Comodules, contramodules and Pontryagin duality

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1 Introduction

Let k be a field, treated as a discrete topological ring. Whenever V is a topological vector space, denote by V^* the vector space $\operatorname{Hom}_k(V, k)$ of continuous linear maps from V to k, given the compact-open topology. Tensor products are taken over k if not otherwise indicated.

Let C be a k-coalgebra [3, §1.1]. In [8], Simson showed that the topological vector space C^* inherits the structure of a pseudocompact algebra and that the contravariant functor $(-)^* = \text{Hom}_k(-, k)$ yields a duality between the category of coalgebras and coalgebra homomorphisms, and the category of pseudocompact algebras and continuous algebra homomorphisms [8, Theorem 3.6]. Consider the following categories:

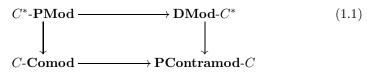
- C-Comod the category of (discrete) left C-comodules and comodule homomorphisms [3, §2.1]. By the fundamental theorem of comodules [3, Theorem 2.1.7], every object of this category is a direct limit of finite dimensional comodules.
- C*-**PMod** the category of pseudocompact left C*-modules and continuous module homomorphisms. Such modules are inverse limits of finite dimensional discrete left C*-modules [2, §1].
- **DMod**-*C*^{*} the category of discrete right *C*^{*}-modules and module homomorphisms. Such modules are direct limits of finite dimensional right *C*^{*}-modules [2, §1].

In the following diagram each arrow is a duality. The image of the object X under either arrow (or its inverse) is X^* as a vector space, imbued with an appropriate additional structure:



The horizontal arrow is an instance of the famous Pontryagin duality [2, Proposition 2.3]. The vertical arrow is due to Simson [8, Theorem 4.3].

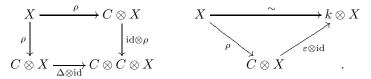
The notion of a contramodule for the coalgebra C was first introduced by Eilenberg and Moore [4, §III.5]. Although they have received far less attention than comodules, they have been further studied by Positselski ([6], for instance) and others ([1, 9, 11], for instance). In this note we define the category **PContramod**-C of pseudocompact right C-contramodules and show that we can complete the diagram above as follows



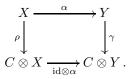
with each arrow or its inverse a duality given on objects by $X \mapsto X^*$ (imbued with appropriate additional structure). One might think of the lower arrow as a Pontryagin duality between comodules and pseudocompact contramodules. As an easy corollary of the main theorem, we will show that the objects of **PContramod**-*C* are inverse limits of finite dimensional right *C*-contramodules. We also extend a very useful natural isomorphism of Takeuchi.

2 Preliminaries

Let (C, Δ, ε) be a coalgebra [3, §1.1]. We define the category *C*-**Comod** explicitly in order to use it later. A left *C*-comodule is a pair (X, ρ) where *X* is a discrete *k*-vector space and $\rho : X \to C \otimes X$ is a linear map such that the following two diagrams commute:



For convenience, we refer to the above diagrams as the "comodule square" and the "comodule triangle", respectively. A homomorphism of left comodules $(X, \rho) \to (Y, \gamma)$ is a linear map $\alpha : X \to Y$ such that the following "comodule homomorphism square" commutes:



Let A,B be discrete vector spaces and let Z be a pseudocompact vector space. The map

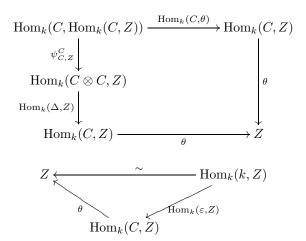
$$\psi^B_{A,Z}$$
: Hom_k(A, Hom_k(B, Z)) \rightarrow Hom_k(A \otimes B, Z)

given by $\psi_{A,Z}^B(\gamma)(a \otimes b) := \gamma(a)(b)$ is an isomorphism. To see this, one writes A and B as direct limits of finite dimensional vector spaces A_i, B_j respectively and Z as an inverse limit of finite dimensional vector spaces Z_k . Using repeatedly the isomorphism of [7, Lemma 5.1.4(b)] (which holds for pseudocompact vector spaces) and applying the tensor-hom adjunction, one obtains isomorphisms

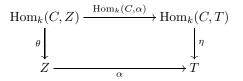
$$\operatorname{Hom}_{k}(A, \operatorname{Hom}_{k}(B, Z)) \cong \varprojlim_{i,j,k} \operatorname{Hom}_{k}(A_{i}, \operatorname{Hom}_{k}(B_{j}, Z_{k}))$$
$$\cong \varprojlim_{i,j,k} \operatorname{Hom}_{k}(A_{i} \otimes B_{j}, Z_{k})$$
$$\cong \operatorname{Hom}_{k}(A \otimes B, Z).$$

The isomorphism $\psi_{A,Z}^B$ is natural in all three variables. Naturality in A and Z comes via the tensor-hom adjunction applied at the level of the finite dimensional vector spaces, while B is the "parameter" (for more on this, see [5, §IV.7]).

Definition 2.1. A pseudocompact right C-contramodule is a pseudocompact vector space Z together with a continuous linear map θ : Hom_k(C, Z) \rightarrow Z such that the following diagrams commute:



We call these diagrams the contramodule square and the contramodule triangle, respectively. A homomorphism of pseudocompact right contramodules $(Z, \theta) \rightarrow$ (T, η) is a continuous linear map $\alpha : Z \rightarrow T$ such that the following "contramodule homomorphism square" commutes:



We denote the category of pseudocompact right contramodules and contramodule homomorphisms by **PContramod**-C.

3 The lower arrow

We define the lower horizontal arrow in (1.1) and prove it is a duality. The right vertical duality is then obtained by composing the others.

From now on we suppress notation, writing (A, B) instead of $\operatorname{Hom}_k(A, B)$ and $\gamma \circ -$ (resp. $-\circ \gamma$) instead of $\operatorname{Hom}_k(A, \gamma)$ (resp. $\operatorname{Hom}_k(\gamma, B)$). Let X be a discrete vector space. We have natural isomorphisms

$$(X, C \otimes X) \xrightarrow{(-)^*} ((C \otimes X)^*, X^*) \xrightarrow{-\circ \psi_{C,k}^X} ((C, X^*), X^*).$$

Given $\rho \in (X, C \otimes X)$, denote the corresponding element $\rho^* \psi_{C,k}^X$ of $((C, X^*), X^*)$ by $\overline{\rho}$.

Theorem 3.1. The assignment defined on objects by $(X, \rho) \mapsto (X^*, \overline{\rho})$ and on morphisms by $\alpha \mapsto \alpha^*$ yields a duality of categories

C-Comod \rightarrow PContramod-C.

Proof. By Pontryagin duality (a special case of [2, Proposition 2.3]) the operation yields a duality from discrete k-vector spaces to pseudocompact vector spaces. It thus suffices to check that (X, ρ) is a comodule if, and only if, $(X^*, \overline{\rho})$ is a contramodule and that $\alpha : (X, \rho) \to (Y, \gamma)$ is a comodule homomorphism if, and only if, $\alpha^* : (Y^*, \overline{\gamma}) \to (X^*, \overline{\rho})$ is a contramodule homomorphism.

Let $\rho: X \to C \otimes X$ be a linear map and consider the following diagram:

$$\begin{array}{c} (C, (C, X^*)) \xrightarrow{\psi_{C,X^*}^C} (C \otimes C, X^*) \xrightarrow{-\circ\Delta} (C, X^*) \\ \psi_{C,k}^X \circ - \downarrow & (1) & \psi_{C\otimes C,k}^X \downarrow & (2) & \downarrow \psi_{C,k}^X \\ (C, (C \otimes X)^*) \xrightarrow{\psi_{C,k}^C \otimes X} (C \otimes C \otimes X)^* \xrightarrow{(\Delta \otimes \mathrm{id})^*} (C \otimes X)^* \\ \rho^* \circ - \downarrow & (3) & (\mathrm{id} \otimes \rho)^* \downarrow & (4) & \downarrow \rho^* \\ (C, X^*) \xrightarrow{\psi_{C,k}^X} (C \otimes X)^* \xrightarrow{-\rho^*} X^* \end{array}$$

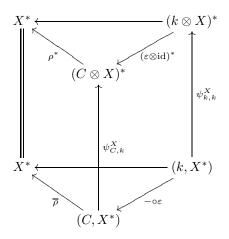
We claim that squares (1), (2) and (3) commute:

- (1) is most easily seen to commute by a simple calculation with elements.
- (2) commutes, being the natural transformation $\psi^X_{-,k}$ applied to the map Δ : $C \to C \otimes C$.
- (3) commutes, being the natural transformation $\psi_{C,k}^-$ applied to the map ρ : $X \to C \otimes X$.

The outer square commutes if, and only if, $(X^*, \overline{\rho}) = (X^*, \rho^* \psi_{C,k}^X)$ satisfies the contramodule square and Square (4) commutes if, and only if (X, ρ) satisfies

the comodule square (by duality). But the morphisms in Square (1) are isomorphisms and hence the outer square commutes if, and only if, Square (4) commutes.

We check next the correspondence between the comodule and contramodule triangles. Consider the following diagram, wherein the horizontal maps are natural isomorphisms.



The square at the back clearly commutes, the left front square commutes by the definition of $\overline{\rho}$ and the right front square commutes by the naturality of $\psi_{-,k}^X$ applied to ε . The vertical maps are isomorphisms, so the upper triangle commutes if, and only if, the lower triangle commutes. But the lower is the contramodule triangle and the upper is the (dual of the) comodule triangle.

Let $\alpha: X \to Y$ be a linear transformation between the comodules (X, ρ) and (Y, γ) . One checks that α is a homomorphism of comodules if, and only if, α^* is a homomorphism of contramodules exactly as above: dualize the coalgebra homomorphism square and join it to the corresponding contramodule homomorphism square using equalities, $\psi_{C,k}^X$ or $\psi_{C,k}^Y$. Observe that the joining squares commute and hence that the comodule homomorphism square commutes if, and only if, the contramodule homomorphism square commutes.

4 Corollaries

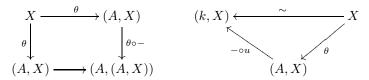
Corollary 4.1. The category **PContramod**-*C* is exactly the category of inverse limits of finite dimensional right C-contramodules.

Proof. This is dual to the fundamental theorem of comodules [3, Theorem 2.1.7]. \Box

A discrete right module for the pseudocompact algebra

$$(A, m: A \widehat{\otimes} A \to A, u: k \to A)$$

(where $\widehat{\otimes} = \widehat{\otimes}_k$ is the completed tensor product [2, §2]) is usually defined to be a discrete vector space X together with a continuous map $X \times A \to X$ (cf. [7, §5.1]) satisfying the obvious axioms. One familiar with coalgebras might complain that the object $X \times A$ lives neither in the discrete nor the pseudocompact category, making it difficult to interpret the multiplication as a map from a tensor product. When dealing with dualities and coalgebras, it is perhaps easier to define a discrete right A-module as a discrete vector space X together with a linear map $\theta: X \to \operatorname{Hom}_k(A, X)$ such that the diagrams



commute, wherein the unmarked arrow is the composition

$$(A,X) \xrightarrow{-\circ m} (A \widehat{\otimes} A,X) \to (A,(A,X))$$

coming from the isomorphism [2, Lemma 2.4]. This definition is equivalent to the standard one. Interpreted this way, a homomorphism from a pseudocompact module to a discrete module

$$\gamma: (Y, \rho: Y \widehat{\otimes} A \to Y) \to (X, \theta: X \to (A, X))$$

is a continuous map $\gamma: Y \to X$ such that

$$\theta\gamma(y)(a) = \gamma\rho(y\widehat{\otimes} a)$$

for each $y \in Y, a \in A$. This equality can be written without elements: γ (as above) is a homomorphism if $\psi(\gamma \rho) = \theta \gamma$, where ψ is the natural isomorphism

$$(Y \widehat{\otimes} A, X) \to (Y, (A, X))$$

again coming from [2, Lemma 2.4]. Observe that, writing $X = Z^*$ for some pseudocompact vector space Z and noting that

$$(A \widehat{\otimes} Z)^* = (\varprojlim_{i,j} A_i \otimes Z_j)^* = \varinjlim_{i,j} A_i^* \otimes Z_j^* = A^* \otimes Z^*,$$

we have

$$(A, X)^* = (A, Z^*)^* = (A \widehat{\otimes} Z)^{**} = (A^* \otimes Z^*)^* = (A^*, Z^{**}) = (A^*, X^*).$$

We can thus dualize the above notion of homomorphism:

Definition 4.2. Let C be a coalgebra, let $(N, \theta : (C, N) \to N)$ be a right pseudocompact C-contramodule and $(M, \mu : M \to M \otimes C)$ be a right C-comodule. A homomorphism $\gamma : (N, \theta) \to (M, \mu)$ is a continuous linear map $N \to M$ such that

$$\overline{\psi}(\gamma\theta) = \mu\gamma_{\pm}$$

where $\overline{\psi}: ((C,N), M) \to (N, M \otimes C)$ is obtained from ψ above by duality.

Let C, D be coalgebras, L a right C-comodule, M a C-D-bicomodule, and N a right pseudocompact D-contramodule. The vector space $\operatorname{Hom}_D(N, M)$ inherits a natural left C-comodule structure (by dualizing twice the discrete right C^* -module $\operatorname{Hom}_{D^*}(M^*, N^*)$, for instance) and so $\operatorname{Hom}_D(N, M)^*$ is a right C-contramodule. We define this way the functor

 $h(M, -) := \operatorname{Hom}_D(-, M)^* : \operatorname{Contramod-}D \to \operatorname{Contramod-}C,$

a variant of the "cohom" functor of Takeuchi [10, §1]. In the category of comodules, the cotensor product functor $M \square_D - : D$ -Comod $\rightarrow C$ -Comod has a left adjoint if, and only if, M is quasi-finite as a left C-comodule, and in this case the corresponding left adjoint is the cohom functor of Takeuchi [10, Proposition 1.10]. By allowing pseudocompact contramodules, we obtain a natural isomorphism without quasi-finiteness conditions on M:

Corollary 4.3. Let C, D be coalgebras, L a right C-comodule, M a C-D-bicomodule, and N a right pseudocompact D-contramodule. There is a natural isomorphism of vector spaces

$$\operatorname{Hom}_{D}(N, L \underset{C}{\Box} M) \cong \operatorname{Hom}_{C}(h(M, N), L).$$

Proof. In light of the discussion above and the fact that

$$L \mathop{\square}_{C} M \cong (L^* \mathop{\widehat{\otimes}}_{C^*} M^*)^*,$$

the result is dual to [2, Lemma 2.4].

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