

ON HOROSPHERIC LIMIT SETS OF KLEINIAN GROUPS

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ABSTRACT. In this paper we partially answer a question of P. Tukia about the size of the difference between the big horospheric limit set and the horospheric limit set of a Kleinian group. We mainly investigate the case of normal subgroups of Kleinian groups of divergence type and show that this difference is of zero conformal measure by using another result obtained here: the Myrberg limit set of a non-elementary Kleinian group is contained in the horospheric limit set of any non-trivial normal subgroup.

1. INTRODUCTION AND STATEMENT OF RESULTS

In [29] Sullivan showed that the conservative part of the action of a Kleinian group G on its limit set coincides up to zero sets of the spherical Lebesgue measure with the horospheric limit set $L_h(G)$ of G , i.e. the set of all limit points at which every horoball contains infinitely many orbit points of the group. Later, Tukia [36] generalised this result by showing that the same conservative part of the group action on the limit set coincides with the so-called big horospheric limit set $L_H(G)$ up to zero sets of any conformal measure of dimension $\delta(G)$ for G . Here, $\delta(G)$ denotes as usual the critical exponent of G and $L_H(G)$ consists of all limit points of G at which there exists a horoball containing infinitely many orbit points of G . For a generalisation of these observations to boundary actions of discrete groups of isometries of Gromov hyperbolic spaces we refer the reader to Kaimanovich's work [15].

All this considered, Tukia [36] asked the very natural question of how big the difference $L_H(G) \setminus L_h(G)$ might be, also in light of the close relationship between so-called Garnett points [29] and this difference of sets (see [36] for more details).

One possible first attempt at answering this question could make use of a stratification of the limit set of a Kleinian group between the radial and the horospheric limit sets in terms of linear escape rates to infinity within the convex core of the corresponding hyperbolic manifold. These ideas have been used by different authors in various slightly different guises and we refer to Section 3 for details. Since both the radial limit set and the big horospheric limit set appear as elements of this stratification, it would seem that it can be used to measure the difference between the big horospheric and the horospheric limit set. However, Proposition 3.3 goes to show that we cannot detect this difference just by changing linear escape rates.

In this paper we follow a different and somewhat surprising idea in order to answer Tukia's question in the case of normal subgroups of Kleinian groups of

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divergence type. We show (see Theorem 5.2) that if N is some non-trivial normal subgroup of the Kleinian group G of divergence type, then $L_H(N) \setminus L_h(N)$ is a nullset w.r.t. the uniquely determined conformal measure μ of dimension $\delta(G)$ for G . This, of course, is also a conformal measure of dimension $\delta(G)$ for N . We obtain this result as a consequence of another observation (see Theorem 5.1) of independent interest, namely, that the Myrberg limit set of a Kleinian group G is always contained in the horospheric limit set of any non-trivial normal subgroup N of G . This in turn is a refinement of the fact proven in [17] that in this situation the radial limit set of G is contained in the big horospheric limit set of N . Of course, one then also needs G to be of divergence type in order to ensure that the Myrberg limit set is of full μ -measure (see e.g. [35], [27] and [8] for more details). The surprising aspect of this approach is that it is an instance where a statement about an essentially non-conservative phenomenon (the difference between the horospheric and big horospheric limit set) is proven using a consequence of ergodicity. The Myrberg limit set can be understood as a qualitative description of the ergodicity of the geodesic flow in the case of divergence type groups. We discuss these notions in Section 2 and the beginning of Section 5.

Clearly, one wants to measure the difference between the horospheric and big horospheric limit set also in a more general case than for normal subgroups of Kleinian groups of divergence type. In view of our previous work [10], we conjecture that the statement of Theorem 5.2 also holds for Kleinian groups for which the convex hull of the limit set admits a uniformly distributed set whose Poincaré series diverges at its critical exponent (see the end of Section 5 and Conjecture 1 for more details).

Having answered Tukia's question for normal subgroups N of groups G of divergence type by considering the Myrberg limit set of G , leaves the question open why one should measure the size of the difference $L_H(N) \setminus L_h(N)$ by a $\delta(G)$ -dimensional conformal measure, as we do, and not a $\delta(N)$ -dimensional one. Recall that $\delta(N)$ may very well be strictly smaller than $\delta(G)$. The answer, at least in our context, is given in Proposition 6.1 where we show that the Hausdorff dimension of $L_M(G)$ is equal to $\delta(G)$, provided G is of divergence type and the strong sublinear growth limit set $\Lambda_*(G)$ is of full measure w.r.t. some Patterson measure of G (see Section 3 and Section 6 for more details). Already Sullivan [28] conjectured that $\Lambda_*(G)$ should be of full Patterson measure for all groups G of divergence type, but we go one step further and conjecture that the Hausdorff dimension of the Myrberg limit set coincides with the critical exponent for all non-elementary Kleinian groups (Conjecture 2 in Section 6). One may be able to prove this by refining a well-known argument of Bishop and Jones [5] showing that the Hausdorff dimension of the radial limit set is equal to the critical exponent for non-elementary groups.

2. PRELIMINARIES

2.1. Limit sets of a Kleinian group. Let (\mathbb{B}^{n+1}, d) , $n \geq 1$, be the unit ball model of $(n+1)$ -dimensional hyperbolic space with the hyperbolic distance d . The n -dimensional unit sphere \mathbb{S}^n is the boundary at infinity of hyperbolic space. *Kleinian groups* are discrete subgroups of the group of orientation preserving isometries of hyperbolic space. The quotient $M_G = \mathbb{B}^{n+1}/G$ of $(n+1)$ -dimensional hyperbolic space through a torsion free Kleinian group, that is, a group without elliptic elements, is an $(n+1)$ -dimensional hyperbolic manifold.

The *limit set* $L(G)$ of a Kleinian group G is the set of accumulation points of an arbitrary G -orbit, and is a closed subset of \mathbb{S}^n . If $L(G)$ consists of more than two points, then it is uncountable and perfect, and G is called *non-elementary*. The hyperbolic convex hull of the union of all geodesics both of whose end points are in $L(G)$ is called the *convex hull of $L(G)$* , and is denoted by $H(L(G))$. The quotient $C(M_G) := H(L(G))/G$ is called the *convex core* of M_G . Equivalently, the convex core is the smallest convex subset of M_G containing all closed geodesics of M_G . A non-elementary Kleinian group G is called *convex cocompact* if the convex core $C(M_G)$ is compact, and *geometrically finite* if some ε -neighbourhood of $C(M_G)$ has finite hyperbolic volume.

A point $\xi \in L(G)$ is a *radial limit point* of G if for any $x \in \mathbb{B}^{n+1}$ and for any geodesic ray towards ξ there is a constant $c \geq 0$ such that infinitely many points of the orbit Gx are within distance c of the given geodesic ray. The set of all radial limit points of G is called the *radial limit set* and is denoted by $L_r(G)$.

A point $\xi \in L(G)$ is a *horospheric limit point* if for any $x \in \mathbb{B}^{n+1}$ every horosphere tangent to \mathbb{S}^n at ξ contains an orbit point gx for some $g \in G$. The set of all horospheric limit points of G is called the *horospheric limit set* and is denoted by $L_h(G)$. A point $\xi \in L(G)$ is an element of the *big horospheric limit set*, denoted $L_H(G)$, if for $x \in \mathbb{B}^{n+1}$ there is some horosphere tangent to \mathbb{S}^n at ξ that contains infinitely many orbit points in Gx . By definition, we have $L_h(G) \subset L_H(G)$.

2.2. The critical exponent and invariant conformal measures. For a Kleinian group G and points $x, z \in \mathbb{B}^{n+1}$, the *Poincaré series* with exponent $s > 0$ is given by

$$P^s(Gx, z) := \sum_{g \in G} e^{-s d(g(x), z)}.$$

The *critical exponent* $\delta = \delta(G)$ of G is

$$\delta(G) := \inf \{s > 0 \mid P^s(Gx, z) < \infty\} = \limsup_{R \rightarrow \infty} \frac{\log \#(B(z, R) \cap Gx)}{R},$$

where $B(z, R)$ is the hyperbolic ball of radius R centred at z and $\#(\cdot)$ denotes the cardinality of a set. By the triangle inequality, δ does not depend on the choice of $x, z \in \mathbb{B}^{n+1}$. If G is non-elementary, then $0 < \delta \leq n$. Also, Roblin [26] showed that the above upper limit is in fact a limit. G is called *of convergence type* if $P^\delta(Gx, z) < \infty$, and *of divergence type* otherwise. It is known that a geometrically finite Kleinian group is of divergence type.

A family of positive finite Borel measures $\{\mu_z\}_{z \in \mathbb{B}^{n+1}}$ on \mathbb{S}^n is called *s-conformal measure* for $s > 0$ if $\{\mu_z\}$ are absolutely continuous to each other and, for each $z \in \mathbb{B}^{n+1}$ and for almost every $\xi \in \mathbb{S}^n$,

$$\frac{d\mu_z}{d\mu_o}(\xi) = |g_z(\xi)|^s,$$

where o is the origin in \mathbb{B}^{n+1} , g_z is a conformal automorphism of \mathbb{B}^{n+1} sending z to o and $|\cdot|$ denotes the linear stretching factor of a conformal map. For a Kleinian group G , if $\{\mu_z\}$ satisfies $g^* \mu_{g(z)} = \mu_z$ (a.e) for every $z \in \mathbb{B}^{n+1}$ and for every $g \in G$, then $\{\mu_z\}$ is called *G-invariant*.

The measure $\mu = \mu_o$ can represent the family $\{\mu_z\}$ and the G -invariance property can be rephrased as follows: for every $g \in G$ and any measurable $A \subset \mathbb{S}^n$ we have

$$\mu(g(A)) = \int_A |g'(\xi)|^s d\mu(\xi).$$

If a positive finite Borel measure μ on \mathbb{S}^n satisfies this condition, we also call μ itself a G -invariant conformal measure of dimension s .

We consider a G -invariant conformal measure of dimension $\delta = \delta(G)$ supported on the limit set $L(G)$. The canonical construction of such a measure due to Patterson [21], [22] is as follows (See also [20]). Assume first that G is of divergence type. For any $s > \delta$, take a weighted sum of Dirac measures on the orbit Gx for some $x \in \mathbb{B}^{n+1}$:

$$\mu_{(x)}^s := \frac{1}{P^s(Gx, o)} \sum_{g \in G} e^{-sd(gx, o)} 1_{gx}.$$

We can choose some sequence $s_n > \delta$ tending to δ such that $\mu_{(x)}^{s_n}$ converges to some measure μ on $\overline{\mathbb{B}^{n+1}}$ in the weak-* sense. Then we see that μ is a G -invariant conformal measure of dimension δ supported on $L(G)$, which is called a *Patterson measure* for G . When G is of convergence type, we need to use a modified Poincaré series $\tilde{P}^s(Gx, o)$ to make it divergent at δ and apply a similar argument. If G is of divergence type, then a G -invariant conformal measure of dimension δ is unique up to multiplication by a positive constant, hence it is *the* Patterson measure.

There is another canonical construction of G -invariant conformal measures, mainly in the case where G is of convergence type. This was introduced briefly by Sullivan in [32] and developed further in [3] (see also [11]). Suppose that the Poincaré series for G converges at dimension $s \geq \delta$. We again consider the weighted sum of Dirac measures $\mu_{(x)}^s$ as above, but here we move the orbit point x to some point $\xi \in \mathbb{S}^n$ at infinity within a Dirichlet fundamental domain for G . We can choose a sequence $x_n \in \mathbb{B}^{n+1}$ tending to ξ such that $\mu_{(x_n)}^s$ converges to some measure μ on $\overline{\mathbb{B}^{n+1}}$ in the weak-* sense. Then we see that μ is a G -invariant conformal measure of dimension s on \mathbb{S}^n , which is called an *ending measure*.

2.3. Ergodicity of the geodesic flow. For a hyperbolic manifold $M_G = \mathbb{B}^{n+1}/G$, the *unit tangent bundle* $T^1M_G = \bigsqcup_{p \in M_G} T_p^1M_G$ is the union of the unit tangent vectors $v \in T_p^1M_G$ at p taken over all $p \in M_G$. Each element of T^1M_G is represented by the pair (v, p) . Let $\tilde{g}_{\xi, z}(t)$ be a geodesic line of unit speed in \mathbb{B}^{n+1} starting from a given point $z \in \mathbb{B}^{n+1}$ towards $\xi \in \mathbb{S}^n$ as $t \rightarrow \infty$. The unit tangent bundle $T^1\mathbb{B}^{n+1}$ of hyperbolic space is also represented by $\mathbb{S}^n \times \mathbb{B}^{n+1} = \{(\xi, z)\}$ through the correspondence of the unit tangent vector

$$\left. \frac{d\tilde{g}_{\xi, z}}{dt} \right|_{t=0} = \tilde{g}'_{\xi, z}(0)$$

to (ξ, z) . For $(v, p) \in T^1M_G$, let $g_{v, p}(t)$ denote the geodesic line such that $g_{v, p}(0) = p$ and $g'_{v, p}(0) = v$. We can assume that this is the projection of some geodesic line $\tilde{g}_{\xi, z}(t)$ under $\mathbb{B}^{n+1} \rightarrow M_G$. In this case, we also use the notation $g_{\xi, z}$ instead of $g_{v, p}$. The *geodesic flow* $\phi_t : T^1M_G \rightarrow T^1M_G$ is a map sending (v, p) to $(g'_{v, p}(t), g_{v, p}(t))$ for each $t \in \mathbb{R}$.

Any conformal measure μ on \mathbb{S} induces a measure $\tilde{\mu}_*$ on the unit tangent bundle $T^1\mathbb{B}^{n+1} = \mathbb{S}^n \times \mathbb{B}^{n+1}$ that is invariant under the geodesic flow (Sullivan [28], see also

[20]). The unit tangent bundle T^1M_G of the hyperbolic manifold M_G is nothing but the quotient of $T^1\mathbb{B}^{n+1}$ by the canonical action of G . If μ is invariant under G , then so is $\tilde{\mu}_*$ and hence it descends to a measure μ_* on T^1M_G .

We say that the geodesic flow ϕ_t is *ergodic* with respect to μ_* if for any measurable subset E of T^1M_G that is invariant under ϕ_t for all $t \in \mathbb{R}$ we have that either $\mu_*(E) = 0$ or $\mu_*(T^1M_G \setminus E) = 0$. Sullivan [28] (and Aaronson and Sullivan [1]) generalised the result of Hopf [13], [14] to show the following (see also [25]).

Theorem 2.1. *Let G be a Kleinian group and μ a G -invariant conformal measure of dimension $\delta(G)$. Then the following conditions are equivalent:*

- (i) $\mu(L_r(G)) = \mu(\mathbb{S}^n)$;
- (ii) the geodesic flow ϕ_t is ergodic with respect to μ_* ;
- (iii) G is of divergence type.

If G is geometrically finite, then the measure μ_* corresponding to the Patterson measure μ is finite. If μ_* is a finite measure, then the geodesic flow ϕ_t is ergodic with respect to μ_* and hence all conditions from Theorem 2.1 hold ([31]). However, there are also large classes of geometrically infinite groups for which μ_* is infinite and these conditions are true ([33], [30], [24] or [1]). Moreover, there are also examples of geometrically infinite groups for which μ_* is a finite measure and the conditions from the theorem hold true ([23]).

3. LIMIT SETS BETWEEN RADIAL AND HOROSPHERIC

For a Kleinian group G , let μ be a G -invariant conformal measure on \mathbb{S}^n and $X \subset \mathbb{S}^n$ a measurable subset that is invariant under G . The action of G is called *conservative* on X with respect to μ if any measurable subset $A \subset X$ with $\mu(A) > 0$ satisfies $\mu(A \cap g(A)) > 0$ for infinitely many $g \in G$. For the n -dimensional spherical measure μ , Sullivan [29] showed that G acts conservatively on the horospheric limit set $L_h(G)$, and that the difference between $L_h(G)$ and $L_H(G)$ is actually of null measure. Later, a characterization of the conservative action for a G -invariant conformal measure μ in general was obtained by Tukia [36]. In particular, if a G -invariant conformal measure μ has no point mass, then the conservative part X , which is the maximal G -invariant measurable subset of \mathbb{S}^n on which G acts conservatively, coincides with the big horospheric limit set $L_H(G)$ up to null sets of μ .

We are interested in the difference $L_H(G) \setminus L_h(G)$, which contains all *Garnett points* originally defined in [29]. For the spherical Lebesgue measure, $L_H(G) \setminus L_h(G)$ is a null set, but Tukia [36] asked how small this difference is as measured by a G -invariant conformal measure. In this section, we will explain why one should expect that the difference between the big horospheric and the horospheric limit set is small.

First we introduce a continuous family of limit sets of a Kleinian group using the approaching order of its orbits. Fix $c > 0$ and $\kappa \in [0, 1]$. For a point $z \in \mathbb{B}^{n+1}$, let $S(z : c, \kappa)$ be the shadow of a hyperbolic ball

$$B\left(z, \frac{\kappa}{1 + \kappa}d(0, z) + c\right)$$

w.r.t. the projection from the origin to \mathbb{S}^n (see [12] for more details). Essentially the same shadow

$$I(z : c, \alpha) := \left\{ \xi \in \mathbb{S}^n \mid \left| \xi - \frac{z}{|z|} \right| < c(1 - |z|)^\alpha \right\}$$

was used in Nicholls [20]; $I(z : c, \alpha)$ corresponds to $S(z : c, \kappa)$ via $\alpha = 1/(1 + \kappa)$. For a Kleinian group G acting on \mathbb{B}^{n+1} , consider the orbit $Gz = \{gz\}_{g \in G}$ of $z \in \mathbb{B}^{n+1}$ and define

$$L_r^{(\kappa)}(G) := \bigcup_{c > 0} \limsup_{g \in G} S(gz : c, \kappa).$$

This is the set of points $\xi \in \mathbb{S}^n$ such that ξ belongs to infinitely many $S(gz : c, \kappa)$ for some $c > 0$. It is not difficult to see that $L_r^{(\kappa)}(G)$ is independent of the choice of z .

When $\kappa = 0$ the set $L_r^{(\kappa)}(G)$ is nothing more than the radial limit set $L_r(G)$, and when $\kappa = 1$, $L_r^{(\kappa)}(G)$ coincides with the big horospheric limit set $L_H(G)$. By moving κ between 0 and 1, we are thus interpolating between the radial limit set and the horospheric limit set.

Related limit sets were introduced by Bishop [4] and Lundh [16]. We set

$$\varphi_\xi(t) := d(g_{\xi,z}(t), g_{\xi,z}(0)),$$

which is the hyperbolic distance in the quotient manifold M_G between $g_{\xi,z}(t)$ and the initial point $g_{\xi,z}(0)$. Alternatively, it is defined as the distance of the orbit Gz from $\tilde{g}_{\xi,z}(t)$ in \mathbb{B}^{n+1} . It is clear that $\varphi_\xi(t) \leq t$. The ratio $\varphi_\xi(t)/t$ measures how rapidly or slowly the geodesic ray $g_{\xi,z}(t)$ escapes to infinity as $t \rightarrow \infty$. For instance, in Bishop [4], $g_{\xi,z}(t)$ is called a *linearly escaping geodesic* if there exists a positive constant $\kappa > 0$ such that $\varphi_\xi(t)/t > \kappa$ for all t . However, here we mainly investigate geodesic rays that are escaping to infinity slowly.

For each $\kappa \in [0, 1]$ we define the following limit set as the set of end points of sublinearly escaping geodesic rays:

$$\Lambda_\kappa(G) := \left\{ \xi \in \mathbb{S}^n \mid \liminf_{t \rightarrow \infty} \frac{\varphi_\xi(t)}{t} \leq \kappa \right\}.$$

The radial limit points correspond to non-escaping geodesic rays and hence $L_r(G)$ is contained in the *sublinear growth limit set* $\Lambda_0(G)$. As an important extremal case, we consider the *strong sublinear growth limit set*

$$\Lambda_*(G) := \left\{ \xi \in \mathbb{S}^n \mid \lim_{t \rightarrow \infty} \frac{\varphi_\xi(t)}{t} = 0 \right\},$$

which is contained in $\Lambda_0(G)$. However, while clearly $L_M(G) \subset L_r(G) \subset \Lambda_0(G)$, the inclusion relation of $L_r(G)$ or $L_M(G)$ to $\Lambda_*(G)$ is not a priori clear (see e.g. Example 6.2).

As it turns out, the limit sets $L_r^{(\kappa)}(G)$ and $\Lambda_\kappa(G)$, while not being coincident, are very similar. Actually, Lundh [16, Th.4.1] proved that

$$L_r^{(\kappa)}(G) = \left\{ \xi \in \mathbb{S}^n \mid \liminf_{t \rightarrow \infty} \left(\frac{1}{1 + \kappa} (\varphi_\xi(t) + t) - t \right) < \infty \right\}$$

for $0 \leq \kappa < 1$, and moreover,

$$\bigcap_{c > 0} \limsup_{g \in G} I(gz : c, (1 + \kappa)^{-1}) = \left\{ \xi \in \mathbb{S}^n \mid \liminf_{t \rightarrow \infty} \left(\frac{1}{1 + \kappa} (\varphi_\xi(t) + t) - t \right) = -\infty \right\}$$

for $0 < \kappa \leq 1$. As a consequence, it was shown in [16, Cor.4.2] that $L_r^{(\kappa)}(G) \subset \Lambda_\kappa(G)$ is always valid, and if $\kappa' < \kappa$ then $\Lambda_{\kappa'}(G) \subset L_r^{(\kappa)}(G)$.

When $\kappa = 1$, $\Lambda_1(G)$ coincides with the entire sphere \mathbb{S}^n since $\varphi_\xi(t)/t \leq 1$ for all $\xi \in \mathbb{S}^n$ and all $t > 0$. Since the horospheric limit set $L_h(G)$ can be represented by $\bigcap_{c>0} \limsup_{g \in G} I(gz : c, 1/2)$, the above result of Lundh implies the following.

Proposition 3.1.

$$L_h(G) = \{\xi \in \mathbb{S}^n \mid \liminf_{t \rightarrow \infty} (\varphi_\xi(t) - t) = -\infty\}.$$

However, a similar dynamical description of the big horospheric limit set $L_H(G)$ is somewhat more involved and we shall deal with this in Section 4.

It is well known [5] that, for any non-elementary Kleinian group G , the Hausdorff dimension of the radial limit set $L_r(G) = L_r^{(0)}(G)$ coincides with the critical exponent $\delta(G)$. The elementary estimate of the Hausdorff dimension here is the one from above, and the corresponding argument can be generalised to prove an upper bound for the Hausdorff dimension $\dim_H L_r^{(\kappa)}(G)$ as follows. This was proved in [12, p.575]. See also [4, Cor.3] and [6, Prop.4.2] for versions of this statement formulated for the limit sets $\Lambda_\kappa(G)$.

Proposition 3.2. *A Kleinian group G satisfies*

$$\dim_H L_r^{(\kappa)}(G) \leq (1 + \kappa) \delta(G)$$

for every $\kappa \in [0, 1]$.

We have already mentioned that $L_r^{(1)}(G) = L_H(G)$ and $\Lambda_1(G) = \mathbb{S}^n$. However, when $\kappa < 1$, the limit sets $L_r^{(\kappa)}(G)$ and $\Lambda_\kappa(G)$ are contained in $L_h(G)$ as the following proposition asserts.

Proposition 3.3. *For any Kleinian group G we have*

$$\bigcup_{0 \leq \kappa < 1} L_r^{(\kappa)}(G) \subset L_h(G) \subset L_r^{(1)}(G) = L_H(G).$$

Proof. By the relationship between $L_r^{(\kappa)}(G)$ and $\Lambda_\kappa(G)$, we see that $\bigcup_{0 \leq \kappa < 1} L_r^{(\kappa)}(G)$ equals $\bigcup_{0 \leq \kappa < 1} \Lambda_\kappa(G)$. For any point $\xi \in \Lambda_\kappa(G)$ with $0 \leq \kappa < 1$, its definition gives

$$\liminf_{t \rightarrow \infty} \frac{\varphi_\xi(t)}{t} \leq \kappa < 1.$$

Since $\varphi_\xi(t)$ is Lipschitz continuous, this implies in particular that

$$\lim_{t \rightarrow \infty} (\varphi_\xi(t) - t) = -\infty.$$

By Proposition 3.1, we conclude that $\xi \in L_h(G)$. □

The statement of Proposition 3.3 goes to show that the difference between $L_h(G)$ and $L_H(G)$ is so small that it cannot be detected by the stratification of the limit set given by the family of sets $L_r^{(\kappa)}(G)$, $\kappa > 0$.

4. DYNAMICAL CHARACTERISATION OF THE BIG HOROSPHERIC LIMIT SET

In the previous section we have seen a dynamical characterisation of horospheric limit points $\xi \in L_h(G)$ in terms of the distance function $\varphi_\xi(t)$ along the geodesic ray towards ξ . The corresponding result for a big horospheric limit point $\xi \in L_H(G)$ does not have such a neat form, but we can think of the following claim for $L_H(G)$ as having a similar flavour as in the case of $L_h(G)$.

For a geodesic ray $\tilde{g}_{\xi,z}$ in \mathbb{B}^{n+1} , the *Busemann function* $b(x)$ for $x \in \mathbb{B}^{n+1}$ is defined by

$$b(x) := \lim_{t \rightarrow \infty} \{d(x, \tilde{g}_{\xi,z}(t)) - t\}.$$

Note that the limit always exists since the function taken the limit is bounded from below and decreasing. A horosphere tangent at ξ is a level set of $b(x)$. For instance, the horosphere passing through z is given by $\{x \in \mathbb{B}^{n+1} \mid b(x) = 0\}$.

Proposition 4.1. *For a Kleinian group G and fixed $z \in \mathbb{B}^{n+1}$, the point $\xi \in \mathbb{S}^n$ belongs to $L_H(G)$ if and only if there exists a sequence $0 = t_0 < t_1 < t_2 < \dots$ converging to infinity and a constant M such that for each $n \in \mathbb{N}$ there is a geodesic segment β_n connecting $g_{\xi,z}(0)$ and $g_{\xi,z}(t_n)$ of length not greater than $t_n + M$ such that the closed curves in the family $g_{\xi,z}([0, t_n]) \cup \beta_n$, $n \geq 0$, are mutually freely non-homotopic to each other.*

Proof. Assume that $\xi \in L_H(G)$. Then there is a horosphere H and a sequence (γ_n) in G such that the $\gamma_n(z)$ are inside of the horoball bounded by H and converge to ξ as $n \rightarrow \infty$. We write $H = \{x \in \mathbb{B}^{n+1} \mid b(x) = M'\}$ for some M' by using the Busemann function b for $\tilde{g}_{\xi,z}$. Set $M = M' + \varepsilon$ for some $\varepsilon > 0$. By the definition of the Busemann function, we can find $t_n > 0$ for each n such that

$$d(\gamma_n(z), \tilde{g}_{\xi,z}(t_n)) - t_n \leq M.$$

By taking the projection of the geodesic connecting $\gamma_n(z)$ and $\tilde{g}_{\xi,z}(t_n)$ to M_G as β_n , we see that this family of closed curves satisfies the required condition.

Conversely, if we have such a sequence of geodesics β_n on M_G , we lift them to \mathbb{B}^{n+1} so that, for each n , one end point is $\tilde{g}_{\xi,z}(t_n)$ on a fixed geodesic ray $\tilde{g}_{\xi,z}$ towards ξ . Then the other end point of the lift of β_n is an orbit of z by G , which lies inside of the horosphere tangent at ξ given by $b(x) = M$. Hence we have that $\xi \in L_H(G)$. \square

As an application of this claim, we can easily explain the result in [17, Th.6] corresponding to Theorem 5.1, which asserts the inclusion relation $L_r(G) \subset L_H(N)$ for a non-trivial normal subgroup N of a non-elementary Kleinian group G . Indeed, for $\xi \in L_r(G)$, choose a geodesic ray $\hat{g}_{\xi,z}$ in M_G that returns infinitely often to some bounded neighbourhood of the initial point. Then its lift $g_{\xi,z}$ to M_N travels within a bounded distance along preimages under the covering transformation $M_N \rightarrow M_G$ of some fixed closed geodesic. If we make a detour at one of these closed geodesics, we can find a geodesic β_n as in Proposition 4.1. We can do this infinitely often in any tail of the geodesic ray, and so $\xi \in L_H(N)$.

By a similar argument, we have the following consequence from Proposition 4.1.

Proposition 4.2. *Let G be a Kleinian group such that the convex core $C(M_G)$ of $M_G = \mathbb{B}^{n+1}/G$ has bounded geometry, that is, there is a constant $M > 0$ such that the injectivity radius at every point of $C(M_G)$ is bounded by M . Then every limit point $\xi \in L(G)$ belongs to $L_H(G)$.*

It was remarked in [11, Prop.5.1] that any G -invariant conformal measure μ does not have an atom at $\xi \in L_H(G)$ unless ξ is a parabolic fixed point. In particular, from the above proposition, we see that a non-parabolic ending measure for G has no atom if the convex core $C(M_G)$ has bounded geometry.

5. THE MYRBERG LIMIT SET IS CONTAINED IN THE HOROSPHERIC LIMIT SET OF A NORMAL SUBGROUP

In this section we answer Tukia's question formulated in the introduction and the beginning of Section 3 about measuring the difference between the big horospheric and the horospheric limit sets for normal subgroups of Kleinian groups of divergence type, and also give a conjecture generalising the statement in this case. The first theorem is the main step towards the answer and is interesting in itself. Before stating it, we need a few preparations.

A point $\xi \in L(G)$ is a *Myrberg limit point* of G if for any distinct limit points $x, y \in L(G)$ and for any geodesic ray β towards ξ there is a sequence $\{g_n\} \subset G$ such that $g_n(\beta)$ converges to the geodesic line connecting x and y . The set of all Myrberg limit points of G is called the *Myrberg limit set* and is denoted by $L_M(G)$. The idea originated in [18] and was introduced as a qualitative version of ergodicity as characterised by Birkhoff's ergodic theorem. Further developments can be found in [34], [19] and [2], and state-of-the-art papers are [35], [27] and [8]. The main result in the latter three papers is that a Kleinian group being of divergence type is equivalent to its Myrberg limit set having full Patterson measure. All geometrically finite Kleinian groups thus have their Myrberg limit set of full measure.

We can also define the Myrberg limit set by using the geodesic flow on T^1M_G . Denote the closed subset of unit tangent vectors that generate geodesic lines staying in the convex core by

$$VC = \{(v, p) \in T^1M_G \mid g_{v,p}(t) \in C(M_G) \text{ for all } t \in \mathbb{R}\}.$$

Then $\xi \in L_M(G)$ if and only if, for $p_0 \in C(M_G)$ and v_ξ being the projection of the tangent vector pointing towards ξ based at some lift of p_0 , the forward orbit $\{\phi_t(v_\xi, p_0) \mid t \in \mathbb{R}\}$ of (v_ξ, p_0) under the geodesic flow contains unit tangent vectors that are arbitrarily close to any element of VC .

We consider normal subgroups N of a Kleinian group G and how properties of limit sets are inherited from G to N . In [17, Th.6], we have seen the inclusion relation $L_r(G) \subset L_H(N)$. In the present paper, as a refinement of this argument, we prove the following theorem.

Theorem 5.1. *Let N be a non-trivial normal subgroup of the Kleinian group G . Then,*

$$L_M(G) \subset L_h(N).$$

We first give a geometric explanation in the manifolds. Since N is a non-trivial normal subgroup of G , it is non-elementary and hence it contains a loxodromic element h . Let c be the closed geodesic in M_N corresponding to h and \hat{c} the projection of c under the normal covering $M_N \rightarrow M_G$, which may be a multi-curve. Take any Myrberg limit point $\xi \in L_M(G)$ and a geodesic ray in \mathbb{B}^{n+1} starting from $z \in \mathbb{B}^{n+1}$ and towards $\xi \in \mathbb{S}^n$. Then its projection to M_G follows \hat{c} infinitely many times within an arbitrarily small tubular neighbourhood.

We consider a lift of this geodesic ray to M_N , which is denoted by $g_{\xi,z}(t)$ ($t \geq 0$). This goes along some of the images of c under the covering transformations of

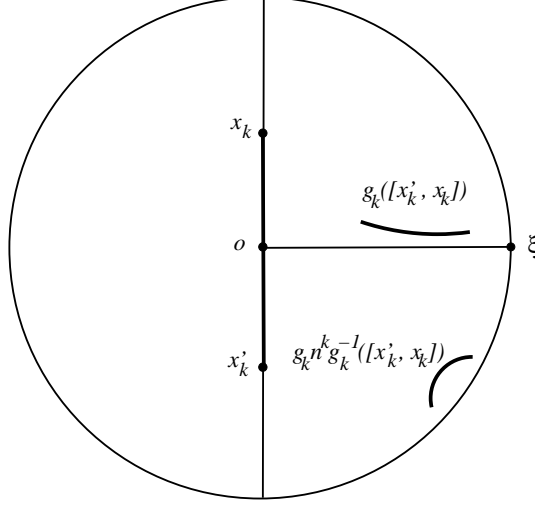


FIGURE 1. The setting of Theorem 5.1.

$M_N \rightarrow M_G$ infinitely many times within arbitrarily small tubular neighbourhoods. This implies that once $g_{\xi,z}(t)$ turns around a copy of c , we can find a geodesic between $g_{\xi,z}(0)$ and $g_{\xi,z}(t)$ which is shorter than t by some uniform length $\ell > 0$. Namely, $\varphi_\xi(t) = d(g_{\xi,z}(t), g_{\xi,z}(0))$ satisfies $\varphi_\xi(t) \leq t - \ell$ at the time t when we finish one round. However, since such detours occur infinitely many times, we have that $\lim_{t \rightarrow \infty} (\varphi_\xi(t) - t) = -\infty$. By Proposition 3.1, this shows that $\xi \in L_h(N)$.

Proof of Theorem 5.1. Consider some arbitrary $\xi \in L_M(G)$. G is assumed to be non-elementary and N to be non-trivial, so N will always contain hyperbolic elements. Let $n \in N$ be one of these and consider the uniquely determined point o on its axis so that the geodesic ray $[o, \xi]$ from o to ξ is orthogonal on the axis of n . For all $k \in \mathbb{N}$ put $x_k := n^k(o)$ and $x'_k := n^{-k}(o)$ and denote the geodesic segment connecting x'_k and x_k by $[x'_k, x_k]$.

Given some arbitrary but fixed $\varepsilon > 0$, we have by the Myrberg property of ξ that there is a sequence $(g_k)_{k \in \mathbb{N}}$ of elements of G so that $g_k(o)$ tends to ξ in the Euclidean metric and, for all $k \in \mathbb{N}$, the geodesic segment $g_k([x'_k, x_k])$ is ε -close to $[o, \xi]$, meaning that any point on $g_k([x'_k, x_k])$ is within distance ε from the geodesic ray $[o, \xi]$.

Since N is normal in G , we know that $g_k n^k g_k^{-1}, g_k n^{-k} g_k^{-1} \in N$ for all $k \in \mathbb{N}$. A priori, it is not clear how $g_k([x'_k, x_k])$ is ‘oriented’ with respect to $[o, \xi]$, but we shall see that this is not important for the argument. For a given $k \in \mathbb{N}$, assuming $g_k(x'_k)$ is closer to o than $g_k(x_k)$, we have that

$$\begin{aligned} d(g_k n^k g_k^{-1}(x_k), g_k(o)) &= d(n^k g_k^{-1}(x_k), o) = d(n^k g_k^{-1}(x_k), n^k(x'_k)) \\ &= d(g_k^{-1}(x_k), x'_k) = d(x_k, g_k(x'_k)) \\ &\asymp_+ d(o, g_k(o)) \end{aligned}$$

and that

$$\begin{aligned} d(g_k n^k g_k^{-1}(x'_k), g_k(x_k)) &= d(n^k g_k^{-1}(x'_k), x_k) = d(n^k g_k^{-1}(x'_k), n^k(o)) \\ &= d(g_k^{-1}(x'_k), o) = d(x'_k, g_k(o)) \\ &\asymp_+ d(o, g_k(x_k)). \end{aligned}$$

Here, the additive comparabilities \asymp_+ , which mean that the difference between the comparable distances is uniformly bounded independently of k , are due to the fact that $[o, \xi]$ is orthogonal on the axis of n and $g_k([x'_k, x_k])$ is ε -close to $[o, \xi]$. If $g_k(x_k)$ is closer to o than $g_k(x'_k)$, then the same argument as above yields that

$$d(g_k n^{-k} g_k^{-1}(x'_k), g_k(o)) \asymp_+ d(o, g_k(o)); \quad d(g_k n^{-k} g_k^{-1}(x_k), g_k(x'_k)) \asymp_+ d(o, g_k(x'_k)).$$

From these estimates, elementary hyperbolic geometry shows that there is some horosphere H tangent at ξ such that either both $g_k n^k g_k^{-1}(x_k)$ and $g_k n^k g_k^{-1}(x'_k)$ or both $g_k n^{-k} g_k^{-1}(x_k)$ and $g_k n^{-k} g_k^{-1}(x'_k)$ lie inside of H for all k . Since

$$d(g_k n^k g_k^{-1}(x_k), g_k n^k g_k^{-1}(x'_k)) = d(g_k n^{-k} g_k^{-1}(x_k), g_k n^{-k} g_k^{-1}(x'_k)) = d(x_k, x'_k) \rightarrow \infty$$

as $k \rightarrow \infty$, the mid points $g_k n^k g_k^{-1}(o)$ or $g_k n^{-k} g_k^{-1}(o)$ of the geodesic segments enter smaller and smaller horospheres tangent at ξ . This can be easily seen if we use the upper half-space model of the hyperbolic space with ξ being infinity. Hence we have that $\xi \in L_h(N)$. \square

As a direct corollary to Theorem 5.1, we obtain that the difference between the big horospheric and the horospheric limit sets of N is a null set for the Patterson measure μ for G when G is of divergence type. Also, since N is normal in G , we have that $L(N) = L(G)$ and that μ is a conformal measure of dimension $\delta(G)$ for N as well. Note that $\delta(N)$ may very well be strictly smaller than $\delta(G)$, in particular, when G is convex cocompact and G/N is non-amenable ([7]). Concerning the investigation of this phenomenon in view of the dimension gap between $L(N)$ and $L_r(N)$, see for instance [10] or the survey [9].

Theorem 5.2. *Let N be a non-trivial normal subgroup of the Kleinian group G , and assume that G is of divergence type. Then,*

$$\mu^{\delta(G)}(L_H(N) \setminus L_h(N)) = 0$$

for some N -invariant conformal measure $\mu^{\delta(G)}$ of dimension $\delta(G)$.

The following class of examples illustrates the statements of Theorem 5.1 and Theorem 5.2 in a non-trivial way.

Example 5.3. Let G_0 and G_1 be Schottky groups with fundamental domains having disjoint complements in hyperbolic space, and define $G := G_0 * G_1$ which is then also a Schottky group. Put $N := \ker(\varphi)$, where $\varphi : G \rightarrow G_1$ is the canonical group homomorphism. Thus, $0 \rightarrow N \rightarrow G \rightarrow G_1 \rightarrow 0$ is a short exact sequence, N is the normal subgroup of G generated by G_0 in G , and $G/N \cong G_1$. Clearly, N is infinitely generated and if we assume that G_1 is freely generated by at least two generators, and is thus non-amenable, then the already mentioned result of Brooks [7] ensures that $\delta(N) < \delta(G)$. For more details on this class of examples see also [12]. Theorem 5.1 now applies and we thus have that $L_M(G) \subset L_h(N)$. We also know on the one hand that in this situation the Hausdorff dimension of $L_M(G)$ coincides with $\delta(G)$ since G is cocompact and thus $L_M(G)$ is of full measure w.r.t. the Patterson measure of G ([35], [27] and [8]) which is known [28] to be proportional

to the $\delta(G)$ -dimensional Hausdorff measure on $L(G)$. On the other hand, we know by [5] that $\dim_H(L_r(N)) = \delta(N)$. We thus know that $L_r(N)$ has strictly smaller Hausdorff dimension than both $L_M(G)$ and $L_h(N)$ which makes the statements of both Theorem 5.1 and Theorem 5.2 meaningful and non-trivial.

Concerning the difference between the big and the small horospheric limit sets in Theorem 5.2, we are considering the situation where N is contained as a normal subgroup in some Kleinian group G , and the limit sets are measured by a conformal measure for G . However, it is desirable to describe the difference between these limit sets only by using the Kleinian group in question itself. Here is an idea how to do this, which makes use of our previous work [10].

We call a discrete G -invariant set $X = \{x_i\}_{i=1}^\infty$ in the convex hull $H(L(G))$ of $L(G)$ *uniformly distributed* if the following two conditions are satisfied:

- (i) There exists a constant $M < \infty$ such that, for every point $z \in H(L(G))$, there is some $x_i \in X$ such that $d(x_i, z) \leq M$;
- (ii) There exists a constant $m > 0$ such that, any distinct points x_i and x_j in X satisfy $d(x_i, x_j) \geq m$.

For a uniformly distributed set X , we define the *extended Poincaré series* with exponent $s > 0$ and reference point $z \in \mathbb{B}^{n+1}$ by

$$P^s(X, z) := \sum_{x \in X} e^{-s d(x, z)}.$$

The *critical exponent* for X is

$$\Delta := \inf\{s > 0 \mid P^s(X, z) < \infty\}.$$

The Poincaré series for X is of *convergence type* if $P^\Delta(X, z) < \infty$, and of *divergence type* otherwise. Moreover, we can define the associated Patterson measure μ_X for X supported on $L(G)$ by a similar construction to the usual case.

As a sufficient condition for the extended Poincaré series $P^s(X, z)$ to be of divergence type, we have the following. A uniformly distributed set X is of *bounded type* if there exists a constant $\rho \geq 1$ such that

$$\frac{\#(X \cap B_R(x))}{\#(X \cap B_R(z))} \leq \rho$$

for every $x \in X$ and for every $R > 0$. In this case, the Δ -dimensional Hausdorff measure of $L(G)$ is positive and $P^\Delta(X, z) = \infty$. For more details see [10].

In view of these similarities to the case where our group in question is a normal subgroup of some Kleinian group of divergence type, we give the following conjecture in analogy to Theorem 5.2.

Conjecture 1. *If G is a Kleinian group whose convex hull $H(L(G))$ admits a uniformly distributed set X so that the extended Poincaré series $P^s(X, z)$ is of divergence type, then*

$$\mu_X(L_H(G) \setminus L_h(G)) = 0$$

for the associated Patterson measure μ_X .

If $L(G) = \mathbb{S}^n$, then we can choose the Lebesgue measure on \mathbb{S}^n as μ_X . In this case, the original result of Sullivan [29] on Garnett points supports the conjecture.

6. THE HAUSDORFF DIMENSION OF THE MYRBERG LIMIT SET

In this section we justify why in the previous section we considered conformal measures of dimension $\delta(G)$ in order to measure the difference between the big horospheric and horospheric limit sets of N . Namely, we will show under a certain assumption that the Hausdorff dimension of $L_M(G)$, and thus, in view of Theorem 5.1, of both $L_h(N)$ and $L_H(N)$, is equal to $\delta(G)$.

Proposition 6.1. *If G is a Kleinian group of divergence type such that the strong sublinear growth limit set $\Lambda_*(G)$ has full measure for the Patterson measure μ of G , then*

$$\dim_H(L_M(G)) = \delta(G).$$

All assumptions on G follow from the condition $\mu_(T_1M_G) < \infty$.*

The method of proof is a generalisation of the argument for the radial limit set given in Sullivan [28]. (See also Nicholls [20, Th.9.3.5].) Note that Sullivan already conjectures in the original paper that for any divergence type group G the strong sublinear growth limit set $\Lambda_*(G)$ is of full measure for the Patterson measure of G . That is, Proposition 6.1 should be valid without the extra assumption on $\Lambda_*(G)$. One can justifiably ask why this is not clear for divergence type groups in general. The reason is that for geometrically infinite groups G , the radial limit set $L_r(G)$ is not necessarily contained in $\Lambda_*(G)$ as the following example shows. However, we expect that $L_M(G) \subset \Lambda_*(G)$ should be true. This can still be regarded as a generalisation of Sullivan's conjecture.

Example 6.2. Let T be a once-punctured torus, and let a and b be simple closed geodesics on T whose intersection number is 1. We cut open T along a to obtain a bordered surface P with one puncture and two geodesic boundary components.

We prepare infinitely many copies of P and paste them one after another along the geodesic boundary components without a twist. The resulting surface is a cyclic cover of T , which is denoted by R . Let $\langle h \rangle$ be the covering transformation group. The lift of b to R , which is a geodesic line invariant under $\langle h \rangle$, is denoted by \tilde{b} . We also take a simple closed geodesic a_0 that is a component of the lift of a and set the intersection of \tilde{b} and a_0 as a base point o . Set $a_n = h^n(a_0)$ and $o_n = h^n(o)$ for every integer n .

We consider the following infinite curve starting at o :

$$\beta_0 = \prod_{k=0}^{\infty} (\tilde{b}[o, o_{(-2)^k}] \cdot a_{(-2)^k} \cdot \tilde{b}[o_{(-2)^k}, o]).$$

Here, $\tilde{b}[x, y]$ denotes the segment in \tilde{b} from x to y . Then, we take the geodesic ray $\beta : [0, \infty) \rightarrow R$ starting from o and going to infinity navigated homotopically by β_0 . Since β returns infinitely many times to some neighbourhood of o , the end point of β gives a radial limit point $\xi \in L_r(G)$ of a Fuchsian group G uniformising R .

On the other hand, ξ does not belong to the strong sublinear growth limit set $\Lambda_*(G)$ of G . To see this, for every $n \geq 1$, let $t_n > 0$ be the arc length parameter of the geodesic ray $\beta = g_{\xi, o}$ such that $\beta(t)$ crosses over $a_{(-2)^n - (-1)^n}$ for the first time. We denote the hyperbolic length by $\ell(\cdot)$ and the hyperbolic distance by $d(\cdot, \cdot)$.

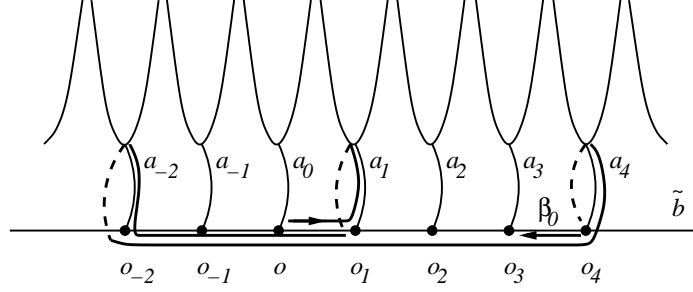


FIGURE 2. The setting of Example 6.2.

Then, we have that

$$\begin{aligned} t_n &\leq \sum_{k=0}^{n-1} \{2\ell(\tilde{b}[o, o_{(-2)^k}]) + \ell(a)\} + \ell(\tilde{b}[o, o_{(-2)^n}]) + \ell(a) \\ &\leq 3 \cdot 2^n \ell(b) + (n+1)\ell(a) \\ &< 3 \cdot 2^n \{\ell(b) + \ell(a)\}. \end{aligned}$$

However, since $d(o, \beta(t_n)) \geq (2^n - 1)d(a_0, a_1)$, we obtain that

$$\frac{d(o, \beta(t_n))}{t_n} > \frac{1}{4} \frac{d(a_0, a_1)}{\ell(b) + \ell(a)}$$

for every $n \geq 1$. This implies that

$$\limsup_{t \rightarrow \infty} \frac{\varphi_\xi(t)}{t} > 0,$$

and hence $\xi \notin \Lambda_*(G)$.

Proof of Proposition 6.1. It is proved in [35], [27] and [8] that for a Kleinian group G of divergence type, the Myrberg limit set $L_M(G)$ has full measure w.r.t. the Patterson measure μ of G . Moreover, by assumption, the sublinear growth limit set $\Lambda_*(G)$ has full μ -measure.

Following the argument from [20, Lemma 9.3.4], we can find a compact subset K of $L_M(G) \cap \Lambda_*(G)$ with $\mu(K) > 0$ satisfying the following property: for any $\varepsilon > 0$ there exists $r_0 > 0$ such that if $\xi \in K$ and $r < r_0$ then

$$\mu(B(\xi, r) \cap K) / r^{\delta(G) - \varepsilon} < A,$$

where A is some absolute constant. From this property, it follows that K has positive $(\delta(G) - \varepsilon)$ -dimensional Hausdorff measure for any $\varepsilon > 0$ (see [20, Th.9.3.5]). Hence $\dim_H L_M(G) \geq \delta(G)$. The converse inequality is clear from $\dim_H L_r(G) = \delta(G)$ and $L_M(G) \subset L_r(G)$, and thus the first statement follows.

To verify the latter statement, it suffices to remark that $\mu_*(T_1 M_G) < \infty$ implies that G is of divergence type and that $\Lambda_*(G)$ has full μ -measure (see [28, Cor.19], [20, Lemma 9.3.3]). \square

Example 6.3. Here it is interesting to mention a class of non-trivial examples for which the statement of Proposition 6.1 applies. In [23] Peigné constructs geometrically infinite Schottky groups G of divergence type which at the same time satisfy

$\mu_*(T_1M_G) < \infty$. For more details we refer the interested reader to the original article [23].

Following a completely different idea of proof, it may be possible to generalise the argument in Bishop and Jones [5] showing that the Hausdorff dimension of the radial limit set coincides with the critical exponent, in order to prove the following conjecture.

Conjecture 2. *For any non-elementary Kleinian group G we have*

$$\dim_H(L_M(G)) = \delta(G).$$

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