CONORMAL VARIETIES ON THE COMINUSCULE GRASSMANNIAN – II

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Abstract. Let X_w be a Schubert subvariety of a cominuscule Grassmannian X, and let $\mu : T^*X \to \mathcal{N}$ be the Springer map from the cotangent bundle of X to the nilpotent cone $\mathcal{N}$. In this paper, we construct a resolution of singularities for the conormal variety $T^*_X X_w$ of X_w in X. Further, for X the usual or symplectic Grassmannian, we compute a system of equations defining $T^*_X X_w$ as a subvariety of the cotangent bundle T^*X set-theoretically. This also yields a system of defining equations for the corresponding orbital varieties $\mu(T^*_X X_w)$. Inspired by the system of defining equations, we conjecture a type-independent equality, namely $T^*_X X_w = \pi^{-1}(X_w) \cap \mu^{-1}(\mu(T^*_X X_w))$. The set-theoretic version of this conjecture follows from this work and previous work for any cominuscule Grassmannian of type A, B, or C.

We work over an *algebraically closed field* k of *good characteristic* (for a definition, see [Car85]). Let G be a connected algebraic group whose Lie algebra $\mathfrak{g}$ is simple.

For $\mathcal{P}$ a conjugacy class of parabolic subgroups of G, we denote by $X^{\mathcal{P}}$ the variety of parabolic subgroups of G whose conjugacy class is $\mathcal{P}$. The cotangent bundle $T^*X^{\mathcal{P}}$ of $X^{\mathcal{P}}$ is given by

$$T^*X^{\mathcal{P}} = \left\{(P, x) \in X^{\mathcal{P}} \times \mathcal{N} \,\middle|\, x \in \mathfrak{u}_P\right\},$$

where $\mathcal{N}$ is the variety of nilpotent elements in $\mathfrak{g}$. The map $\mu : T^*X \to \mathcal{N}$, given by $\mu(P, x) = x$, is the celebrated *Springer map*.

Let $\mathcal{B}$ be the conjugacy class of Borel subgroups of G. The Steinberg variety,

$$\mathcal{Z}^{\mathcal{P}} = \left\{(B, P, x) \in X^{\mathcal{B}} \times X^{\mathcal{P}} \times \mathcal{N} \,\middle|\, x \in \mathfrak{u}_B \cap \mathfrak{u}_P\right\},$$

is reducible. Each irreducible component $\mathcal{Z}^{\mathcal{P}}_w$ of $\mathcal{Z}^{\mathcal{P}}$ is the conormal variety of a G-orbit closure (under the diagonal action) in $X^{\mathcal{B}} \times X^{\mathcal{P}}$.

In this paper, for certain choices of $\mathcal{P}$, namely the ones for which $X^{\mathcal{P}}$ is cominuscule (see Section 1.9), we construct a resolution of singularities for each irreducible component $\mathcal{Z}^{\mathcal{P}}_w \subset \mathcal{Z}^{\mathcal{P}}$. In types A and C, we also provide a system of defining equations, for each component $\mathcal{Z}^{\mathcal{P}}_w$ as a subvariety in $X^{\mathcal{B}} \times X^{\mathcal{P}} \times \mathcal{N}$. This also yields a system of defining equations for certain orbital varieties. We discuss this later in this section.

Before getting into the details, let us first present the irreducible components of $\mathcal{Z}^{\mathcal{P}}$ from an alternate point of view. We fix a Borel subgroup B in G, and a standard parabolic subgroup P corresponding to omitting a cominuscule simple root γ, see Section 1.9. Let X be the variety of 'parabolic subgroups conjugate to P'. We have an isomorphism $X \cong {}^{G}/_{P}$, and further, a G-equivariant isomorphism,

$$G \times^B X \xrightarrow{\sim} X^{\mathcal{B}} \times X,$$

given by $(g, P) \mapsto (gBg^{-1}, gPg^{-1})$.

A B-orbit $C_w \subset X$ is called a Schubert cell, and its closure X_w is called a Schubert variety. The conormal variety $T^*_X X_w$ of X_w in X is simply the closure of the conormal bundle of C_w in X, see Section 1.8.

Consider the map $\mathcal{Z}^{\mathcal{P}}_w \to X^{\mathcal{B}}$, given by $(B, P, x) \mapsto B$. We identify $\mathcal{Z}^{\mathcal{P}}_w$ as a fibre bundle over $X^{\mathcal{B}}$ via this map, with the fibre over the point $B \in X^{\mathcal{B}}$ being precisely the conormal variety $T^*_X X_w$. In particular, we have an isomorphism

$$G \times^B T^*_X X_w \xrightarrow{\sim} \mathcal{Z}^{\mathcal{P}}_w.$$

From this viewpoint, it is clear that the geometry of $T^*_X X_w$ is closely related to the geometry of $\mathcal{Z}^{\mathcal{P}}_w$.

Remark. A similar (and essentially equivalent) statement can be found in [CG97, Proposition 3.3.4]; the proof there is different, leveraging the fact that the Springer map $\mu : T^*X \to \mathcal{N}$ can be identified with the moment map arising from the G-symplectic structure on T^*X.

We now present our main results. Let X_w be a Schubert subvariety of a cominuscule Grassmannian X. In Section 2, we present a variety $\widetilde{Z_{\underline{w}}}$, which is a vector bundle over a Bott-Samelson variety resolving X_w, along with a proper birational B-equivariant map, $\theta_{\underline{w}} : \widetilde{Z_{\underline{w}}} \to T^*_X X_w$.

Theorem A. *The map $\theta_{\underline{w}} : \widetilde{Z_{\underline{w}}} \to T^*_X X_w$ is a B-equivariant resolution of singularities.*

Since the map $\theta_{\underline{w}}$ is B-equivariant, it also yields a resolution of singularities,

$$G \times^B \theta_{\underline{w}} : G \times^B \widetilde{Z_{\underline{w}}} \to \mathcal{Z}^{\mathcal{P}}_w.$$

of the Steinberg component $\mathcal{Z}^{\mathcal{P}}_w$.

Next, we study the system of defining equations for the conormal variety $T^*_X X_w$ inside T^*X. For $i \geq 1$, let $E(i)$ denote a vector space with basis $e_1, \cdots e_i$. We fix a non-degenerate skew-symmetric bilinear form ω on $E(2d)$. Let G either $SL(E(n))$ or $Sp(E(2d), \omega)$, and accordingly, let X be either the *usual Grassmannian*,

$$Gr(d, n) = \{V \subset E(n) \mid \dim V = d\},$$

or the *symplectic Grassmannian*,

$$SGr(2d) = \left\{V \subset E(2d) \,\middle|\, V = V^{\perp}\right\}.$$

The cotangent bundle of X is given by

$$T^*X = \{(V, x) \in X \times \mathcal{N} \mid \operatorname{Im} x \subset V \subset \ker x\},$$

where, recall that $\mathcal{N}$ denotes the corresponding nilpotent cone. Let B be the Borel subgroup which is the stabilizer of the flag $(E(i))_i$ in G. In Theorem B, we provide a system of defining equations for $T^*_X X_w$ in T^*X.

Theorem B. *Consider $(V, x) \in T^*X$. Then $(V, x) \in T^*_X X_w$ if and only if $V \in X_w$, and further, for all $1 \leq j < i \leq l + 1$, we have*

$$\dim({}^{xE(t_i)}/_{E(t_j)}) \leq \begin{cases} r_{i-1} - r_j, \\ c_i - c_{j+1}. \end{cases}$$

The numbers r_i, c_i, t_i are defined in terms of w, see Section 3.14.

A system of defining equations for $\mathcal{Z}_w^{\mathcal{P}}$ in $X^{\mathcal{B}} \times X \times \mathcal{N}$ follows as a consequence. We simply replace the subspaces $E(t_i)$ with the subspace $E'(t_i)$, where $(E'(i))_i$ is the flag fixed by B', the Borel subgroup at the first coordinate in $X^{\mathcal{B}} \times X \times \mathcal{N}$.

Theorem B does not hold for the orthogonal Grassmannian. The key difference between C_n and D_n is the following: Consider the embedding of the Weyl group W into S_{2n}. Then, for C_n, the Bruhat order on W is identical to the order induced by restricting the (type A) Bruhat order on S_{2n}. This is not true for D_n.

In Equation (7.1), we interpret Theorem B in a type-independent manner,

$$T_X^*X_w = \mu^{-1}(\mu(T_X^*X_w)) \cap \pi^{-1}(X_w).$$

Here $\mu : T^*X \to \mathcal{N}$ is the Springer map, and $\pi : T^*X \to X$ is the structure map defining the cotangent bundle.

We conjecture that Equation (7.1) holds for any Schubert variety X_w in any cominuscule Grassmannian X. The containment $\subset$ holds trivially. Besides Theorem B, further evidence in support of this conjecture is provided by Proposition 7.2, which states that Equation (7.1) holds set-theoretically if X_w is smooth, and by Proposition 7.4, which states that Equation (7.1) holds scheme-theoretically if the opposite Schubert variety X_{w_0w} is smooth. Combining these results, we see that Equation (7.1) holds set-theoretically for any cominuscule Grassmannian in types A, B, and C.

Finally, let us discuss orbital varieties, and their relationship with the conormal varieties of Schubert varieties. Consider a G-orbit $\mathcal{N}_{\boldsymbol{\lambda}}^\circ \subset \mathcal{N}$. The irreducible components of the closure $\overline{\mathcal{N}_{\boldsymbol{\lambda}} \cap \mathfrak{u}_B}$ are called orbital varieties. The reader might consult [DR09] for a general survey.

Caution. Some authors define an orbital variety to be an irreducible component of $\mathcal{N}_{\boldsymbol{\lambda}}^\circ \cap \mathfrak{u}_B$, where $\mathcal{N}_{\boldsymbol{\lambda}}^\circ$ is G-orbit in $\mathcal{N}$. What we call an orbital variety here is an orbital variety closure in their language.

The key fact relating orbital varieties with conormal varieties is the following: Given a conjugacy class $\mathcal{P}$, and a Schubert variety $X_w^{\mathcal{P}}$, the image of the conormal variety $T_{X^{\mathcal{P}}}^*X_w^{\mathcal{P}}$ under the Springer map μ is an orbital variety. Conversely, every orbital variety is of the form $\mu(T_{X^{\mathcal{B}}}^*X_w^{\mathcal{B}})$ for some Schubert variety $X_w^{\mathcal{B}} \subset X^{\mathcal{B}}$.

Theorem B yields equations for the corresponding orbital varieties.

Theorem C. *Let G, B, P, X, w, and μ be as in Theorem B. Then*

$$\mu(T_X^*X_w) = \left\{ x \in \mathfrak{u}_B \;\middle|\; x^2 = 0,\ \dim(xE(t_i)/E(t_j)) \leq \begin{cases} r_{i-1} - r_j, \\ c_i - c_{j+1}, \end{cases} \forall\, 1 \leq i < j \leq l \right\}.$$

It is in general an open problem to give a combinatorial description of the inclusion order on orbital varieties. For varieties of matrices x satisfying $x^2 = 0$, this problem was solved by Melnikov [Mel05] in type A, and in types B and C by Melnikov and Barnea [BM17]. In Corollary 6.6, we show how their results can be recovered as a simple consequence of Theorem C.

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1. The Conormal Variety of a Schubert Variety

In this section, we recall some standard results about Schubert varieties, their conormal varieties, and cominuscule Grassmannians.

Let k be an *algebraically closed field* of *good characteristic*, $\mathfrak{g}$ a simple Lie algebra over k, and G a connected algebraic group for which $\mathfrak{g} = \mathrm{Lie}(G)$. We fix a maximal torus T in G, and a Borel subgroup B containing T.

Let Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t} = \mathrm{Lie}(T)$, and let S and Δ^+ be the set of simple roots and positive roots respectively, corresponding to the choice of Borel subalgebra $\mathfrak{b} = \mathrm{Lie}(B)$. For $\alpha \in \Delta$, we will write $\mathfrak{g}_\alpha$ for the corresponding root space.

1.1. **Standard Parabolic Subgroups.** A subgroup $Q \subset G$ is called parabolic if the quotient $X^Q \stackrel{\text{def}}{=} {}^{G}/_{Q}$ is proper. We will say that Q is a *standard* parabolic subgroup if $B \subset Q$.

Let $\{s_\alpha \,|\, \alpha \in S\}$ be the set of simple reflections in the *Weyl group* $W = {}^{N_G(T)}/_{T}$; here $N_G(T)$ is the normalizer of T in G. For any subset $R \subset S$, we have a subgroup $W_R \subset W$, given by $W_R = \langle s_\alpha \,|\, \alpha \in R\rangle$. The subgroup $BW_RB \subset G$, given by,

$$BW_RB = \{b_1 w b_2 \,|\, b_1, b_2 \in B,\, w \in W_R\}\,,$$

is a standard parabolic subgroup; further, the map $R \leftrightarrow BW_RB$ is a bijective correspondence from subsets of S to the standard parabolic subgroups of G.

1.2. **Schubert Varieties.** Let Q be a standard parabolic subgroup of G, corresponding to some subset $S_Q \subset S$. A B-orbit $C_w^Q \subset X^Q$ is called a *Schubert cell.* The pull-back of C_w^Q along the quotient map $G \to {}^{G}/_{Q} = X^Q$ is

$$BwQ = \{bwq \,|\, b \in B,\, q \in Q\}\,.$$

The closure X_w^Q of the Schubert cell C_w^Q is called a *Schubert variety.* The Schubert varieties in $X_w^Q \subset X^Q$ are indexed by $w \in W^Q$, where,

$$W^Q \stackrel{\text{def}}{=} \{w \in W \,|\, w(\alpha) > 0, \forall \alpha \in S_Q\}\,. \tag{1.3}$$

The set W^Q is called the set of minimal representatives of W with respect to Q.

1.4. **Bott-Samelson Varieties.** Let $\underline{w} = (s_1, \cdots, s_r)$ be a minimal word for w, i.e., the s_i are simple reflections such that $w = s_1 \cdots s_r$, and further, there is no sub-sequence of $\underline{w}$ whose product is w.

Let P_i be the standard parabolic subgroup $\overline{Bs_iB}$. The Bott-Samelson variety,

$$\widetilde{X_{\underline{w}}} \stackrel{\text{def}}{=} {}^{P_1 \times^B \cdots \times^B P_r}/_{B},$$

provides a resolution of singularities of X_w^Q via the map $\rho_{\underline{w}}^Q : \widetilde{X_{\underline{w}}} \to X_w^Q$, given by,

$$(p_1, \cdots, p_r) \mapsto p_1 \cdots p_r (\mathrm{mod}\, Q).$$

Let P_i° denote the open set $Bs_iB \subset P_i$. The map $\rho_{\underline{w}}^Q$ induces an isomorphism,

$$(\widetilde{X_{\underline{w}}})^\circ = {}^{P_1^\circ \times^B \cdots \times^B P_r^\circ}/_{B} \xrightarrow{\sim} C_w^Q.$$

1.5. **The Cotangent Bundle T^*X^Q.** The cotangent bundle $\pi : T^*X^Q \to X^Q$ is the vector bundle whose fibre $T^*_pX^Q$ at any point $p \in X^Q$ is precisely the cotangent space of X^Q at p. We call π the *structure map* defining the cotangent bundle.

Recall that the characteristic of k is a good prime. We have (cf. [BK05, Ch. 5]),

$$T^*X^Q = G \times^Q \mathfrak{u}_Q = {}^{(G\times\mathfrak{u}_Q)}/_{Q}, \tag{1.6}$$

where the quotient is with respect to the action $q \cdot (g, x) = (gq, Ad(q^{-1}x))$.

1.7. **The Springer Map.** Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$, i.e.,

$$\mathcal{N} = \{x \in \mathfrak{g} \mid Ad(x) \text{ is nilpotent}\}.$$

The Springer map $\mu^Q : T^*X^Q \to \mathcal{N}$, given by,

$$\mu^Q(g, x) = Ad(g)x,$$

is a proper map. The product map,

$$(\pi, \mu^Q) : T^*X^Q \to X^Q \times \mathcal{N}, \qquad (g, x) \mapsto (g, \mu^Q(x)),$$

is a closed immersion, see [CG97] for details.

1.8. **The Conormal Variety of a Schubert Variety.** Let C^Q_w (resp. X^Q_w) be a Schubert cell (resp. Schubert variety) in X^Q, corresponding to some $w \in W^Q$. The conormal bundle of C^Q_w in X^Q is the vector bundle,

$$\pi^\circ_w : T^*_{X^Q}C^Q_w \to C^Q_w,$$

whose fibre at a point $p \in C^Q_w$ is precisely the annihilator of the tangent subspace $T_pC^Q_w$ in $T^*_pX^Q$, i.e.,

$$(T^*_{X^Q}C^Q_w)_p = \left\{x \in T^*_pX^Q \,\middle|\, x(v) = 0, \forall v \in T_pC^Q_w\right\}.$$

The *conormal variety* $T^*_{X^Q}X^Q_w$ of X^Q_w in X^Q is the closure (in T^*X^Q) of the conormal bundle $T^*_{X^Q}C^Q_w$. The restriction of the structure map $\pi : T^*X^Q \to X^Q$ to the conormal variety induces a *structure map*, $\pi_w : T^*_{X^Q}X^Q_w \to X^Q_w$.

1.9. **Cominuscule Grassmannians.** A simple root $\gamma \in S$ is called *cominuscule* if the coefficient of γ in any positive root is either 0 or 1, i.e.,

$$\alpha \in \Delta^+ \implies 2\gamma \not\leq \alpha.$$

The cominuscule roots for various Dynkin diagrams are labelled in Table 1.10.

Example 1.11. Let $E(n)$ be an n–dimensional vector space. The variety of d–dimensional subspaces of $E(n)$ is called the usual Grassmannian variety,

$$Gr(d, n) \overset{\text{def}}{=} \{V \subset E(n) \mid \dim V = d\}.$$

It is a cominuscule Grassmannian corresponding to the group $G = SL_n$, and the cominuscule root $\alpha_d \in \mathsf{A}_{n-1}$, see Table 1.10.

Example 1.12. Let $E(2d)$ be a $2d$–dimensional vector space, and ω a symplectic form on $E(2d)$. The variety of Lagrangian subspaces in $E(2d)$,

$$SGr(2d) \overset{\text{def}}{=} \left\{V \subset E(2d) \,\middle|\, V = V^\perp\right\},$$

is called the *symplectic Grassmannian*. It is a cominuscule Grassmannian corresponding to the group $G = Sp_{2d}$, and the cominuscule root $\alpha_d \in \mathsf{C}_d$, see Table 1.10.

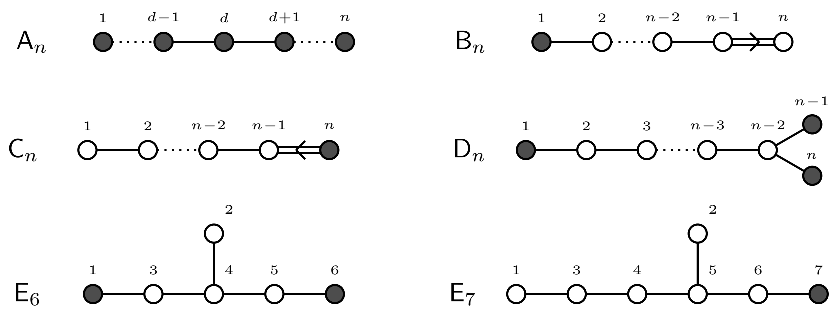

Table 1.10. Dynkin diagrams with cominuscule simple roots marked in black.

2. A Resolution of Singularities of the Conormal Variety

Let G, B, T, Δ, Δ^+, and S be as in the previous section. We fix a cominuscule root $\gamma \in S$. Let P be the standard parabolic subgroup corresponding to $S \backslash \{\gamma\}$, and let $\mathfrak{u}$ be the Lie algebra of the unipotent radical of P. We will denote the variety $X^P = {}^{G}\!/_{P}$ as simply X, and the Schubert varieties X_w^P as simply X_w.

In this section, we study the conormal variety $T_X^* X_w$ of a Schubert variety X_w in X. In particular, we describe the structure of the *conormal bundle* $T_X^* C_w$ in Lemma 2.2, and construct a resolution of singularities of $T_X^* X_w$ in Theorem A.

Lemma 2.1. *For any $w \in W^P$, the subspace $\mathfrak{u}_w \overset{\text{def}}{=} \mathfrak{u} \cap Ad(w^{-1})\mathfrak{u}_B$ is B-stable.*

Proof. The subspaces $\mathfrak{u}$ and $Ad(w^{-1})\mathfrak{u}_B$ are T-stable, and so their intersection $\mathfrak{u}_w$ is also T-stable. Further, since $Ad(w^{-1})\mathfrak{g}_\alpha = \mathfrak{g}_{w^{-1}(\alpha)}$, we have,

$$\mathfrak{u}_w = \bigoplus_{\alpha \geq \gamma} \mathfrak{g}_\alpha \bigcap \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{w^{-1}(\alpha)} = \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $R = \{\alpha \in \Delta \,|\, \alpha \geq \gamma,\, w(\alpha) > 0\}$. Since B is generated by the torus T and the root subgroups U_α, $\alpha \in S$, it suffices to show that $\mathfrak{u}_w$ is U_α-stable for all $\alpha \in S$. This follows from the claim,

$$\alpha \in R, \beta \in S, \alpha + \beta \in \Delta \implies \alpha + \beta \in R,$$

which we now prove. We first consider the case $\beta = \gamma$. For any $\alpha \in R$, we have $\gamma \leq \alpha$, hence $2\gamma \leq \alpha + \beta$. Now, since γ is cominuscule, we have $\alpha + \beta \notin \Delta$.

Next, we consider $\beta \in S \backslash \{\gamma\}$. In this case, since $w \in W^P$, it follows from Equation (1.3) that $w(\beta) > 0$. Now, for any $\alpha \in R$, we have $w(\alpha) > 0$, hence

$$w(\alpha + \beta) = w(\alpha) + w(\beta) > 0.$$

It follows from the definition of R that if $\alpha + \beta \in \Delta$, then $\alpha + \beta \in R$. $\square$

Lemma 2.2. *The conormal bundle $T_X^* C_w \to C_w$ is isomorphic to the vector bundle $BwB \times^B \mathfrak{u}_w \to C_w$, given by $(bw, x) \mapsto bw(\operatorname{mod} P)$.*

Proof. Let $\mathrm{pr} : X_w^B \to X_w$ be restriction of the quotient map ${}^{G}\!/_{B} \to {}^{G}\!/_{P}$ to X_w^B. Since $w \in W^P$, the map pr restricts to an isomorphism of Schubert cells $C_w^B \xrightarrow{\sim} C_w$, see [Kum02]. The claim now follows from the observation (cf. [LS17, §4.3]) that for any $(bw, x) \in T^*X$, we have $(bw, x) \in T^*_X C_w$ if and only if $x \in \mathfrak{u}_w$. □

2.3. **The Subgroup Q.** As a consequence of the previous lemma, we see that $Stab_G(\mathfrak{u}_w)$ is a standard parabolic subgroup. Let Q be any standard parabolic subgroup contained in P that stabilizes $\mathfrak{u}_w$, i.e.,

$$Q \subset Stab_G(\mathfrak{u}_w) \cap P. \tag{2.4}$$

We define a vector bundle $\pi^Q_w : Z^Q_w \to X^Q_w$, where,

$$Z^Q_w = BwQ \times^Q \mathfrak{u}_w, \qquad\qquad \pi^Q_w(g, x) = g(\operatorname{mod} Q).$$

Let $\underline{w} = (s_1, \cdots, s_r)$ be a minimal word for w. Recall the Bott-Samelson variety $\widetilde{X_{\underline{w}}}$ from Section 1.4. We lift Z^Q_w to a vector bundle $\pi_{\underline{w}} : \widetilde{Z_{\underline{w}}} \to \widetilde{X_{\underline{w}}}$ given by,

$$\widetilde{Z_{\underline{w}}} = P_1 \times^B \cdots \times^B P_r \times^B \mathfrak{u}_w,$$

and $\pi_{\underline{w}}(\underline{p}, x) = \underline{p}(\operatorname{mod} B)$, for $\underline{p} \in P_1 \times^B \cdots \times^B P_r$.

Proposition 2.5. *Let τ be the quotient map $G \times^Q \mathfrak{u} \to G \times^P \mathfrak{u}$. Viewing Z^Q_w as a subvariety of $G \times^Q \mathfrak{u}$, we have $\tau(Z^Q_w) \subset T^*_X X_w$. Let $\tau_w : Z^Q_w \to T^*_X X_w$ denote the induced map. We have a commutative diagram,*

$$\begin{array}{ccccc}
\widetilde{Z_{\underline{w}}} & \xrightarrow{\theta^Q_{\underline{w}}} & Z^Q_w & \xrightarrow{\tau_w} & T^*_X X_w \\
\big\downarrow{\scriptstyle \pi_{\underline{w}}} & & \big\downarrow{\scriptstyle \pi^Q_w} & & \big\downarrow{\scriptstyle \pi_w} \\
\widetilde{X_{\underline{w}}} & \xrightarrow{\rho^Q_{\underline{w}}} & X^Q_w & \xrightarrow{\mathrm{pr}} & X_w
\end{array}$$

Here $\mathrm{pr} : X^Q_w \to X_w$ is the restriction of the quotient map $G/Q \to G/P$ to X^Q_w, and $\theta^Q_{\underline{w}} : \widetilde{Z_{\underline{w}}} \to Z^Q_w$ is the map given by $\theta^Q_{\underline{w}}(p_1, \cdots, p_r, x) = (p_1 \cdots p_r, x)$.

Proof. Let $(Z^Q_w)^\circ$ be the restriction of of the vector bundle $\pi^Q_w : Z^Q_w \to X^Q_w$ to the Schubert cell C^Q_w. The quotient map $G/B \to G/Q$ induces an isomorphism $C^B_w \xrightarrow{\sim} C^Q_w$ of Schubert cells. Consequently, the quotient map,

$$Z^\circ_w = BwB \times^B \mathfrak{u}_w \longrightarrow BwQ \times^Q \mathfrak{u}_w, \qquad (bw, x) \mapsto (bw, x), \tag{2.6}$$

is an isomorphism. Observe that this map is the inverse of $\tau|(Z^Q_w)^\circ$, and so,

$$\tau((Z^Q_w)^\circ) = T^*_X C_w \subset T^*_X X_w.$$

Now, since $T^*_X X_w$ is a closed subvariety, it follows that $\tau(Z^Q_w) \subset T^*_X X_w$.

Finally, the commutativity of the diagram is a simple verification based on the formulae defining the various maps. □

Before we prove Theorem A, let us recall some standard results about proper maps, which the reader can find, for example, in [Har77, Ch.2].

Proposition 2.7. *The following properties are true:*

1. *Closed immersions are separated and proper.*
2. *The composition of proper maps is proper.*
3. *If $g : X \to Y$ is a proper map, then $g \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is proper.*
4. *Let $f : Y \hookrightarrow Z$ be a closed immersion. A map $g : X \to Y$ is proper if and only if $f \circ g$ is proper.*

Theorem A. *The maps $\theta^Q_{\underline{w}}$ and τ_w are proper and birational, and the composite map $\theta_{\underline{w}} \stackrel{\text{def}}{=} \tau_w \circ \theta^Q_w$ is a B-equivariant resolution of singularities $\theta_{\underline{w}} : \widetilde{Z_{\underline{w}}} \to T^*_X X_w$. The map $\theta_{\underline{w}}$ is independent of the choice of Q.*

Proof. The birationality of τ_w is a consequence of Equation (2.6). Recall from Section 1.4 that $\rho^Q_{\underline{w}}$ induces an isomorphism $(\widetilde{X_{\underline{w}}})^\circ \xrightarrow{\sim} C^Q_w$. Consequently, $\theta^Q_{\underline{w}}$ induces an isomorphism $(\widetilde{Z_{\underline{w}}})^\circ \xrightarrow{\sim} (Z^Q_w)^\circ$. It follows that $\theta^Q_{\underline{w}}$ is birational.

Consider now the commutative diagram

$$
\begin{array}{ccccc}
\widetilde{Z_{\underline{w}}} & \xrightarrow{\quad\theta^Q_{\underline{w}}\quad} & Z^Q_w & \xrightarrow{\quad\tau_w\quad} & T^*_X X_w \\
\big\downarrow f & & \big\downarrow g & & \big\downarrow h \\
\widetilde{X_{\underline{w}}} \times \mathcal{N} & \xrightarrow{\rho^Q_{\underline{w}} \times \mathrm{id}_{\mathcal{N}}} & X^Q_w \times \mathcal{N} & \xrightarrow{\mathrm{pr} \times \mathrm{id}_{\mathcal{N}}} & X_w \times \mathcal{N},
\end{array} \tag{2.8}
$$

where f, g, h are the closed immersions given by

$$
\begin{aligned}
f(p_1, \cdots, p_r, x) &= (\pi_{\underline{w}}(p_1, \cdots, p_r, x), Ad(p_1 \cdots p_r)x), \\
g(a, x) &= (\pi^Q_w(a, x), Ad(a)x), \\
h(a, x) &= (\pi_w(a, x), Ad(a)x).
\end{aligned}
$$

Observe that the map,

$$(\mathrm{pr} \times \mathrm{id}_{\mathcal{N}}) \circ (\rho^Q_{\underline{w}} \times \mathrm{id}) = \rho_{\underline{w}} \times \mathrm{id}_{\mathcal{N}},$$

is independent of the choice of Q, and therefore, the map $\theta_{\underline{w}} = \tau_w \circ \theta^Q_{\underline{w}}$ is also independent of the choice of Q.

Next, the maps $\rho^Q_{\underline{w}}$ and pr are proper; hence $\rho^Q_{\underline{w}} \times \mathrm{id}_{\mathcal{N}}$ and $\widetilde{X_{\underline{w}}} \times \mathcal{N}$ are proper. Consequently, $\theta^Q_{\underline{w}}$ and τ_w are proper.

Finally, observe that $\widetilde{Z_{\underline{w}}}$, being a vector bundle over the smooth variety $\widetilde{X_{\underline{w}}}$, is itself a smooth variety. Therefore, the map $\theta_{\underline{w}}$ is a resolution of singularities. □

3. The Type A Grassmannian

In this section, we recall the classical theory of Schubert varieties in type A. Further, for any Schubert variety $X_w \subset Gr(d, n)$, we choose a particular standard parabolic subgroup Q satisfying Equation (2.4). This choice of Q allows us to present a uniform proof of Theorem B in Section 5. The primary reference for this section is [LR08].

3.1. **The Root System of SL_n.** Let $E(n)$ be a n–dimensional vector space with privileged basis $\{e_1, \cdots, e_n\}$. The group $G = SL_n$ acts on $E(n)$ by left multiplication with respect to the basis $e_1, \cdots, e_n$. The Lie algebra $\mathfrak{g}$ of G is precisely the set of traceless $n \times n$ matrices, i.e.,

$$\mathfrak{g} = \{x \in \mathrm{Mat}_n(\mathsf{k}) \,|\, \mathrm{trace}(x) = 0\}.$$

Let $\mathfrak{t}$ be the set of diagonal matrices in $\mathfrak{g}$. For $1 \le i \le n$, let $\epsilon_i \in \mathfrak{t}^*$ be the linear functional given by

$$\left\langle \epsilon_i, \sum_{j=1}^{n} a_j E_{j,j} \right\rangle = a_i,$$

where $E_{j,j}$ is the diagonal matrix with entry 1 in the j^{th} position and zero elsewhere.

We fix B to be the set of upper triangular matrices in G. The Lie algebra $\mathfrak{b}$ of B is then precisely the set of upper triangular matrices in $\mathfrak{g}$. The root system Δ of $\mathfrak{g}$ with respect to $(\mathfrak{b}, \mathfrak{t})$ is precisely,

$$\Delta = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\},$$

with the simple root $\alpha_i \in S = \mathsf{A}_{n-1}$ being given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$. The root $\epsilon_i - \epsilon_j$ is positive if and only if $i < j$.

We denote by $E_{i,j}$ the elementary $n \times n$ matrix with 1 in the (i, j) position and 0 elsewhere; and by $[E_{i,j}]$ the one-dimensional subspace of $\mathfrak{g}$ spanned by $E_{i,j}$. Then $[E_{i,j}]$ is precisely the root space corresponding to the root $\epsilon_i - \epsilon_j$.

3.2. **Partial Flag Varieties.** Let $\underline{q} = (q_0, \cdots, q_r)$ be an integer-valued sequence satisfying $0 = q_0 \leq q_1 \leq \cdots \leq q_r = n$.

For $0 \leq i \leq n$, we denote by $E(i)$, the subspace of $E(n)$ with basis $e_1, \cdots, e_i$, and by $E(\underline{q})$, the partial flag $E(q_0) \subset \cdots \subset E(q_r)$. Let Q be the parabolic subgroup of SL_n corresponding to the subset $S_Q = \{\alpha_j \mid j \neq q_i, 1 \leq i \leq r\}$. The variety $X^Q = {}^G/_Q$ is precisely the variety of *partial flags of shape* $\underline{q}$,

$$X^Q = \{F(q_0) \subset \cdots \subset F(q_r) \mid \dim F(q_i) = q_i\}.$$

For brevity, we will denote a partial flag $F(q_0) \subset \cdots \subset F(q_r)$ of shape $\underline{q}$ by $F(\underline{q})$.

As a particular example, let P be the standard parabolic subgroup corresponding to the subset $S\backslash\{\alpha_d\}$. Then $X^d := {}^G/_P$ is precisely,

$$X^d = Gr(d, n) = \{V \mid \dim V = d\}.$$

3.3. **The Weyl Group.** The Weyl group of G is isomorphic to S_n, the symmetric group on n elements. The action of W on Δ is given by the formula,

$$w(\epsilon_i - \epsilon_j) = \epsilon_{w(i)} - \epsilon_{w(j)}.$$

In particular, $w(\epsilon_i - \epsilon_j) > 0$ if and only if $w(i) < w(j)$.

The set of *minimal representatives* with respect to Q is given by,

$$S_n^Q = \{w \in S_n \mid w(q_i + 1) < w(q_i + 2) < \cdots < w(q_{i+1}), \forall\, 0 \leq i \leq r\}. \tag{3.4}$$

3.5. **Schubert Varieties.** For $w \in S_n$, let $\mathrm{m}_w(i, j)$ be the number of non-zero entries in the top left $i \times j$ sub-matrix of the permutation matrix $\sum E_{w(k),k}$, i.e.,

$$\begin{aligned} \mathrm{m}_w(i, j) &\stackrel{\text{def}}{=} \#\{w(1), \cdots, w(j)\} \cap \{1, \cdots, i\} \\ &= \#\{(k, w(k)) \mid k \leq j, w(k) \leq i\}. \end{aligned} \tag{3.6}$$

The Schubert cells, $C_w^Q \subset X^Q$ and $C_w^d \subset X$, are given by,

$$C_w^Q = \left\{F(\underline{q}) \in X^Q \,\middle|\, \dim(F(q_i) \cap E(j)) = \mathrm{m}_w(j, q_i),\, 1 \leq j \leq n, 1 \leq i \leq l\right\},$$
$$C_w^d = \left\{V \in X^d \,\middle|\, \dim(V \cap E(j)) = \mathrm{m}_w(j, d),\, 1 \leq j \leq n\right\},$$

while the Schubert varieties, $X_w^Q \subset X^Q$ and $X_w^d \subset X^d$, are given by

$$\begin{aligned} X_w^Q &= \left\{F(\underline{q}) \in X^Q \,\middle|\, \dim(F(q_i) \cap E(j)) \geq \mathrm{m}_w(j, q_i),\, 1 \leq j \leq n, 1 \leq i \leq l\right\}, \\ X_w^d &= \{V \in X \mid \dim(V \cap E(j)) \geq \mathrm{m}_w(j, d),\, 1 \leq j \leq n\}. \end{aligned} \tag{3.7}$$

In particular, we have $F(\underline{q}) \in X_w^Q$, if and only if $F(q_i) \in X_w^{q_i}$ for all i.

3.8. **The Projection Map.** Suppose $d = q_i$ for some i, and consider the projection map $\mathrm{pr}_d : X^Q \to X^d$, given by $F(\underline{q}) \mapsto F(d)$.

For any $w \in S_n$, we have $\mathrm{pr}_d(X^Q_w) = X^d_w$. Further, if w satisfies

$$\begin{aligned} w(1) > \cdots > w(d), \\ w(d+1) > \cdots > w(n), \end{aligned} \tag{3.9}$$

then $\mathrm{pr}_d^{-1}(X^d_w) = X^Q_w$, see [LR08]. Any $w \in S_n$ satisfying Equation (3.9) is called a *maximal representative* with respect to P.

The following lemmas are easy consequences of standard results on Schubert varieties. They are used repeatedly in the proofs of Propositions 5.6, 5.9 and 6.1.

Lemma 3.10. *Suppose we have integers $0 \le k \le d \le n$, a permutation $w \in S_n$, and a k-dimensional subspace $U \subset E(n)$. If $U \in X^k_w$, then*

$$\dim(U \cap E(i)) \ge \mathrm{m}_w(i,d) - (d-k) \qquad \forall\, 1 \le i \le n.$$

Conversely, suppose the above inequalities hold, and further, $w(1) > \cdots > w(d)$. Then $U \in X^k_w$.

Proof. Observe that for $1 \le i \le n$, we have,

$$\mathrm{m}_w(i,k) \ge \max\{0, \mathrm{m}_w(i,d) - (d-k)\},$$

with equality holding for all i if and only if $w(1) > \cdots > w(d)$. □

Lemma 3.11. *Consider integers $0 \le k \le d \le n$, and a permutation $w \in S_n$, satisfying $w(k+1) > \cdots > w(n)$. Given $U \in X^d$, we have $U \in X^d_w$, if and only if,*

$$\dim(U \cap E(i)) \ge \mathrm{m}_w(i,k) \qquad \forall\, 1 \le i \le n.$$

Proof. There exists $1 \le i \le n$ such that $\dim(U \cap E(i)) = k$. Set $V = U \cap E(i)$. It is clear that $V \in X^k_w$. Now, since the statement of the lemma only involves w via the integers $\mathrm{m}_w(i,k)$ and $\mathrm{m}_w(i,d)$, we may assume without loss of generality that $w(1) > \cdots > w(k)$.

Let Q be the parabolic group corresponding to the sequence $\underline{q} = (k,d)$. Then, we have $\mathrm{pr}_k(V \subset U) = V$ and $\mathrm{pr}_d(V \subset U) = U$. It follows from Section 3.8 that,

$$\mathrm{pr}_k^{-1}(X^k_w) = X^Q_w \implies \mathrm{pr}_d(\mathrm{pr}_k^{-1}(X^k_w)) = X^d_w.$$

Consequently, we obtain $U \in X^d_w$. □

Proposition 3.12. *Consider integers $0 \le k \le d \le m \le n$, and a permutation $w \in S_n$. Given subspaces $U \subset V \subset E(n)$ satisfying* $\dim U = k$, *and*

$$\begin{aligned} \dim(U \cap E(i)) &\ge \mathrm{m}_w(i,d) - (d-k) && \forall\, 1 \le i \le n, \\ \dim(V \cap E(i)) &\ge \mathrm{m}_w(i,m) && \forall\, 1 \le i \le n, \end{aligned}$$

there exists $U' \in X^d_w$ satisfying $U \subset U' \subset V$.

Proof. Set $l = \dim V$. Observe that

$$l = \dim(V \cap E(n)) \ge \mathrm{m}_w(n,m) = m.$$

Let Q' be the parabolic group corresponding to the sequence (k,l), and Q the parabolic group corresponding to the sequence (k,d,l). We have a projection map $\mathrm{pr} : X^Q \to X^{Q'}$, given by $F(k,d,l) \mapsto F(k,l)$.

Since the statement of the proposition only involves w via the integers $\mathrm{m}_w(i,d)$ and $\mathrm{m}_w(i,m)$, we may replace w by any permutation v satisfying

$$\begin{aligned}\{v(1),\cdots,v(d)\} &= \{w(1),\cdots,w(d)\},\\ \{v(d+1),\cdots,v(m)\} &= \{w(d+1),\cdots,w(m)\},\\ \{v(m+1),\cdots,v(n)\} &= \{w(m+1),\cdots,w(n)\},\end{aligned}$$

without changing the statement. In particular, we may assume that

$$\begin{aligned}w(1)>w(2)>\cdots>w(d),\\ w(d+1)>\cdots>w(m),\\ w(m+1)>\cdots>w(n).\end{aligned} \tag{3.13}$$

Using Lemmas 3.10 and 3.11, we deduce that $(U\subset V)\in X_w^{Q'}$. Further, it follows from Equation (3.13) that the projection map $X_w^Q\to X_w^{Q'}$ is surjective, see [LR08]. In particular, there exists a partial flag $F(k,d,l)\in X_w^Q$, for which $F(k)=U$ and $F(l)=V$. This partial flag yields the required subspace $F(d)$, satisfying $U\subset F(d)\subset V$, and $F(d)\in X_w^d$. □

3.14. **The Numbers r_i, c_i.** For integers a,b, let $(a,b]$ denote the sequence,

$$a+1,a+2,\cdots,b.$$

We fix $w\in S_n^P$. Following Equation (3.4), we have,

$$w(1)<w(2)<\ldots<w(d), \qquad w(d+1)<\ldots<w(n).$$

Consequently, $w\in S_n^P$ is uniquely identified by the sequence $w(1),\cdots,w(d)$, which we now write as the following concatenation of contiguous sequences,

$$(t_1',t_1],\ (t_2',t_2],\ \cdots,\ (t_l',t_l].$$

Here the t_i,t_i' are certain integers satisfying,

$$0\le t_1'<t_1<t_2'<\cdots<t_{l-1}<t_l'<t_l\le n,$$

and $\sum(t_i-t_i')=d$. For convenience, we set $t_0=0$ and $t_{l+1}'=n$. The sequence $w(d+1),\cdots,w(n)$ is precisely,

$$(t_0,t_1'],\ (t_1,t_2'],\ \cdots,\ (t_l,\cdots,t_{l+1}'].$$

Consider the partial sums $r_0,\cdots,r_l$, and $c_0,\cdots,c_{l+1}$, given by,

$$r_i \overset{\text{def}}{=} \sum_{1\le j\le i}(t_j-t_j'), \qquad c_i \overset{\text{def}}{=} \sum_{1\le j\le i}(t_j'-t_{j-1}). \tag{3.15}$$

For $1\le i\le l$, we have $t_i=r_i+c_i$. Further, we have $r_l=d$, $c_{l+1}=n-d$, and

$$\begin{aligned}\mathrm{m}_w(t_i,r_j)&=\min\{r_i,r_j\}=r_{\min\{i,j\}},\\ \mathrm{m}_w(t_i,d+c_j)&=r_i+\min\{c_i,c_j\}=r_i+c_{\min\{i,j\}},\end{aligned} \tag{3.16}$$

for all $1\le i,j\le l$. The permutation matrix of w is described in Fig. 3.17.

Proposition 3.18. *Consider the sequence $\underline{q}=(q_0,\cdots,q_{2l+1})$ given by*

$$q_i=\begin{cases}r_i & \text{for } 0\le i\le l,\\ d+c_{i-l} & \text{for } l<i\le 2l+1.\end{cases}$$

Let Q be the standard parabolic subgroup associated to the sequence $\underline{q}$, in the sense of Section 3.2. Then Q satisfies Equation (2.4)*, i.e., $Q\subset Stab_G(\mathfrak{u}_w)\cap P$.*

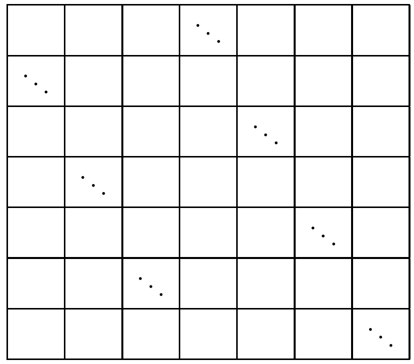

Figure 3.17. The permutation matrix of w. The empty boxes are zero matrices, while the dotted cells are identity matrices of size $r_1, r_2 - r_1, \cdots, r_l - r_{l-1}, c_1, c_2 - c_1, \cdots, c_{l+1} - c_l$, going left to right.

Proof. Observe that $q_l = d$; hence, we have $Q \subset P$. It remains to show that Q stabilizes $\mathfrak{u}_w$. Recall the set R and the subspace $\mathfrak{u}_w$ from Lemma 2.1. It follows from Section 3.14 that

$$\begin{aligned} R &= \{\epsilon_i - \epsilon_j \mid \exists k \text{ such that } i \leq q_k, j > q_{k+l}\}, \\ \mathfrak{u}_w &= \{x \in \mathfrak{g} \mid xE(q_{l+i}) \subset E(q_{i-1}),\ \forall\, 1 \leq i \leq l+1\}. \end{aligned} \tag{3.19}$$

Now, since Q stabilizes the flag $E(\underline{q})$, it also stabilizes $\mathfrak{u}_w$. □

4. The Symplectic Grassmannian

In this section, we recall some facts about Schubert subvarieties of the symplectic Grassmannian. In particular, for any Schubert variety X_w in the symplectic Grassmannian, we choose a particular standard parabolic subgroup Q satisfying Equation (2.4). This choice of Q allows us to present a uniform proof of Theorem B in Section 5. The primary reference for this section is [LR08].

4.1. **The Bilinear Form.** Let $E(2d)$ be a $2d$–dimensional vector space with a privileged basis $\{e_1, \cdots, e_{2d}\}$. For $1 \leq i \leq 2d$, we define,

$$\overline{i} \stackrel{\text{def}}{=} 2d + 1 - i.$$

Consider the non-degenerate skew-symplectic bilinear form ω on $E(2d)$ given by,

$$\omega(e_i, e_j) = \begin{cases} \delta_{i,\overline{j}} & \text{if } i \leq d, \\ -\delta_{i,\overline{j}} & \text{if } i > d. \end{cases}$$

For V a subspace of $E(2d)$, we define,

$$V^{\perp} = \{u \in E(2d) \mid \omega(u, v) = 0,\ \forall\, v \in V\}.$$

A simple calculation yields $E(i)^{\perp} = E(2d - i)$, for $1 \leq i \leq 2d$. Further, as a consequence of the non-degeneracy of ω, we have the formulae,

$$\dim V + \dim V^{\perp} = 2d, \qquad U^{\perp} \cap V^{\perp} = (U + V)^{\perp}, \tag{4.2}$$

and $(V^{\perp})^{\perp} = V$, for any subspaces $U, V \subset E(2d)$.

4.3. **The Symplectic Group Sp_{2d}.** Let $G = Stab_{SL_{2d}}(\omega)$, i.e.,

$$G = \{g \in SL_{2d} \,|\, \omega(gu, gv) = \omega(u, v),\ \forall\, u, v \in E(2d)\}\,.$$

The group G is the symplectic group Sp_{2d}. Its Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{g} = \{x \in \mathfrak{sl}_{2d} \,|\, \omega(xu, v) + \omega(u, xv) = 0,\ \forall\, u, v \in E(2d)\}\,.$$

Let T' (resp. $\mathfrak{t}'$) be the set of diagonal matrices, and B' (resp. $\mathfrak{b}'$) the set of upper triangular matrices in SL_{2d} (resp. $\mathfrak{sl}_{2d}$). The subgroup $T = T' \cap G$ is a maximal torus in G, and the subgroup $B = B' \cap G$ is a Borel subgroup of G.

4.4. **The Root System of Sp_{2d}.** The group G is a simple group with Dynkin diagram C_d. Recall from Section 3.1, the linear functionals $\epsilon_1, \cdots, \epsilon_{2d}$ on $\mathfrak{t}'$. By abuse of notation, we also denote by ϵ_i, the restriction $\epsilon_i|\mathfrak{t}$.

Following [LR08], we present the root system of G with respect to (B, T). The simple root $\alpha_i \in S = \mathsf{C}_d$ is given by

$$\alpha_i = \begin{cases} \epsilon_i - \epsilon_{i+1} & \text{for } 1 \leq i < d, \\ 2\epsilon_d & \text{for } i = d. \end{cases}$$

The set of roots Δ, and the set of positive roots Δ^+, are given by

$$\Delta = \{\pm\epsilon_i \pm \epsilon_j \,|\, 1 \leq i \neq j \leq d\} \sqcup \{\pm 2\epsilon_i \,|\, 1 \leq i \leq d\}\,,$$
$$\Delta^+ = \{\epsilon_i \pm \epsilon_j \,|\, 1 \leq i < j \leq d\} \sqcup \{2\epsilon_i \,|\, 1 \leq i \leq d\}\,.$$

The corresponding root spaces are given by $\mathfrak{g}_{2\epsilon_i} = [E_{i,\bar{\imath}}]$, $\mathfrak{g}_{-2\epsilon_i} = [E_{\bar{\imath},i}]$,

$$\mathfrak{g}_{\epsilon_i+\epsilon_j} = [E_{i,\bar{j}} + E_{j,\bar{\imath}}], \qquad \mathfrak{g}_{-\epsilon_i-\epsilon_j} = [E_{\bar{\imath},j} + E_{\bar{j},i}], \qquad \mathfrak{g}_{\epsilon_i-\epsilon_j} = [E_{i,j} - E_{\bar{j},\bar{\imath}}].$$

4.5. **The Weyl Group.** Let $s_1, \cdots, s_d$ denote the simple reflections in Weyl group W of G, and let $r_1, \cdots, r_{2d-1}$ denote the simple reflections of S_{2d-1}. We have an embedding $W \hookrightarrow S_{2d}$, given by,

$$s_i \mapsto \begin{cases} r_i r_{2d-i} & \text{for } 1 \leq i < d, \\ r_d & \text{for } i = d. \end{cases}$$

Via this embedding, we have,

$$W = \left\{ w \in S_{2d} \,\middle|\, \overline{w(i)} = w(\bar{\imath}),\ 1 \leq i \leq d \right\}.$$

The Bruhat order on S_{2d} induces a partial order on W. This induced order is precisely the Bruhat order on W. Further, by virtue of being a subgroup of S_{2d}, the group W acts on $\mathfrak{sl}_{2d}$. One obtains the action of W on $\mathfrak{g}$ by restricting this action.

4.6. **Standard Parabolic Subgroups.** Let $\underline{q} = (q_0, \cdots, q_r)$ be any integer-valued sequence satisfying $0 = q_0 \leq q_1 \leq \cdots \leq q_r = 2d$, and further, $q_i + q_{r-i} = 2d$ for $1 \leq i \leq r$. Suppose Q' is the standard parabolic subgroup of SL_{2d} corresponding to the subset,

$$\{\alpha_j \in \mathsf{A}_{2d-1} \,|\, j \neq q_i,\ 1 \leq i \leq r-1\}\,.$$

Then $Q = Q' \cap G$ is the parabolic subgroup of G corresponding to the subset,

$$\{\alpha_j \in S \,|\, j \neq q_i,\ 1 \leq i \leq \lceil r/2 \rceil\}\,.$$

The variety $X^Q = {}^{G}\!/\!_{Q}$ is precisely the variety of isotropic flags of shape $\underline{q}$, i.e.,

$$X^Q = \left\{ F(\underline{q}) \in {}^{SL_{2d}}\!/\!_{Q'} \,\middle|\, F(q_i)^{\perp} = F(q_{r-i}) \right\}.$$

As a particular example, let P' be the standard parabolic subgroup of SL_n corresponding to the subset $\mathsf{A}_{2d-1}\backslash\{\alpha_d\}$, and let $P = P' \cap G$. Then P is the standard parabolic corresponding to $S\backslash\{\alpha_d\}$, and further,

$$X = {}^{G}\!/\!_{P} = \left\{ V \subset E(2d) \,\middle|\, V = V^{\perp} \right\}.$$

Observe that the condition $V = V^{\perp}$ ensures that $\dim V = d$, see Equation (4.2).

4.7. **Schubert Varieties.** Consider an element $w \in W$. By viewing w as an element of S_{2d}, we define the numbers $\mathrm{m}_w(i,k)$ precisely as in Equation (3.6). The Schubert cells C_w, C_w^Q are then given by,

$$\begin{aligned} C_w^Q &= \left\{ F(\underline{q}) \in X^Q \,\middle|\, \dim(F(q_i) \cap E(j)) = \mathrm{m}_w(j, q_i),\ 1 \le i \le l, 1 \le j \le n \right\}, \\ C_w &= \left\{ V \in X \,|\, \dim(V \cap E(j)) = \mathrm{m}_w(j,d),\ 1 \le j \le n \right\}, \end{aligned}$$

and the Schubert varieties X_w and X_w^Q are given by,

$$\begin{aligned} X_w^Q &= \left\{ F(\underline{q}) \in X^Q \,\middle|\, \dim\left(F(q_i) \cap E(j)\right) \ge \mathrm{m}_w(j, q_i),\ 1 \le i \le l, 1 \le j \le n \right\}, \\ X_w &= \left\{ V \in X \,|\, \dim(V \cap E(i)) \ge \mathrm{m}_w(i,d),\ 1 \le i \le n \right\}. \end{aligned} \tag{4.8}$$

In particular, any Schubert subvariety of X^Q can be identified (set-theoretically) as the intersection of a Schubert subvariety of ${}^{SL_{2d}}\!/\!_{Q'}$ with ${}^{Sp_{2d}}\!/\!_{Q} \subset {}^{SL_{2d}}\!/\!_{Q'}$.

4.9. **Numerical Redundancy.** By viewing w as an element of S_{2d}, we define the numbers t_i, t_i', r_i, c_i exactly as in Section 3.14 and Equation (3.15). Observe that since $w(\bar{i}) = \overline{w(i)}$ for all $1 \le i \le 2d$, the permutation matrix of w is symmetric across the anti-diagonal, see Fig. 3.17. Consequently, for any $0 \le i \le l$, we have,

$$r_i + c_{l-i} = d, \qquad\qquad t_i + t_{l-i} = 2d. \tag{4.10}$$

In particular, we have $E(t_i)^{\perp} = E(t_{l-i})$.

The conditions defining the Schubert variety $X_w^Q \subset X^Q$, described in Equation (4.8), are not minimal. We describe this redundancy in the next lemma.

Lemma 4.11. *Consider $F(\underline{q}) \in X^Q$. Then $F(\underline{q}) \in X_w^Q$ if and only if*

$$\dim\left(F(q_i) \cap E(j)\right) \ge \mathrm{m}_w(j, q_i), \qquad 1 \le i \le l,\ 1 \le j \le 2d.$$

Proof. Since the permutation matrix of w is symmetric across the anti-diagonal, the number of non-zero entries in the top left $i \times j$ corner of w equals the number of entries in the bottom right $i \times j$ corner. Further, since each row and column of this matrix has precisely one non-zero entry, we have,

$$\begin{aligned} \mathrm{m}_w(i,j) =& \#\left\{ (k, w(k) \,|\, k > 2d-j, w(k) > 2d-i \right\} \\ =& 2d - \#\left(\{(k, w(k)) \,|\, k \le 2d-j\} \cup \{(k, w(k)) \,|\, w(k) \le 2d-i\} \right) \\ =& 2d - \left((2d-j) + (2d-i) - \mathrm{m}_w(2d-i, 2d-j) \right). \end{aligned}$$

Hence, for $1 \le i, j \le 2d$, we have the formula,

$$2d - (i + j - \mathrm{m}_w(i,j)) = \mathrm{m}_w(2d-i, 2d-j).$$

Consider some $F(\underline{q}) \in X^Q$ satisfying the inequalities of the lemma. Given $1 \leq i \leq l$ and $1 \leq j \leq 2d$, we have,

$$\begin{aligned} &\dim(F(q_i) \cap E(j)) \geq \mathrm{m}_w(j, q_i) \\ \implies &\dim(F(q_i) + E(j)) \leq q_i + j - \mathrm{m}_w(j, q_i) \\ \implies &\dim((F(q_i) + E(j))^\perp) \geq 2d - (q_i + j - \mathrm{m}_w(j, q_i)) \\ \implies &\dim(F(q_i)^\perp \cap E(2d-j)) \geq \mathrm{m}_w(2d-j, 2d-q_i). \end{aligned}$$

The final inequality follows from the penultimate as a consequence of Equation (4.2). We see that $F(\underline{q})$ satisfies Equation (4.8), and hence obtain $F(\underline{q}) \in X_w^Q$. □

4.12. **The Subspace $\mathfrak{u}_w$.** Let $\mathfrak{v}$ be the Lie algebra of the unipotent radical of P', and $\mathfrak{u}$ the Lie algebra of the unipotent radical of P. We have

$$\mathfrak{v} = \bigoplus_{i \leq d < j} [E_{i,j}], \qquad \mathfrak{u} = \bigoplus_{1 \leq i < j \leq d} \mathfrak{g}_{\epsilon_i + \epsilon_j} = \bigoplus_{1 \leq i < j \leq d} [E_{i,\bar{j}} + E_{j,\bar{i}}].$$

In particular, we have $\mathfrak{u} = \mathfrak{v} \cap \mathfrak{g}$. Recall the subspace $\mathfrak{u}_w$ from Lemma 2.1. Since $\mathfrak{g}$ is stable under the action of $Ad(w^{-1})$, we have,

$$\mathfrak{u}_w = \mathfrak{u} \cap Ad(w^{-1})\mathfrak{b} = (\mathfrak{v} \cap \mathfrak{g}) \cap Ad(w^{-1})(\mathfrak{b}' \cap \mathfrak{g}) = \mathfrak{v} \cap Ad(w^{-1})\mathfrak{b}' \cap \mathfrak{g}. \tag{4.13}$$

Let $q_0, q_1, \cdots, q_{2l+1}$ be the sequence defined by

$$q_i = \begin{cases} r_i & \text{for } 0 \leq i \leq l, \\ d + c_{i-l} & \text{for } l < i \leq 2l+1. \end{cases}$$

It follows from Equation (4.10) that $q_i + q_{2l-i} = 2d$ for all $1 \leq i \leq 2l$.

Proposition 4.14. *Let Q be the standard parabolic subgroup of G associated to the sequence $\underline{q} = (q_0, \cdots, q_{2l})$, in the sense of Section 4.6. Then Q satisfies Equation* (2.4)*, i.e., $Q \subset Stab_G(\mathfrak{u}_w) \cap P$.*

Proof. It follows from Equation (4.10) that $c_{l+1} = c_l = d$, hence $q_{2l+1} = q_{2l} = 2d$. Therefore, the standard parabolic subgroup $Q' \subset SL_{2d}$ associated to $(q_0, \cdots, q_{2l+1})$ is the same as the standard parabolic subgroup of SL_{2d} associated to $(q_0, \cdots, q_{2l})$.

Next, it follows from Equations (3.19) and (4.13) that

$$\mathfrak{u}_w = \{x \in \mathfrak{g} \,|\, xE(q_{l+i}) \subset E(q_{i-1}), \forall\, 1 \leq i \leq l+1\}. \tag{4.15}$$

Now, since Q' stabilizes $\mathfrak{u}_w$, and since $Q = Q' \cap G$, we have $Q \subset Stab_G(\mathfrak{u}_w)$. Finally, since $q_l = d$, we have $Q \subset P$, hence $Q \subset Stab_G(\mathfrak{u}_w) \cap P$. □

5. Defining Equations for the Conormal Variety in Types A and C

Fix integers $d < n$. Let G be either SL_n or SO_{2d}, let B be the subgroup of upper triangular matrices in G, and let P be the standard parabolic subgroup of G corresponding to the subset $S \backslash \{\alpha_d\}$ of simple roots. As discussed in Sections 3 and 4, the variety $X = {}^{G}/_{P}$ is either the usual Grassmannian $Gr(d, n)$ or the symplectic Grassmannian $SGr(2d)$.

We fix a Schubert variety $X_w \subset X$ corresponding to some $w \in W^P$. In this section, we prove Theorem B, which gives a system of defining equations for the conormal variety $T_X^* X_w$ as a subvariety of T^*X. Let $\pi : T^*X \to X$ be the structure map, and μ the Springer map. Theorem B states that a point $p \in T^*X$ is in $T^*_X X_w$ if and only if $\pi(p) \in X_w$ and $\mu(p)$ satisfies Equation (5.13).

Recall the commutative diagram from Proposition 2.5. We show in Proposition 5.5 that for any point in Z^Q_w, its image under $\mu \circ \tau_w$ satisfies Theorem B. Conversely, we show in Propositions 5.6 and 5.9 that any point in T^*X lying over X_w, and further satisfying Equation (5.13), belongs to $\tau_w(Z^Q_w) = T^*_X X_w$.

5.1. **Combinatorial Description of X_w.** Fix $w \in W^P$. Let the integers $\mathrm{m}_w(i,j)$, r_i, and c_i be as in Equations (3.6) and (3.15) respectively. It follows from Equations (3.7), (3.16) and (4.8) that $F(\underline{q}) \in X^Q_w$ if and only if

$$\begin{aligned} \dim(F(q_i) \cap E(t_j)) &\geq \min\{r_i, r_j\} = r_{\min\{i,j\}} && \forall\, 1 \leq i,j \leq l, \\ \dim(F(q_{i+l}) \cap E(t_j)) &\geq r_j + \min\{c_i, c_j\} = r_j + c_{\min\{i,j\}} && \forall\, 1 \leq i,j \leq l. \end{aligned}$$

In particular, when $i = j$, this yields $F(q_i) \subset E(t_i) \subset F(q_{i+l})$.

5.2. **The Cotangent Bundle.** Let $\pi : T^*X \to X$ be the structure map defining the cotangent bundle, and $\mu : T^*X \to \mathcal{N}$ the Springer map, see Section 1.7. We identify the cotangent bundle T^*X with its image under the closed embedding $(\pi, \mu) : T^*X \hookrightarrow X \times \mathcal{N}$,

$$T^*X = \{(V, x) \in X \times \mathcal{N} \,|\, xE(n) \subset V,\, xV = 0\}\,.$$

5.3. **The Variety Z^Q_w.** For $G = SL_n$, let $\underline{q}$ and Q be as in Proposition 3.18. For $G = Sp_{2d}$, let $\underline{q}$ and Q be as in Proposition 4.14. Recall the variety Z^Q_w from Section 2.3, and the descriptions of $\mathfrak{u}_w$ from Equations (3.19) and (4.15). Using the closed embedding f from Equation (2.8), we obtain,

$$Z^Q_w = \left\{(F(\underline{q}), x) \in X^Q_w \times \mathcal{N} \,\middle|\, xF(q_{i+l}) \subset F(q_{i-1}),\, \forall\, 1 \leq i \leq l+1\right\}. \tag{5.4}$$

Theorem B states that given $(V, x) \in T^*X$, we have $(V, x) \in T^*_X X_w$, if and only if $V \in X_w$, and x satisfies Equation (5.13). The purpose of the following proposition is to show that Equation (5.13) is necessary, i.e., if $(V, x) \in T^*_X X_w$, then x satisfies Equation (5.13).

Proposition 5.5. *For any point $(F(\underline{q}), x) \in Z^Q_w$, we have, for $1 \leq j < i \leq l$,*

$$\dim({}^{xE(t_i)}/{}_{E(t_j)}) \leq \begin{cases} r_{i-1} - r_j, \\ c_i - c_{j+1}. \end{cases}$$

Proof. Consider $(F(\underline{q}), x) \in Z^Q_w$, and integers $1 \leq j < i \leq l$. We see from Section 5.1 that $E(t_i) \subset F(q_{i+l})$, and from Equation (5.4) that $xF(q_{i+l}) \subset F(q_{i-1})$. Consequently, we have $xE(t_i) \subset F(q_{i-1})$, and hence,

$$\begin{aligned} \dim({}^{xE(t_i)}/{}_{E(t_j)}) &\leq \dim({}^{F(q_{i-1})}/{}_{E(t_j)}) \\ &= \dim F(q_{i-1}) - \dim(F(q_{i-1}) \cap E(t_j)) \\ &\leq r_{i-1} - r_j, \end{aligned}$$

where the final inequality follows from Section 5.1. Next, we see from Section 5.1 and Equation (5.4) that $xF(q_{j+l+1}) \subset F(q_j) \subset E(t_j)$. In particular, $F(q_{j+l+1})$ is contained in the kernel of the map,

$$F(q_{i+l}) \to {}^{E(n)}/{}_{E(t_j)}, \qquad v \mapsto xv (\operatorname{mod} E(t_j)).$$

Since the image of this map is precisely ${}^{xF(q_{i+l})}/_{E(t_j)}$, we have,

$$\begin{aligned}\dim({}^{xF(q_{i+l})}/_{E(t_j)}) &\leq \dim F(q_{i+l}) - \dim F(q_{j+l+1})\\ &= q_{i+l} - q_{j+l+1} = c_i - c_{j+1}.\end{aligned}$$

Finally, since $E(t_i) \subset F(q_{i+l})$, we deduce that $\dim({}^{xE(t_i)}/_{E(t_j)}) \leq c_i - c_{j+1}$. ☐

The following two propositions lay the groundwork required to prove that Equation (5.13) is sufficient.

Proposition 5.6. *Consider* $(V, x) \in X_w \times \mathcal{N}$ *satisfying* $\operatorname{Im} x \subset V \subset \ker x$*, and*

$$\dim({}^{xE(t_i)}/_{E(t_j)}) \leq r_{i-1} - r_j, \qquad 0 \leq j < i \leq l+1. \tag{5.7}$$

Then, there exists a sequence of subspaces $V_0 \subset \cdots \subset V_l = V$*, satisfying,*

$$\begin{aligned}\dim V_i &= q_i,\\ xE(t_{i+1}) &\subset V_i \subset E(t_i),\\ \dim(V_i \cap E(t_j)) &\geq \min\{r_i, r_j\} = \mathrm{m}_w(t_j, q_i),\end{aligned} \tag{5.8}$$

for all $1 \leq i, j \leq l$.

Proof. Since $V \in X_w$, it follows from Section 5.1 that $V_l = V$ satisfies Equation (5.8). We construct the subspaces V_i inductively. In particular, given subspaces $V_i, \cdots, V_l$ satisfying Equation (5.8), we construct V_{i-1}.

Applying Equation (5.7) with $j = i - 1$, we have $xE(t_i) \subset E(t_{i-1})$. Further, Equation (5.8) yields $xE(t_i) \subset xE(t_{i+1}) \subset V_i$. Hence, we have,

$$xE(t_i) \subset V_i \cap E(t_{i-1}).$$

Set $U_1 = xE(t_i)$, and $U_2 = V_i \cap E(t_{i-1})$. Applying Equation (5.7) with $j = 0$, we see that $\dim U_1 \leq r_{i-1}$. Let $k = r_{i-1} - \dim U_1$.

Observe that $U_1 \cap E(t_j)$ is the kernel of the quotient map $U_1 \to {}^{U_1}/_{E(t_j)}$. Now, since $\dim({}^{xE(t_i)}/_{E(t_j)}) \leq r_{i-1} - r_j$, we have, for $1 \leq j \leq l$,

$$\begin{aligned}\dim(U_1 \cap E(t_j)) &= \dim U_1 - \dim({}^{U_1}/_{E(t_j)})\\ &\geq (r_{i-1} - k) - (r_{i-1} - r_j) = r_j - k\\ &\geq \min\{r_{i-1}, r_j\} - k\\ &= \mathrm{m}_w(t_j, q_{i-1}) - k.\end{aligned}$$

On the other hand, observe that,

$$\begin{aligned}U_2 \cap E(t_j) &= \begin{cases} V_i \cap E(t_{i-1}) & \text{if } i \leq j,\\ V_i \cap E(t_j) & \text{if } i > j,\end{cases}\\ \implies \dim(U_2 \cap E(t_j)) &\geq \begin{cases} r_{i-1} & \text{if } i \leq j,\\ r_j & \text{if } i > j,\end{cases}\\ &= \min\{r_{i-1}, r_j\} = \mathrm{m}_w(t_j, q_{i-1}).\end{aligned}$$

It now follows from Proposition 3.12 that there exists a subspace V_{i-1} satisfying $xE(t_i) \subset V_{i-1} \subset U_2 \subset E(t_{i-1})$, and Equation (5.8). ☐

Proposition 5.9. *Consider* $(V,x) \in X_w \times \mathcal{N}$ *satisfying* $\operatorname{Im} x \subset V \subset \ker x$*, and*

$$\dim({}^{xE(t_i)}/_{E(t_j)}) \leq c_i - c_{j+1}, \qquad \forall\, 0 \leq j < i \leq l+1. \tag{5.10}$$

Then, there exists a sequence of subspaces $V = V_l \subset \cdots \subset V_{2l+1}$*, satisfying,*

$$\begin{gathered}\dim V_{l+i} = q_{l+i},\\ V_{l+i} \subset \ker x + E(t_i),\\ \dim(V_{l+i} \cap E(t_j)) \geq r_j + \min\{c_i, c_j\} = \mathrm{m}_w(t_j, q_{l+i}).\end{gathered} \tag{5.11}$$

for all $1 \leq i, j \leq l$*.*

Proof. Since $V \in X_w$, it follows from Section 5.1 that $V_l = V$ satisfies Equation (5.11). We construct the subspaces V_{l+i} inductively. In particular, given subspaces $V_l, \cdots, V_{l+i-1}$ satisfying Equation (5.11), we construct V_{l+i}.

We see from Equation (5.11) that $V_{l+i-1} \subset \ker x + E(t_i)$. Set $U = \ker x + E(t_i)$. We first prove that,

$$\dim(U \cap E(t_j)) \geq r_j + \min\{c_i, c_j\} = \mathrm{m}_w(t_j, q_{l+i}) \quad \forall\, 1 \leq j \leq l+1. \tag{5.12}$$

For $j \leq i$, we have $E(t_j) \subset U$, hence $U \cap E(t_j) = E(t_j)$. It follows that,

$$\dim(U \cap E(t_j)) = t_j = r_j + c_j = r_j + \min\{c_i, c_j\}.$$

For $j > i$, consider the map,

$$E(t_j) \to {}^{xE(t_j)}/_{E(t_{i-1})}, \qquad v \mapsto xv (\operatorname{mod} E(t_{i-1})).$$

Since $xE(t_i) \subset E(t_{i-1})$, the subspace $U \cap E(t_j)$ is contained in the kernel of this map. Further, Equation (5.10) states that $\dim({}^{xE(t_j)}/_{E(t_i)}) \leq c_j - c_{i+1}$, hence

$$\begin{aligned}\dim(U \cap E(t_j)) &\geq t_j - (c_j - c_i)\\ &= r_j + c_i = r_j + \min\{c_i, c_j\}.\end{aligned}$$

This finishes the proof of Equation (5.12). It now follows from Equation (5.12), Lemma 3.10, and Proposition 3.12 that there exists a subspace V_{l+i} containing V_{l+i-1}, and further satisfying Equation (5.11). ☐

Theorem B. *Consider* $(V,x) \in T^*X$*. Then* $(V,x) \in T^*_X X_w$ *if and only if* $V \in X_w$*, and further, for all* $1 \leq j < i \leq l+1$*, we have,*

$$\dim({}^{xE(t_i)}/_{E(t_j)}) \leq \begin{cases} r_{i-1} - r_j,\\ c_i - c_{j+1}.\end{cases} \tag{5.13}$$

Proof. Recall the map $\tau_w : Z_w^Q \to T^*_X X_w$ from Proposition 2.5, given by,

$$\tau_w(F(\underline{q}), x) = (F(d), x).$$

The map τ_w is proper and birational (see Theorem A), hence surjective. It follows that $(V,x) \in T^*_X X_w$ if and only if there exists $F(\underline{q}) \in X_w^Q$ such that $F(d) = V$, and $(F(\underline{q}), x) \in Z_w^Q$.

Consider $(V,x) \in T^*_X X_w$. It follows from Theorem A that $V \in X_w$, and from Proposition 5.5 that Equation (5.13) holds. Conversely, consider $(V,x) \in T^*X$ satisfying $V \in X_w$, and Equation (5.13). We will construct $F(\underline{q}) \in X_w^Q$ such that $(F(\underline{q}), x) \in Z_w^Q$, and $\tau_w(F(\underline{q}), x) = (V,x)$.

Using Proposition 5.6, we construct subspaces $V_0, \cdots, V_l = V$ satisfying Equation (5.8). Similarly, we use Proposition 5.9 to construct subspaces $V_{l+1}, \cdots, V_{2l+1}$ satisfying Equation (5.11).

Suppose first that $G = SL_n$. We set,

$$F(\underline{q}) = V_0 \subset V_1 \subset \cdots \subset V_{2l+1}.$$

Observe that $F(d) = V_l = V$. It follows from Equations (5.8) and (5.11) that $F(\underline{q}) \in X_w^Q$. Further, for $1 \leq i \leq l+1$, we have,

$$F(q_{l+i}) \subset \ker x + E(t_i) \implies xF(q_{l+i}) \subset xE(t_i) \subset F(q_{i-1}).$$

This is precisely the condition for $(F(\underline{q}), x)$ to belong to Z_w^Q.

Suppose next that $G = Sp_{2d}$. Let $F(\underline{q})$ be the partial flag given by,

$$F(q_i) = \begin{cases} V_i & \text{for } i \leq l, \\ V_{2l-i}^{\perp} & \text{for } l < i \leq 2l. \end{cases}$$

In particular, we have $F(d) = V_l = V$. It follows from Lemma 4.11 and Proposition 5.6 that $F(\underline{q}) \in X_w^Q$. It remains to show that $(F(\underline{q}), x) \in Z_w^Q$.

For $0 \leq i \leq l$, we have $xE(t_{i+1}) \subset V_i$, hence $\omega(xE(t_{i+1}), V_i^{\perp}) = 0$. It follows from Section 4.3 that $\omega(E(t_{i+1}), x(V_i^{\perp})) = 0$, hence,

$$xF(q_{2l-i}) = x(V_i^{\perp}) \subset E(t_{i+1})^{\perp} = E(t_{l-i-1}).$$

The final equality is a consequence of Equation (4.10). Substituting $i \mapsto l - i$ yields $xE(q_{l+i}) \subset E(t_{i-1})$ for all $0 \leq i \leq l$. It follows that $(F(\underline{q}), x) \in Z_w^Q$, hence $(F(d), x) \in T_X^* X_w$. ☐

6. Orbital Varieties

Let G, B be as in the previous sections. Consider a G-orbit closure $\mathcal{N}_{\boldsymbol{\lambda}} \subset \mathcal{N}$. The irreducible components of $\mathcal{N}_{\boldsymbol{\lambda}} \cap \mathfrak{u}_B$ are called orbital varieties. Orbital varieties are closely related to the conormal varieties of Schubert varieties.

Proposition 6.1 (cf. [Spa82])**.** *Given a standard parabolic subgroup Q, and a Schubert variety $X_w^Q \subset X^Q$, the image of the conormal variety $T^*_{X^Q} X_w^Q$ under the Springer map $\mu^Q : T^*X^Q \to \mathcal{N}$ is an orbital variety.*

For more details on the relationship between conormal varieties and orbital varieties, the reader may consult [DR09, Spa82]. Providing a combinatorial description of the inclusion order on orbital varieties, and providing the defining equations for an orbital variety viewed as a subvariety of $\mathfrak{u}_B$, are both open problems in general. For certain orbital varieties in types A, B, C (those corresponding to the nilpotent orbits satisfying $x^2 = 0$), these problems were solved in [Mel05, BM17].

Suppose G is either SL_{2n} or Sp_{2d}, and P is the standard parabolic group corresponding to $S \backslash \{\alpha_d\}$. We derive, in Theorem C, a system of defining equations for orbital varieties of the form $\mu(T_X^* X_w)$. This is an easy consequence of Theorem B, and recovers some of the results of [Mel05, BM17].

Theorem C. *Let G, B, P, X, w, and μ be as in Theorem B. Then,*

$$\mu(T_X^* X_w) = \left\{ x \in \mathfrak{u}_B \;\middle|\; x^2 = 0,\, \dim({}^{xE(t_i)}/{}_{E(t_j)}) \leq \begin{cases} r_{i-1} - r_j, \\ c_i - c_{j+1}, \end{cases} \quad \forall\, 1 \leq i < j \leq l \right\}.$$

Proof. Consider $x \in \mathfrak{u}_B$ satisfying $x^2 = 0$, and

$$\dim({}^{xE(t_i)}/_{E(t_j)}) \leq \begin{cases} r_{i-1} - r_j, \\ c_i - c_{j+1}, \end{cases} \qquad \forall\, 1 \leq j < i \leq l. \tag{6.2}$$

Substituting $j = 0$ in Equation (6.2), we obtain,

$$\begin{aligned} \dim(\operatorname{Im}(x|E(t_i)) &= \dim(xE(t_i)) \\ &\leq c_i - c_1 \\ \implies \dim(\ker x \cap E(t_i)) &= \dim(\ker(x|E(t_i))) \\ &\geq t_i - (c_i - c_1) \\ &= r_i + c_1 \geq r_i. \end{aligned}$$

Let $k = d - \dim(\operatorname{Im} x)$. Substituting $i = l$ in Equation (6.2) yields,

$$\begin{aligned} \dim({}^{\operatorname{Im} x}/_{E(t_j)}) &\leq r_{l-1} - r_j \leq d - r_j \\ \implies \dim(\operatorname{Im} x + E(t_j)) &\leq (d - r_j) + t_j \\ \implies \dim(\operatorname{Im} x \cap E(t_j)) &= \dim(\operatorname{Im} x) + \dim E(t_j) - \dim(\operatorname{Im} x + E(t_j)) \\ &\geq (d - k) + t_j - (d - r_j + t_j) = r_j - k. \end{aligned}$$

Observe that since $x^2 = 0$, we have $\operatorname{Im} x \subset \ker x$. It now follows from Proposition 3.12 and Section 5.1 that there exists $V \in X_w$ such that,

$$\operatorname{Im} x \subset V \subset \ker x,$$

i.e., $(V, x) \in T^*_X X_w$. Consequently, we have $x \in \mu(T^*_X X_w)$. □

6.3. **Orbital Varieties in Type A.** When $G = SL_n$, the G-orbits in $\mathcal{N}$ are indexed by the partitions of n. For $\boldsymbol{\lambda}$ a partition of n, the irreducible components of $\mathcal{N}_{\boldsymbol{\lambda}} \cap \mathfrak{u}_B$ are indexed by the standard Young tableaux of shape $\boldsymbol{\lambda}$. For T a standard Young tableau, we denote the corresponding orbital variety by $\mathcal{O}_{\mathsf{T}}$.

In this case, the relationship between conormal varieties of Schubert varieties, and orbital varieties, as described in Proposition 6.1, provides a geometric realization of the Robinson-Schensted correspondence.

Proposition 6.4 (cf. [Ste88])**.** *Suppose T is the left Robinson-Schensted tableau of some $w \in S_n$. Then $\mathcal{O}_{\mathsf{T}} = \mu^B(T^*_{X^B} X^B_w)$.*

Proposition 6.5. *Let T be a standard Young tableau with exactly two columns. Then there exists a standard parabolic subgroup $P \subset G$, and a Schubert variety X_w in $X = {}^G/_P$, such that $\mathcal{O}_{\mathsf{T}} = \mu(T^*_X X_w)$.*

Proof. Let k be the number of boxes in the first column of T, and let P be the standard parabolic subgroup of G corresponding to $\mathsf{A}_{n-1} \backslash \{\alpha_k\}$. The longest element w_P of W_P is given by

$$w_P(i) = \begin{cases} k + 1 - i & \text{for } i \leq k, \\ n + 1 - k & \text{for } i > k. \end{cases}$$

Let $a_1, \cdots, a_k$ be the entries in the first column of T, written in increasing order, i.e., top to bottom; and $b_1, \cdots, b_{n-k}$ the entries in the second column, also written in increasing order. We consider the element $w \in S_n$ given by,

$$w(i) = \begin{cases} a_i & \text{for } i \leq k, \\ b_{i-k} & \text{for } i > k. \end{cases}$$

Let $v = ww_P$. Since $w \in S_n^P$, the Schubert variety X_v^B is a fibre bundle over X_w with fibre ${}^B/_P$, and we have a Cartesian diagram,

$$\begin{array}{ccc} X_v^B & \hookrightarrow & X^B \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow \\ X_w^P & \hookrightarrow & X^P. \end{array}$$

The map $X^B \to X^P$ is precisely the quotient map ${}^G/_B \to {}^G/_P$. Consequently, $T^*_{X^B} X_v^B$ is a ${}^B/_P$-fibre bundle over $T^*_X X_w$, with the map $T^*_{X^B} X_v^B \to T^*_X X_w$ being simply the restriction (to $T^*_{X^B} X_v^B \subset T^*X \subset X^B \times \mathcal{N}$) of the map,

$$\mathrm{pr} \times \mathrm{id}_{\mathcal{N}} : X^B \times \mathcal{N} \to X \times \mathcal{N}.$$

This yields us $\mu^B(T^*_{X^B} X_w^B) = \mu(T^*_X X_w)$. Finally, we verify that T is the left RSK-tableau of v, thus obtaining $\mathcal{O}_\mathsf{T} = \mu^B(T^*_{X^B} X_w^B) = \mu(T^*_X X_w)$. □

Corollary 6.6. *Let T be a two-column standard Young tableau. Consider integers $0 \leq j < i \leq n$, and the skew-tableau $\mathsf{T} \backslash \{1, \cdots, j, i+1, \cdots, n\}$. Let T_i^j denote the tableau obtained from this skew-tableau via 'jeu de taquin'. Then,*

$$\mathcal{O}_\mathsf{T} = \left\{ x \in \mathcal{N} \,\middle|\, J(x_i^j) \preceq \mathsf{T}_i^j \right\},$$

where x_i^j is the square sub-matrix of x with corners (t_j+1, t_j+1) and (t_i, t_i), $J(x_i^j)$ denotes the Jordan type of x_i^j, and $\preceq$ denotes the inclusion order on the set of G-orbits $\mathcal{N}_{\boldsymbol{\lambda}}$.

Proof. This statement is proved in [Mel05]. We explain here how it also follows as a consequence of Theorem C and Section 6.3.

Since $x^2 = 0$, we have $(x_i^j)^2 = 0$ for all i, j. Consequently, the inequality $J(x_i^j) \preceq T_i^j$ is equivalent to the inequality $rk(x_i^j) \leq f_i^j$, where f_i^j is the number of boxes in the second column of T_i^j. On the other hand, it follows from Theorem C and Section 6.3 that

$$\mathcal{O}_\mathsf{T} = \left\{ x \in \mathcal{N} \,\middle|\, \mathrm{rk}(x_i^j) \leq g_i^j \right\},$$

for certain integers g_i^j. It is a simple exercise to verify that the integers f_i^j and g_i^j defined here are equal. □

7. A Type Independent Conjecture

In this section, we assume that X is a cominuscule Grassmannian corresponding to some Dynkin diagram. We conjecture, for any Schubert variety $X_w \subset X$, the following equality,

$$T^*_X X_w = \mu^{-1}(\mu(T^*_X X_w)) \cap \pi^{-1}(X_w). \tag{7.1}$$

The question is well-posed in both set-theoretic and scheme-theoretic settings.

Suppose $X_w \subset X$ is a smooth Schubert subvariety. We prove in Proposition 7.2 that Equation (7.1) holds set-theoretically in this case.

Next, let w_0 denote the longest element in the Weyl group W. We show in Proposition 7.4 that if $X_w \subset X$ is a Schubert variety such that the *opposite* Schubert variety X_{w_0w} is smooth, then Equation (7.1) holds scheme-theoretically. This is a straightforward corollary to [LS17, Theorem 1.1].

When X is the usual Grassmannian or the symplectic Grassmannian, the set-theoretic version is a consequence of Theorems B and C. In type B, the only cominuscule Grassmannian is the one corresponding to $G = SO_{2n+1}$, and the cominuscule root α_1. In this cases, one easily verifies that for each $w \in W^P$, either X_w is smooth, or X_{w_0w} is smooth, hence settling the set-theoretic version of our conjecture for all cominuscule Grassmannians in types A, B, and C.

One would like to know in which of these cases Equation (7.1) holds scheme-theoretically, and also whether Equation (7.1) holds for types D and E. If it does, can we find a uniform, type independent proof?

Proposition 7.2. *Suppose X_w is smooth. Then the conormal variety $T^*_X X_w$ satisfies Equation* (7.1) *set-theoretically.*

Proof. A Schubert variety X_w in a cominuscule Grassmannian X is smooth if and only if X_w is homogeneous for some standard parabolic subgroup L, see [BM10].

Suppose X_w is homogeneous for some standard parabolic subgroup L; let S_L be the corresponding subset of S, and w_L the longest word of W supported on S_L. Then w is the minimal representative of w_L in W^P. Further, the subspace $\mathfrak{u}_w \subset \mathfrak{u}$ from Lemma 2.1 is precisely,

$$\mathfrak{u}_w = \bigoplus_{\alpha \geq \gamma,\, Supp(\alpha) \notin S_L} \mathfrak{g}_\alpha$$

In particular, $\mathfrak{u}_w$ is L-stable.

The quotient map ${}^{G}\!/_{B} \to {}^{G}\!/_{P}$ induces an isomorphism ${}^{L}\!/_{B} \xrightarrow{\sim} X_w$, and the conormal variety $T^*_X X_w \to X_w$ is simply the vector bundle $L \times^B \mathfrak{u}_w \to {}^{L}\!/_{B}$. Consequently, we have,

$$\mu(T^*_X X_w) = \{Ad(l_0)x_0 \,|\, l_0 \in L,\, x_0 \in \mathfrak{u}_w\}\,. \tag{7.3}$$

Now, consider some $(l, x) \in G \times^P \mathfrak{u}$, satisfying $\pi(l) \in X_w$ and $\mu(l, x) \in \mu(T^*_X X_w)$. We may assume, without loss of generality, that $l \in L$. As a consequence of Equation (7.3), there exist $l_0 \in L$, and $x_0 \in \mathfrak{u}_w$, such that,

$$\begin{aligned} &\mu(l, x) = Ad(l)x = Ad(l_0)x_0 \\ \implies\ & x = Ad(l^{-1}l_0)x_0. \end{aligned}$$

Now, since $\mathfrak{u}_w$ is L-stable, we have $x \in \mathfrak{u}_w$, hence $(l, x) \in T^*_X X_w$. □

Proposition 7.4. *Suppose the opposite Schubert variety X_{w_0w} is smooth for some $w \in W^P$. Then $T^*_X X_w$ satisfies Equation* (7.1) *scheme-theoretically.*

Proof. Let $\mathcal{D}_0$ denote the Dynkin diagram of G, and let $\mathcal{D}$ be the corresponding extended Dynkin diagram. The loop group $LG = G(\mathsf{k}[t, t^{-1}])$ is an affine Kac-Moody group corresponding to the extended Dynkin diagram $\mathcal{D}$. Let $\mathcal{G}_0$, $\mathcal{G}_d$, and $\mathcal{P}$

be parabolic subgroups of LG corresponding to the subsets $\mathcal{D}\backslash\{\alpha_0\}, \mathcal{D}\backslash\{\alpha_d\}$, and $\mathcal{D}\backslash\{\alpha_0, \alpha_d\}$ respectively.

Following [LS17], there exists an embedding $\phi : T^*_X X_w \to {}^{LG}/_{\mathcal{P}}$ such that $\phi(T^*_X X_w)$ is an open subset of some Schubert subvariety of ${}^{LG}/_{\mathcal{P}}$. Further, we can identify the structure map π and the Springer map μ as the restriction to $\phi(T^*_X X_w)$ of the quotient maps $\pi_d : {}^{LG}/_{\mathcal{P}} \to {}^{LG}/_{\mathcal{G}_d}$ and $\pi_0 : {}^{LG}/_{\mathcal{P}} \to {}^{LG}/_{\mathcal{G}_0}$ respectively.

Now, for any Schubert variety $Y \subset {}^{LG}/_{\mathcal{P}}$, we have the scheme-theoretic equality,

$$Y = \pi_0^{-1}(\pi_0(Y)) \cap \pi_d^{-1}(\pi_d(Y)).$$

From this, we deduce that Equation (7.1) holds for $T^*_X X_w$ scheme-theoretically. □

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