Quasi-shuffle algebras and applications

Michael E. Hoffman
U. S. Naval Academy
Annapolis, MD 21402 USA
meh@usna.edu

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Abstract

Quasi-shuffle algebras have been a useful tool in studying multiple zeta values and related quantities, including multiple polylogarithms, finite multiple harmonic sums, and q-multiple zeta values. Here we show that two ideas previously considered only for multiple zeta values, the interpolated product of S. Yamamoto and the symmetric sum theorem, can be generalized to any quasi-shuffle algebra.

1 Introduction

Multiple zeta values and related quantities, although studied by Euler in the simplest cases, only began to receive systematic attention in the early 1990s. Suddenly they seemed to be everywhere: in high-energy physics, in knot theory, and in theoretical computer science. Many early papers on these quantities emphasized proofs of specific identities, and used methods of analysis. But from the beginning the importance of algebraic structure proved its importance.

In [7] the author recognized multiple zeta values as homomorphic images of quasi-symmetric functions, allowing the use of familiar results on symmetric functions in proving relations of multiple zeta values. This was generalized in [8], which introduced the quasi-shuffle product. (In the same year Li Guo and W. Keigher [5] independently introduced an essentially equivalent construction, but it took several years for the relation between the two to be generally recognized.)

Meanwhile, the circle of ideas around multiple zeta values and multiple polylogarithms continued to expand, and came to include examples that went beyond the framework of [8], particularly q-multiple zeta values. K. Ihara, J. Kajikawa, Y. Ohno and J. Okuda [12] generalized the definition of the quasi-shuffle product to include such cases, but neglected much of the algebraic machinery developed in [8], particularly the Hopf algebra structure and the linear maps induced by formal power series. In 2012 Ihara and the author set out to develop a generalization of the definition used in [8] while retaining the algebraic structures developed in that paper, and indeed extending them. This led to [11], which which presented such a generalization and applied it to an array of examples. We review this construction §2.

The methods introduced in [11] proved especially effective in treating the interpolated multiple zeta values (or r-MZVs) introduced by S. Yamamoto [20], which interpolate between ordinary multiple zeta values ($r = 0$) and multiple zeta-star values $r = 1$). Yamamoto showed that r-MZVs multiply according to an interpolated product; in §3 we define interpolated products on any quasi-shuffle algebra. A quasi-shuffle algebra with the interpolated product has a Hopf algebra structure, generalizing the results of [11].

The algebraic machinery of [11], which allows transparent proofs of many results in [13] and [12], is briefly introduced in §4 and applied to multiple zeta values in §5. We also give a new result for multiple zeta-half values (i.e., r-MZVs with $r = \frac{1}{2}$). The same quasi-shuffle algebra that has the multiple zeta values as homomorphic images also has as images various "exotic" multiple zeta values, such as the multiple t-values [10], the Bessel-function zeta values introduced by T. V. Wakhare and C. Vignat [18], and the Airy multiple zeta values, all discussed in §6.

In §7 we consider a different quasi-shuffle algebra, which has as its image the alternating or "colored" multiple zeta values. Finally, in §8 we show how the symmetric sum theorems given in [6] for multiple zeta values can be generalized to any quasi-shuffle algebra.

2 The basic construction

We begin by reviewing the construction given in [11]. Let A be a countable set A of letters, k a field. We assume there is a commutative, associative product $\diamond$ on kA.

Now let $k\langle A\rangle$ be the noncommutative polynomial algebra over A. So $k\langle A\rangle$ is the vector space over k generated by "words" (monomials) $a_1a_2\cdots a_n$, with $a_i \in A$: for a word $w = a_1\cdots a_n$ we write $\ell(w) = n$ (and we set $\ell(1) = 0$). Define a k-bilinear product $*$ on $k\langle A\rangle$ by making $1 \in k\langle A\rangle$ the identity element for each product, and requiring that $*$ satisfy the relation

$$(aw) * (bv) = a(w * bv) + b(aw * v) + (a \diamond b)(w * v) \tag{1}$$

for all $a, b \in A$ and all monomials w, v in $k\langle A\rangle$. Then $(k\langle A\rangle, *)$ is a commutative algebra. If the product $\diamond$ is identically zero, then $*$ coincides with the usual shuffle product $ш$ on $k\langle A\rangle$. We will need the following lemma in the next section.

Lemma 1. *For letters a, b and words v, w such that $v \neq 1 \neq w$,*

$$a \diamond (v * b) + ba \diamond v = (a \diamond v) * b + a \diamond bv, \tag{2}$$

$$(a \diamond v) * (b \diamond w) = a \diamond (v * (b \diamond w)) + b \diamond ((a \diamond v) * w) - a \diamond b \diamond (v * w), \tag{3}$$

and

$$a(v * (b \diamond w)) + a \diamond (v * bw) + b((a \diamond v) * w) + b \diamond (av * w) = \\ av * (b \diamond w) + (a \diamond v) * bw + 2(a \diamond b)(v * w). \tag{4}$$

Proof. Writing $v = cv'$ for a letter c, Eq. (2) is

$$a \diamond (cv' * b) + ba \diamond cv' = a \diamond cv' * b + a \diamond bcv',$$

or

$$a \diamond c(v' * b) + a \diamond bv + a \diamond b \diamond v + ba \diamond v = a \diamond c(v' * b) + ba \diamond v + a \diamond b \diamond v + a \diamond bv,$$

which is evidently true. Setting also $w = dw'$, the left- and right-hand sides of Eq. (3) are

$$a \diamond c(v' * (b \diamond w)) + b \diamond d((a \diamond v) * w') + a \diamond b \diamond c \diamond d(v' * w')$$

and

$$
\begin{aligned}
&a\diamond c(v'*(b\diamond w)) + a\diamond b\diamond d(v*w') + a\diamond c\diamond b\diamond d(v'*w') \\
&\quad + b\diamond a\diamond c(v'*w) + b\diamond d((a\diamond v)*w') + b\diamond a\diamond c\diamond d(v'*w') \\
&\qquad - a\diamond b\diamond c(v'*w) - a\diamond b\diamond d(v*w') - a\diamond b\diamond c\diamond d(v'*w'),
\end{aligned}
$$

respectively, and these agree after cancellation. Using the same notation, we can rewrite the left-hand side of Eq. (4) as

$$
\begin{aligned}
&a(v*(b\diamond w)) + b((a\diamond v)*w) + a\diamond(c(v'*bw) + b(v*w) + c\diamond b(v'*w)) \\
&\qquad + b\diamond(a(v*w) + d(av*w') + a\diamond d(v*w')) \\
&= a(v*(b\diamond w)) + b((a\diamond v)*w) + a\diamond c(v'*bw) + a\diamond b(v*w) + a\diamond c\diamond b(v'*w) \\
&\qquad + b\diamond a(v*w) + b\diamond d(av*w') + b\diamond a\diamond d(v*w')
\end{aligned}
$$

and the right-hand side of Eq. (4) as

$$
\begin{aligned}
&a(v*(b\diamond w)) + b\diamond d(av*w') + a\diamond b\diamond d(v*w') \\
&\quad + a\diamond c(v'*bw) + b((a\diamond v)*w) + a\diamond c\diamond b(v'*w) + 2a\diamond b(v*w),
\end{aligned}
$$

and these evidently agree. □

If Δ denotes the usual deconcatenation on $k\langle A\rangle$, i.e.,

$$
\begin{aligned}
\Delta(a_1a_2\cdots a_n) = 1\otimes a_1a_2\cdots a_n + a_1\otimes a_2\cdots a_n + \cdots + a_1\cdots a_{n-1}\otimes a_n \\
+ a_1a_2\cdots a_n\otimes 1,
\end{aligned}
$$

then $(k\langle A\rangle, *, \Delta)$ is a Hopf algebra [11, Thm. 4.2]. It is easy to see that it is a bialgebra, and (using the filtration of $k\langle A\rangle$ by word length) it is filtered connected; this makes existence of the antipode automatic.

For a composition $I=(i_1,\dots,i_m)$ of n and a word $w=a_1\cdots a_n$ of $k\langle A\rangle$, define

$$
I[w] = (a_1\diamond\dots\diamond a_{i_1})(a_{i_1+1}\diamond\dots\diamond a_{i_1+i_2})\cdots(a_{i_1+\cdots+i_{m-1}+1}\diamond\dots\diamond a_n).
$$

Let

$$
f = c_1t + c_2t^2 + c_3t^3 + \cdots \in tk[[t]]
$$

be a formal power series. We can define a k-linear map $\Psi_f : k\langle A\rangle \to k\langle A\rangle$ by

$$
\Psi_f(w) = \sum_{I=(i_1,\dots,i_m)\in\mathcal{C}(\ell(w))} c_{i_1}\cdots c_{i_m} I[w], \tag{5}
$$

where $\mathcal{C}(n)$ is the set of compositions of n. Then we have the following result.

Theorem 1. *[[11, Thm. 3.1]] For $f, g \in k[[t]]$ as specified above, $\Psi_f \Psi_g = \Psi_{f \circ g}$.*

Here are some examples. First, it is immediate from equation (5) that Ψ_t is the identity homomorphism of $k\langle A \rangle$. Also, $T = \Psi_{-t}$ sends a word w to $(-1)^{\ell(w)} w$; evidently T is an involution. We note that $\Sigma = \Psi_{\frac{t}{1-t}}$ and $\Sigma^{-1} = \Psi_{\frac{t}{1+t}}$ are given by

$$\Sigma(w) = \sum_{I \in \mathcal{C}(\ell(w))} I[w] \quad \text{and} \quad \Sigma^{-1}(w) = \sum_{I \in \mathcal{C}(\ell(w))} (-1)^{\ell(w) - \ell(I)} I[w],$$

where $\ell(I)$ is the number of parts of the composition I. Evidently $\Sigma(aw) = a\Sigma(w) + a \diamond \Sigma(w)$ for letters a and words w, and (as in [12]) this property can be used to define Σ. While Σ and T are not inverses, it is easy to see from Theorem 1 that $T\Sigma T = \Sigma^{-1}$, from which it follows that ΣT and $T\Sigma$ are involutions.

From [8] we have the (inverse) functions $\exp = \Psi_{e^t - 1}$ and $\log = \Psi_{\log(1+t)}$. As shown in [8, Theorem 2.5], exp is an algebra isomorphism from $(k\langle A \rangle, ⧢)$ to $(k\langle A \rangle, *)$. We have the following identity.

Theorem 2. $\Sigma = \exp T \log T$.

Proof. This follows from Theorem 1, since $\exp T = \Psi_{e^{-t}-1}$, $\log T = \Psi_{\log(1-t)}$, and $\log(1-t)$ composed with $e^{-t} - 1$ gives

$$\frac{1}{1-t} - 1 = \frac{1 - (1-t)}{1-t} = \frac{t}{1-t}.$$

□

3 The interpolated product

For any $r \in k$, define $\Sigma^r = \Psi_{\frac{t}{1-rt}}$; it then follows immediately from Theorem 1 that $\Sigma^r \Sigma^s = \Sigma^{r+s}$, and it is easily seen that

$$\Sigma^r(aw) = a\Sigma^r w + ra \diamond \Sigma^r w$$

for any letter a and word w. We now define the interpolated product $\overset{r}{*}$ by

$$u \overset{r}{*} v = \Sigma^{-r}(\Sigma^r u * \Sigma^r v)$$

for any words u, v. Henceforth we shall treat both concatenation and $\diamond$ has having higher binding than $*$ and $\overset{r}{*}$, so the second identity of Lemma 1 reads

$$a \diamond v * b \diamond w = a \diamond (v * b \diamond w) + b \diamond (a \diamond v * w) - a \diamond b \diamond (v * w).$$

Lemma 2. *Lemma 1 remains true when $*$ is replaced by $\overset{r}{*}$.*

Proof. For each identity, first replace v and w by $\Sigma^r v$ and $\Sigma^r w$ respectively and then apply Σ^{-r} to both sides. After appropriate simplification and (in the case of identity (4)) cancellation, the conclusion follows. $\square$

We now show that the product $\overset{r}{*}$ can be defined inductively by a rule similar to Eqn. (1) for the quasi-shuffle product $*$. This rule was first given by Yamamoto [20] in the case of multiple zeta values.

Theorem 3. *The product $\overset{r}{*}$ can be specified by setting $1 \overset{r}{*} w = w \overset{r}{*} 1 = w$ for any word w, $a \overset{r}{*} b = ab + ba + (1 - 2t)a \diamond b$ for any letters a, b, and*

$$av \overset{r}{*} bw = a(v \overset{r}{*} bw) + b(av \overset{r}{*} w) + (1 - 2r)a \diamond b(v \overset{r}{*} w) \\ + (r^2 - r)a \diamond b \diamond (v \overset{r}{*} w)$$

for any letters a, b and words v, w such that $vw \neq 1$.

Proof. Evidently $1 \overset{r}{*} w = w \overset{r}{*} 1$ for any word w, and for letters a, b we have

$$a \overset{r}{*} b = \Sigma^{-r}(a * b) = \Sigma^{-r}(ab + ba + a \diamond b) = ab - ra \diamond b + ba - rb \diamond a + a \diamond b \\ = ab + ba + (1 - 2r)a \diamond b.$$

Now let a, b be letters, $v \neq 1$ a word. Then

$$av \overset{r}{*} b = \Sigma^{-r}(\Sigma^r(av) * b) = \Sigma^{-r}(a\Sigma^r v * b + ra \diamond \Sigma^r v * b) = \\ \Sigma^{-r}(a(\Sigma^r v * b) + ba\Sigma^r v + a \diamond b\Sigma^r v + ra \diamond \Sigma^r v * b) = \\ a(v \overset{r}{*} b) - ra \diamond (v \overset{r}{*} b) + b\Sigma^{-r} a\Sigma^r v - rb \diamond \Sigma^{-r} a\Sigma^r v + a \diamond bv - ra \diamond b \diamond v + ra \diamond v \overset{r}{*} b \\ = a(v \overset{r}{*} b) + bav + (1-r)a \diamond bv + (r^2 - r)a \diamond b \diamond v - r(a \diamond (v \overset{r}{*} b) + ba \diamond v - a \diamond v \overset{r}{*} b) \\ = a(v \overset{r}{*} b) + bav + (1 - 2r)a \diamond bv + (r^2 - r)a \diamond b \diamond v,$$

where we used Lemma 2 in the last step. Finally, let a,b be letters, v,w words with $v \neq 1 \neq w$. Then

$$\begin{aligned}
&av \overset{r}{*} bw = \Sigma^{-r}(\Sigma^r av * \Sigma^r bw) = \Sigma^{-r}((a\Sigma^r v + ra \diamond \Sigma^r v) * (b\Sigma^r w + rb \diamond \Sigma^r w)) \\
&= \Sigma^{-r}(a\Sigma^r v * b\Sigma^r w + ra\Sigma^r v * b \diamond \Sigma^r w + ra \diamond \Sigma^r v * b\Sigma^r w + r^2 a \diamond \Sigma^r v * b \diamond \Sigma^r w) \\
&= \Sigma^{-r}(a(\Sigma^r v * b\Sigma^r w) + b(a\Sigma^r v * \Sigma^r w) + (a \diamond b)(\Sigma^r v * \Sigma^r w) + ra\Sigma^r v * b \diamond \Sigma^r w \\
&\qquad + ra \diamond \Sigma^r v * b\Sigma^r w + r^2 a \diamond \Sigma^r v * b \diamond \Sigma^r w \\
&= a(v \overset{r}{*} \Sigma^{-r} b\Sigma^r w) - ra \diamond (v \overset{r}{*} \Sigma^{-r} b\Sigma^r w) + b(\Sigma^{-r} a\Sigma^r v \overset{r}{*} w) - rb \diamond (\Sigma^{-r} a\Sigma^r v \overset{r}{*} w) \\
&+ (a \diamond b)(v \overset{r}{*} w) - ra \diamond b \diamond (v \overset{r}{*} w) + r\Sigma^{-r} a\Sigma^r v \overset{r}{*} b \diamond w + ra \diamond v \overset{r}{*} \Sigma^{-r} b\Sigma^r w + r^2 a \diamond v \overset{r}{*} b \diamond w \\
&= a(v \overset{r}{*} bw) - ra(v \overset{r}{*} b \diamond w) - ra \diamond (v \overset{r}{*} bw) + r^2 a \diamond (v \overset{r}{*} b \diamond w) + b(av \overset{r}{*} w) - rb(a \diamond v \overset{r}{*} w) \\
&- rb \diamond (av \overset{r}{*} w) + r^2 b \diamond (a \diamond v \overset{r}{*} w) + (a \diamond b)(v \overset{r}{*} w) - ra \diamond b \diamond (v \overset{r}{*} w) + rav \overset{r}{*} b \diamond w \\
&\qquad - r^2 a \diamond v \overset{r}{*} b \diamond w + ra \diamond v \overset{r}{*} bw - r^2 a \diamond v \overset{r}{*} b \diamond w + r^2 a \diamond v \overset{r}{*} b \diamond w \\
&= a(v \overset{r}{*} bw) + b(av \overset{r}{*} w) + (a \diamond b)(v \overset{r}{*} w) - ra \diamond b \diamond (v \overset{r}{*} w) - ra(v \overset{r}{*} b \diamond w) \\
&- ra \diamond (v \overset{r}{*} bw) - rb(a \diamond v \overset{r}{*} w) - rb \diamond (av \overset{r}{*} w) + rav \overset{r}{*} b \diamond w + ra \diamond v \overset{r}{*} bw \\
&\qquad + r^2(a \diamond (v \overset{r}{*} b \diamond w) + b \diamond (a \diamond v \overset{r}{*} w) - a \diamond v \overset{r}{*} b \diamond w) \\
&= a(v \overset{r}{*} bw) + b(av \overset{r}{*} w) + (1-2r)(a \diamond b)(v \overset{r}{*} w) + (r^2 - r)a \diamond b \diamond (v \overset{r}{*} w),
\end{aligned}$$

where we used Lemma 2 in the last step. □

If $r = 1$, we write $\star$ instead of $\overset{1}{*}$. The product $\star$ has inductive rule

$$av \star bw = a(v \star bw) + b(av \star w) - a \diamond b(v \star w),$$

which is of the same form as Eq. (1). As noted in [11], $T : (k\langle A\rangle, \star) \to (k\langle A\rangle, *)$ and $T : (k\langle A\rangle, *) \to (k\langle A\rangle, \star)$ are isomorphisms. These are special cases of the following result.

Proposition 1. $T : (k\langle A\rangle, \overset{r}{*}) \to (k\langle A\rangle, \overset{1-r}{*})$ *is an isomorphism.*

Proof. First note that $\Sigma^s : (k\langle A\rangle, \overset{r}{*}) \to (k\langle A\rangle, \overset{r-s}{*})$ is an isomorphism, and that $\Sigma^r T = T\Sigma^{-r}$ for all $r \in k$. Then

$$T(u \overset{r}{*} v) = T\Sigma^{-r}(\Sigma^r u * \Sigma^r v) = \Sigma^r T(\Sigma^r u * \Sigma^r v) = \Sigma^r(T\Sigma^r \star T\Sigma^r v) = \\ \Sigma^r(\Sigma^{-r}Tu \star \Sigma^{-r}Tv) = Tu \overset{1-r}{*} Tv$$

for $u, v \in k\langle A\rangle$, and the result follows. □

In what follows, R is the linear function on $k\langle A\rangle$ that reverses words, i.e., $R(a_1a_2\cdots a_n) = a_na_{n-1}\cdots a_1$. We note that R commutes with Ψ_f for all $f \in tk[[t]]$ [11, Prop. 4.3]. The following result generalizes [11, Thm. 4.2].

Theorem 4. *$(k\langle A\rangle, \overset{r}{*}, \Delta)$ is a filtered connected Hopf algebra with antipode $\Sigma^{1-2r}TR$. Also, $\Sigma^r : (k\langle A\rangle, \overset{r}{*}, \Delta) \to (k\langle A\rangle, *, \Delta)$ is a Hopf algebra isomorphism.*

Proof. To see that $(k\langle A\rangle, \overset{r}{*}, \Delta)$ is a Hopf algebra, the main thing to check is that $\Delta(w_1 \overset{r}{*} w_2) = \Delta(w_1) \overset{r}{*} \Delta(w_2)$ for any two words w_1 and w_2. We do this inductively on the word length. We can assume $w_1 \neq 1 \neq w_2$, so let $w_1 = au$ and $w_2 = bv$ for letters a, b. Using Sweedler's notation

$$\Delta(u) = \sum u_{(1)} \otimes u_{(2)}, \quad \Delta(v) = \sum v_{(1)} \otimes v_{(2)},$$

we have

$$\Delta(au) = \sum au_{(1)} \otimes u_{(2)} + 1 \otimes au$$

and

$$\Delta(bv) = \sum bv_{(1)} \otimes v_{(2)} + 1 \otimes bv$$

so that

$$\begin{aligned}\Delta(w_1) \overset{r}{*} \Delta(w_2) = &\sum (au_{(1)} \overset{r}{*} bv_{(1)}) \otimes (u_{(2)} \overset{r}{*} v_{(2)}) + \sum au_{(1)} \otimes (u_{(2)} \overset{r}{*} bv) \\ &+ \sum bv_{(1)} \otimes (au \overset{r}{*} v_{(2)}) + 1 \otimes (au \overset{r}{*} bv) \\ = &\sum a(u_{(1)} \overset{r}{*} bv_{(1)}) \otimes (u_{(2)} \overset{r}{*} v_{(2)}) + \sum b(au_{(1)} \overset{r}{*} v_{(1)}) \otimes (u_{(2)} \overset{r}{*} v_{(2)}) + \\ (1-2r) &\sum a\diamond b(u_{(1)} \overset{r}{*} v_{(1)}) \otimes (u_{(2)} \overset{r}{*} v_{(2)}) + (r^2-r) \sum a\diamond b\diamond(u_{(1)} \overset{r}{*} v_{(1)}) \otimes (u_{(2)} \overset{r}{*} v_{(2)}) \\ + &\sum au_{(1)} \otimes (u_{(2)} \overset{r}{*} w_2) + \sum bv_{(1)} \otimes (w_1 \overset{r}{*} v_{(2)}) + 1 \otimes a(u \overset{r}{*} w_2) + 1 \otimes b(w_1 \overset{r}{*} v) \\ &+ (1-2r)1 \otimes a \diamond b(u \overset{r}{*} v) + (r^2-r)1 \otimes a \diamond b \diamond (u \overset{r}{*} v).\end{aligned}$$

Using the induction hypothesis, this is

$$\begin{aligned}(a \otimes 1)(\Delta(u \overset{r}{*} w_2)) + 1 \otimes a(u \overset{r}{*} w_2) + (b \otimes 1)(\Delta(w_1 \overset{r}{*} v)) + 1 \otimes b(w_1 \overset{r}{*} v) \\ + (1-2r)(a \diamond b \otimes 1)\Delta(u \overset{r}{*} v) + (1-2r)(1 \otimes a \diamond b)\Delta(u \overset{r}{*} v) \\ + (r^2-r)(a \diamond b \otimes 1) \diamond \Delta(u \overset{r}{*} v) + (r^2-r)(1 \otimes a \diamond b) \diamond \Delta(u \overset{r}{*} v)\end{aligned}$$

which can be recognized as

$$\Delta(w_1 \overset{r}{*} w_2) = \Delta(a(u \overset{r}{*} w_2) + b(w_1 \overset{r}{*} v) + (1-2r) a \diamond b (u \overset{r}{*} v) + (r^2 - r) a \diamond b \diamond (u \overset{r}{*} v)).$$

Now $\Sigma^r : (k\langle A\rangle, \overset{r}{*}) \to (k\langle A\rangle, *)$ is an algebra homomorphism by definition, and is also a coalgebra map for Δ [11, Thm. 4.1]. Hence Σ^r is a Hopf algebra isomorphism. Also, if

$$w = \sum_w w_{(1)} \otimes w_{(2)}$$

for a nonempty word, then

$$\Sigma^r w = \sum_w \Sigma^r w_{(1)} \otimes \Sigma^r w_{(2)}$$

and we have

$$\sum_w S_* \Sigma^r w_{(1)} * \Sigma^r w_{(2)} = 0$$

for $S_* = \Sigma T R$ the antipode of the Hopf algebra $(k\langle A\rangle, *, \Delta)$ [11, Thm. 4.2]. Apply Σ^{-r} to get

$$\sum_w \Sigma^{1-r} T R \Sigma^r w_{(1)} \overset{r}{*} w_{(2)} = 0;$$

but this shows that $\Sigma^{1-r} T R \Sigma^r = \Sigma^{1-2r} T R$ is the antipode of $(k\langle A\rangle, \overset{r}{*}, \Delta)$. $\square$

Of course if $r = 0$ the Hopf algebra $(k\langle A\rangle, \overset{r}{*}, \Delta)$ is just $(k\langle A\rangle, *, \Delta)$; the antipode is $\Sigma T R$. If $r = 1$ we get $(k\langle A\rangle, \star, \Delta)$, and the antipode is $\Sigma^{-1} T R = T \Sigma R$. For $r = \frac{1}{2}$ the inductive rule for the product is

$$av \overset{\frac{1}{2}}{*} bw = a(v \overset{\frac{1}{2}}{*} bw) + b(av \overset{\frac{1}{2}}{*} w) - \frac{1}{4} a \diamond b \diamond (v \overset{\frac{1}{2}}{*} w)$$

and the antipode is simply TR.

4 Algebraic formulas

In [13] and [12] there are algebraic formulas involving exp and log. These can be proved systematically from the following result of [11], where for $w \in k\langle A\rangle$ and $f = c_1 t + c_2 t^2 + \cdots \in tk[[t]]$, $f_\bullet(\lambda w)$ denotes

$$\lambda c_1 w + \lambda^2 c_2 w \bullet w + \lambda^3 c_3 w \bullet w \bullet w + \cdots \in k\langle A\rangle[[\lambda]]$$

for $\bullet = *, ⧢, \star, \diamond$.

Theorem 5. *[[11, Thm. 5.1]] For any $f \in tk[[t]]$ and $w \in k\langle A\rangle$,*

$$\Psi_f\left(\frac{1}{1-\lambda w}\right) = \frac{1}{1-f_\diamond(\lambda w)}.$$

We write $\exp_\bullet(\lambda w)$ for $1+f_\bullet(\lambda w)$, $f=e^t-1$, and $\log_\bullet(\lambda w)$ for $f_\bullet(\lambda w)$, $f=\log(1+t)$. By applying Theorem 5 with $f=\log(1-t)$, we get

$$\exp_*(\log_\diamond(1+\lambda z)) = \frac{1}{1-\lambda z}, \tag{6}$$

and by applying it with $f=e^t-1$ we get

$$\exp_*(\lambda z) = \exp\left(\frac{1}{1-\lambda z}\right) = \frac{1}{2-\exp_\diamond(\lambda z)}.$$

Another consequence of Theorem 5 is the following.

Corollary 1. *[[11, Cor. 5.5]] For any $z \in k\langle A\rangle$ and $r \in k$,*

$$\Sigma^r\left(\frac{1}{1-\lambda z}\right) * \frac{1}{1-r\lambda z} = \frac{1}{1-(1-r)z}.$$

5 Multiple zeta values

For positive integers $i_1,\dots,i_k$ with $i_1>1$, the corresponding multiple zeta value is defined by

$$\zeta(i_1,\dots,i_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{i_1} n_2^{i_2}\cdots n_k^{i_k}}.$$

Let $A=\{z_1,z_2,\dots\}$, with the operation $z_i\diamond z_j = z_{i+j}$. The following result can be extracted from [7].

Theorem 6. *The Hopf algebra $(\mathbb{Q}\langle A\rangle, *, \Delta)$ is isomorphic to the algebra* QSym *of quasi-symmetric functions over $\mathbb{Q}$.*

Let $\mathbb{Q}\langle A\rangle^0$ be the subspace of $\mathbb{Q}\langle A\rangle$ generated by 1 and all words that do not begin with z_1. Then $(\mathbb{Q}\langle A\rangle^0, *)$ is a subalgebra of $(\mathbb{Q}\langle A\rangle, *)$. We write QSym^0 for the corresponding subalgebra of QSym. The following fact was proved in [7].

Theorem 7. *The linear function* $\zeta : \mathrm{QSym}^0 \to \mathbb{R}$ *defined by* $\zeta(z_{i_1} \cdots z_{i_k}) = \zeta(i_1, \dots, i_k)$ *is a homomorphism from* QSym^0 *to the reals with their usual multiplication.*

If we take $z = z_k$ in Eq. (6) above, we get

$$\sum_{n\geq 0} \lambda^n z_k^n = \exp_*(\log_\diamond(1+\lambda z_k)) = \exp_*\left(\sum_{j\geq 1} \frac{(-1)^{j-1}\lambda^j z_{kj}}{j}\right),$$

or, after applying ζ,

$$\sum_{n\geq 0} \lambda^n \zeta(\{k\}_n) = \exp\left(\sum_{j\geq 1} \frac{(-1)^{j-1}\lambda^j \zeta(kj)}{j}\right),$$

where $\{k\}_n$ means k repeated n times. If $k = 2$ the right-hand side is

$$\exp\left(\sum_{j\geq 1} \frac{B_{2j}(2\pi)^{2j}}{(2j)(2j)!}\lambda^j\right) = \frac{\sinh(\pi\lambda)}{\pi\lambda},$$

from which follows

$$\zeta(\{2\}_n) = \frac{\pi^{2n}}{(2n+1)!}, \tag{7}$$

and a similar argument gives

$$\zeta(\{4\}_n) = \frac{2^{2n+1}\pi^{4n}}{(4n+2)!}. \tag{8}$$

Two remarkable results about multiple zeta values are (1) the "sum theorem," i.e., the sum of all multiple zeta values of a fixed depth and weight n is just $\zeta(n)$, as in

$$\zeta(4,1,1)+\zeta(3,2,1)+\zeta(3,1,2)+\zeta(2,3,1)+\zeta(2,2,2)+\zeta(2,1,3) = \zeta(6),$$

and, (2) the "duality theorem," i.e., there is an involution $\tau : \mathrm{QSym}^0 \to \mathrm{QSym}^0$ so that $\zeta(\tau(u)) = \zeta(u)$, as in $\zeta(3,1,2) = \zeta(2,3,1)$. To describe τ in terms of our algebraic setup, introduce two noncommuting variables x and y, and set $z_i = x^{i-1}y$. Then QSym^0 is just the subspace of $\mathbb{Q}\langle x, y\rangle$ generated by 1 and words that begin with x and end with y: the function τ is the anti-isomorphism exchanging x and y (so, e.g., $\tau(z_3z_1z_2) = \tau(x^2y^2xy) = xyx^2y^2 = z_2z_3z_1$).

If we let $\zeta^r = \zeta \circ \Sigma^r$, then $\zeta^r(w)$ is exactly the interpolated multiple zeta value as defined by Yamamoto [20]. Thus $\zeta^0(w) = \zeta(w)$ and $\zeta^1(w) = \zeta^\star(w)$ is the multiple zeta-star value defined by

$$\zeta^\star(i_1, \dots, i_k) = \sum_{n_1 \ge n_2 \ge \cdots \ge n_k \ge 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.$$

Yamamoto showed that the interpolated multiple zeta values satisfy the following version of the sum theorem, which is proved another way in [11].

Theorem 8. *If* $n \ge 2$, *then*

$$\sum_{\substack{i_1+\cdots+i_l=n \\ i_1>1}} \zeta^r(i_1, \dots, i_k) = \zeta(n) \sum_{k=0}^{l-1} r^n \binom{n-l-1+k}{k}.$$

Formulas for repeated values $\zeta^r(\{m\}_n)$ can be obtained from those for $\zeta(\{m\}_n)$: from Corollary 1 it follows that if

$$Z(\lambda) = \sum_{n=0}^{\infty} \zeta(\{m\}_n)\lambda^n,$$

then

$$\sum_{n=0}^{\infty} \zeta^r(\{m\}_n)\lambda^n = \frac{Z((1-r)\lambda)}{Z(-r\lambda)}.$$

Hence, e.g.,

$$\sum_{n=0}^{\infty} \zeta^r(\{2\}_n)\lambda^n = \sqrt{\frac{r}{1-r}} \frac{\sinh(\pi\sqrt{(1-r)\lambda})}{\sin(\pi\sqrt{r\lambda})}.$$

For interpolated multiple zeta values ζ^r with $r = \frac{1}{2}$ there is a "totally odd sum theorem." This follows from two known results: the cyclic sum theorem and the two-one theorem. Define the cyclic sum operation on QSym^0 by

$$C(x^{i_1-1}yx^{i_2-1}y\cdots x^{i_k-1}y) = x^{i_1}yx^{i_2-1}y\cdots x^{i_k-1}y \\ + x^{i_2}yx^{i_3-1}y\cdots x^{i_k-1}yx^{i_1-1}y + \cdots + x^{i_k}yx^{i_1-1}y\cdots x^{i_{k-1}-1}y.$$

Then the cyclic sum theorem for multiple zeta-star values [15] asserts that

$$\zeta^\star(\tau C(w)) = (n-1)\zeta(n)$$

for any word $w \in \mathrm{QSym}^0$ of degree $n-1$. The two-one formula [20, 21] gives

$$\zeta^\star((xy)^{j_1}y(xy)^{j_2}y\cdots(xy)^{j_l}y) = 2^l\zeta^{\frac{1}{2}}(x^{2j_1}yx^{2j_2}y\cdots x^{2j_l}y)$$

for any sequence $(j_1,\ldots,j_l)$ of nonnegative integers with $j_1>0$.

Theorem 9. *Let $n>2$, $l<n$ be positive integers of the same parity. Then*

$$\sum_{\substack{a_1+\cdots+a_l=n\\ a_i\ odd,\ a_1>1}} \zeta^{\frac{1}{2}}(a_1,\ldots,a_l) = \frac{n-1}{n-l}\binom{\frac{n+l}{2}-2}{l-1}\frac{\zeta(n)}{2^{l-1}} = \frac{n-1}{\frac{n+l}{2}-1}\binom{\frac{n+l}{2}-1}{l-1}\frac{\zeta(n)}{2^l}.$$

Proof. By the two-one formula

$$\sum_{\substack{a_1+\cdots+a_l=n\\ a_i\text{ odd},\ a_1>1}} \zeta^{\frac{1}{2}}(a_1,\ldots,a_l) = 2^{-l}\sum_{\substack{j_1+\cdots+j_l=\frac{n-l}{2}\\ j_i\geq 0,\ j_1\geq 1}} \zeta^\star((xy)^{j_1}y(xy)^{j_2}y\cdots(xy)^{j_l}y).$$

The latter sum has

$$\binom{\frac{n+l}{2}-2}{l-1} \tag{9}$$

terms. To see this, note that written in the sequence notation each term corresponds to a string

$$\underbrace{2,\ldots,2}_{j_1},1,\underbrace{2,\ldots,2}_{j_2},1,\ldots,\underbrace{2,\ldots,2}_{j_l},1 \tag{10}$$

with $j_i\geq 0$, $j_1\geq 1$, and $\sum_{i=1}^l j_i = \frac{n-l}{2}$. Now the string (10) always starts with 2 and ends with 1, so we can think about the middle part: it has length $\frac{n+l}{2}-2$, and consists of $\frac{n-l}{2}-1$ twos and $l-1$ ones. To specify such a string, we need only give the $l-1$ positions where the ones go; so such strings are counted by the binomial coefficient (9).

Now each word u of the form

$$(xy)^{j_1}y(xy)^{j_2}y\cdots(xy)^{j_l}y \tag{11}$$

with $\sum_{i=1}^l j_i = \frac{n-l}{2}$ and $j_1>0$ has

$$\tau(u) = x^{i_1-1}yx^{i_2-1}y\cdots x^{i_k-1}y$$

with $i_1>2$, $i_2,\ldots,i_k>1$, $i_1+\cdots+i_k=n$, and $k=\frac{n-l}{2}$. These are exactly the words that appear in $C(w)$ for w of the form $x^{a_1-1}yx^{a_2-1}y\cdots x^{a_k-1}y$

with $a_1, \dots a_2 > 1$, $a_1 + \cdots + a_k = n-1$, and $k = \frac{n-1}{2}$. For any such w the expansion of $\tau C(w)$ will have $\frac{n-l}{2}$ terms, so each term $\zeta^\star(u)$ contributes

$$\frac{2}{n-l}(n-1)\zeta(n),$$

and the result follows. (It may happen that $\frac{2}{n-l}\binom{\frac{n+l}{2}-2}{l-1}$ is not an integer, but the preceding sentence is still true since in that case there are duplications in one or more of the images under τC.) □

From the definition of the zeta-half values we get the following corollary of Theorem 9.

Corollary 2. *The sum*

$$\sum_{\substack{a_1+\cdots+a_l=n \\ a_i \text{ odd},\ a_1>1}} \zeta(a_1, \dots, a_l)$$

is a rational linear combination of multiple zeta values of weight n and depth less than l.

In the depth three case we can say more.

Corollary 3. *If n is odd, the sum*

$$\sum_{\substack{a_1+a_2+a_3=n \\ a_i \text{ odd},\ a_1>1}} \zeta(a_1, a_2, a_3)$$

is a polynomial in the ordinary zeta values with rational coefficients.

Proof. By Corollary 2, the sum can be written as a rational linear combination of single and double zeta values of weight n. But double zeta values of odd weight are known to be rational polynomials in the ordinary zeta values, and the conclusion follows. □

6 "Exotic" multiple zeta values

In this section we give some examples of "exotic" homomorphic images of subalgebras of QSym. Our first example involves the multiple t-values as defined in [10]. For positive integers $i_1, \dots, i_k$ with $i_1 > 1$, let

$$t(i_1, \dots, i_k) = \sum_{\substack{n_1 > n_2 > \cdots > n_k \ge 1 \\ n_j \text{ odd}}} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.$$

Then $t : \mathrm{QSym}^0 \to \mathbb{R}$ defined by $t(z_{i_1} \cdots z_{i_k}) = t(i_1, \dots, i_k)$ defines a homomorphism. The multiple t-values have obvious parallels with multiple zeta values; for example, it is evident that $t(n) = (1 - 2^{-n})\zeta(n)$ for $n \ge 2$. Also, paralleling the identities (7) and (8) of the last section we have from [10]

$$t(\{2\}_n) = \frac{\pi^{2n}}{2^{2n}(2n)!}, \quad t(\{4\}_n) = \frac{\pi^{4n}}{2^{2n}(4n)!}. \tag{12}$$

Following Wakhare and Vignat [18], we can take any function G with real zeros $\{a_1, a_2, \dots\}$ such that $\lim_{n\to\infty} |a_n| = \infty$, and define a homomorphism $\zeta_G : S \to \mathbb{R}$ by sending $z_{i_1} \cdots z_{i_l}$ to

$$\zeta_G(i_1, \dots, i_l) = \sum_{n_1 > n_2 > \cdots > n_k \ge 1} \frac{1}{a_{n_1}^{i_1} a_{n_2}^{i_2} \cdots a_{n_l}^{i_l}}$$

for some subalgebra S of QSym that depends on the growth rate of $|a_n|$ with n. Wakhare and Vignat consider the case where a_n is the nth positive zero of the Bessel function J_ν of the first kind of order ν. They obtain the remarkable formulas

$$\zeta_{J_\nu}(\{2\}_n) = \frac{1}{2^{2n} n! (\nu+1)(\nu+2)\cdots(\nu+n)}, \tag{13}$$

$$\zeta_{J_\nu}(\{4\}_n) = \frac{1}{2^{4n} n! (\nu+1)\cdots(\nu+2n)(\nu+1)\cdots(\nu+n)}. \tag{14}$$

We note that since

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z \quad \text{and} \quad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z$$

we have

$$\pi^{|w|} \zeta_{J_{\frac{1}{2}}}(w) = \zeta(w) \quad \text{and} \quad \left(\frac{\pi}{2}\right)^{|w|} \zeta_{J_{-\frac{1}{2}}}(w) = t(w),$$

and thus Eqs. (13) and (14) imply Eqs. (7), (8), and (12) above.

We can also choose $0 > a_1 > a_2 > \cdots$ to be the zeros of the Airy function $\mathrm{Ai}(z)$. Now $\mathrm{Ai}(z)$ has the infinite product expansion [17, p. 18]

$$\mathrm{Ai}(z) = \mathrm{Ai}(0) e^{-\kappa z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}, \tag{15}$$

where

$$\kappa = \left| \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} \right| = \frac{3^{\frac{5}{6}} \Gamma(\frac{2}{3})^2}{2\pi} \approx 0.729011.$$

Starting with Eq. (15), take logarithms and differentiate to get

$$\frac{d}{dz} \log \mathrm{Ai}(z) = -\kappa + \sum_{n=1}^{\infty} \left[\frac{1}{a_n} + \frac{1}{z - a_n} \right].$$

Then evidently

$$\frac{d^k}{dz^k} \log \mathrm{Ai}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{(z - a_n)^k} \tag{16}$$

for $k \geq 2$. Since $\mathrm{Ai}''(z) = z\,\mathrm{Ai}(z)$, we have

$$\frac{d^2}{dz^2} \log \mathrm{Ai}(z) = z - \frac{\mathrm{Ai}'(z)^2}{\mathrm{Ai}(z)^2}. \tag{17}$$

Combining Eq. (16) for $k = 2$ and Eq. (17), we have

$$\sum_{n=1}^{\infty} \frac{-1}{(z - a_n)^2} = z - \frac{\mathrm{Ai}'(z)^2}{\mathrm{Ai}(z)^2}, \tag{18}$$

which at $z = 0$ gives

$$\zeta_{\mathrm{Ai}}(2) = \sum_{n=1}^{\infty} \frac{1}{a_n^2} = \kappa^2. \tag{19}$$

Repeated differentiation of $f(z) = \mathrm{Ai}'(z)/\mathrm{Ai}(z)$ gives the following result, originally due to Crandall [2].

Theorem 10. *For all $n \geq 2$, $\zeta_{\mathrm{Ai}}(n)$ is a rational polynomial in κ of degree n, with leading coefficient 1.*

Also, from Eq. (15) it follows that

$$\mathrm{Ai}(z)\,\mathrm{Ai}(-z) = \mathrm{Ai}(0)^2 \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{a_k^2}\right)$$

and thus that

$$\sum_{n=0}^{\infty} \zeta_{\mathrm{Ai}}(\{2\}_n)(-1)^n z^{2n} = \frac{\mathrm{Ai}(z)\,\mathrm{Ai}(-z)}{\mathrm{Ai}(0)^2} = \\ 1 - \kappa^2 z^2 + \frac{\kappa}{6} z^4 - \frac{1}{60} z^6 + \frac{\kappa^2}{336} z^8 - \frac{\kappa}{6480} z^{10} + \cdots .$$

By comparison with the series [16]

$$\mathrm{Ai}(z)\,\mathrm{Ai}(-z) = \frac{2}{\sqrt{\pi}} \sum_{n\ge 0} \frac{(-1)^n z^{2n}}{12^{\frac{2n+5}{6}} n! \Gamma(\frac{2n+5}{6})}$$

it can be seen that $\zeta_{\mathrm{Ai}}(\{2\}_n)$ is rational if $n \equiv 0 \bmod 3$, a rational multiple of κ^2 if $n \equiv 1 \bmod 3$, and a rational multiple of κ if $n \equiv 2 \bmod 3$. Further formulas for $\zeta_{\mathrm{Ai}}(\{2\}_n)$ and also for $\zeta_{\mathrm{Ai}}(\{4\}_n)$ were given by Wakhare and Vignat [19].

7 Alternating multiple zeta values

Let r be a positive integer, $A = \{z_{m,j} |\ m \in \mathbb{Z}^+, i \in \{0, 1, \dots, r-1\}\}$, with $z_{m,j} \diamond z_{n,k} = z_{m+n,j+k}$, where addition in the second subscript is understood mod r. Then $(\mathbb{Q}\langle A\rangle, *)$ is the "Euler algebra" $\mathfrak{E}_r$ as defined in [8]. If we let $\mathfrak{E}_r^0$ be the subalgebra generated by 1 and all words that do not begin with $z_{1,0}$, then there is a homomorphism $\mathfrak{Z}_r : \mathfrak{E}_r^0 \to \mathbb{C}$ sending $z_{m_1,j_1} \cdots z_{m_k,j_k}$ to

$$\sum_{n_1 > \cdots > n_k \ge 1} \frac{\epsilon^{n_1 j_1} \cdots \epsilon^{n_k j_k}}{n_1^{m_1} \cdots n_k^{m_k}},$$

where $\epsilon = e^{\frac{2\pi i}{r}}$. Of course $\mathfrak{E}_1$ is just QSym, with $\mathfrak{Z}_1 = \zeta$. In the case $r = 2$ the image of $\mathfrak{Z}_r$ is real-valued, and $\mathfrak{Z}_r$ sends a monomial to what is usually called an alternating or "colored" multiple zeta value. In this case we can adapt the sequence notation of multiple zeta values and write, e.g., $\zeta(\bar{1}, 2, \bar{3})$

for $\mathfrak{z}_2(z_{1,1}z_{2,0}z_{3,1})$. Evidently $\zeta(\bar{1}) = -\log 2$ and $\zeta(\bar{k}) = (2^{-k+1}-1)\zeta(k)$ for $k \geq 2$. Generating functions for $\zeta(\{\bar{k}\}_n)$ are discussed already in [1]. A notable case is

$$\sum_{n=0}^{\infty} \zeta(\{\bar{1}\}_n)\lambda^n = \frac{\sqrt{\pi}}{\Gamma(\frac{1-\lambda}{2})\Gamma(1+\frac{\lambda}{2})}. \tag{20}$$

The theory of interpolated products carries over to this case; for example

$$\zeta^r(\bar{1},2,\bar{3}) = \zeta(\bar{1},3,\bar{3}) + r\zeta(\bar{3},\bar{3}) + r\zeta(\bar{1},\bar{5}) + r^2\zeta(6).$$

We can generalize formulas like (20) to interpolated alternating multiple zeta values:

$$\sum_{n=0}^{\infty} \zeta^r(\{\bar{1}\}_n)\lambda^n = \frac{\Gamma(\frac{1+r\lambda}{2})\Gamma(1-\frac{r\lambda}{2})}{\Gamma(\frac{1-(1-r)\lambda}{2})\Gamma(1+\frac{(1-r)\lambda}{2})}.$$

Some results for alternating multiple zeta values can be stated in terms of interpolated values, such as the following one of C. Glanois [4].

Theorem 11. *If $s_1,\dots,s_r$ is a sequence of elements of $\{1,\bar{2},3,\bar{4},5,\dots\}$ with $s_1 \neq 1$, then the interpolated alternating multiple zeta value $\zeta^{\frac{1}{2}}(s_1,\dots,s_r)$ is a rational linear combination of multiple zeta values.*

8 Symmetric sum theorems

The prototypical symmetric sum theorem was proved in [6].

Theorem 12. *[[6, Thm. 2.2]] If $k_1,\dots,k_n \geq 2$, then*

$$\sum_{\sigma\in S_n} \zeta(k_{\sigma(1)},\dots,k_{\sigma(n)}) = \sum_{B=\{B_1,\dots,B_l\}\in\Pi_n} c(B)\prod_{m=1}^{l}\zeta\left(\sum_{j\in B_m} k_j\right)$$

where S_n is the symmetric group on n letters, Π_n is the set of partitions of the set $\{1,\dots,n\}$, and

$$c(B) = (-1)^{k-l}(\operatorname{card} B_1 - 1)!(\operatorname{card} B_2 - 1)!\cdots(\operatorname{card} B_l - 1)!$$

for $B = \{B_1,\dots,B_l\} \in \Pi_n$.

In fact, as noted in [9], this identity can be proved in QSym by Möbius inversion and then (if all the $k_i \geq 2$) transferred to the reals via the homomorphism $\zeta : \mathrm{QSym}^0 \to \mathbb{R}$. But in fact it can be generalized in two ways: first, it is true for *any* quasi-shuffle algebra $(\mathbb{Q}\langle A\rangle, *)$, and second, we can extend it to the interpolated product. The result is as follows.

Theorem 13. *If $u_1, \ldots, u_n \in A$, then in $(\mathbb{Q}\langle A\rangle, \overset{r}{*})$*

$$\sum_{\sigma \in S_k} u_{\sigma(1)} u_{\sigma(2)} \cdots u_{\sigma(k)} = \sum_{B=\{B_1,\ldots,B_l\}\in \Pi_k} c_r(B) u_{B_1} \overset{r}{*} u_{B_2} \overset{r}{*} \cdots \overset{r}{*} u_{B_l}, \tag{21}$$

where $u_{B_i} = \diamond_{j\in B_i} u_j$, $p_a(r) = (1-r)^a - (-r)^a$, and

$$c_r(B) = (-1)^{k-l} \prod_{m=1}^{l} (\operatorname{card} B_m - 1)! p_{\operatorname{card} B_m}(r)$$

for $B = \{B_1, \ldots, B_l\} \in \Pi_k$.

Proof. We write $S(a,b) = ab+ba$, $S(a,b,c) = abc+acb+bac+bca+cab+cba$, and so on, so Eq. (21) is

$$S(u_1, \ldots, u_n) = \sum_{\substack{\text{partitions } \Pi = (P_1,\ldots,P_l) \\ \text{of } \{1,\ldots,n\}}} c_r(\Pi) u_{P_1} \overset{r}{*} u_{P_2} \overset{r}{*} \cdots \overset{r}{*} u_{P_l}.$$

We proceed by induction on n. Take the $\overset{r}{*}$-product of both sides of Eq. (21) with u_{n+1} to get

$$\begin{aligned} S(u_1, \ldots, u_{n+1}) + (1-2r)[S(u_1 \diamond u_{n+1}, u_2, \ldots, u_n) + S(u_1, u_2 \diamond u_{n+1}, \ldots, u_n) \\ + \cdots + S(u_1, \ldots, u_{n-1}, u_n \diamond u_{n+1})] + 2(r^2 - r)[S(u_1 \diamond u_2 \diamond u_{n+1}, u_3, \ldots, u_n) \\ + S(u_1 \diamond u_3 \diamond u_{n+1}, u_2, u_4, \ldots, u_n) + \cdots + S(u_{n-1} \diamond u_n \diamond u_{n+1}, u_1, \ldots, u_{n-2})] \\ = \sum_{\substack{\text{partitions } \Pi = (P_1,\ldots,P_l) \\ \text{of } \{1,\ldots,n\}}} c_r(\Pi) u_{P_1} \overset{r}{*} u_{P_2} \overset{r}{*} \cdots \overset{r}{*} u_{P_l} \overset{r}{*} u_{n+1} \end{aligned}$$

or

$$
\begin{aligned}
S(u_1,\ldots,u_{n+1}) &= -(1-2r)[S(u_1\diamond u_{n+1},u_2,\ldots,u_n)+S(u_1,u_2\diamond u_{n+1},\ldots,u_n) \\
&+\cdots+S(u_1,\ldots,u_{n-1},u_n\diamond u_{n+1})]-2(r^2-r)[S(u_1\diamond u_2\diamond u_{n+1},u_3,\ldots,u_n) \\
&+S(u_1\diamond u_3\diamond u_{n+1},u_2,u_4,\ldots,u_n)+\cdots+S(u_{n-1}\diamond u_n\diamond u_{n+1},u_1,\ldots,u_{n-2})] \\
&+\sum_{\substack{\text{partitions } \Pi=(P_1,\ldots,P_l) \\ \text{of } \{1,\ldots,n+1\} \text{ having} \\ \{n+1\} \text{ as a part}}} c_r(\Pi)u_{P_1}\overset{r}{*}u_{P_2}\overset{r}{*}\cdots\overset{r}{*}u_{P_l}. \qquad (22)
\end{aligned}
$$

We must show that the right-hand side of this equation coincides with

$$
\sum_{\substack{\text{partitions } \Pi=(P_1,\ldots,P_l) \\ \text{of } \{1,\ldots,n+1\}}} c_r(\Pi)u_{P_1}\overset{r}{*}u_{P_2}\overset{r}{*}\cdots\overset{r}{*}u_{P_l}, \qquad (23)
$$

which we shall do by considering whether the cardinality of the part of Π to which $n+1$ belongs is 1, 2, or ≥ 3.

Note that there are three groups on terms on the right-hand side of Eq. (22). If $\{n+1\}$ is a part of Π, the corresponding term in (23) is contributed by the third group of terms on the right-hand side of (22).

Suppose now that $n+1$ belongs to a part of cardinality 2 in $\Pi = (P_1,\ldots,P_l)$, say P_1. The term corresponding to Π in (23) only arises (via the induction hypothesis) from the first group of terms on the right-hand side of (22), and the coefficient of $u_{P_1}\cdots u_{P_l}$ is

$$
\begin{aligned}
&-(1-2r)(-1)^{n-l}(\operatorname{card}P_2-1)!\cdots(\operatorname{card}P_l-1)!p_{\operatorname{card}P_2}(t)\cdots p_{\operatorname{card}P_l}(t) \\
&\qquad=(-1)^{n+1-l}(\operatorname{card}P_1-1)!\cdots(\operatorname{card}P_l-1)!p_{\operatorname{card}P_1}(t)\cdots p_{\operatorname{card}P_l}(t).
\end{aligned}
$$

Finally, suppose $n+1$ belongs to a part P_1 of Π with cardinality $k \geq 3$. The term $u_{P_1}\cdots u_{P_l}$ arises from the first group of terms in $k-1$ ways, contributing coefficient

$$
-(k-1)(1-2r)(-1)^{n-l}p_{k-1}(r)(k-2)!C,
$$

where

$$
C=(\operatorname{card}P_2-1)!\cdots(\operatorname{card}P_l-1)!p_{\operatorname{card}P_2}(t)\cdots p_{\operatorname{card}P_l}(t).
$$

The same term arises from the second group of terms in $\binom{k-1}{2}$ ways, contributing coefficient

$$-\binom{k-1}{2} 2(r^2-r)(-1)^{n-1-l} p_{k-2}(r)(k-3)!C,$$

and it suffices to show

$$(1-2r)p_{k-1}(r) - (r^2-r)p_{k-2}(r) = p_k(r),$$

which is immediate. □

Note that $p_a(0) = 1$ and $p_a(1) = (-1)^{a-1}$, so $c_0(\Pi) = c(\Pi)$ and $c_1(\Pi) = |c(\Pi)|$, making Theorem 13 reduce to

$$\sum_{\sigma \in S_n} u_{\sigma(1)} u_{\sigma(2)} \cdots u_{\sigma(n)} = \sum_{\substack{\text{partitions } \Pi = (P_1, \dots, P_l) \\ \text{of } \{1, \dots, n\}}} c(\Pi) u_{P_1} * u_{P_2} * \cdots * u_{P_l}$$

in the case $r = 0$; if $r = 1$ we get

$$\sum_{\sigma \in S_n} u_{\sigma(1)} u_{\sigma(2)} \cdots u_{\sigma(n)} = \sum_{\substack{\text{partitions } \Pi = (P_1, \dots, P_l) \\ \text{of } \{1, \dots, n\}}} |c(\Pi)| u_{P_1} \star u_{P_2} \star \cdots \star u_{P_l}.$$

Also,

$$p_a\left(\frac{1}{2}\right) = \begin{cases} 0, & \text{if } a \text{ even,} \\ 2^{1-a}, & \text{if } a \text{ odd,} \end{cases}$$

so that only partitions with all parts of odd cardinality appear when $r = \frac{1}{2}$. In fact

$$c_{\frac{1}{2}}(\Pi) = \begin{cases} \left(\frac{1}{2}\right)^{n-l} \prod_{i=1}^{l} (\operatorname{card} P_i - 1)!, & \text{if } \operatorname{card} P_1 \cdots \operatorname{card} P_l \text{ is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

If in Theorem 13 we take $A = \{z_1, z_2, \dots\}$ with $z_i \diamond z_j = z_{i+j}$ and $u_i = z_{k_i}$, $1 \le i \le n$ (with $k_i \ne 1$ for all i), we get

$$\sum_{\sigma \in S_n} \zeta^r(k_{\sigma(1)}, \dots, k_{\sigma(n)}) = \sum_{\substack{\text{partitions } \Pi = (P_1, \dots, P_l) \\ \text{of } \{1, \dots, n\}}} c_r(\Pi) \prod_{j=1}^{l} \zeta\left(\sum_{h \in P_j} k_h\right), \quad (24)$$

generalizing Theorem 12; in fact $r = 0$ gives Theorem 12 and $r = 1$ gives the corresponding result for star-zeta values [6, Thm. 2.1]. Identity (24) holds with t (or ζ_{J_ν} or ζ_{Ai}) in place of ζ.

From Theorem 13 we can obtain a result in terms of integer partitions.

Corollary 4. *If $u \in A$, then in $(\mathbb{Q}\langle A\rangle, \overset{r}{*})$*

$$u^n = \sum_{\lambda \vdash n} \frac{\epsilon_\lambda}{z_\lambda} \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(r) u^{\diamond \lambda_1} \overset{r}{*} \cdots \overset{r}{*} u^{\diamond \lambda_l}$$

where $u^{\diamond n}$ means $\underbrace{u \diamond \cdots \diamond u}_{n}$ and (as in [14]) $\epsilon_\lambda = (-1)^{n-\ell(\lambda)}$ and $z_\lambda = m_1(\lambda)! 1^{m_1(\lambda)} m_2(\lambda)! 2^{m_2(\lambda)} \cdots$, for $m_i(\lambda)$ the multiplicity of i in λ.

Proof. Set $u_1 = \cdots = u_n = u$ in Theorem 13 to get

$$n! u^n = \sum_{\substack{\text{partitions}\\ \Pi=(P_1,\dots,P_l)\\ \text{of } \{1,\dots,n\}}} (-1)^{n-l} (\lambda_1 - 1)! \cdots (\lambda_l - 1)! p_{\lambda_1}(r) \cdots p_{\lambda_l}(r) u_{i\lambda_1} \overset{r}{*} \cdots \overset{r}{*} u_{i\lambda_l},$$

where we write $\lambda_i = \operatorname{card} P_i$. Now the number of set partitions $(P_1, \dots, P_l)$ of $\{1, \dots, n\}$ corresponding to the integer partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of n is

$$\frac{1}{m_1(\lambda)! m_2(\lambda)! \cdots} \binom{n}{\lambda_1} \binom{n - \lambda_1}{\lambda_2} \cdots = \frac{1}{m_1(\lambda)! m_2(\lambda)! \cdots} \frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_l!}.$$

Thus u^n is

$$\sum_{\substack{\text{partitions}\\ \lambda=(\lambda_1,\dots,\lambda_l)\\ \text{of } n}} \frac{(-1)^{n-l} (\lambda_1 - 1)! \cdots (\lambda_l - 1)!}{m_1(\lambda)! m_2(\lambda)! \cdots \lambda_1! \cdots \lambda_l!} p_{\lambda_1}(r) \cdots p_{\lambda_l}(r) u^{\diamond \lambda_1} \overset{r}{*} \cdots \overset{r}{*} u^{\diamond \lambda_l}$$

$$= \sum_{\substack{\text{partitions}\\ \lambda=(\lambda_1,\dots,\lambda_l)\\ \text{of } n}} \frac{\epsilon_\lambda}{z_\lambda} p_{\lambda_1}(r) \cdots p_{\lambda_l}(r) u^{\diamond \lambda_1} \overset{r}{*} \cdots \overset{r}{*} u^{\diamond \lambda_l}.$$

□

Applying ζ^r to the corollary with $u = z_i$, $i \geq 2$, we obtain

$$\zeta(z_i^n) = \sum_{\lambda \vdash n} \frac{\epsilon_\lambda}{z_\lambda} \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(r) \zeta(i\lambda_j).$$

In the cases $r = 0, 1, \frac{1}{2}$, this identity is respectively

$$\zeta(\{i\}_n) = \sum_{\lambda \vdash n} \frac{\epsilon_\lambda}{z_\lambda} \prod_{j=1}^{\ell(\lambda)} \zeta(i\lambda_j) \tag{25}$$

$$\zeta^\star(\{i\}_n) = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \prod_{j=1}^{\ell(\lambda)} \zeta(i\lambda_j) \tag{26}$$

$$\zeta^{\frac{1}{2}}(\{i\}_n) = \sum_{\substack{\lambda \vdash n \\ \text{all parts of } \lambda \text{ odd}}} \frac{1}{2^{n-\ell(\lambda)} z_\lambda} \prod_{j=1}^{\ell(\lambda)} \zeta(i\lambda_j). \tag{27}$$

Eqs. (26) and (25) are homomorphic images of the two parts of [14, Eq. (2.14′)]. Eq. (27) is obtained a different way in [11] (see Eq. (41)).

We note that Eq. (24) applies to alternating multiple zeta values as well, provided we define addition on the set $\mathcal{I} = \{\dots, \bar{2}, \bar{1}, 1, 2, \dots\}$ of indices to agree with usual addition on $\{1, 2, \dots,\}$ and extend it to $\mathcal{I}$ via

$$\begin{aligned} a + \bar{b} = \bar{a} + b &= \overline{a+b} \\ \bar{a} + \bar{b} &= a + b \end{aligned}$$

for positive integers a, b. Thus, e.g.,

$$\begin{aligned} &\zeta^r(\bar{1}, 2, \bar{3}) + \zeta^r(\bar{1}, \bar{3}, 2) + \zeta^r(2, \bar{1}, \bar{3}) + \zeta^r(2, \bar{3}, \bar{1}) + \zeta^r(\bar{3}, \bar{1}, 2) + \zeta^r(\bar{3}, 2, \bar{1}) = \\ &\qquad \zeta(\bar{1})\zeta(2)\zeta(\bar{3}) - (1-2r)(\zeta(\bar{3})^2 + \zeta(\bar{1})\zeta(\bar{5})) + 2(1 - 3r + 3r^2)\zeta(6). \end{aligned}$$

Eqs. (25-27) also hold, provided we interpret $i\lambda_j$ in those formulas as the sum of λ_j copies of $i \in \mathcal{I}$.

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