

Canonical Formalism of Non-Relativistic Theories Coupled to Newton-Cartan Gravity

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Abstract

In this short note we perform canonical analysis of Schrödinger field and non-relativistic electrodynamics coupled to Newton-Cartan gravity. We identify physical degrees of freedom and analyze constraints structure of these theories.

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1 Introduction and Summary

There is a renewed interest in Newton-Cartan (NC) geometry which has been observed in the last years. The first significant paper was [1] which introduced NC to field theory that analyzes strongly correlated electrons. It was further shown in [2, 3] that NC geometry with torsion naturally emerges as the background boundary geometry in holography for $z = 2$ Lifshitz geometries, for relevant works, see [4, 5, 6, 7] and for review and extensive list of references, see [8]. In fact, NC geometry is non-relativistic background geometry to which non-relativistic field theories can be covariantly coupled, see for example [5, 9, 10, 11, 12, 13]. In particular, it was shown in significant paper [11] how non-relativistic electrodynamics can couple to the most general NC geometry with torsion. Further, non-relativistic scalar fields coupled to NC geometry and background electromagnetic field were also analyzed there.

Since these results are very interesting non-relativistic theories in NC background certainly deserve to be studied in more details further. In this short note we focus on canonical analysis of non-relativistic scalar field and non-relativistic electrodynamics coupled to NC geometry. It turns out that this is rather non-trivial problem with interesting property that the constraints explicit depend on time. In more details, we start with the action for Schrödinger field in the background NC geometry and background non-relativistic electromagnetic field. Such an action was derived in [11] with the help of null reduction of complex scalar field in higher dimensional space-time². Then in order to find Hamiltonian form of this action we have to impose important restriction on the NC space-time in the sense that it has to have a notion of foliation by spatial surfaces that are orthogonal to one form τ_μ where τ_μ is known as clock form. This form defines a preferred notion of spatial direction at each point and also arrow of time in the sense that vector field t^μ is said to be future directed if it obeys the condition $\tau_\mu t^\mu > 0$. τ_μ defines a pointwise notion of spatial direction with the help of the vectors w^μ that obey the condition $\tau_\mu w^\mu = 0$. However this notion can be integrated to a local codimension one subspace when τ_μ obeys Frobenius condition $\tau \wedge d\tau = 0$ where $\tau = \tau_\mu dx^\mu$. Then we define causal space-times as space-times where this condition holds everywhere, for more detailed analysis and discussion, see for example [9, 12]. For such space-time we will be able to find Hamiltonian for the Schrödinger field in NC background. However we also find that when we write the complex scalar field in polar form as $\psi = \sqrt{\rho} e^{iS}$ that the momentum conjugate to ρ is zero which is first primary constraint of the theory. Further, in case of the momentum conjugate to S we find that it is determined by second primary constraint that explicitly depends on time. This is very interesting situation that deserves careful treatment. For that reason we perform analysis of constraint systems with explicit time dependent primary constraints in appendix³. Taking into account explicit time dependence of the constraints we will be able to

²Null reduction was studied in some earlier papers [14, 15].

³Discussion of the constraint analysis with explicit time dependence can be found in [17] however the analysis presented there is slightly different from ours.

derive canonical equations of motion that reproduce Lagrangian equations of motion which is a nice consistency check.

As the next step we extend this analysis to the case of non-relativistic electrodynamics in Newton-Cartan background. Since canonical analysis is based on an existence of Lagrangian we start with the non-relativistic electrodynamics action in NC background that is derived using null dimensional reduction [11]. We again restrict to the case of causal space-time and in the first step we determine set of primary constraints which Poisson commute among themselves. This is different situation than in case of the scalar field where the primary constraints were the second class constraints. Then the requirement of the preservation of the primary constraints gives set of secondary constraints which together with the primary constraints form set of the second class constraints. As a result we find that gauge field and corresponding conjugate momenta can be eliminated from the theory at least in principle. We also determine Lagrange multipliers corresponding to the primary constraints using the equations of motion for gauge field and we show that the resulting equations of motion coincide with the equations of motion derived by variation of action.

Let us outline main results derived in this paper. We obtain Hamiltonian form of non-relativistic theories on NC background and we determine physical degrees of freedom. This is very important result since we show that in case of non-relativistic electrodynamics the only physical degree of freedom is the scalar field and conjugate momenta. We also discuss the problem of the constraint structure in case of theories with explicit time dependent constraints.

The structure of this paper is as follows. In the next section (2) we review basic facts about NC geometry and introduce an action for Schrödinger field in the NC background and background non-relativistic electromagnetic field through null dimensional reduction. Then we perform Hamiltonian analysis of this theory and determine structure of constraints. In section (3) we analyze non-relativistic electrodynamics in NC background. We firstly perform canonical analysis of non-relativistic electrodynamics in flat background and then we extend this analysis to the case of non-relativistic electrodynamics in NC background. Finally in appendix (A) we study constrained systems with explicit time dependence and discuss their properties.

2 Hamiltonian Analysis of Schrödinger field in NC background

2.1 Summary of Newton-Cartan Geometry

We start this analysis with the brief review of Newton-Cartan geometry in $d + 1$ dimensions. Newton-Cartan background in $d + 1$ dimensions is given by a set of one forms (τ_μ, e_μ^a) where $a = 1, \dots, d$ and where $\mu, \nu = 0, 1, \dots, d$. We also have one

form M_μ . We define inverse vielbeins v^μ and e_μ^a through the relations

$$v^\mu e_\mu^a = 0, \quad v^\mu \tau_\mu = -1, \quad e_\mu^a \tau_\mu = 0, \quad e_\mu^a e_\mu^b = \delta_a^b. \quad (1)$$

The determinant of the $(d+1) \times (d+1)$ matrix (τ_μ, e_μ^a) is denoted by e . With the help of vierbeins we can construct degenerative "spatial metric"

$$h_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab}, \quad h^{\mu\nu} = e_\mu^a e_\nu^b \delta^{ab}. \quad (2)$$

By definitions, one forms τ_μ, e_μ^a and M_μ transform under diffeomorphism as usual but they also transform under various local transformations: Galilean boosts with λ_a as local parameter, local $SO(d)$ rotations which is parameterized by $\lambda_{ab} = -\lambda_{ba}$ and $U(1)_\sigma$ gauge transformation that is parameterized by σ where we have

$$\begin{aligned} \delta \tau_\mu &= 0, \quad \delta e_\mu^a = \tau_\mu \lambda^a + \lambda_a^b e_\mu^b, \\ \delta v^\mu &= \lambda^a e_a^\mu, \quad \delta e_a^\mu = \lambda_a^b e_b^\mu, \\ \delta M_\mu &= \lambda_a e_\mu^a + \partial_\mu \sigma. \end{aligned} \quad (3)$$

The inverse vielbein e_a^μ is invariant under local Galilean transformations. Note that we have an important relation

$$e_\mu^a e_a^\nu - \tau_\mu v^\nu = \delta_\mu^\nu \quad (4)$$

that implies

$$h_{\mu\nu} h^{\nu\rho} = \delta_\mu^\rho + \tau_\mu v^\rho \quad (5)$$

which will be useful below. It is also useful to define objects that are invariant under local Galilean transformations $\hat{v}^\mu, \hat{e}_a^\mu, \hat{h}^{\mu\nu}$ and Φ defined as

$$\begin{aligned} \hat{v}^\mu &= v^\mu - h^{\mu\nu} M_\nu, \quad \hat{e}_a^\mu = e_a^\mu - M_\nu e_b^\nu \delta^{ba} \tau_\mu, \\ \hat{h}_{\mu\nu} &= h_{\mu\nu} - M_\mu \tau_\nu - M_\nu \tau_\mu, \quad \Phi = -v^\mu M_\mu + \frac{1}{2} h^{\mu\nu} M_\mu M_\nu. \end{aligned} \quad (6)$$

It is important to stress that $\hat{h}_{\mu\nu} \neq \hat{e}_\mu^a \hat{e}_\nu^b \delta_{ab}$. Instead, using the definition of \hat{e}_μ^a given above, we obtain following relation

$$\hat{e}_\mu^a \hat{e}_\nu^b \delta_{ab} = \hat{h}_{\mu\nu} + 2\tau_\mu \tau_\nu \Phi. \quad (7)$$

Finally note that hatted objects obey following the relations

$$\hat{v}^\mu \hat{e}_\mu^a = 0, \quad \hat{v}^\mu \tau_\mu = -1, \quad e_\mu^a \hat{e}_\mu^b = \delta_a^b. \quad (8)$$

After this review of NC geometry we proceed to the Hamiltonian analysis of Schrödinger field.

2.2 Schrödinger Field in NC geometry through Null Dimensional Reduction

We would like to find Hamiltonian formulation of scalar field on Newton-Cartan background with fixed electromagnetic background. The most convenient way how to find such an action is to perform null dimensional reduction, see for example [9, 11]. Let us consider an action for complex scalar field in $d + 2$ dimension in the form

$$I = \int d^{d+2}x \sqrt{-\gamma} \left(-\gamma^{AB} D_A \Psi D_B \Psi^* \right) , \quad (9)$$

where $D_A = \partial_A \Psi - iq A_A \Psi$ and where $A_A, A = 0, \dots, d + 1$ is background electromagnetic field. Let us now consider a background metric which possesses a null isometry that is generated by coordinates ∂_u

$$ds^2 = \gamma_{AB} dx^A dx^B = 2\tau_\mu dx^\mu (du - M_\nu dx^\nu) + h_{\mu\nu} dx^\mu dx^\nu \quad (10)$$

so that

$$\gamma_{\mu u} = \gamma_{u\mu} = \tau_\mu , \quad \gamma_{\mu\nu} = h_{\mu\nu} - \tau_\mu M_\nu - \tau_\nu M_\mu \equiv \hat{h}_{\mu\nu} . \quad (11)$$

Then

$$\sqrt{-\gamma} = e , \quad e = \det (\tau_\mu, e_\mu^a) . \quad (12)$$

Since the metric γ_{AB} is non-singular we can easily find inverse metric with components

$$\gamma^{uu} = 2\Phi , \quad \gamma^{u\mu} = -\hat{v}^\mu , \quad \gamma^{\mu\nu} = h^{\mu\nu} , \quad (13)$$

where \hat{v}^μ and Φ are defined in (6). We further presume that the gauge field has the form $A_A = (A_u, A_\mu) = (\varphi, \bar{A}_\mu - \varphi M_\mu)$. Now with the help of this metric we perform dimensional reduction of the action. To do this we have to presume that all fields do not depend on u . We impose following ansatz for the scalar field Ψ

$$\Psi = e^{imu} \psi \quad (14)$$

and insert it to the action. Then also using (13) we obtain

$$\begin{aligned} I &= \int d^{d+1}x e \left(\hat{v}^\mu (D_\mu \psi)^* i(m - q\varphi) \psi - i(m - q\varphi) \hat{v}^\mu D_\mu \psi \psi^* \right. \\ &\quad \left. - 2\Phi (m - q\varphi)^2 \psi \psi^* - h^{\mu\nu} D_\mu \psi (D_\nu \psi)^* \right) , \end{aligned} \quad (15)$$

where

$$D_\mu \psi = \partial_\mu \psi - iq A_\mu \psi , \quad (D_\mu \psi)^* = \partial_\mu \psi^* + iq A_\mu \psi^* . \quad (16)$$

The action (15) is the action for Schrödinger field of the mass m and charge q in Newton-Cartan background and in the background electromagnetic field where the electromagnetic field has components A_μ . Note that ψ couples to φ through the combination $m - q\varphi$ and hence φ effectively shifts the mass of the scalar field and hence it is natural to call it as mass potential [11]. Our goal is to find Hamiltonian from the action (15).

2.3 Hamiltonian Analysis

We would like to work with real variables rather than with complex ones. For that reason we introduce following parameterization of the scalar field ψ as $\psi = \sqrt{\rho}e^{iS}$ so that the action has the form

$$I^{sch} = \int d^4x e \left(2(m - q\varphi)\rho\hat{v}^\mu\partial_\mu S - 2(m - q\varphi)\hat{v}^\mu A_\mu\rho - 2\Phi(m - q\varphi)\rho - \frac{1}{4\rho}h^{\mu\nu}(D_\mu\rho)^*D_\nu\rho + 2qh^{\mu\nu}\sqrt{\rho}A_\mu\partial_\nu S - \rho h^{\mu\nu}\partial_\mu S\partial_\nu S \right), \quad (17)$$

where

$$D_\mu\rho = \partial_\mu\rho - 2iqA_\mu\rho, \quad (D_\mu\rho)^* = \partial_\mu\rho + 2iqA_\mu\rho. \quad (18)$$

It is clear that the previous action is well defined for general Newton-Cartan background for arbitrary τ_μ apart from the fact that τ_μ has to obey Newton-Cartan compatibility condition. On the other hand in order to have well defined Hamiltonian formulation we have to have a notion of foliation by spatial surfaces that are orthogonal to τ_μ . This is guaranteed when we impose hypersurface orthogonality condition $\tau_{[\mu}\partial_\nu\tau_{\rho]} = 0$ on the whole space-time M . This condition is known as Frobenius condition and for more detailed discussion of causality in Newton-Cartan background, see [9, 12]. Space M that obeys this condition is called as causal. Since τ_μ is nowhere non-zero we can write it as $\tau_0 = e^{-\Phi_L}$ where Φ_L is known as Luttinger potential. In what follows we restrict to such space-time. Since $\tau_i = 0$ we obtain following consequences on the form of the metric $h^{\mu\nu}$ thanks to the condition

$$\tau_\mu h^{\mu\nu} = 0. \quad (19)$$

Explicitly, for $\nu = 0$ this equation implies $\tau_0 h^{00} = 0$ and hence we have to have $h^{00} = 0$ while for $\nu = i$ we have $\tau_\mu h^{\mu i} = \tau_0 h^{0i} = 0$ which again implies that $h^{0i} = 0$. Then the action I^{sch} simplifies considerably

$$I^{sch} = \int d^{d+1}x e \left(2(m - q\varphi)\rho\hat{v}^\mu\partial_\mu S - 2(m - q\varphi)\hat{v}^\mu A_\mu\rho - 2\Phi(m - q\varphi)\rho - \frac{1}{4\rho}h^{ij}(D_i\rho)^*D_j\rho + 2qh^{ij}\sqrt{\rho}A_i\partial_j S - \rho h^{ij}\partial_i S\partial_j S \right), \quad (20)$$

where $\hat{v}^\mu = v^\mu - h^{\mu\nu}M_\nu$ has generally non-zero all its components.

Before we proceed to the Hamiltonian formulation of the theory we derive equa-

tions of motion for ρ and S from (20)

$$\begin{aligned}
& \partial_\mu [e\hat{v}^\mu(m - q\varphi)\rho] + q\partial_i [e\sqrt{\rho}h^{ij}A_j] - \partial_i [e\rho h^{ij}\partial_j S] = 0 , \\
& 2e(m - q\varphi)\hat{v}^\mu\partial_\mu S - 2e(m - q\varphi)\hat{v}^\mu A_\mu - 2e\Phi(m - q\varphi) + \\
& + \frac{e}{4\rho^2}(D_i\rho)^*D_j\rho h^{ij} + D_i^* \left[\frac{e}{4\rho}h^{ij}D_j\rho \right] + D_i \left[\frac{e}{4\rho}h^{ij}(D_j\rho)^* \right] \\
& + \frac{q}{\sqrt{\rho}}eh^{ij}A_i\partial_j S - eh^{ij}\partial_i S\partial_j S = 0 .
\end{aligned} \tag{21}$$

Now we are ready to proceed to the Hamiltonian formalism. From (20) we obtain following conjugate momenta

$$p_S = \frac{\partial \mathcal{L}^{sch}}{\partial(\partial_t S)} = 2(m - q\varphi)\rho\hat{v}^0 , \quad p_\rho = \frac{\partial \mathcal{L}^{sch}}{\partial(\partial_t \rho)} = 0 . \tag{22}$$

From these two equations we see that there are two primary constraints

$$\mathcal{G}_S \equiv p_S - 2e(m - q\varphi)\rho\hat{v}^0 \approx 0 , \quad \mathcal{G}_\rho \equiv p_\rho \approx 0 \tag{23}$$

while the bare Hamiltonian is equal to

$$\begin{aligned}
H_B &= \int d^d \mathbf{x} (p_\rho \partial_t \rho + p_S \partial_t S - \mathcal{L}) = \int d^d \mathbf{x} \mathcal{H}_B , \\
\mathcal{H}_B &= -2e(m - q\varphi)\rho\hat{v}^i\partial_i S + 2e(m - q\varphi)\hat{v}^\mu A_\mu \rho + 2e\Phi(m - q\varphi)\rho + \\
&+ \frac{1}{4\rho}eh^{ij}(D_i\rho)^*D_j\rho - 2qeh^{ij}\sqrt{\rho}A_i\partial_j S + \rho eh^{ij}\partial_i S\partial_j S .
\end{aligned} \tag{24}$$

We see that generally $\mathcal{G}_S \approx 0$ and \mathcal{H}_B explicit depend on time. This is not usual situation and we discuss theory of constraints systems with explicit time dependence in more details in Appendix (A).

As the next step we calculate Poisson bracket between \mathcal{G}_S and \mathcal{G}_ρ and we obtain

$$\{\mathcal{G}_S(\mathbf{x}), \mathcal{G}_\rho(\mathbf{y})\} = -2e(m - q\varphi)\hat{v}^0\delta(\mathbf{x} - \mathbf{y}) \equiv \triangle_{S\rho}(\mathbf{x}, \mathbf{y}) \tag{25}$$

which show that they are two second class constraints. Note that the inverse matrix has the form $\triangle^{\rho S} = -\frac{1}{2e(m - q\varphi)\hat{v}^0}\delta(\mathbf{x} - \mathbf{y})$. As a result we can eliminate $p_\rho = 0$ and $p_S = 0$ from the set of canonical variables when we introduce Dirac bracket between ρ and S defined as

$$\begin{aligned}
\{\rho(\mathbf{x}), S(\mathbf{y})\}_D &= \{\rho(\mathbf{x}), S(\mathbf{y})\} - \int d^d \mathbf{z} d^d \mathbf{z}' \{\rho(\mathbf{x}), \mathcal{G}_S(\mathbf{z})\} \triangle^{S\rho}(\mathbf{z}, \mathbf{z}') \{p_\rho(\mathbf{z}'), S(\mathbf{y})\} - \\
&- \int d^d \mathbf{z} d^d \mathbf{z}' \{\rho(\mathbf{x}), p_\rho(\mathbf{z})\} \triangle^{\rho S}(\mathbf{z}, \mathbf{z}') \{\mathcal{G}_S(\mathbf{z}'), S(\mathbf{y})\} = -\frac{1}{2e\hat{v}^0(m - q\varphi)}\delta(\mathbf{x} - \mathbf{y}) .
\end{aligned} \tag{26}$$

As was explicitly shown in Appendix (A), in the presence of the time dependent constraints the equations of motion for canonical variables have the form ⁴

$$\begin{aligned}
\partial_t \rho &= \{\rho, H_B\}_D - \\
&- \int d^d \mathbf{z} d^d \mathbf{z}' \{\rho, \mathcal{G}_S(\mathbf{z})\} \triangle^{S\rho}(\mathbf{z}, \mathbf{z}') \frac{\partial p_\rho}{\partial t} - \int d^d \mathbf{z} d^d \mathbf{z}' \{\rho, p_\rho(\mathbf{z})\} \triangle^{\rho S}(\mathbf{z}, \mathbf{z}') \frac{\partial \mathcal{G}_S(\mathbf{z}')}{\partial t} = \\
&= -\frac{1}{e\hat{v}^0(m-q\varphi)} \partial_i [(m-q\varphi)e\hat{v}^i \rho] - \frac{1}{e\hat{v}^0(m-q\varphi)} \partial_t [e(m-q\varphi)\hat{v}^0] \rho - \\
&- \frac{q}{e\hat{v}^0(m-q\varphi)} \partial_i [e\sqrt{\rho} h^{ij} A_j] + \frac{1}{e\hat{v}^0(m-q\varphi)} \partial_i [eh^{ij} \rho \partial_j S]
\end{aligned} \tag{27}$$

that can be rewritten into more symmetric form

$$\partial_t [e\hat{v}^0(m-q\varphi)\rho] + \partial_i [e\hat{v}^i(m-q\varphi)\rho] + \partial_i [e\sqrt{\rho} h^{ij} A_j] - \partial_i [eh^{ij} \rho \partial_j S] = 0 \tag{28}$$

that coincides with the first equation of motion given in (21). Let us now proceed to the canonical equation of motion for S

$$\begin{aligned}
\partial_t S &= \{S, H_B\}_D - \int d^d \mathbf{z} d^d \mathbf{z}' \{S, \mathcal{G}_S(\mathbf{z})\} \triangle^{S\rho}(\mathbf{z}, \mathbf{z}') \frac{\partial \mathcal{G}_\rho}{\partial t} \\
&= -\frac{\hat{v}^i}{\hat{v}^0} \partial_i S + \frac{\hat{v}^\mu}{\hat{v}^0} A_\mu + \frac{1}{\hat{v}^0} \Phi - \frac{1}{8\rho^2 \hat{v}^0(m-q\varphi)} h^{ij} (D_i \rho)^* D_j \rho \\
&- \frac{1}{2e\hat{v}^0(m-q\varphi)} D_i \left[\frac{eh^{ij}}{4\rho} D_j \rho \right] - \frac{1}{2e\hat{v}^0(m-q\varphi)} D_i^* \left[\frac{eh^{ij}}{4\rho} D_j^* \rho \right] \\
&- \frac{eq}{2\hat{v}^0(m-q\varphi)\sqrt{\rho}} h^{ij} A_i \partial_j S + \frac{e}{\hat{v}^0(m-q\varphi)} h^{ij} \partial_i S \partial_j S
\end{aligned} \tag{29}$$

that can be again rewritten into the form

$$\begin{aligned}
&2e(m-q\varphi)\hat{v}^\mu \partial_\mu S - 2e(m-q\varphi)\hat{v}^\mu A_\mu - 2e(m-q\varphi)\Phi + \\
&+ \frac{e}{4\rho^2} h^{ij} (D_i \rho)^* D_j \rho + D_i \left[\frac{e}{4\rho} h^{ij} D_j^* \rho \right] + D_i^* \left[\frac{e}{4\rho} h^{ij} D_j \rho \right] + \\
&+ \frac{qe}{\sqrt{\rho}} h^{ij} A_i \partial_j S - eh^{ij} \partial_i S \partial_j S = 0
\end{aligned} \tag{30}$$

that coincides with the second equation of motion given in (21).

In summary, we found the Hamiltonian formulation of Schrödinger field in NC background. We found that the dynamical fields are ρ and S that have non-zero Dirac bracket (26). Then we derived their canonical equations of motion and found that they coincide with the equations of motion derived from Lagrangian.

⁴See equation (77) in Appendix (A).

3 Hamiltonian Formalism for Electromagnetic Field in Newton-Cartan Gravity

In this section we focus on canonical analysis of non-relativistic electromagnetic field in NC background. We start with the simpler case of the action for non-relativistic electrodynamics in flat background.

3.1 Non-Relativistic Electrodynamics through Null Dimensional Reduction

Following [11] we derive an action for non-relativistic electrodynamics by performing a null reduction of the Maxwell action in one higher dimension. More precisely, let us consider $d + 2$ dimensional Maxwell action

$$S = -\frac{1}{4} \int dt du d^d \mathbf{x} F_{AB} \eta^{AC} \eta^{BD} F_{CD} , \quad (31)$$

where $\eta_{AB} dx^A dx^B = 2dt du + dx^i dx^i$. Following [11] we set $A_u = \varphi$, $A_t = -\tilde{\varphi}$, $A_i = a_i$ and presume that all fields do not depend on u . Since the inverse metric has the form $\eta^{tu} = \eta^{ut} = 1$, $\eta^{ij} = \delta^{ij}$ we get

$$F_{AB} \eta^{AC} \eta^{BD} F_{CD} = -2(F_{tu})^2 + F_{ij} F^{ij} - 4F_{iu} F_{tk} = -2(\partial_t \varphi)^2 - 4(\partial_t a_i + \partial_i \tilde{\varphi}) \partial_i \varphi + f_{ij} F^{ij} . \quad (32)$$

As a result we obtain an action for non-relativistic electrodynamics in flat background in the form

$$S = \int dt d^d \mathbf{x} \left(-\frac{1}{4} f_{ij} f^{ij} + (\partial_i \tilde{\varphi} + \partial_t a_i) \partial_i \varphi + \frac{1}{2} (\partial_t \varphi)^2 \right) , \quad (33)$$

where $f_{ij} = \partial_i a_j - \partial_j a_i$. From the action (33) we derive following conjugate momenta⁵

$$\pi^i = \partial_i \varphi , \quad p_{\tilde{\varphi}} = 0 , \quad p_{\varphi} = \partial_t \varphi \quad (34)$$

so that we have following primary constraints

$$\mathcal{G}^i \equiv \pi^i - \partial_i \varphi \approx 0 , \quad p_{\tilde{\varphi}} \approx 0 , \quad (35)$$

together with the bare Hamiltonian in the form

$$H_B = \int d^d \mathbf{x} (\pi^i \partial_t a_i + p_{\varphi} \partial_t \varphi + p_{\tilde{\varphi}} \partial_t \tilde{\varphi} - \mathcal{L}) = \int d^d \mathbf{x} \left(\frac{1}{4} f_{ij} f^{ij} - \partial_i \tilde{\varphi} \partial_i \varphi + \frac{1}{2} p_{\varphi}^2 \right) \quad (36)$$

and consequently the extended Hamiltonian is equal to

$$H_E = H_B + \int d^d \mathbf{x} (\lambda_i \mathcal{G}^i + \lambda^{\tilde{\varphi}} p_{\tilde{\varphi}}) , \quad (37)$$

⁵In this section we do not carry about upper or lower spatial index since they are equivalent in flat background.

where $\lambda_i, \lambda^{\tilde{\varphi}}$ are Lagrange multipliers corresponding to the constraints $\mathcal{G}^i \approx 0$ and $p_{\tilde{\varphi}} \approx 0$. As the next step we have to ensure the preservation of all primary constraints. In case of the constraint $\mathcal{G}^i \approx 0$ we obtain

$$\frac{d\mathcal{G}^i}{dt} = \{\mathcal{G}^i, H_E\} = \partial_k f^{ki} - \partial_i p_\phi \equiv \mathcal{G}_{II}^i \approx 0, \quad (38)$$

where $\mathcal{G}_{II}^i \approx 0$ are secondary constraints. In case of the constraint $p_{\tilde{\varphi}} \approx 0$ we obtain

$$\frac{dp_{\tilde{\varphi}}}{dt} = \{p_{\tilde{\varphi}}, H_E\} = \partial_i \partial^i \varphi = \partial_i \pi^i - \partial_i \mathcal{G}^i \approx \partial_i \pi^i \equiv \mathcal{G}_{\tilde{\varphi}}^{II} \approx 0 \quad (39)$$

which is the generator of gauge transformations. In fact, if we define

$$\mathbf{G}(\Lambda) = \int d^d \mathbf{x} \Lambda \mathcal{G}_{\tilde{\varphi}}^{II} \quad (40)$$

we obtain standard transformation rules

$$\{\mathbf{G}(\Lambda), A_i\} = \partial_i \Lambda, \quad \{\mathbf{G}(\Lambda), f_{ij}\} = 0. \quad (41)$$

Finally we have to ensure the preservation of the constraint $\mathcal{G}_{II}^i \approx 0$. To do this we have to calculate the Poisson bracket between constraints \mathcal{G}^i and \mathcal{G}_{II}^i . After some calculations we obtain

$$\{\mathcal{G}^i(\mathbf{x}), \mathcal{G}_{II}^j(\mathbf{y})\} = -\partial_k \partial^k \delta(\mathbf{x} - \mathbf{y}) \delta^{ij} \equiv \Delta^{ij}(\mathbf{x}, \mathbf{y}). \quad (42)$$

Let us introduce an inverse matrix $D_{ij}(\mathbf{x}, \mathbf{y})$ that obeys the relation

$$\int d^d \mathbf{z} \Delta^{ik}(\mathbf{x}, \mathbf{z}) D_{kj}(\mathbf{z}, \mathbf{y}) = \delta_j^i \delta(\mathbf{x} - \mathbf{y}). \quad (43)$$

Since Δ_{ij} is given in (42) we find that D_{ij} is a solution of the equation

$$\frac{\partial}{\partial x^k} \frac{\partial}{\partial x^k} D_{ij}(\mathbf{x}, \mathbf{y}) = -\delta_{ij} \delta(\mathbf{x} - \mathbf{y}). \quad (44)$$

As the next step we determine canonical equations of motion for φ and p_φ

$$\begin{aligned} \partial_t \varphi &= \{\tilde{\varphi}, H_E\} = \{\tilde{\varphi}, H_B\} + \left\{ \varphi, \int d^d \mathbf{z} \lambda^i \mathcal{G}_i(\mathbf{z}) \right\} = p_\varphi \\ \partial_t p_\varphi &= \{p_\varphi, H_E\} = \{p_\varphi, H_B\} + \left\{ p_{\tilde{\varphi}}, \int d^d \mathbf{z} \lambda^i \mathcal{G}_i(\mathbf{z}) \right\} = -\partial_k \partial_k \tilde{\varphi} - \partial_i \lambda^i. \end{aligned} \quad (45)$$

Finally the equation of motion for a_i has the form

$$\partial_t a_i = \{a_i, H_E\} = \int d^d \mathbf{z} \lambda_j(\mathbf{z}) \{a_i, \mathcal{G}^j(\mathbf{z})\} = \lambda_i \quad (46)$$

so that the equation of motion for p_φ can be written as

$$\partial_t p_\varphi = -\partial_k(\partial_k \tilde{\varphi} + \partial_t a_k) = \partial_k \tilde{E}_k , \quad (47)$$

where $\tilde{E}_k = -\partial_t a_k - \partial_k \tilde{\varphi}$. If we perform partial time derivation of the first equation in (45) and use the second one we obtain

$$\partial_t^2 \varphi = \partial_t p_\varphi = \partial_k \tilde{E}_k \quad (48)$$

with agreement with the equation (2.11) in [11]. Further, if we apply the partial derivative ∂_i on the first equation in (45) we obtain

$$\partial_i \partial_t \varphi = \partial_i p_\varphi = \partial_k f^{ki} \quad (49)$$

with agree with the second equation in (2.3) in [11]. In the same way we find that the divergence of $\mathcal{G}^i = 0$ implies

$$\partial_i \mathcal{G}^i = \mathcal{G}_{\tilde{\varphi}}^{II} - \partial_i \partial^i \varphi \approx -\partial_i \partial^i \tilde{\varphi} = 0 \quad (50)$$

that agrees with the first equation in (2.3) [11]. Finally we should determine the Lagrange multiplier λ_i using the requirement of the preservation of the constraint \mathcal{G}_{II}^i but this is not necessary since we know that $\lambda_i = \partial_t a_i$. On the other hand since $\mathcal{G}^i \approx 0$ and $\mathcal{G}_{II}^i \approx 0$ are two second class constraints they can be explicitly solved for π^i and a_i . In other words there is only one dynamical variable which is φ and its conjugate momentum p_φ .

3.2 Null Reduction of Maxwellian Electromagnetism in NC Background

We determine action for electromagnetic field in Newton-Cartan background again with the help of null dimensional reduction, following [11]. We start with the action for electromagnetic field in $d + 2$ dimensions that has the form

$$S = -\frac{1}{4} \int d^{d+2}x \sqrt{-\gamma} F_{AB} \gamma^{AC} \gamma^{BD} F_{CD} . \quad (51)$$

Our goal is to dimensionally reduce this action along null isometry so that we will presume that A_M do not depend on u . We further write $A_M = (A_u, A_\mu)$ and define $A_\mu \equiv \varphi$. Since the gauge field transforms under $U(1)$ transformations as

$$A'_A = A_A + \partial_A \Lambda \quad (52)$$

it is clear that φ is invariant under gauge transformation since Λ does not depend on u . On the other hand the gauge field A_μ transform as

$$A'_\mu = A_\mu + \partial_\mu \Lambda . \quad (53)$$

In order to perform null dimensional reductions we use the components of metric inverse given in (13) and we obtain the action in the form

$$S = \int d^{d+1}x e \left(-\frac{1}{4} F_{\mu\nu} h^{\mu\rho} h^{\nu\sigma} F_{\rho\sigma} - \Phi \partial_\mu \varphi h^{\mu\nu} \partial_\nu \varphi + \frac{1}{2} (\hat{v}^\mu \partial_\mu \varphi)^2 - \hat{v}^\nu F_{\nu\mu} h^{\mu\sigma} \partial_\sigma \varphi \right) , \quad (54)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \quad (55)$$

It is convenient to use slightly different form of the action which depends on v^μ instead of \hat{v}^μ . Following [11] we introduce vector field \bar{A}_μ defined as

$$A_\mu = \bar{A}_\mu - \varphi M_\mu . \quad (56)$$

Performing this substitution in the action (54) we find

$$S = \int d^{d+1}x e \left(-\frac{1}{4} h^{\mu\rho} h^{\nu\sigma} \bar{F}_{\mu\nu} \bar{F}_{\rho\sigma} - h^{\mu\nu} v^\rho \bar{F}_{\rho\nu} \partial_\mu \varphi + \frac{1}{2} (v^\mu \partial_\mu \varphi)^2 \right) , \quad (57)$$

where

$$\bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu - \varphi (\partial_\mu M_\nu - \partial_\nu M_\mu) . \quad (58)$$

The action (57) will be the starting point for the Hamiltonian formulation of the theory. As we argued above we restrict to causal space-time with non-zero τ_0 only. Then v^μ has generally non-zero all components with $v^0 = -\tau_0$. In case of causal space-time the action has the form

$$S = \int d^{d+1}x e \left(-\frac{1}{4} \bar{F}_{ij} h^{ik} h^{jl} \bar{F}_{kl} - \Phi \partial_i \varphi h^{ij} \partial_j \varphi + \frac{1}{2} (v^\mu \partial_\mu \varphi)^2 - v^\nu \bar{F}_{\nu i} h^{ij} \partial_j \varphi \right) . \quad (59)$$

Note that from this action we also obtain equations of motion in the form

$$\begin{aligned} & 2\partial_i [e \Phi h^{ij} \partial_j \varphi] - \partial_\mu [e v^\mu v^\nu \partial_\nu \varphi] + \partial_j [e v^\nu \bar{F}_{\nu i} h^{ij}] + \\ & + \frac{1}{2} e (\partial_i M_j - \partial_j M_i) h^{ik} h^{jl} \bar{F}_{kl} + e v^0 (\partial_0 M_i - \partial_i M_0) h^{ij} \partial_j \varphi + e v^k (\partial_k M_i - \partial_i M_k) h^{ij} \partial_j \varphi = 0 , \\ & \partial_j [e h^{il} \bar{F}_{lk} h^{kj}] + \partial_0 [e v^0 h^{ij} \partial_j \varphi] + \partial_k [e v^k h^{ij} \partial_j \varphi] - \partial_k [e v^i h^{kj} \partial_j \varphi] = 0 , \\ & \partial_i [e v^0 h^{ij} \partial_j \varphi] = 0 . \end{aligned} \quad (60)$$

Let us now proceed to the canonical analysis. From (59) we obtain following conjugate momenta

$$\begin{aligned} \pi^i &= \frac{\partial \mathcal{L}}{\partial(\partial_t \bar{A}_i)} = -e v^0 h^{ij} \partial_j \varphi , \quad \pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_t \bar{A}_0)} \approx 0 , \\ p_\varphi &= \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} = e v^0 (v^\mu \partial_\mu \varphi) \end{aligned} \quad (61)$$

so that we have following explicitly time dependent primary constraints

$$\mathcal{G}^i \equiv \pi^i + ev^0 h^{ij} \partial_j \varphi \approx 0 \quad (62)$$

together with familiar constraint $\mathcal{G}^0 \equiv \pi^0 \approx 0$. Further, with the help of (61), we obtain the bare Hamiltonian in the form

$$\begin{aligned} H_B = & \int d^d \mathbf{x} \left(e \frac{1}{4} \bar{F}_{ij} h^{ik} h^{jl} \bar{F}_{kl} + e \Phi \partial_i \varphi h^{ij} \partial_j \varphi + ev^k \bar{F}_{ki} h^{ij} \partial_j \varphi + \right. \\ & \left. + \frac{1}{2e} (\tau_0)^2 p_\varphi^2 + \tau_0 v^i \partial_i \varphi p_\varphi - A_0 \partial_i \pi^i + \pi^i \varphi (\partial_0 M_i - \partial_i M_0) \right) . \end{aligned} \quad (63)$$

Now we have to analyze the requirement of the preservation of primary constraints $\mathcal{G}^i \approx 0$, $\mathcal{G}^0 \approx 0$. Note that the extended Hamiltonian has the form

$$H_E = H_B + \int d^d \mathbf{x} (\lambda_i \mathcal{G}^i + \lambda_0 \pi^0) . \quad (64)$$

In case of $\mathcal{G}^0 \approx 0$ we obtain that the requirement of its preservation during the time development of the system implies standard Gauss law constraint

$$\mathcal{G}^{II} \equiv \partial_i \pi^i \approx 0 , \quad (65)$$

while in case of \mathcal{G}^i we get

$$\begin{aligned} \frac{d\mathcal{G}^i}{dt} &= \frac{\partial \mathcal{G}^i}{\partial t} + \{\mathcal{G}^i, H_E\} \\ &= \partial_t (ev^0 h^{ij}) \partial_j \varphi - \partial_k [eh^{ik} \bar{F}_{kl} h^{lj}] + \partial_k (ev^k h^{in} \partial_n \varphi) - \\ &\quad - \partial_m (ev^i h^{mn} \partial_n \varphi) + ev^0 h^{ij} \partial_j \left[\frac{1}{e} (\tau_0)^2 p_\varphi \right] + ev^0 h^{ij} \partial_j [\tau_0 v^m \partial_m \varphi] \equiv \mathcal{G}_{II}^i \approx 0 , \end{aligned} \quad (66)$$

where we used the fact that $\{\mathcal{G}^i(\mathbf{x}), \mathcal{G}^j(\mathbf{y})\} = 0$. We see that the requirement of the preservation of the constraints $\mathcal{G}^i \approx 0$ implies the second set of the constraints $\mathcal{G}_{II}^i \approx 0$. It is again easy to see that $\mathcal{G}^i \approx 0$, $\mathcal{G}_{II}^j \approx 0$ are two sets of second class constraints with rather complicated Poisson bracket between them. Then it is difficult to determine Lagrange multipliers λ_i from the requirement of the preservation of the constraints $\mathcal{G}_{II}^i \approx 0$ during the time evolution of the system. On the other hand, as we will show below, these Lagrange multipliers can be determined with the help of the equations of motion for \bar{A}_i . Further, it is easy to see that \mathcal{G}^0 and $\mathcal{G}^{II} \approx 0$ are first class constraints where $\mathcal{G}^{II} \approx 0$ is generator of gauge transformations.

As we argued above $\mathcal{G}^i \approx 0$, $\mathcal{G}_{II}^i \approx 0$ are two sets of second class constraints where $\mathcal{G}^i = 0$ can be solved for π^i while $\mathcal{G}_{II}^i = 0$ can be solved for \bar{A}_i at least in principle. On the other hand when we try to write equations of motion for φ and

p_φ it is convenient to express Lagrange multiplier λ_i as a function of non-dynamical variable \bar{A}_i using its equation of motion

$$\partial_t \bar{A}_i = \{ \bar{A}_i, H_E \} = \partial_i A_0 + \lambda_i + \varphi(\partial_0 M_i - \partial_i M_0) \quad (67)$$

that implies that $\lambda_i = \bar{F}_{0i}$. Then we can write canonical equations of motion for φ and p_φ as

$$\begin{aligned} \partial_t \varphi &= \{ \varphi, H_B \} + \int d^d \mathbf{z} \lambda_i \{ \varphi, \mathcal{G}^i(\mathbf{z}) \} = \frac{1}{e(v^0)^2} p_\varphi + \tau_0 v^i \partial_i \varphi, \\ \partial_t p_\varphi &= \{ p_\varphi, H_B \} + \int d^d \mathbf{z} \lambda_i(\mathbf{z}) \{ p_\varphi, \mathcal{G}^i(\mathbf{z}) \} = \\ &= \frac{1}{2} e(\partial_i M_j - \partial_j M_i) h^{ik} h^{jl} \bar{F}_{kl} - \pi^i (\partial_0 M_i - \partial_i M_0) + \partial_m (\lambda_j h^{jm} e) + \\ &+ 2\partial_i [e \Phi h^{ij} \partial_j \varphi] + \partial_j [e v^k \bar{F}_{ki} h^{ij}] + \partial_i [\tau_0 v^i p_\varphi] + e v^k (\partial_k M_i - \partial_i M_k) h^{ij} \partial_j \varphi \\ &\approx \frac{1}{2} e(\partial_i M_j - \partial_j M_i) h^{ik} h^{jl} \bar{F}_{kl} + v^0 e h^{ij} \partial_j \varphi (\partial_0 M_i - \partial_i M_0) + \partial_m (\bar{F}_{0j} h^{jm} e v^0) + \\ &+ 2\partial_i [e \Phi h^{ij} \partial_j \varphi] + \partial_j [e v^k \bar{F}_{ki} h^{ij}] + \partial_i [\tau_0 v^i p_\varphi] + e v^k (\partial_k M_i - \partial_i M_k) h^{ij} \partial_j \varphi. \end{aligned} \quad (68)$$

If we combine these two equations together we obtain

$$\begin{aligned} \partial_t (e(v^0)^2 \partial_t \varphi) &= \partial_t p_\varphi - \partial_t (e v^0 v^i \partial_i \varphi) \\ &= \frac{1}{2} e(\partial_i M_j - \partial_j M_i) h^{ik} h^{jl} \bar{F}_{kl} + e v^0 h^{ij} \partial_j \varphi (\partial_0 M_i - \partial_i M_0) + \partial_m (\bar{F}_{0j} h^{jm} e v^0) \\ &+ 2\partial_i [e \Phi h^{ij} \partial_j \varphi] + \partial_j [e v^k \bar{F}_{ki} h^{ij}] - \partial_i [e v^i v^0 \partial_t \varphi] + e v^k (\partial_k M_i - \partial_i M_k) h^{ij} \partial_j \varphi \end{aligned} \quad (69)$$

that coincides with the first equation in (60). Further, it is easy to see that the second equation in (60) coincides with the secondary constraint $\mathcal{G}_{II}^i = 0$. Finally, the last equation in (60) is equivalent to the combination of the primary constraint \mathcal{G}^{II} and \mathcal{G}^i since

$$\mathcal{G}^{II} = \partial_i [\mathcal{G}^i - e v^0 h^{ij} \partial_j \varphi] = -\partial_i [e v^0 h^{ij} \partial_j \varphi] = 0. \quad (70)$$

In summary, we have shown that canonical equations of motion and constraints reproduce Lagrangian equations of motion. We have also determined physical degrees of freedom of non-relativistic electrodynamics and we have shown that there are only two phase space physical degrees left corresponding to φ and p_φ .

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A Appendix: Systems with explicit time dependent constraints

Let us consider phase space system with variables $p_m, q^m, m = 1, \dots, N$, bare Hamiltonian H_B and set of primary constraints $\phi_j = \phi_j(p, q, t), j = 1, \dots, J$, that explicitly depend on time t . Then the phase space action with primary constraints included has the form

$$S = \int dt (p_m \dot{q}^m - H_B - \lambda^j \phi_j) , \quad (71)$$

where λ^j are independent variables known as Lagrange multipliers. Variation of the action with respect to p_m, q^m and λ^j we obtain following set of equations of motion

$$\begin{aligned} \dot{q}^m - \frac{\partial H_B}{\partial p_m} - \lambda^j \frac{\partial \phi_j}{\partial p_m} &= 0 , \\ -\dot{p}_m - \frac{\partial H_B}{\partial q^m} - \lambda^j \frac{\partial \phi_j}{\partial q^m} &= 0 , \\ \phi_j &\approx 0 . \end{aligned} \quad (72)$$

Introducing standard Poisson bracket they can be written as

$$\begin{aligned} \dot{q}^m &= \{q^m, H_B\} + \lambda^j \{q^m, \phi_j\} , \\ \dot{p}_m &= \{p_m, H_B\} + \lambda^j \{p_m, \phi_j\} , \quad \phi_j = 0 . \end{aligned} \quad (73)$$

In order to determine Lagrange multipliers λ^j we demand that the constraints $\phi_j = 0$ are preserved during the time evolution of the system. Note that it is clear from the form of the equations of motion written above that we have to *firstly* calculate Poisson bracket between canonical variables and ϕ_j and then we can impose the condition $\dot{\phi}_j = 0$. This is the reason why we write $\phi_j \approx 0$ instead of $\phi_j = 0$. Now the time evolution of the constraint ϕ_i is equal to

$$\begin{aligned} \dot{\phi}_i &= \frac{\partial \phi_i}{\partial t} + \frac{\partial \phi_i}{\partial q^m} \dot{q}^m + \frac{\partial \phi_i}{\partial p_m} \dot{p}_m = \\ &= \frac{\partial \phi_i}{\partial t} + \{\phi_i, H_B\} + \{\phi_i, \phi_j\} \lambda^j . \end{aligned} \quad (74)$$

If we impose the condition that the constraint ϕ_i is preserved during the time evolution of the system we find that the conditions $\dot{\phi}_i = 0$ provide J equations for J unknown λ^i . Let us now presume non-degenerative case when $\{\phi_i, \phi_j\} = \Delta_{ij}$ is non-singular matrix so that it has an inverse $\Delta^{jk}, \Delta_{ij} \Delta^{jk} = \delta_i^k$. Then (74) can be solved as

$$\lambda^i = -\Delta^{ik} \left(\frac{\partial \phi_k}{\partial t} + \{\phi_k, H_B\} \right) . \quad (75)$$

As a result we find that the time evolution of the phase space variables q^m and p_m is governed by equations

$$\begin{aligned}\dot{q}^m &= \{q^m, H_B\} - \{q^m, \phi_i\} \Delta^{ij} \{\phi_j, H_B\} - \{q^m, \phi_i\} \Delta^{ij} \frac{\partial \phi_j}{\partial t} , \\ \dot{p}_m &= \{p_m, H_B\} - \{p_m, \phi_i\} \Delta^{ij} \{\phi_j, H_B\} - \{p_m, \phi_i\} \Delta^{ij} \frac{\partial \phi_j}{\partial t} , \\ \phi_j &= 0\end{aligned}\tag{76}$$

that can be written in an equivalent form

$$\begin{aligned}\dot{q}^m &= \{q^m, H_B\}_D - \{q^m, \phi_i\} \Delta^{ij} \frac{\partial \phi_j}{\partial t} , \\ \dot{p}_m &= \{p_m, H_B\}_D - \{p_m, \phi_i\} \Delta^{ij} \frac{\partial \phi_j}{\partial t} , \\ \phi_j &= 0 ,\end{aligned}\tag{77}$$

where we introduced Dirac bracket between two phase space functions defined as $\{X, Y\}_D = \{X, Y\} - \{X, \phi_i\} \Delta^{ij} \{\phi_j, Y\}$.

A.1 Secondary time dependent constraints

Let us now consider situation when the primary constraints $\phi_j(p, q, t)$ have weakly vanishing Poisson bracket among themselves. Then the requirement of their preservation during the time development of the system has the form

$$\frac{d\phi_i}{dt} = \frac{\partial \phi_i}{\partial t} + \{\phi_i, H_B\} + \lambda^j \{\phi_i, \phi_j\} \approx \frac{\partial \phi_i}{\partial t} + \{\phi_i, H_B\} \equiv \phi_i^{II}(p, q, t) \approx 0 , \tag{78}$$

where now $\phi_i^{II}(p, q, t)$ are secondary constraints. Finally we have to ensure the preservation of the secondary constraints which implies

$$\frac{d\phi_i^{II}}{dt} = \frac{\partial \phi_i^{II}}{\partial t} + \{\phi_i^{II}, H_B\} + \lambda^j \{\phi_i^{II}, \phi_j\} \approx 0 . \tag{79}$$

We will presume that the matrix $\{\phi_i^{II}, \phi_j\} \equiv \Delta_{ij}$ is non-singular and hence the equation above can be solved for λ^i as

$$\lambda^i = -\Delta^{ij} \left(\frac{\partial \phi_j^{II}}{\partial t} + \{\phi_j^{II}, H_B\} \right) \tag{80}$$

and hence the equation of motion for p^m, q_m have the form

$$\frac{dq^m}{dt} = \{q^m, H_B\} + \lambda^j \{q^m, \phi_j\} = \{q^m, H_B\} - \{q^m, \phi_i\} \Delta^{ij} \left(\frac{\partial \phi_j^{II}}{\partial t} + \{\phi_j^{II}, H_B\} \right) \tag{81}$$

and equivalent one for p_m .

Let us outline results of the analysis performed in Appendix. We have shown that in case of the explicit time dependent constraints, either primary or secondary, there are additional terms in the equations of motion for canonical variables which are proportional to explicit time derivative of these constraints. These terms are crucial for the equivalence between Lagrangian and Hamiltonian equations of motion.

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