DIMER MODEL: FULL ASYMPTOTIC EXPANSION OF THE PARTITION FUNCTION

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Dedicated to Ludvig Faddeev

ABSTRACT. We give a complete rigorous proof of the full asymptotic expansion of the partition function of the dimer model on a square lattice on a torus for general weights z_h, z_v of the dimer model and arbitrary dimensions of the lattice m, n. We assume that m is even and we show that the asymptotic expansion depends on the parity of n. We review and extend the results of Ivashkevich, Izmailian, and Hu [6] on the full asymptotic expansion of the partition function of the dimer model, and we give a rigorous estimate of the error term in the asymptotic expansion of the partition function.

1. INTRODUCTION

1.1. **Dimer Model on a Square Lattice.** We consider the dimer model on a square lattice $\Gamma_{m,n} = (V_{m,n}, E_{m,n})$ on the torus $\mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}^2/(m\mathbb{Z} \times n\mathbb{Z})$ (periodic boundary conditions), where $V_{m,n}$ and $E_{m,n}$ are the sets of vertices and edges of $\Gamma_{m,n}$, respectively. A *dimer* on $\Gamma_{m,n}$ is a set of two neighboring vertices $\langle x, y \rangle$ connected by an edge. A *dimer configuration* σ on $\Gamma_{m,n}$ is a set of dimers $\sigma = \{\langle x_i, y_i \rangle, \ i = 1, \ldots, \frac{mn}{2}\}$ which cover $V_{m,n}$ without overlapping. An example of a dimer configuration is shown in Fig. 1. An obvious necessary condition for a configuration to exist is that at least one of m, n is even, and so we assume that m is even, $m = 2m_0$.

To define a weight of a dimer configuration, we split the full set of dimers in a configuration σ into two classes: horizontal and vertical, with respective weights $z_h, z_v > 0$. If we denote the total number of horizontal and vertical dimers in σ by $N_h(\sigma)$ and $N_v(\sigma)$, respectively, then the *dimer configuration weight* is

$$w(\sigma) = \prod_{i=1}^{\frac{mn}{2}} w(x_i, y_i) = z_h^{N_h(\sigma)} z_v^{N_v(\sigma)}, \tag{1.1}$$

where $w(x_i, y_i)$ denotes the weight of the dimer $\langle x_i, y_i \rangle \in \sigma$. We denote by $\Sigma_{m,n}$ the set of all dimer configurations on $\Gamma_{m,n}$. The *partition function* of the dimer model is given by

$$Z = \sum_{\sigma \in \Sigma_{m,n}} w(\sigma). \tag{1.2}$$

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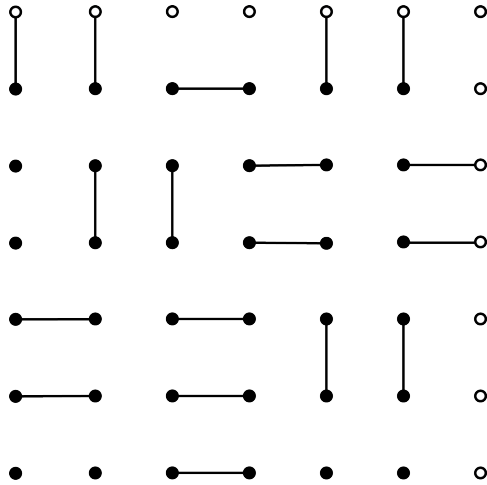[†]bradelwood@gmail.com

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FIGURE 1. Example of a dimer configuration on a square 6×6 lattice on the torus.

Notice that if all the weights are set equal to one, then Z simply counts the number of dimer configurations, or perfect matchings, on $\Gamma_{m,n}$.

Our goal is to evaluate the full asymptotic series expansion of the partition function Z as $m, n \to \infty$. The free energy of the dimer model on the square lattice was obtained in the papers of Kasteleyn [8] and Temperley and Fisher [16]. Our work is based on the Kasteleyn's expression of the partition function Z on a torus as a linear combination of 4 Pfaffians developed in the works [8], [9], [10] (see also the works of Galluccio and Loebl [5], Tesler [17], and Cimasoni and Reshetikhin [3]). The constant term in the asymptotic of the partition function was obtained by Ferdinand [4] (see also the work of Kenyon, Sun and Wilson [11]).

The asymptotic expansion of the partition function on a torus was developed by Ivashke-vich, Izmailian, and Hu [6] and our calculations use their ideas. Ivashkevich, Izmailian, and Hu considered the case when $z_h = z_v$ and n is even. In the present work we extend their calculations to arbitrary weights z_h, z_v and to odd n. It is worth noticing that the asymptotic expansions for even and odd values of n are different. We give a complete rigorous proof of the asymptotic expansion of the partition function, with an estimate of the error term. The asymptotic expansion of the partition function is expressed in terms of the classical Jacobi theta functions, Dedekind eta function, and Kronecker double series. The work [6] has been further extended by Izmailian, Oganesyan, and Hu [7] to the dimer model on a square lattice with various boundary conditions for both even and odd n. Our result for the dimer model on a torus coincides with the one in [7] for even n, and for odd n it coincides except for the value of the elliptic nome in formula (2.14) below. The difference in the value of the elliptic nome for even and odd n is explained after formula (5.6) in Section 5 below.

It follows from (1.1), (1.2) that the partition function Z is a homogeneous polynomial of the variables z_h, z_v, and it can be written as

$$Z(z_h, z_v) = \sum_{\sigma \in \Sigma_{m,n}} z_h^{N_h(\sigma)} z_v^{N_v(\sigma)} = z_h^{\frac{mn}{2}} Z(1, \zeta), \tag{1.3}$$

where

$$\zeta = \frac{z_v}{z_h} > 0, \tag{1.4}$$

so without loss of generality we may assume that

$$z_h = 1, \quad z_v = \zeta, \tag{1.5}$$

and we will evaluate the full asymptotic series expansion of the partition function $Z(1, \zeta)$ as $m, n \to \infty$.

To formulate our main result we have to introduce and remind some special functions and operators.

1.2. Function $g(x)$. Introduce the function

$$g(x) = \ln \left(\zeta \sin(\pi x) + \sqrt{1 + \zeta^2 \sin^2(\pi x)} \right), \tag{1.6}$$

where $\zeta > 0$ is defined in (1.4). Observe that $g(x)$ has the following properties:

(1) $g(-x) = -g(x)$,
(2) $g(x + 1) = -g(x)$,
(3) $g(x)$ is real analytic on $[0, 1]$ and

$$g(x) = \sum_{p=0}^{\infty} g_{2p+1} x^{2p+1}, \tag{1.7}$$

where

$$g_1 = \pi \zeta, \quad g_3 = -\frac{\pi^3 \zeta(\zeta^2 + 1)}{6}, \quad g_5 = \frac{\pi^5 \zeta(\zeta^2 + 1)(9\zeta^2 + 1)}{120}, \quad \ldots . \tag{1.8}$$

(4) $g(x) \geq C_0 x$ on the segment $0 \leq x \leq \frac{1}{2}$, with some $C_0 > 0$.

The constant C_0 in the latter inequality can depend on ζ. In what follows we assume that ζ is fixed and we do not indicate the dependence of various constants C_k on ζ. Unless otherwise is stated, the constants C_k can be different in different inequalities.

Observe that since $g(x)$ is analytic at $x = 0$, we have that

$$|g_{2p+1}| \leq C\xi^p, \tag{1.9}$$

with some $C, \xi > 0$.

1.3. Differential Operator Δ_p. Let $\mathcal{S}_p$ be the set of collections of positive integers $(p_1, \ldots, p_r; q_1, \ldots, q_r)$, $1 \leq r \leq p$, such that

$$\mathcal{S}_p = \left\{ (p_1, \ldots, p_r; q_1, \ldots, q_r) \mid 0 < p_1 < \ldots < p_r; \ p_1 q_1 + \ldots + p_r q_r = p \right\}. \tag{1.10}$$

Introduce the differential operator

$$\Delta_p = \sum_{\mathcal{S}_p} \frac{(g_{2p_1+1})^{q_1} \ldots (g_{2p_r+1})^{q_r}}{q_1! \ldots q_r!} \frac{d^q}{d\lambda^q}, \quad q = q_1 + \ldots + q_r - 1. \tag{1.11}$$

Observe that

$$\Delta_1 = g_3, \quad \Delta_2 = \frac{g_3^2}{2} \frac{d}{d\lambda} + g_5, \quad \Delta_3 = \frac{g_3^3}{3!} \frac{d^2}{d\lambda^2} + g_3 g_5 \frac{d}{d\lambda} + g_7, \quad \ldots . \tag{1.12}$$

1.4. **Kronecker's Double Series.** The Kronecker double series of order p with parameters α, β is defined as

$$K_p^{\alpha,\beta}(\tau) = -\frac{p!}{(-2\pi i)^p} \sum_{(j,k)\neq(0,0)} \frac{e(j\alpha + k\beta)}{(k+\tau j)^p}, \qquad (1.13)$$

where

$$e(x) = e^{-2\pi i x}. \qquad (1.14)$$

We will use the following Kronecker double series with parameters $(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}), (\frac{1}{2}, 0)$, respectively:

$$K_p^{\frac{1}{2},\frac{1}{2}}(\tau) = -\frac{p!}{(-2\pi i)^p} \sum_{(j,k)\neq(0,0)} \frac{(-1)^{j+k}}{(k+\tau j)^p},$$

$$K_p^{0,\frac{1}{2}}(\tau) = -\frac{p!}{(-2\pi i)^p} \sum_{(j,k)\neq(0,0)} \frac{(-1)^k}{(k+\tau j)^p}, \qquad (1.15)$$

$$K_p^{\frac{1}{2},0}(\tau) = -\frac{p!}{(-2\pi i)^p} \sum_{(j,k)\neq(0,0)} \frac{(-1)^j}{(k+\tau j)^p}.$$

We will use it for τ pure imaginary and $p \geq 4$. Then the double series are absolutely convergent.

1.5. **Dedekind Eta Function.** The Dedekind eta function is defined as

$$\eta = \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{k=1}^{\infty} \left(1 - e^{2\pi i \tau k}\right) = q^{\frac{1}{12}} \prod_{k=1}^{\infty} \left(1 - q^{2k}\right), \qquad (1.16)$$

where

$$q = e^{\pi i \tau} \qquad (1.17)$$

is the elliptic nome.

1.6. **Jacobi Theta Functions.** There are four Jacobi theta functions:

$$\theta_1(z, q) = 2 \sum_{k=0}^{\infty} (-1)^k q^{\left(k+\frac{1}{2}\right)^2} \sin\left((2k+1)z\right),$$

$$\theta_2(z, q) = 2 \sum_{k=0}^{\infty} q^{\left(k+\frac{1}{2}\right)^2} \cos\left((2k+1)z\right),$$

$$\theta_3(z, q) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2} \cos(2kz), \qquad (1.18)$$

$$\theta_4(z, q) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos(2kz),$$

where $q = e^{\pi i \tau}$ is elliptic nome.

We have the following identities (see, e.g., [18]):

$$\theta_2 = \theta_2(0, q) = \frac{2\eta^2(2\tau)}{\eta(\tau)},$$

$$\theta_3 = \theta_3(0, q) = \frac{\eta^5(\tau)}{\eta^2(2\tau)\eta^2(\frac{\tau}{2})}, \tag{1.19}$$

$$\theta_4 = \theta_4(0, q) = \frac{\eta^2(\frac{\tau}{2})}{\eta(\tau)}.$$

Also (see, e.g., [6]),

$$K_4^{0,\frac{1}{2}}(\tau) = \frac{1}{30}\left(\frac{7}{8}\theta_2^8 - \theta_3^4\theta_4^4\right),$$

$$K_4^{\frac{1}{2},0}(\tau) = \frac{1}{30}\left(\frac{7}{8}\theta_4^8 - \theta_2^4\theta_3^4\right), \tag{1.20}$$

$$K_4^{\frac{1}{2},\frac{1}{2}}(\tau) = \frac{1}{30}\left(\frac{7}{8}\theta_3^8 + \theta_2^4\theta_4^4\right).$$

2. Main Result: Full Asymptotic Expansion of the Dimer Model Partition Function

2.1. Pfaffians.
We would like to evaluate the asymptotic expansion of the dimer model partition function Z on the square lattice, $\Gamma_{m,n}$, of dimensions $m \times n$, with periodic boundary conditions where $m, n \to \infty$ under the assumption that there exist positive constants $C_2 > C_1$ such that

$$C_1 \le \frac{m}{n} \le C_2. \tag{2.1}$$

As shown by Kasteleyn [8, 9, 10], the partition function Z can be written in terms of four Pfaffians as

$$Z = \frac{1}{2}\left(-\operatorname{Pf} A_1 + \operatorname{Pf} A_2 + \operatorname{Pf} A_3 + \operatorname{Pf} A_4\right), \tag{2.2}$$

where A_1, A_2, A_3, A_4 are the antisymmetric Kasteleyn matrices with periodic-periodic, periodic-antiperiodic, antiperiodic-periodic, and antiperiodic-antiperiodic boundary conditions, respectively. Their determinants are given by the double product formulae as

$$\det A_i = \prod_{j=0}^{\frac{m}{2}-1} \prod_{k=0}^{n-1} \left[4\left(\sin^2\frac{2\pi(j+\alpha_i)}{m} + \zeta^2\sin^2\frac{2\pi(k+\beta_i)}{n}\right)\right], \tag{2.3}$$

with

$$(\alpha_1, \beta_1) = (0, 0), \quad (\alpha_2, \beta_2) = (0, 1/2), \quad (\alpha_3, \beta_3) = (1/2, 0), \quad (\alpha_4, \beta_4) = (1/2, 1/2). \tag{2.4}$$

These double product formulae are obtained by diagonalizing the matrices A_i (see [8, 13, 14]). The Pfaffian of a square antisymmetric matrix A is related to its determinant through the classical identity:

$$(\operatorname{Pf} A)^2 = \det A. \tag{2.5}$$

Observe that $\det A_1 = 0$ due to the factor $j = k = 0$ in (2.3), hence

$$\operatorname{Pf} A_1 = 0, \tag{2.6}$$

and for odd n, $\det A_2 = 0$, due to the factor $j = 0$, $k = \frac{n-1}{2}$, hence

$$\mathrm{Pf}\, A_2 = 0, \quad \text{if } n \text{ is odd.} \tag{2.7}$$

In addition,

$$\mathrm{Pf}\, A_3 = \mathrm{Pf}\, A_4, \quad \text{if } n \text{ is odd} \tag{2.8}$$

(see [2]). As shown in [11],[2],

$$\begin{aligned}
\mathrm{Pf}\, A_2 > 0, \quad &\text{if } n \text{ is even,} \\
\mathrm{Pf}\, A_3 > 0, \quad \mathrm{Pf}\, A_4 > 0 \quad &\text{for all } n,
\end{aligned} \tag{2.9}$$

hence from (2.3) we obtain that

$$\mathrm{Pf}\, A_i = \prod_{j=0}^{\frac{m}{2}-1} \prod_{k=0}^{n-1} \left[4 \left(\sin^2 \frac{2\pi(j+\alpha_i)}{m} + \zeta^2 \sin^2 \frac{2\pi(k+\beta_i)}{n} \right) \right]^{1/2}. \tag{2.10}$$

Combining (2.2) with (2.6), (2.7), (2.8), we obtain that

$$\begin{aligned}
Z &= \frac{1}{2} \left(\mathrm{Pf}\, A_2 + \mathrm{Pf}\, A_3 + \mathrm{Pf}\, A_4 \right), \quad \text{if } n \text{ is even,} \\
Z &= \mathrm{Pf}\, A_3, \quad \text{if } n \text{ is odd.}
\end{aligned} \tag{2.11}$$

2.2. **Main Result.** Before stating the main theorem, let us introduce some additional notations. Denote

$$S = mn, \qquad \nu = \frac{m}{n}. \tag{2.12}$$

We set

$$\tau = \begin{cases} i\zeta\nu, & \text{if } n \text{ is even,} \\ \dfrac{i\zeta\nu}{2}, & \text{if } n \text{ is odd,} \end{cases} \tag{2.13}$$

so that the elliptic nome is equal to

$$q = e^{\pi i \tau} = \begin{cases} e^{-\pi\zeta\nu}, & \text{if } n \text{ is even,} \\ e^{\frac{-\pi\zeta\nu}{2}}, & \text{if } n \text{ is odd.} \end{cases} \tag{2.14}$$

For brevity we also denote

$$\eta = \eta(\tau), \qquad \theta_k = \theta_k(0, q), \quad k = 2, 3, 4, \tag{2.15}$$

where $\eta(\tau)$ is the Dedekind eta function, and $\theta_k(z, q)$ are the Jacobi theta functions. The main result is the following asymptotic expansion of the partition function Z in powers of S^{-1}, derived by Ivashkevich et al. in [6] in the case $\zeta = 1$ and n is even. We give a complete rigorous proof of the asymptotic expansion for any $\zeta > 0$ and for n both even and odd.

Theorem 2.1. *If n is even, then as $m, n \to \infty$ under condition* (2.1), *we have that*

$$Z = e^{SF} \left(C^{(2)} e^{R^{(2)}} + C^{(3)} e^{R^{(3)}} + C^{(4)} e^{R^{(4)}} \right), \tag{2.16}$$

where

$$F = \frac{1}{\pi} \int_0^{\zeta} \frac{\arctan x}{x}\, dx, \tag{2.17}$$

$$C^{(2)} = \frac{\theta_4^2}{2\eta^2}, \quad C^{(3)} = \frac{\theta_2^2}{2\eta^2}, \quad C^{(4)} = \frac{\theta_3^2}{2\eta^2}, \tag{2.18}$$

and $R^{(j)}$, $j = 2, 3, 4$, admit the asymptotic expansions

$$R^{(j)} \sim \sum_{p=1}^{\infty} \frac{R_p^{(j)}}{S^p}, \quad j = 2, 3, 4, \tag{2.19}$$

with

$$R_p^{(j)} = -\frac{2^{2p+1}\nu^{p+1}}{p+1} \Delta_p \left[K_{2p+2}^{\beta_j, \alpha_j} \left(\frac{i\nu\lambda}{\pi} \right) \right] \Big|_{\lambda = \pi\zeta}, \tag{2.20}$$

where α_j, β_j are defined in (2.4). In particular, by (1.12) and (1.20),

$$R_1^{(2)} = -\frac{2\nu^2 g_3}{15} \left(\frac{7}{8}\theta_4^8 - \theta_2^4\theta_3^4 \right), \quad R_1^{(3)} = -\frac{2\nu^2 g_3}{15} \left(\frac{7}{8}\theta_2^8 - \theta_3^4\theta_4^4 \right),$$

$$R_1^{(4)} = -\frac{2\nu^2 g_3}{15} \left(\frac{7}{8}\theta_3^8 + \theta_2^4\theta_4^4 \right). \tag{2.21}$$

Furthermore, if n is odd, then as $m, n \to \infty$ under condition (2.1), we have that

$$Z = Ce^{SF+R}, \tag{2.22}$$

where F is given in (2.17),

$$C = \frac{\theta_2}{\eta}, \tag{2.23}$$

and R admits the asymptotic expansions

$$R \sim \sum_{p=1}^{\infty} \frac{R_p}{S^p}, \tag{2.24}$$

with

$$R_p = -\frac{\nu^{p+1}}{p+1} \Delta_p \left[K_{2p+2}^{0, \frac{1}{2}} \left(\frac{i\nu\lambda}{2\pi} \right) \right] \Big|_{\lambda = \pi\zeta}. \tag{2.25}$$

By (1.12) and (1.20),

$$R_1 = -\frac{\nu^2 g_3}{60} \left(\frac{7}{8}\theta_2^8 - \theta_3^4\theta_3^4 \right). \tag{2.26}$$

Remark: As noticed by Kasteleyn [8], the free energy F in (2.17) can be expressed in terms of the Euler dilogarithm function

$$\mathrm{L}_2(z) = -\int_0^z \frac{\ln(1-s)\,ds}{s} \tag{2.27}$$

as

$$F(\zeta) = (2i)^{-1} \left[\mathrm{L}_2(i\zeta) - \mathrm{L}_2(-i\zeta) \right]. \tag{2.28}$$

The proof of Theorem 2.1 will be given in Sections 3–5.

3. ASYMPTOTIC BEHAVIOR OF $\operatorname{Pf} A_2$ FOR EVEN n

Since $\sin^2(x + \pi) = \sin^2 x$, we can rewrite $\operatorname{Pf} A_2$ in (2.10) for even n as

$$\operatorname{Pf} A_2 = \prod_{j=0}^{\frac{m}{2}-1} \prod_{k=0}^{\frac{n}{2}-1} \left[4 \left(\sin^2 \frac{2j\pi}{m} + \zeta^2 \sin^2 \frac{(2k+1)\pi}{n} \right) \right]. \tag{3.1}$$

Using the Chebyshev type identity (see e.g. [8]),

$$\prod_{j=0}^{\frac{m}{2}-1} \left[4 \left(u^2 + \sin^2 \frac{2j\pi}{m} \right) \right] = \left[\left(u + \sqrt{1+u^2} \right)^{\frac{m}{2}} - \left(-u + \sqrt{1+u^2} \right)^{\frac{m}{2}} \right]^2, \tag{3.2}$$

equation (3.1) is reduced to

$$\operatorname{Pf} A_2 = \prod_{k=0}^{\frac{n}{2}-1} \left[\left(u_k + \sqrt{1+u_k^2} \right)^{\frac{m}{2}} - \left(-u_k + \sqrt{1+u_k^2} \right)^{\frac{m}{2}} \right]^2, \tag{3.3}$$

where

$$u_k = \zeta \sin(\pi x_k) \geq 0, \quad x_k = \frac{2k+1}{n}. \tag{3.4}$$

Observe that

$$\left(u_k + \sqrt{1+u_k^2} \right) \left(-u_k + \sqrt{1+u_k^2} \right) = 1, \tag{3.5}$$

hence

$$\operatorname{Pf} A_2 = B_{m,n}^{(2)} C_{m,n}^{(2)}, \tag{3.6}$$

where

$$B_{m,n}^{(2)} = \prod_{k=0}^{\frac{n}{2}-1} \left(u_k + \sqrt{1+u_k^2} \right)^m,$$

$$C_{m,n}^{(2)} = \prod_{k=0}^{\frac{n}{2}-1} \left[1 - \frac{1}{\left(u_k + \sqrt{1+u_k^2} \right)^m} \right]^2. \tag{3.7}$$

Respectively,

$$\ln(\operatorname{Pf} A_2) = G_{m,n}^{(2)} + H_{m,n}^{(2)}, \tag{3.8}$$

with

$$G_{m,n}^{(2)} = m \sum_{k=0}^{\frac{n}{2}-1} \ln \left(u_k + \sqrt{1+u_k^2} \right) = m \sum_{k=0}^{\frac{n}{2}-1} g \left(\frac{2k+1}{n} \right),$$

$$H_{m,n}^{(2)} = 2 \sum_{k=0}^{\frac{n}{2}-1} \ln \left[1 - \frac{1}{\left(u_k + \sqrt{1+u_k^2} \right)^m} \right]. \tag{3.9}$$

The function $g(x)$, defined in (1.6), is real analytic, and we will evaluate an asymptotic series expansion of $G_{m,n}^{(2)}$ for large n by using an Euler–Maclaurin type formula and the Bernoulli polynomials $B_k(x)$ (see [1] or Appendix C).

3.1. Evaluation of $G_{m,n}^{(2)}$.

Lemma 3.1. *As $n, m \to \infty$ under condition (2.1), we have that $G_{m,n}^{(2)}$ admits the following asymptotic expansion:*

$$G_{m,n}^{(2)} \sim SF + \frac{\gamma}{6} - m \sum_{p=1}^{\infty} \frac{B_{2p+2}\left(\frac{1}{2}\right) g_{2p+1}}{(p+1)\left(\frac{n}{2}\right)^{2p+1}}, \quad \gamma = \pi\nu\zeta. \tag{3.10}$$

Proof. From (3.9) we have that

$$G_{m,n}^{(2)} = m\, G_n^{(2)}, \quad G_n^{(2)} = \sum_{k=0}^{\frac{n}{2}-1} g\left(\frac{2k+1}{n}\right). \tag{3.11}$$

Using the Euler-Maclaurin formula (C.5), we obtain that $G_n^{(2)}$ is expanded in the asymptotic series in powers of $\frac{1}{n}$ as

$$G_n^{(2)} \sim \frac{n}{2} \int_0^1 g(x)\, dx + \sum_{p=1}^{\infty} \frac{B_p\left(\frac{1}{2}\right)}{\left(\frac{n}{2}\right)^{p-1} p!} [g^{(p-1)}(1) - g^{(p-1)}(0)]. \tag{3.12}$$

From (1.7) and the equation $g(x+1) = -g(x)$ we obtain that

$$\begin{aligned} g^{(2p)}(0) &= g^{(2p)}(1) = 0, \\ g^{(2p+1)}(0) &= (2p+1)!g_{2p+1}, \quad g^{(2p+1)}(1) = -(2p+1)!g_{2p+1}. \end{aligned} \tag{3.13}$$

Now, (3.12) becomes

$$\begin{aligned} G_n^{(2)} &\sim \frac{n}{2} \int_0^1 g(x)\, dx + \sum_{p=1}^{\infty} \frac{B_{2p}\left(\frac{1}{2}\right)}{\left(\frac{n}{2}\right)^{2p-1}(2p)!} [g^{(2p-1)}(1) - g^{(2p-1)}(0)] \\ &\sim \frac{n}{2} \int_0^1 g(x)\, dx - \sum_{p=1}^{\infty} \frac{2B_{2p}\left(\frac{1}{2}\right)(2p-1)!g_{2p-1}}{\left(\frac{n}{2}\right)^{2p-1}(2p)!} \\ &\sim \frac{n}{2} \int_0^1 g(x)\, dx - \sum_{p=0}^{\infty} \frac{B_{2p+2}\left(\frac{1}{2}\right) g_{2p+1}}{(p+1)\left(\frac{n}{2}\right)^{2p+1}}. \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.11), we obtain that

$$G_{m,n}^{(2)} \sim \frac{mn}{2} \int_0^1 g(x)\, dx - m \sum_{p=0}^{\infty} \frac{B_{2p+2}\left(\frac{1}{2}\right) g_{2p+1}}{(p+1)\left(\frac{n}{2}\right)^{2p+1}}. \tag{3.15}$$

Since $B_2(\frac{1}{2}) = -\frac{1}{12}$ and $g_1 = \pi\zeta$, we obtain that

$$G_{m,n}^{(2)} \sim SF + \frac{\gamma}{6} - m \sum_{p=1}^{\infty} \frac{B_{2p+2}\left(\frac{1}{2}\right) g_{2p+1}}{(p+1)\left(\frac{n}{2}\right)^{2p+1}}, \quad \gamma = \pi\nu\zeta. \tag{3.16}$$

where

$$F = \frac{1}{2} \int_0^1 \ln \left(\zeta \sin(\pi x) + \sqrt{1 + \zeta^2 \sin^2(\pi x)} \right) dx. \tag{3.17}$$

As shown by Kasteleyn [8],

$$\frac{1}{2} \int_0^1 \ln \left(\zeta \sin(\pi x) + \sqrt{1 + \zeta^2 \sin^2(\pi x)} \right) dx = \frac{1}{\pi} \int_0^\zeta \frac{\arctan x}{x} dx, \tag{3.18}$$

hence Lemma 3.1 follows. $\square$

Next, we evaluate $H_{m,n}^{(2)}$ in (3.9).

3.2. Evaluation of $H_{m,n}^{(2)}$.

Lemma 3.2. *As $n, m \to \infty$ under condition (2.1), we have that $H_{m,n}^{(2)}$ admits the following asymptotic expansion:*

$$H_{m,n}^{(2)} \sim A^{(2)} + B^{(2)}, \tag{3.19}$$

with

$$A^{(2)} = 4 \sum_{k=0}^\infty \ln \left(1 - e^{-\gamma(2k+1)} \right), \quad \gamma = \pi \nu \zeta,$$

$$B^{(2)} = m \sum_{p=1}^\infty \frac{B_{2p+2} \left(\frac{1}{2} \right) g_{2p+1}}{(p+1) \left(\frac{n}{2} \right)^{2p+1}} + \sum_{p=1}^\infty \frac{R_p^{(2)}}{S^p}, \tag{3.20}$$

where

$$R_p^{(2)} = -\frac{2^{2p+1} \nu^{p+1}}{p+1} \Delta_p \left[K_{2p+2}^{\frac{1}{2},0} \left(\frac{i\nu\lambda}{\pi} \right) \right] \Big|_{\lambda = \pi \zeta}. \tag{3.21}$$

Proof. From (3.9), (3.4), and (1.6) we have that

$$H_{m,n}^{(2)} = 2 \sum_{k=0}^{\frac{n}{2}-1} \ln \left[1 - e^{-mg(x_k)} \right], \quad x_k = \frac{2k+1}{n}. \tag{3.22}$$

Since $g(x) \geq C_0 x$ on the segment $[0, \frac{1}{2}]$, for some $C_0 > 0$, we have that

$$e^{-mg(x_k)} \leq e^{-C_0 \nu(2k+1)}, \quad \nu = \frac{m}{n}, \tag{3.23}$$

hence the sum in (3.22) is estimated from above by a geometric series, and for any $L > 0$ there is $R > 0$ such that

$$H_{m,n}^{(2)} = 4 \sum_{k=0}^{R \ln n} \ln \left[1 - e^{-mg(x_k)} \right] + \mathcal{O}(n^{-L}), \tag{3.24}$$

so that in our calculations we can restrict k to $k \leq R \ln n$.

Following [6], let us expand the logarithm in (3.24) into the Taylor series

$$H_{m,n}^{(2)} = -4 \sum_{k=0}^{R\ln n} \sum_{j=1}^{\infty} \frac{e^{-mjg(x_k)}}{j} + \mathcal{O}(n^{-L}).$$
(3.25)

Observe that

$$mjg(x_k) \geq C_0(2k+1)j,$$
(3.26)

hence the series in j converges exponentially and for any $L > 0$ there is $R > 0$ such that

$$H_{m,n}^{(2)} = -4 \sum_{k=0}^{R\ln n} \sum_{j=1}^{R\ln n} \frac{e^{-mjg(x_k)}}{j} + \mathcal{O}(n^{-L}).$$
(3.27)

Expanding now $g(x)$ into power series (1.7), we obtain that

$$e^{-mjg(x_k)} = e^{-mj\pi\zeta x_k} \exp\left[-mj\left(\sum_{p=1}^{\infty} g_{2p+1} x_k^{2p+1}\right)\right].$$
(3.28)

Since $S = mn$ and $\nu = \frac{m}{n}$, we have that $n^{2p} = \frac{S^p}{\nu^p}$. Hence,

$$mx_k^{2p+1} = m\left(\frac{2k+1}{n}\right)^{2p+1} = \frac{(2k+1)^{2p+1}\nu^{p+1}}{S^p}$$
(3.29)

and

$$e^{-mjg(x_k)} = e^{-(2k+1)j\gamma} \exp\left[-j\sum_{p=1}^{\infty} \frac{g_{2p+1}(2k+1)^{2p+1}\nu^{p+1}}{S^p}\right], \quad \gamma = \pi\nu\zeta.$$
(3.30)

Denote

$$a_p = -(2k+1)j\nu g_{2p+1}, \quad x = \frac{(2k+1)^2\nu}{S}.$$
(3.31)

Then formula (3.30) simplifies to

$$e^{-mjg(x_k)} = e^{-(2k+1)j\gamma} \exp\left(\sum_{p=1}^{\infty} a_p x^p\right).$$
(3.32)

Substituting this expression into (3.27), we obtain that

$$H_{m,n}^{(2)} = -4 \sum_{k=0}^{R\ln n} \sum_{j=1}^{R\ln n} \frac{e^{-(2k+1)j\gamma}}{j} \exp\left(\sum_{p=1}^{\infty} a_p x^p\right) + \mathcal{O}(n^{-L}).$$
(3.33)

Expanding the exponent into the Taylor series, we obtain that

$$\exp\left(\sum_{p=1}^{\infty} a_p x^p\right) = 1 + \sum_{p=1}^{\infty} b_p x^p,$$
(3.34)

with

$$b_p = \sum_{\mathcal{S}_p} \frac{(a_{p_1})^{q_1} \ldots (a_{p_r})^{q_r}}{q_1! \ldots q_r!},$$
(3.35)

where $\mathcal{S}_p$ is defined in (1.10) (see [6] and Appendix A). Thus,

$$H_{m,n}^{(2)} = -4 \sum_{k=0}^{R\ln n} \sum_{j=1}^{R\ln n} \frac{e^{-(2k+1)j\gamma}}{j} \left(1 + \sum_{p=1}^{\infty} b_p x^p\right) + \mathcal{O}(n^{-L}), \tag{3.36}$$

or

$$H_{m,n}^{(2)} = A_n^{(2)} + B_n^{(2)} + \mathcal{O}(n^{-L}), \tag{3.37}$$

where

$$A_n^{(2)} = -4 \sum_{k=0}^{R\ln n} \sum_{j=1}^{R\ln n} \frac{e^{-(2k+1)j\gamma}}{j},$$

$$B_n^{(2)} = -4 \sum_{k=0}^{R\ln n} \sum_{j=1}^{R\ln n} \sum_{p=1}^{\infty} \frac{e^{-(2k+1)j\gamma}}{j} b_p x^p, \quad x = \frac{(2k+1)^2 \nu}{S}. \tag{3.38}$$

We have that

$$A_n^{(2)} = -4 \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \frac{e^{-(2k+1)j\gamma}}{j} + \mathcal{O}(n^{-L}) = 4 \sum_{k=0}^{\infty} \ln(1 - e^{-(2k+1)\gamma}) + \mathcal{O}(n^{-L}). \tag{3.39}$$

We write now $B_n^{(2)}$ as

$$B_n^{(2)} = B_{n,K}^{(2)} + R_{n,K}^{(2)}, \quad B_{n,K}^{(2)} = -4 \sum_{k=0}^{R\ln n} \sum_{j=1}^{R\ln n} \sum_{p=1}^{K-1} \frac{e^{-(2k+1)j\gamma}}{j} b_p x^p,$$

$$R_{n,K}^{(2)} = -4 \sum_{k=0}^{R\ln n} \sum_{j=1}^{R\ln n} \sum_{p=K}^{\infty} \frac{e^{-(2k+1)j\gamma}}{j} b_p x^p, \tag{3.40}$$

and we would like to estimate the error term $R_{n,K}^{(2)}$. To that end we will prove the following lemma:

Lemma 3.3. (Error term estimate) *Fix any $\varepsilon > 0$. Then as $S \to \infty$,*

$$R_{n,K}^{(2)} = \mathcal{O}(S^{-K(1-\varepsilon)}). \tag{3.41}$$

Remark: Remind that $S = mn = \nu n^2$, where $C_1 \le \nu \le C_2$, hence $S \to \infty$ implies that $n \to \infty$.

Proof. Let us estimate b_p. From (3.31) and (1.9) we have that

$$|a_p| = (2k+1)j\nu|g_{2p+1}| \le C_1(2k+1)j\xi^p, \tag{3.42}$$

hence

$$\left|\sum_{p=1}^{\infty} a_p z^p\right| \le C_2(2k+1)j|z|, \quad |z| \le (2\xi)^{-1}, \quad z \in \mathbb{C}. \tag{3.43}$$

This implies that

$$\left|\sum_{p=1}^{\infty} a_p z^p\right| \le C_2, \quad \text{if} \quad |z| \le \min\left\{(2\xi)^{-1}, [(2k+1)j]^{-1}\right\}, \quad z \in \mathbb{C}, \tag{3.44}$$

hence

$$\left| \exp \left(\sum_{p=1}^{\infty} a_p z^p \right) \right| \leq C_3 = e^{C_2}, \quad \text{if} \quad |z| \leq \min \left\{ (2\xi)^{-1}, [(2k+1)j]^{-1} \right\}, \quad z \in \mathbb{C}. \quad (3.45)$$

By the Cauchy integral formula,

$$\frac{f^{(p)}(0)}{p!} = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{f(z)\,dz}{z^{p+1}}, \quad (3.46)$$

applied to $f(z) = \exp \left(\sum_{p=1}^{\infty} a_p z^p \right)$ and $\rho = \min \left\{ (2\xi)^{-1}, [(2k+1)j]^{-1} \right\}$, it follows that

$$|b_p| \leq C_3 \big[(2k+1)j \big]^p \quad \text{if} \quad (2k+1)j \geq \xi, \quad (3.47)$$

and

$$|b_p| \leq C_3 \xi^p \quad \text{if} \quad (2k+1)j \leq \xi. \quad (3.48)$$

Using these estimates of b_p, we will now prove (3.41).

As $n \to \infty$, we may assume that $R \ln n > \xi$, and we partition $R_{n,K}^{(2)}$ as follows:

$$\begin{aligned}
R_{n,K}^{(2)} = &\sum_{j,k:\,(2k+1)j\leq\xi} \left(\sum_{p=K}^{\infty} \frac{e^{-(2k+1)j\gamma}}{j} |b_p| x^p \right) \\
&+ \sum_{j,k:\,j,k\leq R \ln n;\,(2k+1)j>\xi} \left(\sum_{p=K}^{\infty} \frac{e^{-(2k+1)j\gamma}}{j} |b_p| x^p \right), \quad x = \frac{(2k+1)^2 \nu}{S}.
\end{aligned} \quad (3.49)$$

In the first term there are only finitely many possible values of j and k, and by (3.48),

$$\sum_{j,k:\,(2k+1)j\leq\xi} \left(\sum_{p=K}^{\infty} \frac{e^{-(2k+1)j\gamma}}{j} |b_p| x^p \right) \leq \sum_{p=K}^{\infty} (C_4 S^{-1})^p = \mathcal{O}(S^{-K(1-\varepsilon)}). \quad (3.50)$$

Consider now the second term in (3.49). Using estimate (3.47), we obtain that

$$\sum_{j,k:\,j,k\leq R \ln n;\,(2k+1)j>\xi} \left(\sum_{p=K}^{\infty} \frac{e^{-(2k+1)j\gamma}}{j} |b_p| x^p \right) \leq \sum_{p=K}^{\infty} \frac{c_p}{S^p}, \quad (3.51)$$

with

$$0 < c_p \leq C_6 \sum_{k=0}^{R \ln n} \sum_{j=1}^{R \ln n} e^{-(2k+1)j\gamma} \big[(2k+1)^3 j\nu \big]^p \leq \big[C_7 \nu (R \ln n)^4 \big]^p, \quad (3.52)$$

hence

$$\sum_{p=K}^{\infty} \frac{c_p}{S^p} \leq \sum_{p=K}^{\infty} \left[\frac{C_7 (R \ln n)^4}{n^2} \right]^p = \mathcal{O}(S^{-K(1-\varepsilon)}). \quad (3.53)$$

Thus,

$$\sum_{j,k:\,j,k\leq R \ln n;\,(2k+1)j>\xi} \left(\sum_{p=K}^{\infty} \frac{e^{-(2k+1)j\gamma}}{j} |b_p| x^p \right) = \mathcal{O}(S^{-K(1-\varepsilon)}), \quad (3.54)$$

and (3.41) is proved. $\qquad\square$

Next, we would like to replace $R \ln n$ in $B_{n,K}^{(2)}$ in (3.40) by ∞. Denote

$$B^{(2)}(p) = -4 \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \frac{e^{-(2k+1)j\gamma}}{j} b_p x^p, \quad B_n^{(2)}(p) = -4 \sum_{k=0}^{R \ln n} \sum_{j=1}^{R \ln n} \frac{e^{-(2k+1)j\gamma}}{j} b_p x^p. \tag{3.55}$$

Then using estimate (3.47) of b_p, we obtain that

$$|B^{(2)}(p) - B_n^{(2)}(p)| = 4 \left| \sum_{j,k:\ j>0,\ k\geq 0;\ \max\{j,k\}>R \ln n} \frac{e^{-(2k+1)j\gamma}}{j} b_p x^p \right|$$

$$\leq C_0 S^{-p} \left| \sum_{j,k:\ j>0,\ k\geq 0;\ \max\{j,k\}>R \ln n} \frac{e^{-(2k+1)j\gamma}}{j} [(2k+1)^3 j\nu]^p \right|. \tag{3.56}$$

We have that

$$\sum_{k=R \ln n}^{\infty} k^{3p} e^{-k\gamma} \leq C_1(p) n^{-R\gamma/2}, \tag{3.57}$$

hence from (3.56) we obtain that

$$|B^{(2)}(p) - B_n^{(2)}(p)| \leq C_2(p) n^{-R\gamma/2}. \tag{3.58}$$

From Lemma 3.3, (3.39), and (3.58) we obtain an asymptotic expansion of $H_{m,n}^{(2)}$ in powers of S^{-1} as

$$H_{m,n}^{(2)} \sim A^{(2)} + B^{(2)}, \tag{3.59}$$

with

$$A^{(2)} = 4 \sum_{k=0}^{\infty} \ln(1 - e^{-(2k+1)\gamma}), \quad B^{(2)} = \sum_{p=1}^{\infty} d_p S^{-p}, \tag{3.60}$$

where

$$d_p = -4\nu^p \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \frac{e^{-(2k+1)j\gamma} b_p (2k+1)^{2p}}{j}. \tag{3.61}$$

Here b_p is given by equation (3.35), and it satisfies estimate (3.48), which shows that the series over k, j in the latter formula is convergent. We can transform d_p as follows.

Substituting expression (3.31) for $a_{p_1}, \ldots, a_{p_r}$ into (3.35), we obtain that

$$b_p = \sum_{\mathcal{S}_p} \frac{(g_{2p_1+1})^{q_1} \ldots (g_{2p_r+1})^{q_r} [-(2k+1) j\nu]^{q_1+\ldots+q_r}}{q_1! \ldots q_r!}. \tag{3.62}$$

We can simplify the latter expression using operator Δ_p in (1.11). Namely, we have that

$$\Delta_p \left[e^{-(2k+1)j\nu\lambda} \right] \Big|_{\lambda=\pi\zeta} = e^{-(2k+1)j\gamma}$$

$$\times \sum_{\mathcal{S}_p} \frac{(g_{2p_1+1})^{q_1} \ldots (g_{2p_r+1})^{q_r} [-(2k+1)j\nu]^{q_1+\ldots+q_r-1}}{q_1! \ldots q_r!}, \tag{3.63}$$

hence

$$b_p = e^{(2k+1)j\gamma} [-(2k+1)j\nu] \Delta_p \left[e^{-(2k+1)j\nu\lambda} \right] \Big|_{\lambda=\pi\zeta}. \tag{3.64}$$

Returning back to formula (3.61), we obtain that

$$d_p = 4\nu^{p+1} \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} (2k+1)^{2p+1} \Delta_p \left[e^{-(2k+1)j\nu\lambda} \right] \Big|_{\lambda=\pi\zeta}. \tag{3.65}$$

To relate d_p to the Kronecker double series, introduce the function

$$F_p^{(2)}(\lambda) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \left(k + \frac{1}{2} \right)^{2p+1} e^{-2\nu j\lambda\left(k+\frac{1}{2}\right)}. \tag{3.66}$$

Then

$$d_p = 2^{2p+3} \nu^{p+1} \Delta_p \left[F_p^{(2)}(\lambda) \right] \Big|_{\lambda=\pi\zeta}. \tag{3.67}$$

The function $F_p^{(2)}(\lambda)$ can be expressed in terms of the Kronecker double series of a complex argument. More precisely, from equation (D.10) in Appendix D we have that

$$F_p^{(2)}(\lambda) = \frac{B_{2p+2}\left(\frac{1}{2}\right) - K_{2p+2}^{\frac{1}{2},0}\left(\frac{i\nu\lambda}{\pi}\right)}{4(p+1)}. \tag{3.68}$$

Furthermore, since the free term in the operator Δ_p in (1.11) is equal to g_{2p+1}, we obtain that

$$\Delta_p F_p^{(2)}(\lambda) = \frac{B_{2p+2}\left(\frac{1}{2}\right)}{4(p+1)} g_{2p+1} - \frac{\Delta_p K_{2p+2}^{\frac{1}{2},0}\left(\frac{i\nu\lambda}{\pi}\right)}{4(p+1)}, \tag{3.69}$$

and therefore,

$$\begin{aligned}
B^{(2)} &= \sum_{p=1}^{\infty} \frac{2^{2p+1}\nu^{p+1} B_{2p+2}\left(\frac{1}{2}\right)}{S^p(p+1)} g_{2p+1} - \sum_{p=1}^{\infty} \frac{2^{2p+1}\nu^{p+1}}{S^p(p+1)} \Delta_p \left[K_{2p+2}^{\frac{1}{2},0}\left(\frac{i\nu\lambda}{\pi}\right) \right] \Big|_{\lambda=\pi\zeta} \\
&= m \sum_{p=1}^{\infty} \frac{B_{2p+2}\left(\frac{1}{2}\right)}{(p+1)\left(\frac{n}{2}\right)^{2p+1}} g_{2p+1} + \sum_{p=1}^{\infty} \frac{R_p^{(2)}}{S^p},
\end{aligned} \tag{3.70}$$

(recall that $S = \nu n^2$), and this completes the proof of Lemma 3.2.   □

3.3. Evaluation of $\ln(\mathrm{Pf}\, A_2)$.

To evaluate $\ln(\mathrm{Pf}\, A_2)$, substitute (3.10) and (3.19) into (3.8) to obtain

$$\begin{aligned}
\ln(\mathrm{Pf}\, A_2) &\sim \left(SF + \frac{\gamma}{6} - m \sum_{p=1}^{\infty} \frac{B_{2p+2}\left(\frac{1}{2}\right)}{(p+1)\left(\frac{n}{2}\right)^{2p+1}} g_{2p+1} \right) \\
&\quad + \left(4 \sum_{k=0}^{\infty} \ln(1 - e^{-(2k+1)\gamma}) + m \sum_{p=1}^{\infty} \frac{B_{2p+2}\left(\frac{1}{2}\right)}{(p+1)\left(\frac{n}{2}\right)^{2p+1}} g_{2p+1} + \sum_{p=1}^{\infty} \frac{R_p^{(2)}}{S^p} \right) \\
&= SF + \frac{\gamma}{6} + \sum_{k=1}^{\infty} \ln\left(1 - e^{-(2k-1)\gamma}\right)^4 + \sum_{p=1}^{\infty} \frac{R_p^{(2)}}{S^p}, \quad \gamma = \pi\nu\zeta.
\end{aligned} \tag{3.71}$$

Note that the series containing the Bernoulli polynomials $B_{2p+2}(\frac{1}{2})$ cancel out. If we let the elliptic nome be equal to

$$q = e^{-\gamma}, \tag{3.72}$$

then by using (1.16) and (1.19), we obtain that

$$
e^{\frac{\gamma}{6}} \prod_{k=1}^{\infty} (1 - e^{-(2k-1)\gamma})^4 = q^{-1/6} \prod_{k=1}^{\infty} (1 - q^{2k-1})^4 = \left[q^{-1/24} \prod_{k=1}^{\infty} \frac{(1 - q^k)}{(1 - q^{2k})} \right]^4
$$
$$
= \left[\frac{q^{1/24} \prod_{k=1}^{\infty} (1 - q^k)}{q^{1/12} \prod_{k=1}^{\infty} (1 - q^{2k})} \right]^4 = \left[\frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)} \right]^4 = \frac{\theta_4^2}{\eta^2} .
$$
(3.73)

Therefore, equation (3.71) implies that

$$
\mathrm{Pf}\, A_2 = e^{SF} \frac{\theta_4^2}{\eta^2} e^{R^{(2)}}, \quad R^{(2)} \sim \sum_{p=1}^{\infty} \frac{R_p^{(2)}}{S^p} .
$$
(3.74)

4. Asymptotic expansions of $\mathrm{Pf}\, A_3$ and $\mathrm{Pf}\, A_4$ for even n

The asymptotic expansions of $\mathrm{Pf}\, A_3$ and $\mathrm{Pf}\, A_4$ for even n can be obtained in the same way as the one of $\mathrm{Pf}\, A_2$. Let us briefly discuss them.

From formula (2.10) we have that

$$
\mathrm{Pf}\, A_3 = \prod_{j=0}^{\frac{m}{2}-1} \prod_{k=0}^{n-1} \left[4 \left(\sin^2 \frac{2\pi(j + \frac{1}{2})}{m} + \zeta^2 \sin^2 \frac{2\pi k}{n} \right) \right]^{1/2} .
$$
(4.1)

Using that $\sin^2(x + \pi) = \sin^2 x$, we can rewrite the latter formula for even n as

$$
\mathrm{Pf}\, A_3 = \prod_{j=0}^{\frac{m}{2}-1} \prod_{k=0}^{\frac{n}{2}-1} \left[4 \left(\sin^2 \frac{2\pi(j + \frac{1}{2})}{m} + \zeta^2 \sin^2 \frac{2\pi k}{n} \right) \right] .
$$
(4.2)

Using the Chebyshev type identity (see [8]),

$$
\prod_{j=0}^{\frac{m}{2}-1} \left[4 \left(u^2 + \sin^2 \frac{(2j+1)\pi}{m} \right) \right] = \left[\left(u + \sqrt{1 + u^2} \right)^{\frac{m}{2}} + \left(-u + \sqrt{1 + u^2} \right)^{\frac{m}{2}} \right]^2 ,
$$
(4.3)

we obtain that

$$
\mathrm{Pf}\, A_3 = \prod_{k=0}^{\frac{n}{2}-1} \left[\left(u_k + \sqrt{1 + u_k^2} \right)^{\frac{m}{2}} + \left(-u_k + \sqrt{1 + u_k^2} \right)^{\frac{m}{2}} \right]^2 .
$$
(4.4)

where

$$
u_k = \zeta \sin(\pi x_k) \geq 0, \quad x_k = \frac{2k}{n} ,
$$
(4.5)

which implies that

$$
\ln(\mathrm{Pf}\, A_3) = G_{m,n}^{(3)} + H_{m,n}^{(3)} ,
$$
(4.6)

where

$$G_{m,n}^{(3)} = m \sum_{k=0}^{\frac{n}{2}-1} \ln\left(u_k + \sqrt{1+u_k^2}\right) = m \sum_{k=0}^{\frac{n}{2}-1} g\left(\frac{2k}{n}\right),$$

$$H_{m,n}^{(3)} = 2 \sum_{k=0}^{\frac{n}{2}-1} \ln\left[1 + \frac{1}{\left(u_k + \sqrt{1+u_k^2}\right)^m}\right]. \tag{4.7}$$

Using the Euler–Maclaurin formula, we obtain the following asymptotic expansion:

$$G_{m,n}^{(3)} \sim SF - \frac{\gamma}{3} - m \sum_{p=1}^{\infty} \frac{B_{2p+2}(0)\, g_{2p+1}}{(p+1)\left(\frac{n}{2}\right)^{2p+1}}, \quad \gamma = \pi\nu\zeta. \tag{4.8}$$

Next, we obtain an asymptotic expansion of $H_{m,n}^{(3)}$:

$$H_{m,n}^{(3)} \sim A^{(3)} + B^{(3)}, \tag{4.9}$$

with

$$A^{(3)} = 4\sum_{k=1}^{\infty} \ln\left(1 + e^{-2k\gamma}\right) + 2\ln 2,$$

$$B^{(3)} = m\sum_{p=1}^{\infty} \frac{B_{2p+2}(0)}{(p+1)\left(\frac{n}{2}\right)^{2p+1}} g_{2p+1} + \sum_{p=1}^{\infty} \frac{R_p^{(3)}}{S^p}, \tag{4.10}$$

where

$$R_p^{(3)} = -\frac{2^{2p+1}\nu^{p+1}}{p+1} \Delta_p \left[K_{2p+2}^{0,\frac{1}{2}}\left(\frac{i\nu\lambda}{\pi}\right)\right]\Bigg|_{\lambda=\pi\zeta}. \tag{4.11}$$

Substituting (4.8) and (4.9) into (4.6) we obtain that

$$\ln(\mathrm{Pf}\, A_3) \sim SF - \frac{\gamma}{3} + 4\sum_{k=1}^{\infty} \ln\left(1 + e^{-2k\gamma}\right) + 2\ln 2 + \sum_{p=1}^{\infty} \frac{R_p^{(3)}}{S^p}. \tag{4.12}$$

Let $q = e^{-\gamma}$. Since

$$4e^{-\frac{\gamma}{3}} \prod_{k=1}^{\infty} \left(1 + e^{-2k\gamma}\right)^4 = 4q^{1/3} \prod_{k=1}^{\infty} (1+q^{2k})^4 = 4\left[q^{1/12} \prod_{k=1}^{\infty} \frac{(1-q^{4k})}{(1-q^{2k})}\right]^4$$

$$= 4\left[\frac{q^{1/6} \prod_{k=1}^{\infty}(1-q^{4k})}{q^{1/12} \prod_{k=1}^{\infty}(1-q^{2k})}\right]^4 = 4\left[\frac{\eta(2\tau)}{\eta(\tau)}\right]^4 = \frac{\theta_2^2}{\eta^2}, \tag{4.13}$$

we obtain that

$$\mathrm{Pf}\, A_3 = e^{SF} \frac{\theta_2^2}{\eta^2} e^{R^{(3)}}, \quad R^{(3)} \sim \sum_{p=1}^{\infty} \frac{R_p^{(3)}}{S^p}. \tag{4.14}$$

Let us turn to $\mathrm{Pf}\, A_4$. Since $\sin^2(x+\pi) = \sin^2 x$, we can rewrite $\mathrm{Pf}\, A_4$ in (2.10) for even n as

$$\mathrm{Pf}\, A_4 = \prod_{j=0}^{\frac{m}{2}-1} \prod_{k=0}^{\frac{n}{2}-1} \left[4\left(\sin^2 \frac{(2j+1)\pi}{m} + \zeta^2 \sin^2 \frac{(2k+1)\pi}{n}\right)\right]. \tag{4.15}$$

Using identity (4.3), we obtain that

$$\mathrm{Pf}\, A_4 = \prod_{k=0}^{\frac{n}{2}-1} \left[\left(u_k + \sqrt{1+u_k^2} \right)^{\frac{m}{2}} + \left(-u_k + \sqrt{1+u_k^2} \right)^{\frac{m}{2}} \right]^2, \tag{4.16}$$

where

$$u_k = \zeta \sin(\pi x_k) \geq 0, \quad x_k = \frac{2k+1}{n}, \tag{4.17}$$

which implies that

$$\ln(\mathrm{Pf}\, A_4) = G_{m,n}^{(4)} + H_{m,n}^{(4)}, \tag{4.18}$$

where

$$G_{m,n}^{(4)} = m \sum_{k=0}^{\frac{n}{2}-1} \ln\left(u_k + \sqrt{1+u_k^2} \right) = m \sum_{k=0}^{\frac{n}{2}-1} g\left(\frac{2k+1}{n} \right),$$

$$H_{m,n}^{(4)} = 2 \sum_{k=0}^{\frac{n}{2}-1} \ln\left[1 + \frac{1}{\left(u_k + \sqrt{1+u_k^2} \right)^m} \right]. \tag{4.19}$$

Using the Euler–Maclaurin formula, we obtain the following asymptotic expansion:

$$G_{m,n}^{(4)} \sim SF + \frac{\gamma}{6} - m \sum_{p=1}^{\infty} \frac{B_{2p+2}\left(\frac{1}{2}\right) g_{2p+1}}{(p+1)\left(\frac{n}{2}\right)^{2p+1}}, \quad \gamma = \pi\nu\zeta, \tag{4.20}$$

and then, similar to Lemma 3.2, we obtain that

$$H_{m,n}^{(4)} \sim A^{(4)} + B^{(4)}, \tag{4.21}$$

with

$$A^{(4)} = 4 \sum_{k=0}^{\infty} \ln\left(1 + e^{-\gamma(2k+1)} \right),$$

$$B^{(4)} = m \sum_{p=1}^{\infty} \frac{B_{2p+2}\left(\frac{1}{2}\right)}{(p+1)\left(\frac{n}{2}\right)^{2p+1}} g_{2p+1} + \sum_{p=1}^{\infty} \frac{R_p^{(4)}}{S^p}, \tag{4.22}$$

where

$$R_p^{(4)} = -\frac{2^{2p+1}\nu^{p+1}}{p+1} \Delta_p \left[K_{2p+2}^{\frac{1}{2},\frac{1}{2}} \left(\frac{i\nu\lambda}{\pi} \right) \right] \Big|_{\lambda=\pi\zeta}. \tag{4.23}$$

Substituting (4.20) and (4.21) into (4.18), we obtain that

$$\mathrm{Pf}\, A_4 = e^{SF} \frac{\theta_3^2}{\eta^2} e^{R^{(4)}}, \quad R^{(4)} \sim \sum_{p=1}^{\infty} \frac{R_p^{(4)}}{S^p}. \tag{4.24}$$

Substituting equations (3.74), (4.14), and (4.24) into (2.11), we obtain the asymptotic formula for Z, (2.16), for even n.

5. Asymptotic behavior of Pf A_3 for odd n

From equation (2.10) we have that

$$\text{Pf } A_3 = \prod_{j=0}^{\frac{m}{2}-1} \prod_{k=0}^{n-1} \left[4 \left(\sin^2 \frac{(2j+1)\pi}{m} + \zeta^2 \sin^2 \frac{2\pi k}{n} \right) \right]^{1/2}. \tag{5.1}$$

For odd n, using the identity $\sin^2(x + \pi) = \sin^2 x$, we can rewrite the latter formula as

$$\text{Pf } A_3 = \prod_{j=0}^{\frac{m}{2}-1} \prod_{k=0}^{n-1} \left[4 \left(\sin^2 \frac{(2j+1)\pi}{m} + \zeta^2 \sin^2 \frac{\pi k}{n} \right) \right]^{1/2}. \tag{5.2}$$

Indeed, if we take $0 \le k \le \frac{n-1}{2}$ in (5.1), then we obtain factors with $\sin^2 \frac{2\pi k}{n}$, while if we take $k = \frac{n+1}{2} + k'$, $0 \le k' \le \frac{n-3}{2}$, then we obtain factors with

$$\sin^2 \frac{2\pi}{n} \left(\frac{n+1}{2} + k' \right) = \sin^2 \frac{\pi(2k'+1)}{n}.$$

Combining these two cases, we obtain (5.2).

From (5.2), using identity (4.3), we obtain that

$$\text{Pf } A_3 = \prod_{k=0}^{n-1} \left[\left(u_k + \sqrt{1 + u_k^2} \right)^{\frac{m}{2}} + \left(-u_k + \sqrt{1 + u_k^2} \right)^{\frac{m}{2}} \right], \tag{5.3}$$

where

$$u_k = \zeta \sin(\pi x_k) \ge 0, \quad x_k = \frac{k}{n}, \tag{5.4}$$

which implies that

$$\ln(\text{Pf } A_3) = G_{m,n}^{(3)} + H_{m,n}^{(3)}, \tag{5.5}$$

where

$$G_{m,n}^{(3)} = \frac{m}{2} \sum_{k=0}^{n-1} \ln \left(u_k + \sqrt{1 + u_k^2} \right) = \frac{m}{2} \sum_{k=0}^{n-1} g \left(\frac{k}{n} \right),$$

$$H_{m,n}^{(3)} = \sum_{k=0}^{n-1} \ln \left[1 + \frac{1}{\left(u_k + \sqrt{1 + u_k^2} \right)^m} \right]. \tag{5.6}$$

Formulae (5.3)-(5.6) are similar to (4.4)-(4.7) for even n, but $x_k = \frac{2k}{n}$ for even n, while $x_k = \frac{k}{n}$ for odd n. This leads to the difference in elliptic nome (2.14) for even and odd n.

Using the Euler–Maclaurin formula, we obtain that

$$G_{m,n}^{(3)} \sim SF - \frac{\gamma}{12} - \frac{m}{2} \sum_{p=1}^{\infty} \frac{B_{2p+2}(0)}{(p+1)n^{2p+1}} g_{2p+1}, \quad \gamma = \pi \nu \zeta, \tag{5.7}$$

and similar to the even case, we obtain the asymptotic expansion of $H_{m,n}^{(3)}$ as

$$H_{m,n}^{(3)} \sim A^{(3)} + B^{(3)}, \tag{5.8}$$

with

$$A^{(3)} = 2 \sum_{k=1}^{\infty} \ln(1 + e^{-\gamma k}),$$

$$B^{(3)} = \frac{m}{2} \sum_{p=1}^{\infty} \frac{B_{2p+2}(0)g_{2p+1}}{(p+1)n^{2p+1}} + \sum_{p=1}^{\infty} \frac{R_p^{(3)}}{S^p},$$

(5.9)

where

$$R_p^{(3)} = -\frac{\nu^{p+1}}{p+1} \Delta_p \left[K_{2p+2}^{0,\frac{1}{2}} \left(\frac{i\nu\lambda}{2\pi} \right) \right] \Bigg|_{\lambda=\pi\zeta}.$$

(5.10)

Combining (5.5), (5.7), and (5.8), we obtain that

$$\ln(\mathrm{Pf}\, A_3) \sim SF - \frac{\gamma}{12} + 2 \sum_{k=1}^{\infty} \ln(1 + e^{-\gamma k}) + \sum_{p=1}^{\infty} \frac{R_p^{(3)}}{S^p} + \ln 2.$$

(5.11)

Let now

$$q = e^{\pi i \tau} = e^{-\frac{\gamma}{2}},$$

(5.12)

then

$$2e^{-\frac{\gamma}{12}} \prod_{k=1}^{\infty} \left(1 + e^{-\gamma k} \right)^2 = 2q^{1/6} \prod_{k=1}^{\infty} (1 + q^{2k})^2 = 2 \left[\prod_{k=1}^{\infty} \frac{q^{1/6}(1 - q^{4k})}{q^{1/12}(1 - q^{2k})} \right]^2$$

$$= 2 \left[\frac{\eta(2\tau)}{\eta(\tau)} \right]^2 = \frac{\theta_2}{\eta},$$

(5.13)

hence

$$\ln Z = \ln(\mathrm{Pf}\, A_3) \sim SF + \ln \frac{\theta_2}{\eta} + \sum_{p=1}^{\infty} \frac{R_p^{(3)}}{S^p}.$$

(5.14)

This finishes the proof of Theorem 2.1.

Appendix A. Exponent of a Taylor Series

Proposition A.1. *We have that*

$$\exp \left(\sum_{p=1}^{\infty} a_p x^p \right) = 1 + \sum_{p=1}^{\infty} b_p x^p,$$

(A.1)

where

$$b_p = \sum_{\mathcal{S}_p} \frac{(a_{p_1})^{q_1} \ldots (a_{p_r})^{q_r}}{q_1! \ldots q_r!},$$

(A.2)

and $\mathcal{S}_p$ is the set of collections of positive integers $(p_1, \ldots, p_r; q_1, \ldots, q_r)$, $1 \le r \le p$, such that

$$\mathcal{S}_p = \left\{ (p_1, \ldots, p_r; q_1, \ldots, q_r) \, \big| \, 0 < p_1 < \ldots < p_r; \ p_1 q_1 + \ldots + p_r q_r = p \right\}.$$

(A.3)

The series in (A.1) are understood as formal ones.

Proof. Expanding the exponent into the Taylor series, we obtain that

$$\exp\left(\sum_{p=1}^{\infty} a_p x^p\right) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\sum_{p=1}^{\infty} a_p x^p\right)^k. \tag{A.4}$$

By the multinomial formula,

$$\frac{1}{k!}\left(\sum_{p=1}^{\infty} a_p x^p\right)^k = \sum_{0<p_1<...<p_r,\ q_1>0,...,q_r>0:\ q_1+...+q_r=k} \frac{(a_{p_1} x^{p_1})^{q_1}\ldots(a_{p_r} x^{p_r})^{q_r}}{q_1!\ldots q_r!}, \tag{A.5}$$

hence

$$\exp\left(\sum_{p=1}^{\infty} a_p x^p\right) = 1 + \sum_{k=1}^{\infty}\ \sum_{0<p_1<...<p_r,\ q_1>0,...,q_r>0:\ q_1+...+q_r=k} \frac{(a_{p_1} x^{p_1})^{q_1}\ldots(a_{p_r} x^{p_r})^{q_r}}{q_1!\ldots q_r!}$$
$$= 1 + \sum_{r=1}^{\infty}\ \sum_{0<p_1<...<p_r,\ q_1>0,...,q_r>0} \frac{(a_{p_1} x^{p_1})^{q_1}\ldots(a_{p_r} x^{p_r})^{q_r}}{q_1!\ldots q_r!}. \tag{A.6}$$

Combining terms with $p_1 q_1 + \ldots + p_r q_r = p$, we obtain formulae (A.1), (A.2). $\square$

Appendix B. Bernoulli's Polynomials

Bernoulli's polynomials are defined recursively by the equations,

$$B_k'(x) = k B_{k-1}(x), \quad \int_0^1 B_k(x)\,dx = 0, \quad k = 1, 2, \ldots; \quad B_0(x) = 1. \tag{B.1}$$

In particular,

$$B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2}. \tag{B.2}$$

The Bernoulli periodic functions $\widehat{B}_k(x)$ are defined by the periodicity condition $\widehat{B}_k(x+1) = \widehat{B}_k(x)$ and by the condition $\widehat{B}_k(x) = B_k(x)$ for $0 \le x \le 1$. Their Fourier series is equal to

$$\widehat{B}_k(x) = -\frac{k!}{(-2\pi i)^k} \sum_{\ell \neq 0} \frac{e^{-2\pi i \ell x}}{\ell^k}. \tag{B.3}$$

For $k \ge 2$ the Fourier series is absolutely convergent, and for $k = 1$ it converges in $L^2[0,1]$.

The generating function of the Bernoulli polynomials is

$$G(\lambda; x) := \frac{\lambda e^{\lambda x}}{e^\lambda - 1} = \sum_{k=0}^{\infty} \frac{\lambda^k B_k(x)}{k!}. \tag{B.4}$$

Substituting (B.3) into (B.4), we obtain that for $0 \le x \le 1$ and $|\lambda| < 2\pi$,

$$\frac{\lambda e^{\lambda x}}{e^{\lambda} - 1} - 1 - \lambda \left(x - \frac{1}{2} \right) = \sum_{k=2}^{\infty} \frac{\lambda^k B_k(x)}{k!} = -\sum_{k=2}^{\infty} \sum_{\ell \neq 0} \frac{\lambda^k}{(-2\pi i)^k} \frac{e^{-2\pi i \ell x}}{\ell^k}$$

$$= -\sum_{\ell \neq 0} e^{-2\pi i \ell x} \sum_{k=2}^{\infty} \left(\frac{\lambda}{-2\pi i \ell} \right)^k = -\sum_{\ell \neq 0} e^{-2\pi i \ell x} \left(\frac{\lambda}{-2\pi i \ell} \right)^2 \frac{1}{1 + \frac{\lambda}{2\pi i \ell}} \tag{B.5}$$

$$= \frac{\lambda^2}{4\pi^2} \sum_{\ell \neq 0} \frac{e^{-2\pi i \ell x}}{\ell \left(\ell + \frac{\lambda}{2\pi i} \right)} \, .$$

Taking $\lambda = 2\pi i z$, where $|z| < 1$, $x = \alpha$, and $\ell = k$, we obtain that

$$\frac{2\pi i z e^{2\pi i z \alpha}}{e^{2\pi i z} - 1} = 1 + 2\pi i z \left(\alpha - \frac{1}{2} \right) - z^2 \sum_{k \neq 0} \frac{e^{-2\pi i k \alpha}}{k(k+z)} \, , \tag{B.6}$$

or

$$\frac{e^{2\pi i z \alpha}}{e^{2\pi i z} - 1} = \frac{1}{2\pi i z} + \left(\alpha - \frac{1}{2} \right) - \frac{z}{2\pi i} \sum_{k \neq 0} \frac{e^{-2\pi i k \alpha}}{k(k+z)} \, , \tag{B.7}$$

We can rewrite the latter equation as

$$\frac{e^{2\pi i z \alpha}}{e^{2\pi i z} - 1} = \frac{1}{2\pi i z} + \left(\alpha - \frac{1}{2} \right) - \frac{1}{2\pi i} \sum_{k \neq 0} e^{-2\pi i k \alpha} \left(\frac{1}{k} - \frac{1}{k+z} \right) \, , \tag{B.8}$$

Appendix C. The Euler–Maclaurin Formula

Let $f(x)$ be an analytic function on the interval $[a, b]$. We partition the interval $[a, b]$ into N equal intervals of the length

$$h = \frac{b - a}{N} \, . \tag{C.1}$$

Let

$$x_k = a + kh + \alpha h, \quad k = 0, 1, \dots, N - 1, \tag{C.2}$$

where $0 \le \alpha \le 1$. Then the Euler–Maclaurin formula with a remainder is

$$\sum_{k=0}^{N-1} f(x_k) = \frac{1}{h} \int_a^b f(x) \, dx + \sum_{p=1}^{\ell} \frac{B_p(\alpha) h^{p-1}}{p!} [f^{(p-1)}(b) - f^{(p-1)}(a)] + R_\ell(\alpha), \tag{C.3}$$

where $B_p(\alpha)$ is the Bernoulli polynomial and the remainder $R_\ell(\alpha)$ can be written as

$$R_\ell(\alpha) = \frac{h^\ell}{\ell!} \int_0^1 \widehat{B}_\ell(\alpha - \tau) \left[\sum_{k=0}^{N-1} f^{(\ell)}(a + kh + \tau h) \right] d\tau, \tag{C.4}$$

where $\widehat{B}_\ell(x)$ is the periodic Bernoulli function.

Thus, the Euler-Maclaurin formula gives an asymptotic series,

$$\sum_{k=0}^{N-1} f(x_k) \sim \frac{1}{h} \int_a^b f(x) \, dx + \sum_{p=1}^{\infty} \frac{B_p(\alpha) h^{p-1}}{p!} [f^{(p-1)}(b) - f^{(p-1)}(a)]. \tag{C.5}$$

In general, since both $B_p(x)$ and $f^{(p)}(x)$ grow like $p!$, the series on the right in (C.5) diverges.

APPENDIX D. KRONECKER'S DOUBLE SERIES OF PURE IMAGINARY ARGUMENT

A classical reference to the Kronecker double series is the book of Weil [19]. In this Appendix we review and specify some results of Ivashkevich, Izmailian, and Hu [6]. Let us consider the Kronecker double series with parameters $(\alpha, \beta) = (\frac{1}{2}, 0)$ as defined in (1.15) with argument $\tau = 2ir\zeta$ and $(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2})$ as defined in (1.15) with argument $\tau = ir\zeta$. Observe that in all cases, if p is odd, the terms (j, k) and $(-j, -k)$ cancel each other. Hence $K_{2p-1}^{\frac{1}{2},0}(\tau) = K_{2p-1}^{\frac{1}{2},\frac{1}{2}}(\tau) = K_{2p-1}^{0,\frac{1}{2}}(\tau) = 0$ for $p = 1, 2, \ldots$. Therefore, we will take p to be even. Let us first consider the case $(\alpha, \beta) = (\frac{1}{2}, 0)$.

D.1. Case $(\alpha, \beta) = (\frac{1}{2}, 0)$. From (1.15), we have that

$$K_{2p}^{\frac{1}{2},0}(\tau) = \frac{(-1)^{p+1}(2p)!}{(2\pi)^{2p}} \sum_{(j,k)\neq(0,0)} \frac{(-1)^k}{(k+\tau j)^{2p}}. \tag{D.1}$$

Separating terms with $j = 0$, we obtain that

$$K_{2p}^{\frac{1}{2},0}(\tau) = \frac{(-1)^{p+1}(2p)!}{(2\pi)^{2p}} \sum_{k\neq0} \frac{(-1)^k}{k^{2p}} + \frac{(-1)^{p+1}(2p)!}{(2\pi)^{2p}} \sum_{j\neq0} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+\tau j)^{2p}}. \tag{D.2}$$

The first term is just the Fourier series for the Bernoulli polynomial $B_{2p}(x)$ evaluated at $x = \frac{1}{2}$. Let us transform the second term. Since the terms (j, k) and $(-j, -k)$ give the same contribution, we can write that

$$K_{2p}^{\frac{1}{2},0}(\tau) = B_{2p}\left(\frac{1}{2}\right) + \frac{2(-1)^{p+1}(2p)!}{(2\pi)^{2p}} \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+\tau j)^{2p}}. \tag{D.3}$$

When $z = iy$, $y > 0$, and $\alpha = 1/2$, identity (B.8) reads

$$\frac{e^{-\pi y}}{1 - e^{-2\pi y}} = \frac{1}{2\pi y} + \frac{1}{2\pi i} \sum_{k\neq0} (-1)^k \left(\frac{1}{k} - \frac{1}{k+iy}\right). \tag{D.4}$$

Expanding the left hand side into the geometric series, we obtain that

$$\sum_{k=0}^{\infty} e^{-2\pi y(k+\frac{1}{2})} = \frac{1}{2\pi y} + \frac{1}{2\pi i} \sum_{k\neq0} (-1)^k \left(\frac{1}{k} - \frac{1}{k+iy}\right). \tag{D.5}$$

Differentiating this identity $(2p - 1)$ times with respect to y, we obtain that

$$\sum_{k=0}^{\infty} \left[-2\pi\left(k + \frac{1}{2}\right)\right]^{2p-1} e^{-2\pi y(k+\frac{1}{2})} = -\frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} \frac{(-i)^{2p-1}(2p-1)!(-1)^k}{(k+iy)^{2p}}, \tag{D.6}$$

or equivalently,

$$\frac{(-1)^p(2p)!}{(2\pi)^{2p}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+iy)^{2p}} = 2p \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right)^{2p-1} e^{-2\pi y(k+\frac{1}{2})}. \tag{D.7}$$

Using this formula in (D.3) with $y = r\zeta j$, we obtain that

$$K_{2p}^{\frac{1}{2},0}(\tau) = B_{2p}\left(\frac{1}{2}\right) - 4p\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)^{2p-1}e^{-2r\lambda j(k+\frac{1}{2})}. \tag{D.8}$$

Thus, we have the following proposition:

Proposition D.1. *We have that*

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)^{2p-1}e^{-2r\lambda j(k+\frac{1}{2})} = \frac{B_{2p}\left(\frac{1}{2}\right) - K_{2p}^{\frac{1}{2},0}(\tau)}{4p}. \tag{D.9}$$

Applying it for $K_{2p+2}^{\frac{1}{2},0}(\tau)$, we obtain that

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)^{2p+1}e^{-2r\lambda j(k+\frac{1}{2})} = \frac{B_{2p+2}\left(\frac{1}{2}\right) - K_{2p+2}^{\frac{1}{2},0}(\tau)}{4(p+1)}. \tag{D.10}$$

D.2. **Case** $(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2})$. From (1.15), we have that

$$K_{2p}^{\frac{1}{2},\frac{1}{2}}(\tau) = \frac{(-1)^{p+1}(2p)!}{(2\pi)^{2p}}\sum_{(j,k)\neq(0,0)}\frac{(-1)^{k+j}}{(k+\tau j)^{2p}}. \tag{D.11}$$

Separating terms with $j = 0$, we obtain that

$$K_{2p}^{\frac{1}{2},\frac{1}{2}}(\tau) = \frac{(-1)^{p+1}(2p)!}{(2\pi)^{2p}}\sum_{k\neq 0}\frac{(-1)^{k}}{k^{2p}} + \frac{(-1)^{p+1}(2p)!}{(2\pi)^{2p}}\sum_{j\neq 0}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+j}}{(k+\tau j)^{2p}}. \tag{D.12}$$

The first term is just the Fourier series for the Bernoulli polynomial $B_{2p}(x)$ evaluated at $x = \frac{1}{2}$. Let us transform the second term. Since the terms (j, k) and $(-j, -k)$ give the same contribution, we can write that

$$K_{2p}^{\frac{1}{2},\frac{1}{2}}(\tau) = B_{2p}\left(\frac{1}{2}\right) + \frac{2(-1)^{p+1}(2p)!}{(2\pi)^{2p}}\sum_{j=1}^{\infty}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k+j}}{(k+\tau j)^{2p}}. \tag{D.13}$$

Now from identity (D.7), with $y = r\zeta j$, we obtain that

$$K_{2p}^{\frac{1}{2},\frac{1}{2}}(\tau) = B_{2p}\left(\frac{1}{2}\right) - 4p\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}(-1)^{j}\left(k+\frac{1}{2}\right)^{2p-1}e^{-2r\lambda j(k+\frac{1}{2})}. \tag{D.14}$$

Thus, we have the following proposition:

Proposition D.2. *We have that*

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}(-1)^{j}\left(k+\frac{1}{2}\right)^{2p-1}e^{-2r\lambda j(k+\frac{1}{2})} = \frac{B_{2p}\left(\frac{1}{2}\right) - K_{2p}^{\frac{1}{2},\frac{1}{2}}(\tau)}{4p}. \tag{D.15}$$

Applying it for $K_{2p+2}(\tau)$, we obtain that

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}(-1)^{j}\left(k+\frac{1}{2}\right)^{2p+1}e^{-2r\lambda j(k+\frac{1}{2})} = \frac{B_{2p+2}\left(\frac{1}{2}\right) - K_{2p+2}^{\frac{1}{2},\frac{1}{2}}(\tau)}{4(p+1)}. \tag{D.16}$$

D.3. Case $(\alpha, \beta) = (0, \frac{1}{2})$**.** From (1.15), we have that

$$K_{2p}^{0,\frac{1}{2}}(\tau) = \frac{(-1)^{p+1}(2p)!}{(2\pi)^{2p}} \sum_{(j,k)\neq(0,0)} \frac{(-1)^j}{(k+\tau j)^{2p}} . \tag{D.17}$$

Separating terms with $j = 0$, we obtain that

$$K_{2p}^{0,\frac{1}{2}}(\tau) = \frac{(-1)^{p+1}(2p)!}{(2\pi)^{2p}} \sum_{k\neq 0} \frac{1}{k^{2p}} + \frac{(-1)^{p+1}(2p)!}{(2\pi)^{2p}} \sum_{j\neq 0} \sum_{k=-\infty}^{\infty} \frac{(-1)^j}{(k+\tau j)^{2p}} . \tag{D.18}$$

The first term is just the Fourier series for the Bernoulli polynomial $B_{2p}(x)$ evaluated at $x = 0$. Let us transform the second term. Since the terms (j, k) and $(-j, -k)$ give the same contribution, we can write that

$$K_{2p}^{0,\frac{1}{2}}(\tau) = B_{2p}(0) + \frac{2(-1)^{p+1}(2p)!}{(2\pi)^{2p}} \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^j}{(k+\tau j)^{2p}} . \tag{D.19}$$

When $z = iy$, $y > 0$, and $\alpha = 0$, identity (B.8) reads

$$\frac{1}{1 - e^{-2\pi y}} = \frac{1}{2\pi y} + \frac{1}{2} + \frac{1}{2\pi i} \sum_{k\neq 0} \left(\frac{1}{k} - \frac{1}{k+iy} \right) . \tag{D.20}$$

Expanding the left hand side into the geometric series, we obtain that

$$\sum_{k=0}^{\infty} e^{-2\pi yk} = \frac{1}{2\pi y} + \frac{1}{2} + \frac{1}{2\pi i} \sum_{k\neq 0} \left(\frac{1}{k} - \frac{1}{k+iy} \right) . \tag{D.21}$$

Differentiating this identity $(2p - 1)$ times with respect to y, we obtain that

$$\sum_{k=0}^{\infty} [-2\pi k]^{2p-1} e^{-2\pi yk} = -\frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} \frac{(-i)^{2p-1}(2p-1)!}{(k+iy)^{2p}} , \tag{D.22}$$

or equivalently,

$$\frac{(-1)^p(2p)!}{(2\pi)^{2p}} \sum_{k=-\infty}^{\infty} \frac{1}{(k+iy)^{2p}} = 2p \sum_{k=0}^{\infty} k^{2p-1} e^{-2\pi yk} . \tag{D.23}$$

Using this formula in (D.19) with $y = r\zeta j$, we obtain that

$$K_{2p}^{0,\frac{1}{2}}(\tau) = B_{2p}(0) - 4p \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} (-1)^j k^{2p-1} e^{-2\lambda rjk} . \tag{D.24}$$

Thus, we have the following proposition:

Proposition D.3. *We have that*

$$\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} (-1)^j k^{2p-1} e^{-2\lambda rjk} = \frac{B_{2p}(0) - K_{2p}^{0,\frac{1}{2}}(\tau)}{4p} . \tag{D.25}$$

Applying it for $K_{2p+2}^{0,\frac{1}{2}}(\tau)$, we obtain that

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}(-1)^j k^{2p+1} e^{-2\lambda rjk} = \frac{B_{2p+2}(0) - K_{2p+2}^{0,\frac{1}{2}}(\tau)}{4(p+1)}. \tag{D.26}$$

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