$ -Whittaker functions*.

3.2. **Twisted  $q$ -Whittaker functions.** The involution  $\iota$  acts on the  $q$ -Whittaker functions  $W_{qY}$  and the dual  $q$ -Whittaker functions  $W'_{qY}$  to produce the twisted  $q$ -Whittaker functions  $W_{qY}^*$  and the twisted dual  $q$ -Whittaker functions  $W'_{qY}{}^*$ ,

$$(3.4) \quad \iota \cdot W_{qY}(\mathbf{x}) = W_{qY}^*(\mathbf{x}), \quad \iota \cdot W'_{qY}(\mathbf{x}) = W'_{qY}{}^*(\mathbf{x}),$$

where  $W_{qY}^*(\mathbf{x})$  and  $W'_{qY}{}^*(\mathbf{x})$  are defined by expanding  $W_{qY}(\mathbf{x})$  and  $W'_{qY}(\mathbf{x})$  in the power sum functions  $p_n(\mathbf{x})$  and acting on the latter as in Equation 3.1.

3.2.1. *Remark.* In the limit  $q \rightarrow 0$ , both  $W_{qY}(\mathbf{x})$  and  $W'_{qY}(\mathbf{x})$  reduce to the same Schur function  $s_Y(\mathbf{x})$  labelled by the Young diagram  $Y$ , and their twisted versions  $W_{qY}^*(\mathbf{x})$  and  $W'_{qY}{}^*(\mathbf{x})$  reduce to the same Schur function  $s_{Y'}(\mathbf{x})$  labelled by the transpose Young diagram  $Y'$ .

3.3. **More Cauchy identities.** Starting from the identity,

$$(3.5) \quad \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(1-q^n)} p_n(\mathbf{x}) p_n(\mathbf{y}) \right) = \prod_{n=0}^{\infty} \frac{1}{(1-\mathbf{x}\mathbf{y}q^n)},$$

and the identities obtained by applying the involution 3.1 on the power sum functions  $p_n(\mathbf{x})$  in the Cauchy identity 2.19,

<sup>30</sup> Ch. I, p. 21 [27].

$$(3.6) \quad \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(1-q^n)} \iota \cdot p_n(\mathbf{x}) p_n(\mathbf{y}) \right) = \\ \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(1-q^n)} p_n(\mathbf{x}) \iota \cdot p_n(\mathbf{y}) \right) = \prod_{n=0}^{\infty} (1 + \mathbf{x} \mathbf{y} q^n),$$

and,

$$(3.7) \quad \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(1-q^n)} \iota \cdot p_n(\mathbf{x}) \iota \cdot p_n(\mathbf{y}) \right) = \prod_{n=0}^{\infty} \frac{1}{(1 - \mathbf{x} \mathbf{y} q^n)},$$

we obtain,

$$(3.8) \quad \sum_Y W_{q^{Y/Y_1}}^*(\mathbf{x}) W'_{q^{Y/Y_2}}(\mathbf{y}) = \prod_{n=0}^{\infty} (1 + \mathbf{x} \mathbf{y} q^n) \sum_Y W_{q^{Y_2/Y}}^*(\mathbf{x}) W'_{q^{Y_1/Y}}(\mathbf{y}),$$

$$(3.9) \quad \sum_Y W_{q^{Y/Y_1}}^*(\mathbf{x}) W'_{q^{Y/Y_2}}(\mathbf{y}) = \prod_{n=0}^{\infty} \left( \frac{1}{1 - \mathbf{x} \mathbf{y} q^n} \right) \sum_Y W_{q^{Y_2/Y}}^*(\mathbf{x}) W'_{q^{Y_1/Y}}(\mathbf{y})$$

Replacing  $q$  with  $q'$ ,

$$(3.10) \quad \sum_Y W_{q'^{Y/Y_1}}(\mathbf{x}) W'_{q'^{Y/Y_2}}(\mathbf{y}) = \prod_{n=0}^{\infty} (1 - \mathbf{x} \mathbf{y} q'^{n+1}) \sum_Y W_{q'^{Y_2/Y}}(\mathbf{x}) W'_{q'^{Y_1/Y}}(\mathbf{y}),$$

$$(3.11) \quad \sum_Y W_{q'^{Y/Y_1}}^*(\mathbf{x}) W'_{q'^{Y/Y_2}}(\mathbf{y}) = \prod_{n=0}^{\infty} \left( \frac{1}{1 + \mathbf{x} \mathbf{y} q'^{n+1}} \right) \sum_Y W_{q'^{Y_2/Y}}^*(\mathbf{x}) W'_{q'^{Y_1/Y}}(\mathbf{y}),$$

$$(3.12) \quad \sum_Y W_{q'^{Y/Y_1}}^*(\mathbf{x}) W'_{q'^{Y/Y_2}}(\mathbf{y}) = \prod_{n=0}^{\infty} (1 - \mathbf{x} \mathbf{y} q'^{n+1}) \sum_Y W_{q'^{Y_2/Y}}^*(\mathbf{x}) W'_{q'^{Y_1/Y}}(\mathbf{y}),$$

where we used

$$(3.13) \quad \exp \left( \pm \sum_{n=1}^{\infty} \frac{1}{1 - q'^n} \frac{x^n y^n}{n} \right) = \exp \left( \mp \sum_{n=1}^{\infty} \frac{q'^n}{1 - q'^n} \frac{x^n y^n}{n} \right) = \prod_{n=0}^{\infty} (1 - \mathbf{x} \mathbf{y} q'^{n+1})^{\pm 1},$$

to express the factors in Cauchy identities in terms of  $q$ .

#### 4. $pqt$ -FREE BOSONS AND $pqt$ -VERTEX OPERATORS

We recall basic facts related to Saito's  $pqt$ -Heisenberg algebras and  $pqt$ -vertex operators and note that Saito's deformation parameter  $p$  appears on equal footing with Macdonald's parameter  $q$ . Setting  $p = t$ , all dependence on  $p$  and on  $t$  disappears, and the remaining parameter  $q$  deforms Schur functions into  $q$ -Whittaker functions.

**4.1. Two  $pqt$ -Heisenberg algebras.** Saito's free-boson realization of the elliptic extension of the Ding-Iohara-Miki algebra is based on two  $pqt$ -Heisenberg algebras<sup>31</sup>,

$$(4.1) \quad [a_m, a_n] = m (1 - p^{|m|}) \left( \frac{1 - q^{|m|}}{1 - t^{|m|}} \right) \delta_{m+n,0}, \quad [b_m, b_n] = -m (1 - p'^{|m|}) \left( \frac{1 - q'^{|m|}}{1 - t'^{|m|}} \right) \delta_{m+n,0},$$

where  $p', \dots$ , stand for  $1/p, \dots$   $a_n$  and  $b_n$ ,  $n = 1, 2, \dots$  act as creation operators on the left vacuum state  $\langle 0|$ , and as annihilation operators on the right vacuum state  $|0\rangle$ , while  $a_n$  and  $b_n$ ,  $n = -1, -2, \dots$  act as annihilation operators on the left vacuum state  $\langle 0|$ , and as creation operators on the right vacuum state  $|0\rangle$ .

**4.1.1. Remark.** One should consider the writing of the second Heisenberg algebra in Equation 4.1 as short-hand notation. When performing computations, and particularly expansions in the various parameters, one should work in terms of the variables  $p < 1$ ,  $q < 1$ , and  $t < 1$ , rather than their inverses, which also makes it clear that the minus sign on the right hand side of the  $b$ -operator commutator is due to notation, and that both algebras have the same signature.

**4.2. Two-boson  $pqt$ -vertex operators.** From the  $pqt$ -Heisenberg algebras, Saito defines two-boson  $pqt$ -vertex operators<sup>32</sup>,

$$(4.2) \quad \phi_{\pm}^{pqt}(x) = \exp \left( \sum_{n=1}^{\infty} \left( \frac{1}{1-p^n} \right) \left( \frac{1-t^n}{1-q^n} \right) \frac{x^{\mp n}}{n} a_{\pm n} \right) \exp \left( \sum_{n=1}^{\infty} \left( \frac{1}{1-p'^n} \right) \left( \frac{1-t'^n}{1-q'^n} \right) \frac{x'^{\mp n}}{n} b_{\pm n} \right),$$

where  $x', \dots$ , stands for  $1/x, \dots$

**4.3. Two equivalent specializations.** In Equations 4.1 and 4.2,  $p$  is Saito's elliptic deformation parameter,  $q$  and  $t$  are Macdonald parameters,  $p$  and  $q$  appear on equal footing, and the following two specializations lead to identical results, up to a renaming of the parameters. One can either **1.** Set  $q \rightarrow t$ , so that all dependence on  $q$  and on  $t$  disappears, to work in terms of Schur functions that are deformed by Saito's elliptic deformation parameter  $p$ , or **2.** set  $p \rightarrow t$ , so that all dependence on  $p$  and on  $t$  disappears, to work in terms of Schur functions that are deformed by Macdonald's parameter  $q$ , which are  $q$ -Whittaker functions. In the following, we choose specialization **2** because that makes it clear that we are dealing with  $q$ -Whittaker functions whose properties follow directly from those of Macdonald functions.

<sup>31</sup> Our notation is slightly different from, but equivalent to Saito's notation.

<sup>32</sup> Equations (3.9) and (3.8) respectively, in [34], but in different notation. In particular, our  $\phi_{+}^{pqt}(x)$  and  $\phi_{-}^{pqt}(x)$  are Saito's  $\phi^*(p; x)$  and  $\phi(p; x)$ , respectively, and our  $b_n$  is Saito's  $-b_n$ .

## 5. $q$ -FREE BOSONS AND $q$ -VERTEX OPERATORS

We set  $q \rightarrow t$ , in Saito's  $pqt$ -Heisenberg algebras and  $pqt$ -vertex operators, so that all dependence on these parameters disappears, and we end up with pairs of expressions that depend on  $q$  and on  $1/q$ .

**5.1. Two  $q$ -Heisenberg algebras.** Consider the  $q$ -Heisenberg algebras, which are specializations of Saito's  $pqt$ -Heisenberg algebras,

$$(5.1) \quad [a_m, a_n] = m (1 - q^{|m|}) \delta_{m+n,0}, \quad [b_m, b_n] = -m (1 - q'^{|m|}) \delta_{m+n,0}, \quad [a_m, b_n] = 0,$$

and the  $q$ -vertex operators, which are specializations of Saito's  $pqt$ -vertex operators,

$$(5.2) \quad \phi_{q\pm}(x) = \exp \left( \sum_{n=1}^{\infty} \left( \frac{1}{1-q^n} \right) \frac{x^{\mp n}}{n} a_{\pm n} \right) \exp \left( \sum_{n=1}^{\infty} \left( \frac{1}{1-q'^n} \right) \frac{x'^{\mp n}}{n} b_{\pm n} \right)$$

**5.1.1. Remark.** Because the oscillators in Equation 5.2 have the same signs, when  $\phi_{\pm}$  acts on a pair of Young diagrams  $(Y_1, Y_2)$ , it generates a new pair of Young diagrams  $(W_1, W_2)$ , such that  $Y_1$  and  $W_1$  are interlacing, and  $Y_2$  and  $W_2$  are also be interlacing.

**5.2. One-boson  $q$ -vertex operators.** We define the one-boson  $q$ -vertex operators<sup>33</sup>,

$$(5.3) \quad \Gamma_{qa\pm}(x) = \exp \left( \sum_{n=1}^{\infty} \left( \frac{1}{1-q^n} \right) \frac{x^{\mp n}}{n} a_{\pm n} \right), \quad \Gamma_{q'b\pm}(x) = \exp \left( \sum_{n=1}^{\infty} \left( \frac{1}{1-q'^n} \right) \frac{x'^{\mp n}}{n} b_{\pm n} \right),$$

which factorize the two-boson  $q$ -vertex operators as<sup>34</sup>,

$$(5.4) \quad \phi_{+q}(x) = \Gamma_{qa+}(x) \Gamma_{q'b+}(x'), \quad \phi_{-q}(x) = \Gamma_{qa-}(x) \Gamma_{q'b-}(x'),$$

and satisfy the commutation relations<sup>35</sup>,

$$(5.5) \quad \Gamma_{qa+}(x') \Gamma_{qa-}(y) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{1-q^n} \frac{x^n y^n}{n} \right) \Gamma_{qa-}(y) \Gamma_{qa+}(x') = \frac{1}{(xy|q)_{\infty}} \Gamma_{qa-}(y) \Gamma_{qa+}(x'),$$

<sup>33</sup> The exponentials in definitions in Equation 5.3 differ by a minus sign from those typically used in the literature. The latter were followed in [16]. This amounts to a redefinition of the creation and annihilation operators  $a_n, n \in \mathbb{Z}$ , which does not change the Heisenberg algebra.

<sup>34</sup>  $\Gamma_{qa\pm}(x)$  and  $\Gamma_{q'b\pm}(x)$  are defined in exactly the same way apart from the fact that  $\Gamma_{qa\pm}(x)$  depends on the  $a$ -oscillators and the parameter  $q$ , while  $\Gamma_{q'b\pm}(x)$  depends on the  $b$ -oscillators and the inverse parameter  $1/q$ .

<sup>35</sup> To prove the commutation relations in Equations 5.5 and 5.6, we assume that  $q < 1$ , so that  $q' > 1$ , so that all expansions and resummations must be made with respect to  $q$ .

$$\begin{aligned}
(5.6) \quad & \Gamma_{q'b+}(x) \Gamma_{q'b-}(y') = \\
& \exp\left(-\sum_{n=1}^{\infty} \frac{1}{1-q'^n} \frac{x'^n y'^n}{n}\right) \Gamma_{q'b-}(y') \Gamma_{q'b+}(x) = (x' y' | q')_{\infty} \Gamma_{q'b-}(y') \Gamma_{q'b+}(x) = \\
& \exp\left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \frac{x'^n y'^n}{n}\right) \Gamma_{q'b-}(y') \Gamma_{q'b+}(x) = \frac{1}{(x' y' q | q)_{\infty}} \Gamma_{q'b-}(y') \Gamma_{q'b+}(x),
\end{aligned}$$

so that,

$$(5.7) \quad \langle 0 | \phi_{q+}(\mathbf{x}') \phi_{q-}(\mathbf{y}) | 0 \rangle = \frac{1}{\theta_q(\mathbf{x}, \mathbf{y})}$$

**5.3. An inverse vertex operator.** We also need to define the inverse vertex operator,

$$(5.8) \quad \Gamma'_{q'b-}(x) = \exp\left(-\sum_{n=1}^{\infty} \left(\frac{1}{1-q'^n}\right) \frac{x^n}{n} b_{-n}\right),$$

which satisfies the commutation relation,

$$\begin{aligned}
(5.9) \quad & \Gamma_{q'b+}(x) \Gamma'_{q'b-}(y') = \\
& \exp\left(+\sum_{n=1}^{\infty} \frac{1}{1-q'^n} \frac{x'^n y'^n}{n}\right) \Gamma'_{q'b-}(y') \Gamma_{q'b+}(x) = \frac{1}{(x' y' | q')_{\infty}} \Gamma'_{q'b-}(y') \Gamma_{q'b+}(x) = \\
& \exp\left(-\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \frac{x'^n y'^n}{n}\right) \Gamma'_{q'b-}(y') \Gamma_{q'b+}(x) = (x' y' q | q)_{\infty} \Gamma'_{q'b-}(y') \Gamma_{q'b+}(x)
\end{aligned}$$

$\Gamma'_{q'b-}(x)$ , as well as the vertex operator obtained from it by the action of the involution  $\iota$ , will be the only inverse vertex operators that we need.

## 6. THE POWER-SUM/HEISENBERG CORRESPONDENCE. THE $a$ -HEISENBERG ALGEBRA

*Starting from the  $q$ -Whittaker Cauchy identities, we obtain identities that involve operator-valued  $q$ -Whittaker functions that act on  $q$ -Whittaker states. In this section, we consider the  $a$ -Heisenberg algebra that depends on  $a_{\pm n}$  only.*

**6.1. An isomorphism.** Comparing the inner product of power-sum symmetric functions in the  $q$ -Whittaker basis, Equation 2.5, and the inner product of the right and left-states, Equation 2.9, we deduce that the power-sum symmetric function basis is isomorphic to the Fock space spanned by the left-states  $\langle a_Y |$ , as well as that spanned by the right-states  $| a_Y \rangle$ , where  $Y$  is a partition. In the left states, on which the operators  $a_n, n = 1, 2, \dots$ , act as creation operators, we have the correspondence,

$$(6.1) \quad p_n(\mathbf{x}) \rightleftharpoons a_n, \quad n \geq 1$$

In the right states, on which the operators  $a_n, n = -1, -2, \dots$ , act as creation operators, we have the correspondence,

$$(6.2) \quad p_n(\mathbf{x}) \rightleftharpoons a_{-n}, \quad n \geq 1$$

**6.2. Operator-valued  $q$ -Whittaker functions and Cauchy identities.** Since the power-sum symmetric functions form a complete basis, we expand the  $q$ -Whittaker functions in terms of the power-sum symmetric functions, then formally replace the latter with Heisenberg generators to obtain operator-valued  $q$ -Whittaker functions that act either on left-or on right-states that are labeled by  $q$ -Whittaker functions..

**6.3. The action of vertex operators on  $q$ -Whittaker states.** From Equations 2.6 and 2.19,

$$(6.3) \quad \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{1-q^n}\right) p_n(\mathbf{x}) p_n(\mathbf{y})\right) \sum_Y W_{q^{Y_1/Y}}(\mathbf{x}) W'_{q^{Y_2/Y}}(\mathbf{y}) \\ = \sum_Y W_{q^{Y/Y_2}}(\mathbf{x}) W'_{q^{Y/Y_1}}(\mathbf{y})$$

6.3.1. *The action of  $\Gamma_{q^{a_+}}$  on a left-state.* Using the power-sum/Heisenberg correspondence, Equation 6.2, on  $p_n(\mathbf{x})$  in Equation 6.3, we obtain the operator-valued  $q$ -Whittaker Cauchy identity,

$$(6.4) \quad \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{1-q^n}\right) a_n p_n(\mathbf{y})\right) \sum_Y W_{q^{Y_1/Y}}(\mathbf{a}_+) W'_{q^{Y_2/Y}}(\mathbf{y}) \\ = \sum_Y W_{q^{Y/Y_2}}(\mathbf{a}_+) W'_{q^{Y/Y_1}}(\mathbf{y})$$

From the definition of the  $\Gamma_{q^{a_+}}$  vertex operators, Equation 5.3,

$$(6.5) \quad \Gamma_{q^{a_+}}(\mathbf{y}') \sum_Y W_{q^{Y_1/Y}}(\mathbf{a}_+) W'_{q^{Y_2/Y}}(\mathbf{y}) = \sum_Y W_{q^{Y/Y_2}}(\mathbf{a}_+) W'_{q^{Y/Y_1}}(\mathbf{y})$$

where we have used the notation in section 1.7.10. Acting with each side of Equation 6.10 on a left vacuum state,

$$(6.6) \quad \sum_Y \langle W_{q^{Y_1/Y}} | W'_{q^{Y_2/Y}}(\mathbf{y}) \Gamma_{q^{a_+}}(\mathbf{y}') \rangle = \sum_Y \langle W_{q^{Y/Y_2}} | W'_{q^{Y/Y_1}}(\mathbf{y}) \rangle,$$

where  $\langle W_{q^{Y_1/Y_2}} |$  is a state in the free-boson Fock space obtained by the action of the operator-valued  $q$ -Whittaker function labelled by the skew Young diagram  $Y_1/Y_2$ , that is, by definition,

$$(6.7) \quad \langle \emptyset | W_{q^{Y_1/Y_2}} (\mathbf{a}_+) = \langle W_{q^{Y_1/Y_2}} |$$

Setting  $Y_2 = \emptyset$  in Equation 6.6, we force  $Y = \emptyset$  in the sum on the left hand side,

$$(6.8) \quad \langle W_{q^{Y_1}} | \Gamma_{qa+} (\mathbf{y}') = \sum_Y \langle W_{q^Y} | W'_{q^{Y/Y_1}} (\mathbf{y})$$

6.3.2. *The action of  $\Gamma_{qa-}$  on a right-state.* Using the power-sum/Heisenberg correspondence, Equation 6.2, on  $p_n (\mathbf{y})$  in Equation 6.3, we obtain the operator-valued  $q$ -Whittaker Cauchy identity,

$$(6.9) \quad \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{1}{1-q^n} \right) p_n (\mathbf{x}) a_{-n} \right) \sum_Y W_{q^{Y_1/Y}} (\mathbf{x}) W'_{q^{Y_2/Y}} (\mathbf{a}_-) \\ = \sum_Y W_{q^{Y/Y_2}} (\mathbf{x}) W'_{q^{Y/Y_1}} (\mathbf{a}_-)$$

From the definition of the  $\Gamma_{qa-}$  vertex operators in Equation 5.3,

$$(6.10) \quad \Gamma_{qa-} (\mathbf{x}) \sum_Y W_{q^{Y_1/Y}} (\mathbf{x}) W'_{q^{Y_2/Y}} (\mathbf{a}_-) = \sum_Y W_{q^{Y/Y_2}} (\mathbf{x}) W'_{q^{Y/Y_1}} (\mathbf{a}_-)$$

Acting with each side of Equation 6.10 on a right vacuum state,

$$(6.11) \quad \Gamma_{qa-} (\mathbf{x}) \sum_Y W_{q^{Y_1/Y}} (\mathbf{x}) |W'_{q^{Y_2/Y}}\rangle = \sum_Y W_{q^{Y/Y_2}} (\mathbf{x}) |W'_{q^{Y/Y_1}}\rangle,$$

where  $|W'_{q^{Y_1/Y_2}}\rangle$  is a state in the free boson Fock space obtained by the action of the operator-valued  $q$ -Whittaker function labelled by the skew Young diagram  $Y_1/Y_2$ , that is, by definition,

$$(6.12) \quad W'_{q^{Y_1/Y_2}} (\mathbf{a}_-) | \emptyset \rangle = |W'_{q^{Y_1/Y_2}}\rangle$$

Setting  $Y_1 = \emptyset$  in Equation 6.11, we force  $Y = \emptyset$  in the sum on the left hand side,

$$(6.13) \quad \Gamma_{qa-} (\mathbf{x}) |W'_{q^{Y_2}}\rangle = \sum_Y W_{q^{Y/Y_2}} (\mathbf{x}) |W'_{q^Y}\rangle$$

6.3.3. *The action of  $\Gamma_{qa-}$  on a left-state, and  $\Gamma_{qa+}$  on a right-state.* Using Equations 6.8 and 6.13, then Equation 2.19,

$$(6.14) \quad \langle W_{qY_1} | \Gamma_{qa+}(\mathbf{x}') \Gamma_{qa-}(\mathbf{y}) | W'_{qY_2} \rangle = \\ = \sum_Y W'_{qY/Y_1}(\mathbf{x}) W_{qY/Y_2}(\mathbf{y}) = \prod_{n=0}^{\infty} \left( \frac{1}{1 - \mathbf{x}\mathbf{y}q^n} \right) \sum_Y W'_{qY_2/Y}(\mathbf{x}) W_{qY_1/Y}(\mathbf{y})$$

Using the  $q$ -vertex operator commutation relation, Equation 5.5, then inserting a complete set of orthonormal states,

$$(6.15) \quad \langle W_{qY_1} | \Gamma_{qa+}(\mathbf{x}') \Gamma_{qa-}(\mathbf{y}) | W'_{qY_2} \rangle = \\ \prod_{n=0}^{\infty} \left( \frac{1}{1 - \mathbf{x}\mathbf{y}q^n} \right) \langle W_{qY_1} | \Gamma_{qa-}(\mathbf{y}) \Gamma_{qa+}(\mathbf{x}') | W'_{qY_2} \rangle = \\ \prod_{n=0}^{\infty} \left( \frac{1}{1 - \mathbf{x}\mathbf{y}q^n} \right) \sum_Y \langle W_{qY_1} | \prod_{j=1}^{\infty} \Gamma_{qa-}(y_j) | W'_{qY} \rangle \langle W_{qY} | \prod_{i=1}^{\infty} \Gamma_{qa+}(x'_i) | W'_{qY_2} \rangle$$

Comparing the right hand sides of Equations 6.14 and 6.15, we obtain the two identities,

$$(6.16) \quad \left( \langle W_{qY_1} | \Gamma_{qa-}(\mathbf{y}) \right) | W'_{qY_2} \rangle = W_{qY_1/Y_2}(\mathbf{y}), \\ \langle W_{qY_1} | \left( \Gamma_{qa+}(\mathbf{x}') | W'_{qY_2} \rangle \right) = W'_{qY_2/Y_1}(\mathbf{x}),$$

where the brackets indicate the state acted on by the vertex operators. Since the states  $\langle W_{qY_1} |$  form a basis of left-states, the states  $| W'_{qY_2} \rangle$  form a basis of right-states, and given the  $q$ -inner product Equation 2.5,

$$(6.17) \quad \langle W_{qY_1} | \Gamma_{qa-}(\mathbf{y}) = \sum_Y \langle W_{qY} | \alpha_Y(\mathbf{y}), \quad \Gamma_{qa+}(\mathbf{x}') | W'_{qY_2} \rangle = \sum_Y \beta_Y(\mathbf{x}) | W'_{qY_2} \rangle,$$

where  $\alpha_Y(\mathbf{y})$  and  $\beta_Y(\mathbf{x})$  are expansion coefficients that carry the dependence on  $\mathbf{x}$  and  $\mathbf{y}$ , while the expansion is in the set of Young diagrams  $Y$ . Using Equation 6.16, we determine  $\alpha_Y(\mathbf{y})$  and  $\beta_Y(\mathbf{x})$ ,

$$(6.18) \quad \langle W_{qY_1} | \Gamma_{qa-}(\mathbf{y}) = \sum_Y \langle W_{qY} | W_{qY_1/Y}(\mathbf{y}), \\ \Gamma_{qa+}(\mathbf{x}') | W'_{qY_1} \rangle = \sum_Y W'_{qY_1/Y}(\mathbf{x}) | W'_{qY} \rangle$$

**6.4. The action of vertex operators on twisted  $q$ -Whittaker states.** Replacing the  $q$ -Whittaker functions with their twisted version, we obtain the following identities for the action of the vertex operators,

$$(6.19) \quad \left( \langle W_{qY_1}^* | \Gamma_{qa-}(\mathbf{y}) \rangle | W_{qY_2}^* \rangle \right) = W_{qY_1/Y_2}^*(\mathbf{y}),$$

$$\langle W_{qY_1}^* | \left( \Gamma_{qa+}(\mathbf{x}') | W_{qY_2}^* \rangle \right) = W_{qY_2/Y_1}^*(\mathbf{x}),$$

$$(6.20) \quad \langle W_{qY_1}^* | \Gamma_{qa-}(\mathbf{y}) \rangle = \sum_Y \langle W_{qY}^* | W_{qY_1/Y}^*(\mathbf{y}) \rangle,$$

$$\Gamma_{qa+}(\mathbf{x}') | W_{qY_1}^* \rangle = \sum_Y W_{qY_1/Y}^*(\mathbf{x}) | W_{qY}^* \rangle$$

## 7. THE POWER-SUM/HEISENBERG CORRESPONDENCE. THE $b$ -HEISENBERG ALGEBRA

Starting from the  $q$ -Whittaker Cauchy identities, we obtain identities that involve operator-valued  $q$ -Whittaker functions that act on  $q$ -Whittaker states. In this section, we consider the  $b$ -Heisenberg algebra that depends on  $b_{\pm n}$  only.

**7.1. An isomorphism.** Due to the extra minus sign in the commutator of  $b_{\pm n}$  bosons, the correspondence between the power-sum functions and oscillators is given by,

$$(7.1) \quad p_n(\mathbf{x}) \rightleftharpoons b_n, \quad n \geq 1,$$

for the power-sum functions in the left states, and

$$(7.2) \quad p_n(\mathbf{x}) \rightleftharpoons -b_{-n}, \quad n \geq 1,$$

for the power-sum functions in the right states<sup>36</sup>.

**7.2. The action of operator-valued  $q$ -Whittaker functions on states.** From the Cauchy identity for skew  $q$ -Whittaker functions, Equation 2.19, with  $q$  replaced by  $q'$ , and use Equation 2.6 to obtain,

$$(7.3) \quad \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{1}{1-q'^n} \right) p_n(\mathbf{x}) p_n(\mathbf{y}) \right) \sum_Y W_{q'Y_1/Y}(\mathbf{x}) W'_{q'Y_2/Y}(\mathbf{y})$$

$$= \sum_Y W_{q'Y/Y_2}(\mathbf{x}) W'_{q'Y/Y_1}(\mathbf{y})$$

<sup>36</sup> The minus sign on the right hand side of Equation 7.2 follows from the minus sign on the right hand side of the commutator of the  $b$ -oscillators, Equation 4.1, and the fact that we write all expressions that involve  $b$ -oscillators in terms of  $q'$  to maintain similarity to those that involve  $a$ -oscillators.

7.2.1. *The action of  $\Gamma_{qb_+}$  on a left-state.* Using the power-sum/Heisenberg correspondence, Equation 7.2, on the power sum function  $p_n(\mathbf{x})$ , on both sides of Equation 7.3, we introduce free-boson mode operators that act as creation operators on a left-state, to obtain the operator-valued  $q$ -Whittaker Cauchy identity,

$$(7.4) \quad \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{1-q'^n}\right) b_n p_n(\mathbf{y})\right) \sum_Y W_{q'Y_1/Y}(\mathbf{b}_+) W'_{q'Y_2/Y}(\mathbf{y}) \\ = \sum_Y W_{q'Y/Y_2}(\mathbf{b}_+) W'_{q'Y/Y_1}(\mathbf{y}),$$

From the definition of the  $\Gamma_{q'b_+}$  vertex operators, Equation 5.3,

$$(7.5) \quad \Gamma_{q'b_+}(\mathbf{y}') \sum_Y W_{q'Y_1/Y}(\mathbf{b}_+) W'_{q'Y_2/Y}(\mathbf{y}) = \sum_Y W_{q'Y/Y_2}(\mathbf{b}_+) W'_{q'Y/Y_1}(\mathbf{y})$$

Acting with each side of Equation 7.5 on a left vacuum state,

$$(7.6) \quad \sum_Y \langle W_{q'Y_1/Y} | W'_{q'Y_2/Y}(\mathbf{y}) \Gamma_{q'b_+}(\mathbf{y}') = \sum_Y \langle W_{q'Y/Y_2} | W'_{q'Y/Y_1}(\mathbf{y}),$$

where  $\langle W_{q'Y_1/Y_2} |$  is a state in the free-boson Fock space obtained by the action of the operator-valued  $q$ -Whittaker function labelled by the skew Young diagram  $Y_1/Y_2$ ,

$$(7.7) \quad \langle \emptyset | W_{q'Y_1/Y_2}(\mathbf{b}_+) = \langle W_{q'Y_1/Y_2} |$$

Setting  $Y_2 = \emptyset$  in Equation 7.6, we force  $Y = \emptyset$  in the sum on the left hand side,

$$(7.8) \quad \langle W_{q'Y_1} | \Gamma_{q'b_+}(\mathbf{y}') = \sum_Y \langle W_{q'Y} | W'_{q'Y/Y_1}(\mathbf{y})$$

7.2.2. *The action of  $\Gamma'_{qb_-}$  on a right-state.* Using the power-sum/Heisenberg correspondence, Equation 7.2, on  $p_n(\mathbf{y})$ , to introduce free-boson mode operators that act as creation operators on a right-state,

$$(7.9) \quad \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{1-q'^n}\right) p_n(\mathbf{x}) b_{-n}\right) \sum_Y W_{q'Y_1/Y}(\mathbf{x}) W'_{q'Y_2/Y}(\mathbf{b}_-) \\ = \sum_Y W_{q'Y/Y_2}(\mathbf{x}) W'_{q'Y/Y_1}(\mathbf{b}_-)$$

From the definition of the  $\Gamma'_{qb_-}$  vertex operator, Equation 5.8,

$$(7.10) \quad \Gamma'_{q'b-}(\mathbf{x}) \sum_Y W_{q'Y_1/Y}(\mathbf{x}) W'_{q'Y_2/Y}(\mathbf{b}_-) = \sum_Y W_{q'Y/Y_2}(\mathbf{x}) W'_{q'Y/Y_1}(\mathbf{b}_-)$$

Acting with each side of Equation 7.10 on a right vacuum state,

$$(7.11) \quad \Gamma'_{q'b-}(\mathbf{x}) \sum_Y W_{q'Y_1/Y}(\mathbf{x}) |W'_{q'Y_2/Y}\rangle = \sum_Y W_{q'Y/Y_2}(\mathbf{x}) |W'_{q'Y/Y_1}\rangle,$$

where  $|W'_{q'Y_1/Y_2}\rangle$  is a state in the free boson Fock space obtained by the action of the operator-valued  $q$ -Whittaker function labelled by the skew Young diagram  $Y_1/Y_2$ ,

$$(7.12) \quad W'_{q'Y_1/Y_2}(\mathbf{b}_-) |\emptyset\rangle = |W'_{q'Y_1/Y_2}\rangle$$

Setting  $Y_1 = \emptyset$  in Equation 7.11, we force  $Y = \emptyset$  in the sum on the left hand side,

$$(7.13) \quad \Gamma'_{q'b-}(\mathbf{x}) |W'_{q'Y_2}\rangle = \sum_Y W_{q'Y/Y_2}(\mathbf{x}) |W'_{q'Y}\rangle$$

7.2.3. *The action of  $\Gamma'_{q'b-}$  on a left-state, and  $\Gamma_{q'b+}$  on a right-state.* Using Equations 7.8 and 7.13, then Equation 2.19,

$$(7.14) \quad \langle W_{q'Y_1} | \Gamma_{q'b+}(\mathbf{x}') \Gamma'_{q'b-}(\mathbf{y}) | W'_{q'Y_2} \rangle = \sum_Y W'_{q'Y/Y_1}(\mathbf{x}) W_{q'Y/Y_2}(\mathbf{y}) = \prod_{n=0}^{\infty} \left( \frac{1}{1 - \mathbf{x}\mathbf{y}q^n} \right) \sum_Y W'_{q'Y_2/Y}(\mathbf{x}) W_{q'Y_1/Y}(\mathbf{y})$$

Using the  $q$ -vertex operator commutation relation, Equation 5.9, the left hand side of Equation 7.14 can be re-written as,

$$(7.15) \quad \langle W_{q'Y_1} | \Gamma_{q'b+}(\mathbf{x}') \Gamma'_{q'b-}(\mathbf{y}) | W'_{q'Y_2} \rangle = \prod_{n=0}^{\infty} \left( \frac{1}{1 - \mathbf{x}\mathbf{y}q^n} \right) \langle W_{q'Y_1} | \Gamma'_{q'b-}(\mathbf{y}) \Gamma_{q'b+}(\mathbf{x}') | W'_{q'Y_2} \rangle = \prod_{n=0}^{\infty} \left( \frac{1}{1 - \mathbf{x}\mathbf{y}q^n} \right) \sum_Y \langle W_{q'Y_1} | \Gamma'_{q'b-}(\mathbf{y}) | W'_{q'Y} \rangle \langle W_{q'Y} | \Gamma_{q'b+}(\mathbf{x}') | W'_{q'Y_2} \rangle$$

Comparing the right hand side of Equation 7.14 and that of Equation 7.15,

$$(7.16) \quad \left( \langle W_{q'Y_1} | \Gamma'_{q'b-}(\mathbf{y}) \right) | W'_{q'Y_2} \rangle = W_{q'Y_1/Y_2}(\mathbf{y}), \\ \langle W_{q'Y_1} | \left( \Gamma_{q'b+}(\mathbf{x}') | W'_{q'Y_2} \rangle \right) = W'_{q'Y_2/Y_1}(\mathbf{x}),$$

where the brackets indicate the state acted on by the vertex operators. Since the states  $\langle W_{qY_1} |$  form a basis of left-states, and  $|W'_{qY_2}\rangle$  form a basis of right-states, and given the  $q$ -inner product Equation 2.5 with  $q$  replaced by  $q'$ ,

$$(7.17) \quad \langle W_{q'Y_1} | \Gamma'_{q'b-}(\mathbf{y}) = \sum_Y \langle W_{q'Y} | \alpha_Y(\mathbf{y}), \quad \Gamma_{qb+}(\mathbf{x}' | W'_{qY_2}\rangle = \sum_Y \beta_Y(\mathbf{x}) | W'_{q'Y_2}\rangle,$$

where  $\alpha_Y(\mathbf{y})$  and  $\beta_Y(\mathbf{x})$  are expansion coefficients that carry the dependence on the variables  $(\mathbf{x})$  and  $(\mathbf{y})$ , while the expansion is in the set of Young diagrams  $Y$ . Using Equation 6.16, we determine  $\alpha_Y(\mathbf{y})$  and  $\beta_Y(\mathbf{x})$ ,

$$(7.18) \quad \langle W_{q'Y_1} | \Gamma'_{q'b-}(\mathbf{y}) = \sum_Y \langle W_{q'Y} | W_{q'Y_1/Y}(\mathbf{y}), \\ \Gamma_{qb+}(\mathbf{x}' | W'_{q'Y_1}\rangle = \sum_Y W'_{q'Y_1/Y}(\mathbf{x}) | W'_{q'Y}\rangle$$

**7.3. Twisted  $q$ -Whittaker function identities.** Replacing the  $q$ -Whittaker functions with their twisted versions,

$$(7.19) \quad (\langle W_{q'Y_1}^* | \Gamma'_{q'b-}(\mathbf{y}) \rangle | W_{q'Y_2}^* \rangle = W_{q'^*Y_1/Y_2}(\mathbf{y}), \\ \langle W_{q'Y_1}^* | (\Gamma_{qb+}(\mathbf{x}' | W_{q'Y_2}^* \rangle) = W_{q'Y_2/Y_1}^*(\mathbf{x}),$$

$$(7.20) \quad \langle W_{q'Y_1}^* | \Gamma'_{q'b-}(\mathbf{y}) = \sum_Y \langle W_{q'Y}^* | W_{q'Y_1/Y}^*(\mathbf{y}), \\ \Gamma_{qb+}(\mathbf{x}' | W_{q'Y_1}^* \rangle = \sum_Y W_{q'Y_1/Y}^*(\mathbf{x}) | W_{q'Y}^* \rangle$$

Applying the twist  $\iota$ , Equation 3.1, on  $p_n(\mathbf{y})$  on both sides of Equation 7.16, renaming the variables, and using,

$$(7.21) \quad \iota \cdot \Gamma'_{q'b-}(x) = \Gamma_{q'b-}(-x),$$

$$(7.22) \quad (\langle W_{q'Y_1} | \Gamma_{q'b-}(-\mathbf{x}) \rangle | W_{q'Y_2}\rangle = W_{q'Y_1/Y_2}^*(\mathbf{x}), \\ \langle W_{q'Y_1} | \Gamma_{q'b-}(-\mathbf{x}) = \sum_Y \langle W_{q'Y} | W_{q'Y_1/Y}^*(\mathbf{x})$$

## 8. $q$ -WHITTAKER PAIRS AND OPERATOR-VALUED IDENTITIES

We define pairs of  $q$ -Whittaker functions, and derive their Cauchy identities.

### 8.1. $q$ -Whittaker and twisted $q$ -Whittaker pairs.

$$(8.1) \quad \mathbf{W}_{qY/R}(\mathbf{x}) = W_{qY_1/R_1}(\mathbf{x}) W_{q'Y_2/R_2}^*(-\mathbf{x}'), \quad \mathbf{W}'_{qY/R}(\mathbf{x}) = W'_{qY_1/R_1}(\mathbf{x}) W'_{q'Y_2/R_2}(\mathbf{x}')$$

$$(8.2) \quad \mathbf{W}_{qY/R}^*(\mathbf{x}) = W_{qY_1/R_1}^*(\mathbf{x}) W_{q'Y_2/R_2}(-\mathbf{x}'), \quad \mathbf{W}'_{qY/R}{}^*(\mathbf{x}) = W'_{qY_1/R_1}{}^*(\mathbf{x}) W'_{q'Y_2/R_2}{}^*(\mathbf{x}')$$

8.2. **Cauchy identities for  $q$ -Whittaker pairs.** From the Cauchy identities in Equations 2.19, 3.11, 3.9, and applications of the involution  $\iota$ ,

$$(8.3) \quad \sum_Y \mathbf{W}_{qY/R}(\mathbf{x}) \mathbf{W}'_{qY/S}(\mathbf{y}) = \frac{1}{\theta_q(\mathbf{x}\mathbf{y})} \sum_W \mathbf{W}_{qS/W}(\mathbf{x}) \mathbf{W}'_{qR/W}(\mathbf{y})$$

$$(8.4) \quad \sum_Y \mathbf{W}_{qY/R}^*(\mathbf{x}) \mathbf{W}'_{qY/S}{}^*(\mathbf{y}) = \frac{1}{\theta_q(\mathbf{x}\mathbf{y})} \sum_W \mathbf{W}_{qS/W}^*(\mathbf{x}) \mathbf{W}'_{qR/W}{}^*(\mathbf{y})$$

$$(8.5) \quad \sum_Y \mathbf{W}_{qY/R}(\mathbf{x}) \mathbf{W}'_{qY/S}{}^*(\mathbf{y}) = \theta_q(-\mathbf{x}\mathbf{y}) \sum_W \mathbf{W}_{qS/W}(\mathbf{x}) \mathbf{W}'_{qR/W}{}^*(\mathbf{y})$$

$$(8.6) \quad \sum_Y \mathbf{W}_{qY/R}^*(\mathbf{x}) \mathbf{W}'_{qY/S}(\mathbf{y}) = \theta_q(-\mathbf{x}\mathbf{y}) \sum_W \mathbf{W}_{qS/W}^*(\mathbf{x}) \mathbf{W}'_{qR/W}(\mathbf{y})$$

## 9. THE ELLIPTIC VERTEX

We construct an elliptic extension of the refined topological vertex using  $q$ -Whittaker functions, and a twisted version of the same vertex using twisted  $q$ -Whittaker functions.

We construct the elliptic vertex  $\mathcal{E}_{Y_1 Y_2 Y_3}(x, y)$  in five steps.

**9.1. Step 1. From  $Y_3$  to an infinite sequence of vertex operators.** We consider the *finite* Young diagram  $Y_3$  that labels the preferred leg of the vertex that we wish to construct, see Figure 9.1, position it as in Figure 9.2, and consider its *infinite profile*, which consists of upward and downward segments ( $/, \backslash$ ), that (scanning the profile from  $-\infty$  all the way to the left, to  $\infty$  all the way to the right) are all upward sufficiently far to the left, and all downward sufficiently far to the right, as indicated in Figure 9.3. We map this infinite profile to a Maya diagram [29], that consists white and black stones ( $\circ, \bullet$ ), then we map the latter to an infinite sequence of  $q$ -vertex operators, that we denote by  $\prod_{\text{Maya}(Y_3)} \phi_{q\pm}$ <sup>37</sup>, using the bijections<sup>38</sup>,

$$(9.1) \quad / \Leftrightarrow \circ \Leftrightarrow \phi_{q-}, \quad \backslash \Leftrightarrow \bullet \Leftrightarrow \phi_{q+}$$

<sup>37</sup> The vertex operators  $\phi_{q\pm}$  are defined in Equation 5.4 in terms of one-boson vertex operators that are *not* inverted.

<sup>38</sup> We could have skipped the intermediate step of mapping to a Maya diagram, but we prefer to keep because it can be useful in related contexts.

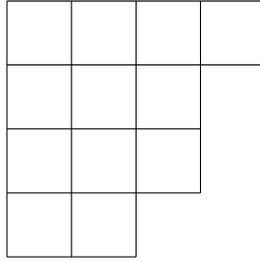


FIGURE 9.1. A Young diagram  $Y = (4, 3, 3, 2)$

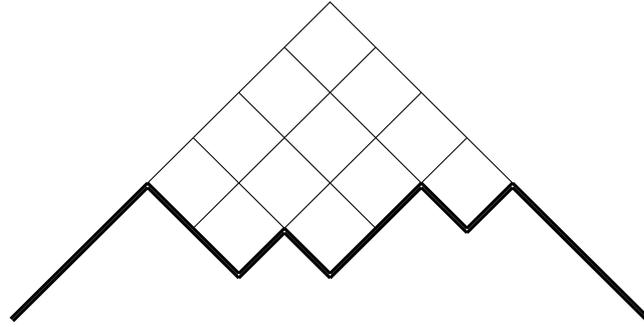


FIGURE 9.2. A tilted Young diagram  $Y$  and its infinite profile indicated by a heavy line.

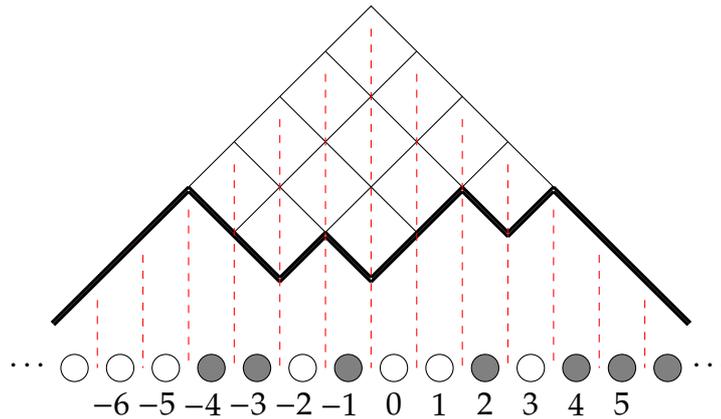


FIGURE 9.3. The tilted Young diagram, its infinite profile, and the corresponding Maya diagram, which gives the Young diagram/Maya diagram correspondence for  $Y = (4, 4, 3, 1)$ . The integer below a stone is its position in the Maya diagram. The apex of the inverted Young diagram is located between positions  $-1$  and  $0$ .

**9.2. Step 2. Choosing the arguments of the vertex operators.** We choose the arguments of the  $q$ -vertex operators to be,

$$(9.2) \quad \phi_{q+} \left( x^{-i} y^{y'_{3,i}} \right), \quad \phi_{q-} \left( y^{j-1} x^{-y_{3,j}} \right),$$

where  $y_{3,i}$  is the length of the  $i$ -column of the Young diagram  $Y_3$  that labels the preferred leg of the vertex, and  $y'_{3,j}$  is the length of the  $j$ -column of the transpose Young diagram  $Y'_3$ .

9.2.1. *Example.* The Young diagram/Maya diagram correspondence in Figure 9.3 leads to the vertex-operator sequence,

$$(9.3) \quad \left( \prod_{\text{Maya}(Y_3)} \phi_{q\pm} \right) = \cdots \phi_{q+}(x^{-5}) \phi_{q-}(x^{-4}) \phi_{q-}(yx^{-4}) \phi_{q+}(x^{-4}y^2) \phi_{q-}(y^2x^{-3}) \\ \phi_{q+}(x^{-3}y^3) \phi_{q+}(x^{-2}y^3) \phi_{q-}(y^3x^{-1}) \phi_{q+}(x^{-1}y^4) \phi_{q-}(y^4) \cdots$$

**9.3. Step 3. From the infinite sequence of vertex operators to an expectation value.** We evaluate the sequence  $\prod_{\text{Maya}(Y_3)} \phi_{q\pm}$  between a left-state that corresponds to a  $q$ -Whittaker pair labelled by a pair of Young diagrams  $\mathbf{Y}_1$ <sup>39</sup>

and a right-state labelled by a dual  $q$ -Whittaker pair labelled by a pair of Young diagrams  $\mathbf{Y}_2$ ,

$$(9.4) \quad \mathcal{E}_{\mathbf{Y}_1 \mathbf{Y}_2 Y_3}^{\text{unnorm}}(x, y) = \langle \mathbf{W}_{q\mathbf{Y}_1} | \left( \prod_{\text{Maya}(Y_3)} \phi_{q\pm} \right) | \mathbf{W}'_{q\mathbf{Y}_2} \rangle$$

To evaluate the expectation value in Equation 9.4, we ‘order’ the sequence in Equation 9.4 *via* an infinite number of commutations that put all  $\phi_{q+}$  vertex operators on the right, and all  $\phi_{q-}$  vertex operators on the left. From Equation 5.7,

$$(9.5) \quad \phi_{q+}(x^{-i}y^{y'_{3,i}}) \phi_{q-}(y^{j-1}x^{-y_{3,j}}) = \\ \prod_{m,n=1}^{\infty} \left( \frac{1}{\theta_q(x^{-y_{3,n}+m}y^{-y'_{3,m}+n})} \right) \phi_{q-}(y^{j-1}x^{-y_{3,j}}) \phi_{q+}(x^{-i}y^{y'_{3,i}}),$$

where  $y_{3,i}^+ = y_{3,i} + 1$ , and  $y_{3,i}$  is the length of the  $i$ -column in  $Y_3$ . Since  $\phi_{q+}$  is attached to a segment  $/$  in the extended profile of  $Y_3$ , and  $\phi_{q-}$  is attached to an adjacent segment  $\backslash$  to the right of the former, the commutation relation, Equation 9.5 describes replacing the adjacent pair  $/\backslash$  with the pair  $\swarrow$ , adding a cell to  $Y_3$ , to generate a Young diagram that is larger by one cell. The exponents that appear in the factor on the right hand side of Equation 9.5 have simple interpretations,

$$(9.6) \quad y_{3,i} - j = L_{\square}, \quad y'_{3,j} - i = A_{\square},$$

where  $A_{\square}$  and  $L_{\square}$  are the arm-length and the leg-length of the cell  $\square$  that is added to  $Y_3$  *via* the commutation in Equation 9.5, to generate a larger Young diagram, that is  $\square \notin Y_3$ . Inserting the sequence  $\prod_{\text{Maya}(Y_3)} \phi_{q\pm}$  between a left-state  $\langle \mathbf{W}_{q\mathbf{Y}_1} |$ , and a right-state  $| \mathbf{W}'_{q\mathbf{Y}_2} \rangle$ , then commuting the (infinitely-many)  $\phi_{q+}$  vertex operators to the right of the  $\phi_{q-}$  vertex operators,

<sup>39</sup> The  $q$ -Whittaker pair that labels the left states were created by the action of pairs of one-boson vertex operators, one of which, that which depends on  $b$ -oscillators is inverted.

$$(9.7) \quad \langle \mathbf{W}_{q\Upsilon_1} | \left( \prod_{i=1}^{\infty} \phi_{q+} (x^{-i} y^{y'_{3,i}}) \right) \left( \prod_{j=1}^{\infty} \phi_{q-} (y^{j-1} x^{-y_{3,j}}) \right) | \mathbf{W}'_{q\Upsilon_2} \rangle =$$

$$\left( \prod_{\square \notin Y_3} \frac{1}{\theta_q (x^{-L_{\square}} y^{-A_{\square}^+} q^n)} \right) \langle \mathbf{W}_{q\Upsilon_1} | \left( \prod_{j=1}^{\infty} \phi_{q-} (y^{j-1} x^{-y_{3,j}}) \right) \left( \prod_{i=1}^{\infty} \phi_{q+} (x^{-i} y^{y'_{3,i}}) \right) | \mathbf{W}'_{q\Upsilon_2} \rangle$$

9.4. **Step 4. From an expectation value to the unnormalized elliptic vertex.** Using the identities,

$$(9.8) \quad \langle \mathbf{W}_{q\Upsilon_1} | \phi_{q-} (\mathbf{y}) = \sum_{\Upsilon} \langle \mathbf{W}_{q\Upsilon} | \mathbf{W}_{q\Upsilon_1/\Upsilon} (\mathbf{y}), \phi_{q+} (\mathbf{x}') | \mathbf{W}'_{q\Upsilon_1} \rangle = \sum_{\Upsilon} \mathbf{W}'_{q\Upsilon_1/\Upsilon} (\mathbf{x}) | \mathbf{W}'_{q\Upsilon} \rangle,$$

in Equation 9.7, we obtain the *unnormalized* elliptic vertex,

$$(9.9) \quad \mathcal{E}_{Y_1 Y_2 Y_3}^{unnorm} (\mathbf{x}, \mathbf{y}) = \langle \mathbf{W}_{q\Upsilon_1} | \left( \prod_{i=1}^{\infty} \phi_{q+} (x^{-i} y^{y'_{3,i}}) \right) \left( \prod_{j=1}^{\infty} \phi_{q-} (y^{j-1} x^{-y_{3,j}}) \right) | \mathbf{W}'_{q\Upsilon_2} \rangle =$$

$$\left( \prod_{\square \notin Y_3} \frac{1}{\theta_q (x^{-L_{\square}} y^{-A_{\square}^+})} \right) \sum_{\Upsilon} \mathbf{W}_{\Upsilon_1/\Upsilon} (y^{J-1} x^{-Y_3}) \mathbf{W}'_{\Upsilon_2/\Upsilon} (x^{\iota} y^{-Y'_3}),$$

where  $\iota = (1, 2, \dots)$ ,  $J = (1, 2, \dots)$ , and the arguments in  $\mathbf{W}_{\Upsilon_1/\Upsilon} (y^{J-1} x^{-Y_3})$  and  $\mathbf{W}'_{\Upsilon_2/\Upsilon} (x^{\iota} y^{-Y'_3})$ , should be understood in the sense of Section 1.7.8.

9.5. **Step 5. From the unnormalized to the normalized elliptic topological vertex.** To normalize the expression in Equation 9.9 such that  $\mathcal{E}_{\emptyset \emptyset \emptyset} = 1$ , we divide it by the elliptic version of the  $xy$ -refined  $q$ -MacMahon partition function, that is,

$$(9.10) \quad \mathcal{M}(x, y, q) = \prod_{i,j=1}^{\infty} \frac{1}{\theta_q (x^i y^{j-1})},$$

Using the identity,

$$(9.11) \quad \left( \prod_{\square \notin Y_3} \frac{1}{1 - x^{-L_{\square}} y^{-A_{\square}^+} q^n} \right) \left( \prod_{i,j=1}^{\infty} \frac{1}{1 - x^i y^{j-1} q^n} \right)^{-1} = \left( \prod_{\square \in Y_3} \frac{1}{1 - x^{L_{\square}^+} y^{A_{\square}} q^n} \right),$$

which follows from Equations 2.8 and 2.11 in [6]. The final, normalised elliptic vertex  $\mathcal{E}_{Y_1 Y_2 Y_3} (x, y)$  is,

$$(9.12) \quad \mathcal{E}_{Y_1 Y_2 Y_3} (x, y) = \frac{\mathcal{E}_{Y_1 Y_2 Y_3}^{unnorm} (x, y)}{\mathcal{E}_{\emptyset \emptyset \emptyset}^{unnorm} (x, y)},$$

where,

$$(9.13) \quad \mathcal{E}_{Y_1 Y_2 Y_3}(x, y) = \prod_{\square \in Y_3} \frac{1}{\theta_q(x^{L_{\square Y_3}^+} y^{A_{\square Y_3}^+})} \sum_Y \mathbf{W}_{qY_1/Y}(y^{J-1} x^{-Y_3}) \mathbf{W}'_{qY_2/Y}(x^t y^{-Y_3'}),$$

$\iota = (1, 2, \dots)$ , and the arguments in  $\mathbf{W}_{Y_1/Y}(y^{J-1} x^{-Y_3})$  and  $\mathbf{W}'_{qY_2/Y}(x^t y^{-Y_3'})$  are in the sense of Section 1.7.8.

**9.6. The twisted version of the vertex.** We define the twisted version of the vertex  $\mathcal{E}^*$ , in the same way that we defined  $\mathcal{E}$ , but with the choice of arguments in Step 2 changed to,

$$(9.14) \quad \phi_{q+}(x^{-j+1} y^{y'_{3,i}}), \quad \phi_{q-}(y^i x^{-y_{3,i}}),$$

and calculate the expectation value in the twisted basis, Step 3, as <sup>40</sup>,

$$(9.15) \quad \mathcal{E}_{Y_1 Y_2 Y_3}^{\star \text{unnorm}}(x, y) = \langle \mathbf{W}_{qY_1}^{\star} | \left( \prod_{\text{Maya}(Y_3)} \phi_{q\pm} \right) | \mathbf{W}_{qY_2}^{\star} \rangle$$

A parallel calculation leads to the following expression for the twisted version of the elliptic vertex,

$$(9.16) \quad \mathcal{E}_{Y_1 Y_2 Y_3}^{\star}(x, y) = \prod_{\square \in Y_3} \frac{1}{\theta_q(x^{L_{\square Y_3}^+} y^{A_{\square Y_3}^+})} \sum_Y \mathbf{W}_{qY_1/Y}^{\star}(y^t x^{-Y_3}) \mathbf{W}'_{qY_2/Y}^{\star}(x^{J-1} y^{-Y_3'})$$

**9.7. The  $q \rightarrow 0$  limit.** In the  $q \rightarrow 0$  limit, the  $q$ -Whittaker function  $W_{qY_1/Y_2}(x)$  reduces to the Schur function  $s_{Y_1/Y_2}(\mathbf{x})$ . As we can see from the expression of vertex operators, equation 5.3,  $\Gamma_{q', b\pm}(x')$  goes to 1 in the  $q \rightarrow 0$  limit and,

$$(9.17) \quad W_{q', Y_1/Y_2}^{\star} \rightarrow \delta_{Y_1, Y_2'}$$

in this limit. This trivializes the  $q'$ -dependent part of the partition function, as long as we consider toric diagrams with trivial external legs. We can effectively drop the  $q'$ -part in the elliptic vertex when we compute the partition function in the limit. Therefore, we can simply replace,

$$(9.18) \quad \mathbf{W}_{qY/R}(\mathbf{x}) \rightarrow s_{Y_1/R_1}(\mathbf{x}), \quad \mathbf{W}'_{qY/R}(\mathbf{x}) \rightarrow s_{Y_1/R_1}(\mathbf{x}), \quad \theta_p(x) \rightarrow (1-x),$$

in this limit, and the elliptic vertex reduces to the refined vertex in [23],

<sup>40</sup> We can alternatively use the vertex operators,  $\iota \cdot \phi_{q+}(x^{-j+1} y^{y'_{3,i}})$ , and  $\iota \cdot \phi_{q-}(y^i x^{-y_{3,i}})$ , and the usual  $q$ -Whittaker basis to obtain the same result.

$$(9.19) \quad \mathcal{E}_{Y_1 Y_2 Y_3}(x, y) \rightarrow \mathcal{R}_{Y'_{1A} Y_{2A} Y_3}(y, x), \quad \mathcal{E}_{Y_1 Y_2 Y_3}^*(x, y) \rightarrow \mathcal{R}_{Y_{1A} Y'_{2A} Y'_3}(x, y)$$

**9.8. Equivalence with the elliptic Awata-Feigin-Shiraishi vertex.** The vertex operators used to construct  $\mathcal{E}$  are equivalent to those used to construct the elliptic Awata-Feigin-Shiraishi vertex constructed in [41]. To see this, we focus on the simple case of  $\mathcal{E}_{Y_1 Y_2 \emptyset}(x, y)$ . In this case, the corresponding AFS vertex is the normal-ordered product <sup>41</sup>,

$$(9.20) \quad \Phi_{\emptyset}(1) =: \phi_{-q}(x^J) \phi_{+q}(y^{-t+1}) :$$

The infinite products of two-boson vertex operators on the right hand side of Equation 9.20 can be evaluated in the form,

$$(9.21) \quad \begin{aligned} \phi_{+q}(y^{-t+1}) &= \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(1-y^n)(1-q^n)} a_n \right) \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(1-y'^n)(1-q'^n)} b_n \right), \\ \phi_{-q}(x^J) &= \exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(1-x'^n)(1-q^n)} a_{-n} \right) \exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(1-x^n)(1-q'^n)} b_{-n} \right) \end{aligned}$$

Setting,

$$(9.22) \quad \begin{aligned} a_n &= a_n^{Saito}, \quad a_{-n} = \left( \frac{1-x'^n}{1-y'^n} \right) a_{-n}^{Saito}, \quad n = 1, 2, \dots, \\ b_n &= -b_n^{Saito}, \quad b_{-n} = - \left( \frac{1-x^n}{1-y^n} \right) b_{-n}^{Saito}, \quad n = 1, 2, \dots, \end{aligned}$$

where  $a_n^{Saito}$  and  $b_n^{Saito}$  are the  $pqt$ -Heisenberg generators, Equation 4.1, used in [34–36] and in [41], with the parameters reset as,

$$(9.23) \quad p \rightarrow q, \quad q \rightarrow y', \quad t \rightarrow x',$$

where the parameters on the left are Saito's and the parameters on the right are those used in the present work, we obtain

$$(9.24) \quad \begin{aligned} \Phi_{\emptyset}(1) &= \\ &: \exp \left( \sum_{n \neq 0} \frac{1}{n} \frac{1}{(1-y^n)(1-q^{|n|})} a_n^{Saito} \right) \exp \left( - \sum_{n \neq 0} \frac{1}{n} \frac{1}{(1-y'^n)(1-q'^{|n|})} b_n^{Saito} \right) : \end{aligned}$$

<sup>41</sup> We use the notation in Equation 1.7 for infinite products of vertex operators, the definition of the two-boson vertex operators in terms of single-boson vertex operators, Equation 5.4, the definition of the single-boson vertex operators, Equation 5.3.



FIGURE 10.1. The assignment of the elliptic vertex and its twisted version to toric diagrams. The preferred legs are shown as thick horizontal lines.

which is the correct expression for the elliptic AFS vertex for  $Y_3 = \emptyset$  [41]. With the mapping in Equation 9.22, one can repeat the argument in [4] to prove the equivalence. We do not do this here as it is not the main point of this paper.

## 10. THE 6D STRIP PARTITION FUNCTION

We compute the 4-vertex strip partition function obtained by gluing four elliptic vertices, and show that the result is a 6D strip partition function.

**10.1. The rules of gluing.** Drawing web diagrams, we set all horizontal legs to be the preferred. Having done that, the set of all vertices splits into two disjoint subsets, one with all preferred legs pointing to the right, and the other with all preferred legs pointing to the left, as in Figure 10.1. We take the first subset to consist of elliptic vertices, and the second to consist of twisted vertices. We glue vertices only to twisted vertices and *vice versa*.

**10.1.1. The choice of Kähler parameters.** We form strips by gluing elliptic vertex non-preferred legs. Each of these elliptic vertex non-preferred legs is effectively a pair of refined vertex non-preferred legs, and each of the latter is assigned a Young diagram. Since the assigned Young diagrams are independent, we can in principle use a pair of independent Kähler parameters  $(Q_A, Q_B)$  to glue. However, to recover the 6D partition functions computed in the literature, including [21, 24, 31], we need to choose the Kähler parameter pair as  $(Q, Q')$ , where  $Q' = 1/Q$ , consistently with all choices of all parameter pairs made so far. At this stage, we have no justification for this choice other than reproducing the known 6D partition functions.

**10.2. Example. The 4-vertex strip partition functions.** The partition function of the 4-vertex strip in Figure 10.2 is,

$$\begin{aligned}
 (10.1) \quad Z_{4\text{-strip}} &= \sum_{Y_{1,2,3}} \prod_{\ell=1}^3 (-Q_\ell)^{|Y_\ell|} \mathcal{E}_{\emptyset Y_1 W_1}(x, y) \mathcal{E}_{Y_1 Y_2 V_1}^*(x, y) \mathcal{E}_{Y_2 Y_3 W_2}(x, y) \mathcal{E}_{Y_3 \emptyset V_2}^*(x, y) \\
 &= \prod_{k=1}^2 \frac{1}{\theta_q(x^{L_{\square, W_k}^+} y^{A_{\square, W_k}}) \theta_q(x^{L_{\square, V_k}} y^{A_{\square, V_k}^+})} \sum_{Y_{1,2,3}} \sum_{R_{1,2}} \prod_{\ell=1}^3 (-Q_\ell)^{|Y_\ell|} \\
 &\quad \mathbf{W}'_{qY_1}(x^t y^{-W'_1}) \left( \mathbf{W}_{qY_1/R_1}^*(y^J x^{-V_1}) \mathbf{W}'_{qY_2/R_1}(x^{t-1} y^{-V'_1}) \right) \\
 &\quad \left( \mathbf{W}_{qY_2/R_2}(y^{t-1} x^{-W_2}) \mathbf{W}'_{qY_3/R_2}(x^J y^{-W'_2}) \right) \mathbf{W}_{qY_3}^*(y^J x^{-V_2})
 \end{aligned}$$

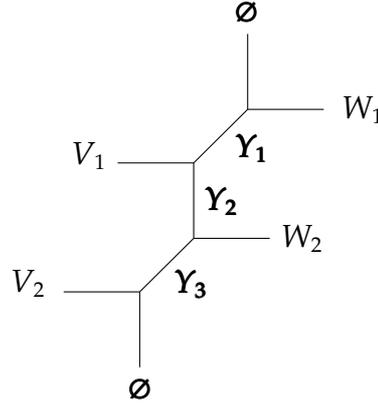


FIGURE 10.2. The strip obtained by gluing 4 vertices along their non-preferred legs.  $V_1, V_2, W_1$  and  $W_2$  are single diagrams.  $Y_1, Y_2$  and  $Y_3$  are pairs of Young diagrams.  $\emptyset$  is a pair of empty Young diagrams.

where  $|\mathbf{Y}| = |Y_A| - |Y_B|$  because of our choice of Kähler parameters, see section 10.1.1.

**10.3. Using the Cauchy identities.** We compute the strip partition function in Equation 10.1 in 4 steps using the Cauchy identities derived in section 7.

**10.3.1. Step 1.** Using the identity  $Q^{|\mathbf{Y}| - |\mathbf{X}|} \mathbf{W}_{\mathbf{Y}/\mathbf{X}}(\mathbf{x}) = \mathbf{W}_{\mathbf{Y}/\mathbf{X}}(Q\mathbf{x})$ , where  $\mathbf{X}$  and  $\mathbf{Y}$  are partition pairs, and  $\mathbf{x}$  is a set of variables, which follows from the properties of the  $q$ -Whittaker functions that make  $\mathbf{W}$ , and the Cauchy identity 8.6 to perform the sum over  $Y_1$  and over  $Y_3$ <sup>42</sup>,

$$\begin{aligned}
(10.2) \quad Z_{4\text{-strip}} &= \prod_{k=1}^2 \frac{1}{\theta_q \left( x^{L_{\square, W_k}^+} y^{A_{\square, W_k}} \right) \theta_q \left( x^{L_{\square, V_k}} y^{A_{\square, V_k}^+} \right)} \sum_{Y_2} \sum_{R_1 R_3} (-Q_2)^{|\mathbf{Y}_2|} \times \\
&\quad \prod_{i,j=1}^{\infty} \theta_q \left( Q_1 x^{i-V_{1,j}} y^{j-W'_{1,i}} \right) \mathbf{W}'_{qR_1} \left( -Q_1 x^t y^{-W'_1} \right) \mathbf{W}'_{qY_2/R_1} \left( x^{t-1} y^{-V'_1} \right) \\
&\quad \mathbf{W}_{qY_2/R_3} \left( y^{t-1} x^{-W_2} \right) \prod_{i,j=1}^{\infty} \theta_q \left( Q_3 x^{i-V_{2,j}} y^{j-W'_{2,i}} \right) \mathbf{W}_{qR_3}^* \left( -Q_3 y^J x^{-V_2} \right) \\
&= \prod_{k=1}^2 \frac{1}{\theta_q \left( x^{L_{\square, W_k}^+} y^{A_{\square, W_k}} \right) \theta_q \left( x^{L_{\square, V_k}} y^{A_{\square, V_k}^+} \right)} \\
&\quad \prod_{i,j=1}^{\infty} \theta_q \left( Q_1 x^{i-V_{1,j}} y^{j-W'_{1,i}} \right) \prod_{i,j=1}^{\infty} \theta_q \left( Q_3 x^{i-V_{2,j}} y^{j-W'_{2,i}} \right) \\
&\quad \sum_{Y_2} \sum_{R_1 R_3} (-Q_2)^{|\mathbf{Y}_2|} \mathbf{W}'_{qR_1} \left( -Q_1 x^t y^{-W'_1} \right) \mathbf{W}'_{qY_2/R_1} \left( x^{t-1} y^{-V'_1} \right) \\
&\quad \mathbf{W}_{qY_2/R_3} \left( y^{t-1} x^{-W_2} \right) \mathbf{W}_{qR_3}^* \left( -Q_3 y^J x^{-V_2} \right)
\end{aligned}$$

<sup>42</sup> To relate the notation  $Y_1$  and  $Y_3$  for the Young diagrams that are summed out to that of the Young diagrams that are introduced, we use  $R_1$  and  $R_3$  for the latter, and there is no  $R_2$

10.3.2. *Step 2.* Using the Cauchy identity 8.5 to perform the sum over  $\mathbf{Y}_2$ ,

$$(10.3) \quad Z_{4\text{-strip}} = \prod_{k=1}^2 \frac{1}{\theta_q \left( x^{L_{\square, W_k}^+} y^{A_{\square, W_k}} \right) \theta_q \left( x^{L_{\square, V_k}} y^{A_{\square, V_k}^+} \right)} \\ \prod_{i,j=1}^{\infty} \theta_q \left( Q_1 x^{i-V_{1,j}} y^{j-W'_{1,i}} \right) \prod_{i,j=1}^{\infty} \theta_q \left( Q_2 x^{i-1-W_{2,j}} y^{j-1-V'_{1,i}} \right) \prod_{i,j=1}^{\infty} \theta_q \left( Q_3 x^{i-V_{2,j}} y^{j-W'_{2,i}} \right) \\ \sum_{\mathbf{S}} \sum_{\mathbf{R}_{1,3}} (-Q_2)^{|\mathbf{R}_1|} \mathbf{W}'_{q\mathbf{R}_1} \left( -Q_1 x^t y^{-W'_1} \right) \mathbf{W}_{q\mathbf{R}_1/\mathbf{S}} \left( y^{t-1} x^{-W_2} \right) \\ \mathbf{W}'_{q\mathbf{R}_3/\mathbf{S}} \left( -Q_2 x^{t-1} y^{-V'_1} \right) \mathbf{W}_{q\mathbf{R}_3} \left( -Q_3 y^j x^{-V_2} \right),$$

10.3.3. *Step 3.* Using the Cauchy identities 8.3 and 8.4 to sum over  $\mathbf{R}_1$  and over  $\mathbf{R}_3$ ,

$$(10.4) \quad Z_{4\text{-strip}} = \prod_{k=1}^2 \frac{1}{\theta_q \left( x^{L_{\square, W_k}^+} y^{A_{\square, W_k}} \right) \theta_q \left( x^{L_{\square, V_k}} y^{A_{\square, V_k}^+} \right)} \\ \frac{\prod_{i,j=1}^{\infty} \theta_q \left( Q_1 x^{i-V_{1,j}} y^{j-W'_{1,i}} \right) \prod_{i,j=1}^{\infty} \theta_q \left( Q_2 x^{i-1-W_{2,j}} y^{j-1-V'_{1,i}} \right) \prod_{i,j=1}^{\infty} \theta_q \left( Q_3 x^{i-V_{2,j}} y^{j-W'_{2,i}} \right)}{\prod_{i,j=1}^{\infty} \theta_q \left( Q_{12} x^{i-W_{2,j}} y^{j-1-W'_{1,i}} \right) \prod_{i,j=1}^{\infty} \theta_q \left( Q_{23} x^{i-1-V_{2,j}} y^{j-V'_{1,i}} \right)} \\ \sum_{\mathbf{S}} \mathbf{W}'_{q\mathbf{S}} \left( Q_{12} x^t y^{-W'_1} \right) \mathbf{W}_{q\mathbf{S}} \left( -Q_{23} y^j x^{-V_2} \right),$$

where  $Q_{ij} = Q_i Q_j$

10.3.4. *Step 4.* Using the Cauchy identity 8.6 to perform the sum over  $\mathbf{S}$ ,

$$(10.5) \quad Z_{4\text{-strip}} = \prod_{k=1}^2 \frac{1}{\theta_q \left( x^{L_{\square, W_k}^+} y^{A_{\square, W_k}} \right) \theta_q \left( x^{L_{\square, V_k}} y^{A_{\square, V_k}^+} \right)} \\ \frac{N'_{W_1 V_1} \left( Q_1 y \right) N'_{V_1 W_2} \left( Q_2 x' \right) N'_{W_2 V_2} \left( Q_3 y \right)}{N'_{W_1 W_2} \left( Q_{12} \right) N'_{V_1 V_2} \left( Q_{23} y x' \right)} N'_{W_1 V_2} \left( Q_{123} y \right), \\ N'_{Y_1 Y_2} \left( Q \right) = \prod_{i,j=1}^{\infty} \theta_q \left( Q y^{-Y'_{1,j}+i-1} x^{-Y_{2,i}+j} \right), \quad Q_{123} = Q_1 Q_2 Q_3,$$

which agrees with that in [21, 24, 31].

10.3.5. *Remark.* The product  $N'_{Y_1 Y_2} \left( Q \right)$  in Equation 10.5 is  $N_{Y_1 Y_2} \left( Q | q_1, q_2, q \right)$ ,  $q_1 = y'$ ,  $q_2 = x$ , in [6], up to a factor,

$$(10.6) \quad N'_{Y_1 Y_2} \left( Q \right) = \left( \prod_{i,j=1}^{\infty} \theta_q \left( Q y^{i-1} x^j \right) \right) N_{Y_1 Y_2} \left( Q | y', x, q \right)$$

10.3.6. *Remark.* In computing the 6D instanton partition function in Equation 10.5, we used the variables  $x^{\iota} y^{-Y'_3}$  and  $y^{J^{-1}} x^{-Y_3}$  in Equation 9.13 instead of the variables  $x^{-\rho} y^{-Y'_3}$  and  $y^{-\rho} x^{-Y_3}$  in [23], where  $x^{-\rho} y^Y = (x^{1/2} y^{y_1}, x^{3/2} y^{y_2}, x^{5/2} y^{y_3}, \dots)$ . To compensate for these differences when comparing our results with those obtained using the refined topological vertex of [23], we need to rewrite the Kähler parameters in Equation 10.5 as,

$$(10.7) \quad Q_{2i} = (yx')^{1/2} Q_{2i}^{IKV}, \quad Q_{2i-1} = (xy')^{1/2} Q_{2i-1}^{IKV}, \quad i = 1, 2, \dots,$$

where  $Q_i^{IKV}, i = 1, 2, \dots$ , is identified with the corresponding Kähler parameter in [23].

10.3.7. *More general strip partition functions.* Strip partition functions with more vertices can be calculated, using the same Cauchy identities as in the 4-vertex case. The result of these computations is that the partition function of an  $N$ -vertex strip,  $N = 6, 8, \dots$ , computed using elliptic topological vertices, with empty top and bottom external legs, is equal to the partition function of the same strip computed using refined topological vertices, with the top and bottom legs identified. Gluing copies of these strips, computed either way, along their horizontal preferred legs, we obtain 6D instanton partition function.

## 11. COMMENTS

11.1. **The  $q \rightarrow 0$  limit.** In the limit  $q \rightarrow 0$ , all  $\theta_q(x) \rightarrow 1 - x$ , and the 6D partition function in Equation 10.5 reduces to the 5D partition function of the corresponding toric diagram computed using the refined topological vertex. From the viewpoint of compactification,  $q \rightarrow 0$  corresponds to the length of the compactification circle going to infinity, which forces the vertical external legs to be trivial.

11.2. **The  $R \rightarrow 0$  limit.** Another interesting limit is obtained by taking the radius of the  $M$ -theory circle  $R \rightarrow 0$ , while keeping  $q$  finite, to obtain a 5D partition function [14]. A study of this 5D partition function, its relation to that obtained in the limit  $q \rightarrow 0$ , and the possible interplay of these two limits is beyond the scope of the present work.

11.3. **The other involution and other Cauchy identities.** In addition to the Cauchy identities that involve  $q$ -Whittaker functions and their twisted versions, there are identities that involve  $q$ -Whittaker functions and Hall-Littlewood functions. These identities are obtained by the action of the Macdonald involution  $\omega$ , Equation 3.3, that changes the  $q$ -Whittaker functions labelled by a Young diagram  $Y$  and a parameter  $q$  to a Hall-Littlewood function labelled by the transpose diagram  $Y'$  and the same parameter  $q$  [9]. We did not consider  $\omega$  and the resulting Cauchy identities because they do not lead to the 6D instanton partition functions that we wish to compute. Instead, we considered the involution  $\iota$ , Equation 3.1 that leads to the twisted  $q$ -Whittaker functions and the Cauchy identities that lead to the 6D instanton partition functions. In this sense, our construction of  $\mathcal{E}$  aimed in a specific direction and is not the only construction possible. It is not clear what the construction that uses  $\omega$  gives <sup>43</sup>.

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<sup>43</sup> One can make a similar remark regarding our choice of the Kähler parameters in section 10.1.1, which was motivated by producing the 6D partition functions in the literature, including [21, 24, 31], and only that.

**11.4. The 2D interpretation.** In addition to their interpretation as 6D instanton partition functions, the partition functions obtained by gluing copies of  $\mathcal{E}$  and  $\mathcal{E}^*$  have a natural interpretation as elliptic deformations of 2D conformal blocks. We expect that these 2D elliptic conformal blocks are related to the  $n$ -point local height probabilities in off-critical exactly-solved statistical mechanical models, with elliptic Boltzmann weights, studied in [25, 26].

### 11.5. Two extensions that need physical interpretation.

11.5.1. *Both methods can work in parallel.* Using the elliptic vertex can be combined with the trace method. One can compute an elliptic partition function using the elliptic vertex, then take the trace over the all possible external states on the doubled non-preferred legs. It is not clear what the result means.

11.5.2. *The Macdonald parameter  $t$ .* One advantage of using  $\mathcal{E}$ , as opposed to taking traces as in [21] is that it makes it obvious that there is room for one parameter, namely the second Macdonald parameter  $t$ . We have not switched this parameter on because it does not appear in the 6D instanton partition function results of [21]. We could have easily switched  $t$  on, but we would have no interpretation for what that means <sup>44</sup>.

**11.6. The Clavelli-Shapiro trace reduction method.** In [10], Clavelli and Shapiro proposed a method to reduce the evaluation of a trace of exponentials of Heisenberg generators, of the type that appears in string theory and in the present work, to the evaluation of a single vacuum expectation value of exponentials of *two* Heisenberg generators <sup>45</sup>. Given a single Heisenberg annihilation operator  $a$  and its conjugate creation operator  $a^\dagger$ , a product  $\mathcal{O}(a, a^\dagger)$  of exponentials in  $a$  and  $a^\dagger$ , and a parameter  $x < 1$ , we wish to evaluate the trace,

$$(11.1) \quad \text{Tr} \left( x^{a^\dagger a} \mathcal{O}(a, a^\dagger) \right) = \sum_{n=0}^{\infty} \langle n | x^{a^\dagger a} \mathcal{O}(a, a^\dagger) | n \rangle, \quad [a, a^\dagger] = 1,$$

where  $\langle n |$  and  $| n \rangle$  are the state created by the action of  $n$  copies of  $a$  and of  $a^\dagger$  on the vacuum states  $\langle 0 |$  and  $| 0 \rangle$  respectively. In [10], Clavelli and Shapiro noticed that by introducing a second Heisenberg algebra, generated by  $b$  and  $b^\dagger$ , that commutes with the first, generated by  $a$  and  $a^\dagger$ , the infinite sum over states on the right hand side of the first Equation 11.1 becomes,

$$(11.2) \quad \sum_{n=0}^{\infty} \langle n | x^{a^\dagger a} \mathcal{O}(a, a^\dagger) | n \rangle = \langle 0 | e^{b a} x^{a^\dagger a} \mathcal{O}(a, a^\dagger) e^{a^\dagger b^\dagger} | 0 \rangle,$$

and the trace has been reduced to computing a single expectation value of operators in a pair of Heisenberg algebras. Following that, Clavelli and Shapiro show that the right hand side of Equation 11.2 can be written in the form,

<sup>44</sup> In [37], Sulkowski showed that starting from the topological string partition function on  $C^3$  then switching on the Macdonald parameter  $t$  produces the topological string partition function of the conifold, with the parameter  $t$  parameterizing the size of the sphere  $P^1$ . We expect that introducing  $t$  in more general topological string partition functions will have a related effect.

<sup>45</sup> Apart from a slight change of notation, this outline follows section C.1, p. 522, of [10].

$$(11.3) \quad \langle 0 | e^{ba} x^{a^+ a} \mathcal{O}(a, a^+) e^{a^+ b^+} | 0 \rangle = \frac{1}{1-x} \langle 0 | \mathcal{O}(c, c^+) | 0 \rangle,$$

$$c = \frac{a}{1-x} + b^+, \quad c^+ = a^+ - \frac{b}{1-1/x},$$

and the trace of an operator that depends on a single Heisenberg algebra, and a weight (propagator-type) parameter  $x$  is reduced to a vacuum expectation value of the same operator that now depends on two Heisenberg algebras that are deformed in a specific way using the parameter  $x$ . The above single-mode result readily extends to the case of a Heisenberg algebra with infinitely-many modes<sup>46</sup>, and the conclusion is that traces over exponentials of free fields can be re-written as vacuum expectation values in twice the number of free fields.

11.6.1. *The elliptic vertex versus taking traces, and similarities with the Clavelli-Shapiro method.* The relation between the result in terms of a trace and the same result in terms of no trace but twice the number of fields is identical to the relation between the computation of the 6D instanton partition functions in terms of the refined vertex and taking traces in [21], and the computation of the same objects in terms of the elliptic vertex proposed in the present work, which is basically a deformation of a doubling of the refined vertex. Even the deformation of the pair of Heisenberg algebras in Equation 11.3 is identical to that that appears in the present work, and in so far as these computations are concerned, using the elliptic vertex as in the present work is related to taking traces as in [24, 31] *via* a Clavelli-Shapiro trace reduction. By using the elliptic vertex, the effect of the compactification is local and can be traced to the deformation of each vertex in the strip, unlike in the case of taking traces. For that reason, it is possible that, while both methods lead to the same 6D partition functions, using the elliptic vertex may be more suited to studies of the algebraic structures that underlie the elliptic deformation, particularly in 2D integrable models.

#### ACKNOWLEDGEMENTS

We thank F Benini, J-E Bourguine, P Gavrylenko, A Hanay, C Kozcaz, Kimyoung Lee, M Manabe, Y Matsuo, V Mitev, E Pomoni, J Shiraishi, Y Tachikawa, O Warnaar, Jian-Feng Wu, F Yagi, and G Zafrir for discussions on this work and on related topics. We thank the organizers of ‘Combinatorics, Statistical Mechanics, and Conformal Field Theory’ for hospitality at the Australian Mathematical Research Institute, MATRIX, Creswick, Victoria, the organizers of ‘Supersymmetric Quantum Field Theories in the Non-perturbative Regime’, the Galileo Galilei Institute for Theoretical Physics, Arcetri, Firenze, and OF thanks Prof A Dabholkar of the Abdus Salam International Center for Theoretical Physics, Trieste, Italy, Profs K Lechner, M Matone and D Sorokin of the Physics Department, University of Padova, Italy for hospitality at various stages of this work. OF is supported by a Special Studies Program grant from the Faculty of Science, University of Melbourne, and the Australian Research Council. RDZ is supported by JSPS fellowship for young students.

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<sup>46</sup> Equation C.4, page 522, in [10].

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