Some properties of various graphs associated with finite groups

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Abstract

In this paper we have investigated some properties of the power graph and commuting graph associated with a finite group, using their tree-numbers. Among other things, it has been shown that the simple group $L_2(7)$ can be characterized through the tree-number of its power graph. Moreover, the classification of groups with power-free decomposition is presented. Finally, we have obtained an explicit formula concerning the tree-number of commuting graphs associated with the Suzuki simple groups.

Keywords: Power graph, commuting graph, tree-number, simple group.

1 Notation and Definitions

We will consider finite undirected simple graphs $\Gamma = (V_\Gamma, E_\Gamma)$, where $V_\Gamma \neq \emptyset$ and E_Γ are the vertex set and edge set of Γ, respectively. A *clique* (or a *complete set*) in Γ is a subset of V_Γ consisting of pairwise adjacent vertices (we do not

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require that it be a maximal complete set). Especially, a complete graph is a graph in which the vertex set is a complete set. A *coclique* (*edgeless graph* or *independent set*) in Γ is a set of pairwise nonadjacent vertices. The *independence number*, denoted by $\alpha(\Gamma)$, is the size of the largest coclique in Γ.

A spanning tree for a graph Γ is a subgraph of Γ which is a tree and contains all the vertices of Γ. The *tree-number* (or *complexity*) of a graph Γ, denoted by $\kappa(\Gamma)$, is the number of spanning trees of Γ (0 if Γ is disconnected), see [18]. The famous Cayley formula shows that the complexity of the complete graph with n vertices is given by n^{n-2} (Cayley's formula).

In this paper, we shall be concerned with some graphs arising from *finite* groups. Two well known graphs associated with groups are commuting graphs and power graphs, as defined more precisely below. Let G be a finite group and X a nonempty subset of G:

- The *power graph* $\mathcal{P}(G, X)$, has X as its vertex set with two distinct elements of X joined by an edge when one is a *power* of the other.
- The *commuting graph* $\mathcal{C}(G, X)$, has X as its vertex set with two distinct elements of X joined by an edge when they *commute* in G.

Clearly, power graph $\mathcal{P}(G, X)$ is a subgraph of $\mathcal{C}(G, X)$. In the case $X = G$, we will simply write $\mathcal{C}(G)$ and $\mathcal{P}(G)$ instead of $\mathcal{C}(G, G)$ and $\mathcal{P}(G, G)$, respectively. Power and commuting graphs have been considered in the literature, see for instance [1, 4, 5, 8, 13, 17]. In particular, in [13, Lemma 4.1], it is shown that $\mathcal{P}(G) = \mathcal{C}(G)$ if and only if G is a cyclic group of prime power order, or a generalized quaternion 2-group, or a Frobenius group with kernel a cyclic p-group and complement a cyclic q-group, where p and q are distinct primes. Obviously, when $1 \in X$, the power graph $\mathcal{P}(G, X)$ and the commuting graph $\mathcal{C}(G, X)$ are connected, and we can talk about the complexity of these graphs. For convenience, we put $\kappa_{\mathcal{P}}(G, X) = \kappa(\mathcal{P}(G, X))$, $\kappa_{\mathcal{P}}(G) = \kappa(\mathcal{P}(G))$, $\kappa_{\mathcal{C}}(G, X) = \kappa(\mathcal{C}(G, X))$ and $\kappa_{\mathcal{C}}(G) = \kappa(\mathcal{C}(G))$. Also, instead of $\kappa_{\mathcal{P}}(G, X)$ and $\kappa_{\mathcal{C}}(G, X)$, we simply write $\kappa_{\mathcal{P}}(X)$ and $\kappa_{\mathcal{C}}(X)$, if it does not lead to confusion. All groups under discussion in this paper are finite and our group theoretic notation is mostly standard and follows that in [7].

2 General Lemmas

We first establish some notation which will be used repeatedly in the sequel. Given a graph Γ, we denote by $\mathbf{A}_\Gamma$ and $\mathbf{D}_\Gamma$ the adjacency matrix and the diagonal matrix of vertex degrees of Γ, respectively. The Laplacian matrix of G is defined as $\mathbf{L}_\Gamma = \mathbf{D}_\Gamma - \mathbf{A}_\Gamma$. Clearly, $\mathbf{L}_\Gamma$ is a real symmetric matrix and its eigenvalues are nonnegative real numbers. The Laplacian spectrum of Γ is

$$\mathrm{Spec}(\mathbf{L}_\Gamma) = (\mu_1(\Gamma), \mu_2(\Gamma), \ldots, \mu_n(\Gamma)),$$

where $\mu_1(\Gamma) \geqslant \mu_2(\Gamma) \geqslant \cdots \geqslant \mu_n(\Gamma)$, are the eigenvalues of L_Γ arranged in weakly decreasing order, and $n = |V(\Gamma)|$. Note that, $\mu_n(\Gamma)$ is 0, because each row sum of $\mathbf{L}_\Gamma$ is 0. Instead of $\mathbf{A}_\Gamma$, $\mathbf{L}_\Gamma$, and $\mu_i(\Gamma)$ we simply write $\mathbf{A}$, $\mathbf{L}$, and μ_i if it does not lead to confusion.

For a graph with n vertices and Laplacian spectrum $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n = 0$ it has been proved [2, Corollary 6.5] that:

$$\kappa(\Gamma) = \frac{\mu_1\mu_2\cdots\mu_{n-1}}{n}. \qquad (1)$$

The vertex–disjoint union of the graphs Γ_1 and Γ_2 is denoted by $\Gamma_1 \oplus \Gamma_2$. Define the *join* of Γ_1 and Γ_2 to be $\Gamma_1 \vee \Gamma_2 = (\Gamma_1^c \oplus \Gamma_2^c)^c$. Evidently this is the graph formed from the vertex–disjoint union of the two graphs Γ_1, Γ_2, by adding edges joining every vertex of Γ_1 to every vertex of Γ_2. Now, one may easily prove the following (see also [15]).

Lemma 2.1 *Let Γ_1 and Γ_2 be two graphs on disjoint sets with m and n vertices, respectively. If*

$$\mathrm{Spec}(\mathbf{L}_{\Gamma_1}) = (\mu_1(\Gamma_1), \mu_2(\Gamma_1), \ldots, \mu_m(\Gamma_1)),$$

and

$$\mathrm{Spec}(\mathbf{L}_{\Gamma_2}) = (\mu_1(\Gamma_2), \mu_2(\Gamma_2), \ldots, \mu_n(\Gamma_2)),$$

then the following hold:

(1) *The eigenvalues of Laplacian matrix $\mathbf{L}_{\Gamma_1 \oplus \Gamma_2}$ are:*

$$\mu_1(\Gamma_1), \ \ldots, \ \mu_m(\Gamma_1), \ \mu_1(\Gamma_2), \ \ldots, \ \mu_n(\Gamma_2).$$

(2) *The eigenvalues of Laplacian matrix $\mathbf{L}_{\Gamma_1 \vee \Gamma_2}$ are:*

$$m+n, \ \mu_1(\Gamma_1)+n, \ \ldots, \ \mu_{m-1}(\Gamma_1)+n, \ \mu_1(\Gamma_2)+m, \ \ldots, \ \mu_{n-1}(\Gamma_2)+m, \ 0.$$

Two Examples. (1) Consider the complete bipartite graph $K_{a,b} = K_a^c \vee K_b^c$. Then, by Lemma 2.1 (2), the eigenvalues of Laplacian matrix $\mathbf{L}_{K_a^c \vee K_b^c}$ are:

$$a+b, \ \underbrace{b, \ b, \ldots, \ b}_{(a-1)-\text{times}}, \underbrace{a, \ a, \ldots, \ a}_{(b-1)-\text{times}}, \ 0.$$

Using Eq. (1) we get $\kappa(K_{a,b}) = b^{a-1}a^{b-1}$.

(2) A graph Γ is a *split graph* if its vertex set can be partitioned into a clique C and an independent set I, where $V_\Gamma = C \uplus I$ is called a *split partition* of Γ (see [9]). Now, consider the split graph $\Gamma = K_a \vee K_b^c$. Again, by Lemma 2.1 (2), the eigenvalues of Laplacian matrix $\mathbf{L}_{K_a \vee K_b^c}$ are:

$$a+b, \ \underbrace{a+b, \ a+b, \ \ldots, \ a+b}_{(a-1)-\text{times}}, \underbrace{a, \ a, \ \ldots, \ a}_{(b-1)-\text{times}}, \ 0,$$

and it follows from Eq. (1) that $\kappa(\Gamma) = (a+b)^{a-1}a^{b-1}$.

Next lemma determines the complete commuting graphs and power graphs.

Lemma 2.2 *Let G be a finite group. Then, we have*

(1) *The commuting graph $\mathcal{C}(G)$ is complete iff G is an abelian group.*

(2) *The power graph $\mathcal{P}(G)$ is complete iff G is a cyclic p-group for some prime p* (see Theorem 2.12 in [6]).

An immediate consequence of Lemma 2.2 is that $\kappa_{\mathcal{P}}(\mathbb{Z}_{p^n}) = p^{n(p^n-2)}$ and if $X \subset G$ is a commuting set, then $\kappa_{\mathcal{C}}(G, X) = |X|^{|X|-2}$. We will use these facts without further references.

Lemma 2.3 *Let M and N be subgroups of a group G such that $M \cap N = 1$. Let $x \in M$ and $y \in N$ be two arbitrary nontrivial elements. Then, x and y are nonadjacent in $\mathcal{P}(G)$ as two vertices. In particular, if $m > 1$ and $X = \cup_{j=1}^{m} G_j$, where $1 < G_j < G$ and $G_i \cap G_j = 1$ for $i \neq j$, then we have*

$$\mathcal{P}(X^{\#}) = \bigoplus_{j=1}^{m} \mathcal{P}\big(G_j^{\#}\big),$$

where $X^{\#} = X \setminus \{1\}$ and $G_j^{\#} = G_j \setminus \{1\}$.

Proof. It is easy to see that $\langle x\rangle \cap \langle y\rangle \subseteq M \cap N = 1$, which forces $\langle x\rangle \nsubseteq \langle y\rangle$ and $\langle y\rangle \nsubseteq \langle x\rangle$. $\Box$

As an immediate consequence of Lemma 2.3, we obtain

Corollary 2.4 *Let G be a finite group. Then the following hold:*

(1) *If G has even order, then* $\mathrm{Inv}(G)$, *the set of involutions of G, forms an independent set of $\mathcal{P}(G)$.*

(2) *Any pair of elements in G with relatively prime orders forms an independent set of $\mathcal{P}(G)$, especially we have $\alpha(\mathcal{P}(G)) \geqslant |\pi(G)|$, where $\pi(G)$ denotes the set of all prime divisors of $|G|$.*

A *universal* vertex is a vertex of a graph that is adjacent to all other vertices of the graph. The following lemma [5, Proposition 4] determines the set of universal vertices of the power graph of G, in general case.

Lemma 2.5 *Let G be a finite group and S the set of universal vertices of the power graph $\mathcal{P}(G)$. Suppose that $|S| > 1$. Then one of the following occurs:*

(a) *G is cyclic of prime power order, and $S = G$;*

(b) *G is cyclic of non-prime-power order n, and S consists of the identity and the generators of G, so that $|S| = 1 + \phi(n)$;*

(c) *G is generalized quaternion, and S contains the identity and the unique involution in G, so that $|S| = 2$.*

Lemma 2.6 [16, Theorem 3.4] *If $H_1, H_2, \ldots, H_t$ are nontrivial subgroups of a group G such that $H_i \cap H_j = \{1\}$, for each $1 \leqslant i < j \leqslant t$, then we have $\kappa_{\mathcal{P}}(G) \geqslant \kappa_{\mathcal{P}}(H_1)\kappa_{\mathcal{P}}(H_2)\cdots\kappa_{\mathcal{P}}(H_t)$.*

Lemma 2.7 [16, Lemma 6.1] *Let G be a finite nonabelian simple group and let p be a prime dividing the order of G. Then G has at least $p^2 - 1$ elements of order p, or equivalently, there is at least $p+1$ cyclic subgroups of order p in G.*

3 Main Results

3.1 Power Graphs

We begin with some elementary but useful results of power graphs. Before stating the results, we need to introduce some additional notation. Let X be a nonempty subset of $G^{\#}$, the set of nonidentity elements of G. We denote by 1_X the bipartite graph with partite sets $\{1\}$ and X. Let ϕ denote Euler's totient function, so that $\phi(n) = |\mathbb{Z}_n^{\times}|$. We will preserve these notation throughout this section.

Lemma 3.1 *Let G be a group and H be a proper subgroup of G. If m is the order of an element of $G \setminus H$, then we have*

$$\kappa_{\mathcal{P}}(G) \geqslant (\phi(m)+1)^{\phi(m)-1}\,\kappa_{\mathcal{P}}(H).$$

In particular, $\kappa_{\mathcal{P}}(G) \geqslant \kappa_{\mathcal{P}}(H)$, with equality if and only if G is a Frobenius group with kernel H and complement C of order 2.

Proof. Let x be an element in $G \setminus H$ of order m. Set $\Omega_x = \{y \mid \langle y\rangle = \langle x\rangle\} \cup \{1\}$. Clearly, $\Omega_x \cap H = 1$ and $|\Omega_x| = \phi(m)+1$. Note that the induced subgraph $\mathcal{P}(G,\Omega_x)$ is a complete graph, that is $\mathcal{P}(G,\Omega_x) = K_{\phi(m)+1}$. Now, if T_{Ω_x} and T_H are two arbitrary spanning trees of $\mathcal{P}(G,\Omega_x)$ and $\mathcal{P}(G,H)$, respectively, then $T_G = T_{\Omega_x} \cup T_H \cup 1_{G\setminus(\Omega_x\cup H)}$ is a spanning tree of $\mathcal{P}(G)$. Thus, by product rule the number of such spanning trees of $\mathcal{P}(G)$ is equal to

$$\kappa_{\mathcal{P}}(\Omega_x)\cdot\kappa_{\mathcal{P}}(H)\cdot 1 = (\phi(m)+1)^{\phi(m)-1}\kappa_{\mathcal{P}}(H). \quad \text{(by Cayley's formula)}$$

This shows that the following inequality holds:

$$\kappa_{\mathcal{P}}(G) \geqslant (\phi(m)+1)^{\phi(m)-1}\kappa_{\mathcal{P}}(H),$$

as required. Finally, since for each positive integer m, $(\phi(m)+1)^{\phi(m)-1} \geqslant 1$, it follows that $\kappa_{\mathcal{P}}(G) \geqslant \kappa_{\mathcal{P}}(H)$.

The preceding argument suggests how to construct a spanning tree of $\mathcal{P}(G)$ through a spanning tree of $\mathcal{P}(G,H)$. In fact, if T_H is a spanning tree of $\mathcal{P}(G,H)$, then $T_G = T_H \cup 1_{G\setminus H}$ is a spanning tree of $\mathcal{P}(G)$, which leads again to the inequality $\kappa_{\mathcal{P}}(G) \geqslant \kappa_{\mathcal{P}}(H)$. Moreover, the equality holds if and only if $\mathcal{P}(G, G\setminus H)$ is a null graph and there are no edges between vertices in $G \setminus H$ and $H^{\#}$. We argue under these conditions that G is a Frobenius group with kernel H and complement C of order 2. We first observe that every element in $G \setminus H$ is an involution. Also, for all $x \in G \setminus H$ and $h \in H$, $xh \in G \setminus H$ and so $(xh)^2 = 1$, or equivalently $x^{-1}hx = h^{-1}$. This shows that H is a normal subgroup of G and the cyclic subgroup $C = \langle x \rangle$ of order 2 acts on H by conjugation which induces a fixed-point-free automorphism of H. Hence, $G = HC$ is a Frobenius group with kernel H and complement C, as required. □

A group G from a class $\mathcal{F}$ is said to be recognizable in $\mathcal{F}$ by $\kappa_{\mathcal{P}}(G)$ (shortly, $\kappa_{\mathcal{P}}$-recognizable in $\mathcal{F}$) if every group $H \in \mathcal{F}$ with $\kappa_{\mathcal{P}}(H) = \kappa_{\mathcal{P}}(G)$ is isomorphic to G. In other words, G is $\kappa_{\mathcal{P}}$-recognizable in $\mathcal{F}$ if $h_{\mathcal{F}}(G) = 1$, where $h_{\mathcal{F}}(G)$ is the (possibly infinite) number of pairwise non-isomorphic groups $H \in \mathcal{F}$ with $\kappa_{\mathcal{P}}(H) = \kappa_{\mathcal{P}}(G)$. We denote by $\mathcal{S}$ the classes of all finite simple groups. In the sequel, we show that the simple group $L_2(7) \cong L_3(2)$ is $\kappa_{\mathcal{P}}$-recognizable group in class $\mathcal{S}$, in other words $h_{\mathcal{S}}(L_2(7)) = 1$.

Theorem 3.2 *The simple group $L_2(7)$ is $\kappa_{\mathcal{P}}$-recognizable in the class $\mathcal{S}$ of all finite simple groups, that is, $h_{\mathcal{S}}(L_2(7)) = 1$.*

Proof. Let $G \in \mathcal{S}$ with $\kappa_{\mathcal{P}}(G) = \kappa_{\mathcal{P}}(L_2(7)) = 2^{84} \cdot 3^{28} \cdot 7^{40}$ (see [12, Theorem 4.1]). We have to prove that G is isomorphic to $L_2(7)$. Clearly, G is nonabelian, since otherwise $G \cong \mathbb{Z}_p$ for some prime p, and so $\kappa_{\mathcal{P}}(G) = \kappa_{\mathcal{P}}(\mathbb{Z}_p) = p^{p-2}$, which is a contradiction. Now, we claim that $\pi(G) \subseteq \{2, 3, 5, 7\}$. Suppose $p \in \pi(G)$ and $p \geqslant 11$. Let c_p be the number of cyclic subgroups of order p in G. By Lemma 2.7, $c_p \geqslant p + 1$, because G is a nonabelian simple group. Therefore, from Lemma 2.6, we deduce that

$$\kappa_{\mathcal{P}}(G) \geqslant \kappa_{\mathcal{P}}(\mathbb{Z}_p)^{c_p} \geqslant \kappa_{\mathcal{P}}(\mathbb{Z}_p)^{p+1} = p^{(p-2)(p+1)} \geqslant 11^{108} > \kappa_{\mathcal{P}}(G),$$

which is a contradiction. This shows that $\pi(G) \subseteq \{2, 3, 5, 7\}$, as claimed.

By results collected in [21, Table 1], G is isomorphic to one of the groups $A_5 \cong L_2(4) \cong L_2(5)$, $A_6 \cong L_2(9)$, $S_4(3) \cong U_4(2)$, $L_2(7) \cong L_3(2)$, $L_2(8)$, $U_3(3)$, A_7, $L_2(49)$, $U_3(5)$, $L_3(4)$, $A_8 \cong L_4(2)$, A_9, J_2, A_{10}, $U_4(3)$, $S_4(7)$, $S_6(2)$ or $O_8^+(2)$. In all cases, except A_5 and $L_2(7)$, G contains a subgroup H which is isomorphic to A_6 (see [7]). But then, $\kappa_{\mathcal{P}}(G) \geqslant \kappa_{\mathcal{P}}(H) = 2^{180} \cdot 3^{40} \cdot 5^{108}$, a contradiction. If G is isomorphic to A_5, then $\kappa_{\mathcal{P}}(G) = 2^{20} \cdot 3^{10} \cdot 5^{18}$, which contradicts the assumption. Thus G is isomorphic to $L_2(7)$, as required. □

3.2 Power-Free Decompositions

A generalization of split graphs was introduced and investigated under the name (m, n)-graphs in [3]. A graph Γ is an (m, n)-graph if its vertex set can be partitioned into m cliques $C_1, \ldots, C_m$ and n independent sets $I_1, \ldots, I_n$. In this situation,

$$V_\Gamma = C_1 \uplus C_2 \uplus \cdots \uplus C_m \uplus I_1 \uplus I_2 \uplus \cdots \uplus I_n,$$

is called an (m, n)-split partition of Γ. Thus, (m, n)-graphs are a natural generalization of split graphs, which are precisely $(1, 1)$-graphs.

Accordingly, we are motivated to make the following definition.

Definition 3.3 Let G be a group and $n \geqslant 1$ an integer. We say that G has an *n-power-free decomposition* if it can be partitioned as a disjoint union of a cyclic p-subgroup C of maximal order and n nonempty subsets $B_1, B_2, \ldots, B_n$:

$$G = C \uplus B_1 \uplus B_2 \uplus \cdots \uplus B_n, \tag{2}$$

such that the B_i's are independent sets in $\mathcal{P}(G)$ and $|B_i| > 1$, for each i. If $n = 1$, we simply say $G = C \uplus B_1$ is a *power-free decomposition* of G.

Since C is a cyclic p-subgroup of maximal order in Definition 3.3, C is a clique, and so Eq (2) is a $(1, n)$-split partition of $\mathcal{P}(G)$. Note that, there are some finite groups which do not have an n-power-free decomposition, for any n, for example one can consider cyclic groups (see Proposition 3.5). On the other hand, the structure of groups G which have a power-free decomposition is obtained (see Theorem 3.8).

Lemma 3.4 *Suppose G has an n-power-free decomposition:*

$$G = C \uplus B_1 \uplus B_2 \uplus \cdots \uplus B_n,$$

where C is a cyclic p-subgroup of G. Then the following statements hold:

(a) *If $b \in G \setminus C$, then $\phi(o(b)) \leqslant n$. In particular, we have*

$$\pi(G) \subseteq \pi((n+1)!) \cup \{p\}.$$

(b) *If $p \notin \pi((n+1)!)$, then C is normal and $C_C(b) = 1$ for each $b \in G \setminus C$. In particular, $Z(G) = 1$.*

(c) *The set of universal vertices of $\mathcal{P}(G)$ is contained in C.*

Proof. (a) The first statement follows immediately from the fact that the set of generators of cyclic group $\langle b \rangle$, which has $\phi(o(b))$ elements, forms a complete set in the $\mathcal{P}(G, G \setminus C)$, and hence each B_i contains at most one of the generators. The second statement is also clear, because for each $q \in \pi(G) \setminus \{p\}$, there exists an element $b \in G \setminus C$ of order q, and so by first part $\phi(q) = q - 1 \leqslant n$, or $q \leqslant n + 1$.

(b) Assume the contrary. Let $C = \langle x \rangle$ with $o(x) = p^m > 1$. Then, there exists $b \in G \setminus C$ such that $x^b \notin C$. By part (a) it follows that $\phi(o(x^b)) \leqslant n$. Since $o(x^b) = o(x)$, $\phi(o(x^b)) = \phi(o(x)) = \phi(p^m) = p^{m-1}(p-1)$, and so we obtain

$$p - 1 \leqslant p^{m-1}(p-1) \leqslant n.$$

This forces $p \leqslant n + 1$, which contradicts the hypothesis.

Let $b \in G \setminus C$. Suppose c in C is not the identity and commutes with b. Replacing b by an appropriate power, we may assume without loss that p divides $o(bc)$. Thus, we conclude that $p - 1$ divides $\phi(o(bc))$. Since $bc \in G \setminus C$, by part (a) we have $\phi(o(bc)) \leqslant n$. Thus, it follows that $p - 1 \leqslant n$, which is a contradiction. This shows that $C_C(b) = 1$, as required.

(c) It is clear from Definition 3.3. □

As the following result shows that there are some examples of groups for which there does not exist any n-power-free decomposition.

Proposition 3.5 *Any cyclic group has no n-power-free decomposition.*

Proof. Assume the contrary and let $G = \langle x \rangle$ be a cyclic group with an n-power-free decomposition:

$$G = C \uplus B_1 \uplus B_2 \uplus \cdots \uplus B_n,$$

for some $n \geqslant 1$, where $C \subset G$ is a cyclic p-subgroup. Clearly, x is a universal vertex in $\mathcal{P}(G)$, and so by Lemma 3.4 (c), $x \in C$. But then $C = G$, which is a contradiction. The proof is complete. □

Proposition 3.6 *The generalized quaternion group Q_{2^n}, $n \geqslant 3$, has a 2-power-free decomposition.*

Proof. With the following presentation:

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, x^y = x^{-1} \rangle,$$

we may choose $C = \langle x \rangle$, and

$$B_1 = \{y, xy, \ldots, x^{2^{n-2}-1}y\}, \ B_2 = \{x^{2^{n-2}}y, x^{2^{n-2}+1}y, \ldots, x^{2^{n-1}-1}y\}.$$

Then $Q_{2^n} = C \uplus B_1 \uplus B_2$ is a 2-power-free decomposition, and this completes the proof. □

Given a group G, $1 \in G$ is a universal vertex of the power graph $\mathcal{P}(G)$. Now, as an immediate corollary of Lemma 2.5 and Propositions 3.5 and 3.6, we have the following.

Corollary 3.7 *Let G be a group, S the set of universal vertices of the power graph $\mathcal{P}(G)$, and $|S| > 1$. Then G has an n-power-free decomposition iff G is isomorphic to a generalized quaternion group.*

Theorem 3.8 *The following conditions on a group G are equivalent:*

(a) *G has a power-free decomposition, $G = C \uplus B$, where C is a cyclic p-subgroup of G.*

(b) *One of the following statements holds:*

  (1) *$p = 2$ and G is an elementary abelian 2-group of order $\geqslant 4$.*
  (2) *$p = 2$ and G is the dihedral group D_{2^m} of order 2^m, for some integer $m \geqslant 3$.*
  (3) *$p > 2$ and G is the dihedral group D_{2p^n} (a Frobenius group) of order $2p^n$ with a cyclic kernel of order p^n.*

Proof. (a) ⇒ (b). Suppose $G = C \uplus B$ is a power-free decomposition of G, where $C \subset G$ is a cyclic p-subgroup of maximal order. It follows by Lemma 3.4 (a) that every element $b \in B$ is an involution, and also $|G| = 2^m p^n$, for some odd prime p and $m \geqslant 1$, $n \geqslant 0$. We shall treat the cases $n = 0$ and $n \geqslant 1$, separately.

Case 1. $n = 0$. In this case, G is a 2-group. If $|C| \leqslant 2$, then G is an elementary abelian 2-group and (1) holds. We may now assume that $|C| > 2$. Put $C = \langle x \rangle$. Then, for every b in B, x^b is not an involution and so $x^b \in C$, which shows that C is a normal subgroup of G. Thus G/C is an elementary abelian 2-group by the previous paragraph.

We now claim that $[G : C] = 2$. To prove this, we assume that $[G : C] = 2^t$, where $t \geqslant 2$. Let $I = \mathrm{Inv}(G)$ be the set of involutions of G. Then, we have $I = B \cup \{z\}$, where z is the unique involution in C, and so

$$|I| = |B| + 1 = |G| - |C| + 1 = |G| - \frac{|G|}{2^t} + 1 = \left(\frac{2^t - 1}{2^t}\right)|G| + 1 \geqslant \frac{3}{4}|G| + 1,$$

which forces G to be an abelian group. We recall that, a finite group is abelian if at least $3/4$ of its elements have order two. But then, if $b \in B$, then bx is not an involution and also $bx \notin C$, which is a contradiction.

Let b be an involution in B. Then $G = \langle x, b \rangle$. Since $bx \in B$, bx is an involution, and thus $bxb = x^{-1}$, which implies that G is a dihedral group and (2) follows.

Case 2. $n \geqslant 1$. In this case, $|G| = 2^m p^n$ where $m, n \geqslant 1$, and C is a cyclic p-group of maximal order. As in previous case $C = \langle x \rangle$ is a normal subgroup of G and G/C is an elementary abelian 2-group. Note that, G does not contain an element of order $2p$, and so $C = C_G(C)$. Moreover, since

$$G/C = N_G(C)/C_G(C) \hookrightarrow \mathrm{Aut}(C),$$

and $\mathrm{Aut}(C)$ is a cyclic group of order $\phi(p^n) = p^{n-1}(p-1)$, we conclude that $|G/C| = 2$. Therefore, if b is an involution in G, then $G = \langle x, b \rangle = \langle x \rangle \rtimes \langle b \rangle$, and since b acts on $\langle x \rangle$ fixed-point-freely, G is a Frobenius group of order $2p^n$ with cyclic kernel C of order p^n, and (3) follows.

(b) ⇒ (a). Obviously. □

3.3 Commuting Graphs

In this section, we consider the problem of finding the tree-number of the commuting graphs associated with a family of finite simple groups. The Suzuki groups $\mathrm{Sz}(q)$, an infinite series of simple groups of Lie type, were defined in [19, 20] as subgroups of the groups $\mathrm{L}_4(q)$, with $q = 2^{2n+1} \geqslant 8$. In what follows, we shall give an explicit formula for $\kappa_{\mathcal{C}}(\mathrm{Sz}(q))$. Let $G = \mathrm{Sz}(q)$, where $q = 2^{2n+1}$. We begin with some well-known facts about the simple group G. These results have been obtained by Suzuki [19, 20]:

(1) Let $r = 2^{n+1}$. Then $|G| = q^2(q-1)(q^2+1) = q^2(q-1)(q-r+1)(q+r+1)$, and $\mu(G) = \{4, q-1, q-r+1, q+r+1\}$. For convenience, we write $\alpha_q = q - r + 1$ and $\beta_q = q + r + 1$.

(2) Let P be a Sylow 2-subgroup of G. Then P is a 2-group of order q^2 with $\exp(P) = 4$, which is a TI-subgroup, and $|N_G(P)| = q^2(q-1)$.

(3) Let $A \subset G$ be a cyclic subgroup of order $q-1$. Then A is a TI-subgroup and the normalizer $N_G(A)$ is a dihedral group of order $2(q-1)$.

(4) Let $B \subset G$ be a cyclic subgroup of order α_q. Then B is a TI-subgroup and the normalizer $N_G(B)$ has order $4\alpha_q$.

(5) Let $C \subset G$ be a cyclic subgroup of order β_q. Then C is a TI-subgroup and the normalizer $N_G(C)$ has order $4\beta_q$.

We recall that, in general, a subgroup $H \leqslant G$ is a *TI-subgroup* (trivial intersection subgroup) if for every $g \in G$, either $H^g = H$ or $H \cap H^g = \{1\}$.

Lemma 3.9 $\kappa_{\mathcal{C}}(P) = 2^{(q-1)^2} q^{(q^2+q-3)}$.

Proof. By Theorem VIII.7.9 of [10] and Lemma XI.11.2 of [11], $Z(P)$ is an elementary abelian 2-group of order q and every element outside $Z(P)$ has order 4. Observe that P is the centralizer in G of all of the nontrivial elements of $Z(P)$. If $x \in P \setminus Z(P)$, then $\langle Z(P), x\rangle \leqslant C_G(x)$. In the proof of Lemma XI.11.7 of [11], we see that the elements of order 4 in G lie in two conjugacy classes. This implies that $|C_G(x)| = 2|Z(P)|$, from which we deduce that $C_G(x) = \langle Z(P), x\rangle$. Then for all $x, y \in P \setminus Z(P)$ either $C_G(x) = C_G(y)$ or $C_G(x) \cap C_G(y) = Z(P)$. Hence, $\{C_G(x) | x \in P \setminus Z(P)\}$ forms a partition of P for which the intersection of pairwise centralizers is $Z(P)$. This shows that

$$\mathcal{C}(P) = K_q \vee (\underbrace{K_q \oplus K_q \oplus \cdots \oplus K_q}_{q-1}).$$

Moreover, by Lemma 2.1, the eigenvalues of Laplacian matrix $L_{\mathcal{C}(P)}$ are:

$$q^2, \underbrace{q^2,\ q^2,\ \ldots,\ q^2}_{q-1},\ \underbrace{2q,\ 2q,\ \ldots,\ 2q}_{(q-1)^2},\ \underbrace{q,\ q,\ \ldots,\ q}_{q-2},\ 0.$$

It follows immediately using Eq. (1) that

$$\kappa_{\mathcal{C}}(P) = 2^{(q-1)^2} q^{(q^2+q-3)},$$

as required. □

Theorem 3.10 *Let $q = 2^{2n+1}$, where $n \geqslant 1$ is an integer. Then, we have*

$$\kappa_{\mathcal{C}}(\mathrm{Sz}(q)) = \left(2^{(q-1)^2} q^{(q^2+q-3)}\right)^{q^2+1} (q-1)^{(q-3)a} (\alpha_q)^{(\alpha_q-2)b} (\beta_q)^{(\beta_q-2)c},$$

where $a = q^2(q^2+1)/2$, $b = q^2(q-1)\beta_q/4$ and $c = q^2(q-1)\alpha_q/4$.

Proof. Let $G = \mathrm{Sz}(q)$, where $q = 2^{2n+1} \geqslant 8$. As already mentioned, G contains a Sylow 2-subgroup P of order q^2 and cyclic subgroups A, B, and C, of orders $q-1$, α_q and β_q, respectively. Moreover, every two distinct conjugates of them intersect trivially and every element of G is a conjugate of an element in $P \cup A \cup B \cup C$. Looking at the proof of Lemma 11.6, we see that the cyclic subgroups A, B, and C, are the centralizers of their nonidentity elements, while P is the centralizer in G of all of the nontrivial elements of $Z(P)$. Let

$$\begin{aligned} G &= N_P x_1 \cup \cdots \cup N_P x_p &= N_A y_1 \cup \cdots \cup N_A y_a \\ &= N_B z_1 \cup \cdots \cup N_B z_b &= N_C t_1 \cup \cdots \cup N_C t_c, \end{aligned}$$

be coset decompositions of G by $N_P = N_G(P)$, $N_A = N_G(A)$, $N_B = N_G(B)$ and $N_C = N_G(C)$, where $p = [G : N_P] = q^2 + 1$, $a = [G : N_A] = q^2(q^2+1)/2$, $b = [G : N_B] = q^2(q-1)\beta_q/4$ and $c = [G : N_C] = q^2(q-1)\alpha_q/4$. Then, we have

$$G = P^{x_1} \cup \cdots \cup P^{x_p} \cup A^{y_1} \cup \cdots \cup A^{y_a} \cup B^{z_1} \cup \cdots \cup B^{z_b} \cup C^{t_1} \cup \cdots \cup C^{t_c}.$$

This shows that

$$\begin{aligned} \mathcal{C}(G) &= K_1 \vee \big(p\, \mathcal{C}(P^{\#}) \oplus a\, \mathcal{C}(A^{\#}) \oplus b\, \mathcal{C}(B^{\#}) \oplus c\, \mathcal{C}(C^{\#})\big) \\ &= K_1 \vee \big(p\, \mathcal{C}(P^{\#}) \oplus aK_{(q-1)-1} \oplus bK_{\alpha_q - 1} \oplus cK_{\beta_q - 1}\big)\,, \end{aligned}$$

and so

$$\kappa_{\mathcal{C}}(G) = \kappa_{\mathcal{C}}(P)^p \cdot \kappa_{\mathcal{C}}(K_{q-1})^a \cdot \kappa_{\mathcal{C}}(K_{\alpha_q})^b \cdot \kappa_{\mathcal{C}}(K_{\beta_q})^c.$$

Now, Lemma 3.9 and Cayley's formula yield the result. □

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References

[1] J. H. Abawajy, A. V. Kelarev and M. Chowdhury, Power graphs: a survey, *Electronic J. Graph Theory and Applications*, 1(2) (2013), 125–147.

[2] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, London, 1974.

[3] A. Brandstädt, Partitions of graphs into one or two independent sets and cliques, *Discrete Math.*, 152(1-3) (1996), 47–54.

[4] J. R. Britnell and N. Gill, Perfect commuting graphs, *J. Group Theory*, 20(1) (2017), 71--102.

[5] P. J. Cameron, The power graph of a finite group. II, *J. Group Theory*, 13(6) (2010), 779–783.

[6] I. Chakrabarty, S. Ghosh and M. K. Sen, Undirected power graphs of semigroups, *Semigroup Forum*, 78 (2009), 410–426.

[7] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson R A, *Atlas of Finite Groups*, Oxford Clarendon Press, 1985.

[8] A. K. Das and D. Nongsiang, On the genus of the commuting graphs of finite nonabelian groups, *Int. Electron. J. Algebra*, 19 (2016), 91–109.

[9] S. Földes and P. L. Hammer, Split graphs, *Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La.,* 1977*)*, 311–315.

[10] B. Huppert and N. Blackbrun, *Finite Groups II*, Springer-Verlag, Berlin, 1982.

[11] B. Huppert and N. Blackbrun, *Finite Groups III*, Springer-Verlag, Berlin, 1982.

[12] S. Kirkland, A. R. Moghaddamfar, S. Navid Salehy, S. Nima Salehy and M. Zohourattar, The complexity of power graphs associated with finite groups, to appear in Contributions to Discrete Mathematics.

[13] A. Mahmoudifar and A. R. Moghaddamfar, Commuting graphs of groups and related numerical parameters, *Comm. Algebra*, 45(7)(2017), 3159–3165.

[14] A. Mahmoudifar, A. R. Moghaddamfar and F. Salehzadeh, Tree-numbers of commuting graphs associated with finite groups, Submitted for publication.

[15] R. Merris, Laplacian graph eigenvectors, *Linear Algebra Appl.*, 278 (1998), 221–236.

[16] A. R. Moghaddamfar, S. Rahbariyan, S. Navid Salehy and S. Nima Salehy, The number of spanning trees of power graphs associated with specific groups and some applications, *Ars Combinatoria*, 2017.

[17] A. R. Moghaddamfar, S. Rahbariyan, and W. J. Shi, Certain properties of the power graph associated with a finite group, *J. Algebra Appl.*, 13(7) (2014), 1450040, 18 pp.

[18] H. Sachs, On the number of spanning trees, *Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975)*, pp. 529–535. Congressus Numerantium, No. XV, Utilitas Math., Winnipeg, Man., 1976.

[19] M. Suzuki, A new type of simple groups of finite order, *Proc. Nat. Acad. Sci. U.S.A.*, 46 (1960), 868–870.

[20] M. Suzuki, On a class of doubly transitive groups, *Ann. of Math.*, 75 (1) (1962), 105–145.

[21] A. V. Zavarnitsine, Finite simple groups with narrow prime spectrum, *Sib. Elektron. Mat. Izv.*, 6 (2009), 1–12.