EXPONENTIAL ERGODICITY AND CONVERGENCE FOR GENERALIZED REFLECTED BROWNIAN MOTION

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ABSTRACT. In this paper we provide convergence analysis for a class of Brownian queues in tandem by establishing an exponential drift condition. A consequence is the uniform exponential ergodicity for these multidimensional diffusions, including the O'Connell-Yor process. A list of open problems are also presented.

Key words : Brownian queue, exponential ergodicity, Foster-Lyapunov stability, O'Connell-Yor process, reflected Brownian motion.
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1. INTRODUCTION AND MAIN RESULTS

This paper is concerned with the convergence to global equilibrium for a large class of multidimensional Brownian diffusions. We consider the following d-dimensional stochastic differential equation (SDE):

$$dX_t = dB_t + \left(\mu + \sum_{i=1}^{d} U'(n_i \cdot X_t) r_i \right) dt \quad \text{and} \quad X_0 \in \mathbb{R}^d, \tag{1.1}$$

where

- $(B_t; t \geq 0)$ is a d-dimensional Brownian motion with covariance matrix Γ;
- $\mu := (\mu_1, \ldots, \mu_d)^T \in \mathbb{R}^d$ is a vector of drifts;
- $n_1, \ldots, n_d \in \mathbb{S}^{d-1}$ are unit vectors;
- $r_1, \ldots, r_d \in \mathbb{R}^d$ are vectors of reflection and $R := (r_i)_{1 \leq i \leq d}$ is the reflection matrix;
- $U : \mathbb{R} \to \mathbb{R}$ is a smooth potential function such that $U' \geq 0$.

The SDE (1.1) was previously considered by O'Connell and Ortmann [34], the strong solution to which is called a *generalized reflected Brownian motion* parametrized by (Γ, μ, R, U), or simply $GRBM(\Gamma, \mu, R, U)$. In the sequel, let $||\Gamma|| := \sup\{||\Gamma x||; ||x|| = 1\}$ be the spectral norm of Γ.

1.1. **Motivations.** The study of GRBMs is motivated from both queueing theory and interacting particle systems. The simplest model in queueing networks is the $M/M/1$ *queue*, where customers arrive according to a Poisson process with rate λ, and are served according to exponential times with rate θ. It is well known that if $\lambda < \theta$, then the number of customers converges to a geometric random variable with parameter λ/θ. *Queues in tandem* are

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consecutive $M/M/1$ queues. Harrison [20, 21] proposed an approximate model of queues in the *heavy traffic limit*, as the *traffic intensity* λ/θ tends to 1. That is,

$$X_t = B_t + \mu t + RL_t \in [0, \infty)^d, \tag{1.2}$$

where $(B_t; t \geq 0)$, μ and R are defined as above, and $L = (L_t^i; t \geq 0)_{1 \leq i \leq d}$ is the local time process satisfying that for all $1 \leq i \leq d$,

- L^i is continuous and non-decreasing with $L_0^i = 0$,
- L^i increases only at times t such that $X_t^i = 0$.

Call the strong solution to (1.2) a *semimartingale reflected Brownian motion* parametrized by (Γ, μ, R), or simply $SRBM(\Gamma, \mu, R)$. The limit theorems were proved in [37]. The SDE (1.2) was also called the *semimartingale reflected Brownian motion in the orthant* by Harrison and Reiman [22, 23]. Let Q be obtained by replacing each entry of $I - R$ by its absolute value. Dupuis and Ishii [15] proved that if $||Q|| < 1$, then (1.2) has a unique strong solution. Taylor and Williams [47] showed that (1.2) has a weak solution which is unique in law if and only if R is completely-$\mathcal{S}$. The positive recurrence of SRBMs was explored by Dupuis and Williams [16], Chen [12], and Bramson, Dai and Harrison [8, 9, 13]. The exponential ergodicity of SRBMs was studied by Budhiraja and Lee [10], Sarantsev [43], and Blanchet and Chen [3]. See also Williams [49] for a survey. Recently, SRBMs appear to be a useful tool in the study of rank-dependent Brownian systems, see [2, 24, 42, 44, 46].

The local time process L is often not easy to study. A common technique, called the *penalty method*, is to substitute the local time process with some drift term which pushes the process inside the domain. The choice $U(y) = -e^{-y}$ in (1.1) corresponds to the *generalized Brownian queue in tandem*, introduced by O'Connell and Yor [35]. This extends earlier work on heavy traffic queues in tandem by Glynn and Whitt [18], and Harrison and Williams [19]. By introducing a parametric family of potentials $U_\beta(y) = -\frac{1}{\beta}e^{-\beta y}$ and letting $\beta \to \infty$, we get the SRBM as weak limit of GRBMs, following the work of Lions and Sznitman [27]. In the particle system literature, the O'Connell-Yor process is also called the *semi-discrete Brownian polymer*, which generalizes the low-density limit of a totally asymmetric simple exclusion process (*Brownian TASEP*). This polymer model was proved to be exactly solvable [33], and belongs to the KPZ universality class [6, 7].

O'Connell and Ortmann [34, Corollary 4.11] proved that under the *generalized skew-symmetry condition*

$$r_{ij} + r_{ji} = 2\Gamma_{ij} \quad \text{for } 1 \leq i \neq j \leq d, \tag{1.3}$$

and under sufficient regularity for U, $GRBM(\Gamma, \mu, R, U)$ has a product-form stationary distribution

$$p(x) = \exp\left\{ 2 \left[\sum_{i=1}^{d} U(x_i) + (2\Gamma - R)^{-1}\mu \cdot x \right] \right\}, \tag{1.4}$$

provided that $\int_{\mathbb{R}^d} p(x)dx < \infty$. This result is an analog of Williams [48, 49] regarding SRBMs. Kang and Ramanan [25] characterized stationary distributions of reflected diffusions with state-dependent drifts. Lépingle [26] considered a two-dimensional GRBM with logarithmic potential.

The formula (1.4) suggests that the GRBM converge exponentially to its stationary distribution. The intuition comes from the Poincaré inequality, see [1, 11] for connections between functional inequalities and rate of convergence for Markov processes. However,

- the GRBM defined by (1.1) is not necessarily time-reversible, or symmetric;
- the stationary distribution of the GRBM has an explicit form only under the generalized skew-symmetric condition (1.3).

So we cannot apply the Poincaré inequality directly. A natural question is whether the rate of convergence is exponential under general conditions. The main tool is stochastic stability theory of Markov processes which we recall in Section 2.

1.2. **Main results.** For any signed Borel measure μ on $\mathbb{R}^d$, we define the total variation norm by

$$||\mu||_{TV} := \sup_{|g| \leq 1} \int_{\mathbb{R}^d} g \, d\mu.$$

Definition 1.1. *Assume that a $\mathbb{R}^d$-valued Markov process $(Z_t; t \geq 0)$ with transition kernel P^t has a unique stationary distribution π. If there exist C, $\delta > 0$ and $W : \mathbb{R}^d \to [1, \infty)$ such that for all $x \in \mathbb{R}^d$ and $t \geq 0$,*

$$||P^t(x, \cdot) - \pi(\cdot)||_{TV} \leq C \, W(x) \exp(-\delta t), \tag{1.5}$$

then $(Z_t; t \geq 0)$ is said to be W-uniformly ergodic with exponent δ.

Note that the SDE (1.1) does not always have a unique strong solution unless we impose additional conditions on the input data (Γ, μ, R, U). See [40, Section V] for background on solutions to SDEs. It is well known that the SDE (1.1) has a strong solution which is pathwise unique under the following conditions.

(1) Γ is strictly positive definite. That is, there exists $\lambda > 0$ such that

$$\xi^T \Gamma \xi \geq \lambda ||\xi||^2 \quad \text{for all } \xi \in \mathbb{R}^d. \tag{1.6}$$

(2) U' is locally Lipschitz. That is, there exists $K_R > 0$ such that

$$|U'(y) - U'(z)| \leq K_R |y - z| \quad \text{for all } |y| \leq R \text{ and } |z| \leq R. \tag{1.7}$$

(3) There exists $K > 0$ such that

$$x \cdot \left(\mu + \sum_{i=1}^d U'(x_i) r_i \right) \leq K(1 + ||x||^2) \quad \text{for all } x = (x_1, \cdots, x_d) \in \mathbb{R}^d. \tag{1.8}$$

Moreover, the unique strong solution is Feller continuous and strong Markov. The infinitesimal generator of $\text{GRBM}(\Gamma, \mu, R, U)$ is given by

$$\mathcal{L} := \frac{1}{2} \sum_{i,j=1}^d \Gamma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d \left(\mu_i + \sum_{j=1}^d U'(x_j) r_{ij} \right) \frac{\partial}{\partial x_i}. \tag{1.9}$$

By proper scaling, we assume that the diagonal entries of the reflection matrix R are all equal to 1, i.e. $r_{ii} = 1$ for $1 \leq i \leq d$. To prove the rate of convergence for a GRBM, we make the following assumptions on the input data.

Assumption 1.2. *(1) The covariance matrix Γ is strictly positive definite.*

(2) *The reflection matrix R is such that $r_{ii} = 1$ for $1 \leq i \leq d$, $r_{i,i-1} = -1$ for $2 \leq i \leq d$ and $r_{ij} = 0$ for $|i - j| \geq 2$. That is,*

$$R = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}, \tag{1.10}$$

(3) *The potential U satisfies that*
   (a) *$U' \geq 0$ is continuous on $\mathbb{R}$, and is decreasing on $\mathbb{R}_-$,*
   (b) *$U'(y) \to 0$ as $y \to \infty$, and $U'(y) \to \infty$ as $y \to -\infty$,*
   (c) *For $a > b > 0$, $U'(ay)/U'(by) \to \infty$ as $y \to -\infty$.*

The form (1.10) of the reflection matrix comes from the application to queues in tandem, and the gap process of the Brownian TASEP. In particular, Assumption 1.2 is satisfied with $\Gamma = I$ and $U(x) = -e^{-x}$, corresponding to the O'Connell-Yor process. See Section 4 for further discussion.

Under Assumption 1.2, the infinitesimal generator of $\mathrm{GRBM}(\Gamma, \mu, R, U)$ simplifies to

$$\mathcal{L} := \frac{1}{2} \sum_{i,j=1}^{d} \Gamma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \left(\mu_i + U'(x_i) - U'(x_{i-1}) \right) \frac{\partial}{\partial x_i}, \tag{1.11}$$

with the convention $x_0 = \infty$. For a measurable set $C \subset \mathbb{R}^d$, write 1_C for the indicator function of C. For $r > 0$, let $\mathcal{B}_r := \{x \in \mathbb{R}^d; \|x\| \leq r\}$ be the closed ball of radius r centered at the origin. A function $V : \mathbb{R}^d \to [1, \infty)$ is said to be norm-like if it is at least twice continuously differentiable, and $V(x) \to \infty$ as $\|x\| \to \infty$.

The main result below establishes an *exponential drift condition* for $\mathrm{GRBM}(\Gamma, \mu, R, U)$.

Theorem 1.3. *Let $\mathcal{L}$ be defined by (1.11). Assume that the input data (Γ, μ, R, U) satisfy Assumption 1.2, and $\mu = (\mu_1, \ldots, \mu_d) < 0$. Then there exist $b < \infty$, and a norm-like function $V : \mathbb{R}^d \to [1, \infty)$ such that for arbitrary small $\epsilon > 0$,*

$$\mathcal{L}V \leq -\frac{1}{2d\|\Gamma\|} \left(\min_{1 \leq i \leq d} |\mu_i|^2 - \epsilon \right) V + b1_{\mathcal{B}_r} \quad \text{for } r \text{ large enough.} \tag{1.12}$$

The proof of Theorem 1.3, which involves an induction over d, will be given in Section 3. The function V is called the *Lyapunov function*. The main difficulty is to prove the estimate (1.12) for $x := (x_1, \ldots, x_d) < 0$, in which case $\mu_i + U'(x_i) - U'(x_{i-1})$ can either be large positive or large negative since $U'(y) \to \infty$ as $y \to -\infty$. Note that this is the case for the O'Connell-Yor process. The key to analysis is to estimate $\beta_d(x)$ defined by (3.4), and the special form (1.10) of R and the monotonicity of U' lead to manageable estimates via telescopic sums (3.6)–(3.7).

A consequence of Theorem 1.3 is the uniform exponential ergodicity for $\mathrm{GRBM}(\Gamma, \mu, R, U)$, which we prove in Section 2. As a byproduct, we obtain a tail estimate of the stationary distribution of $\mathrm{GRBM}(\Gamma, \mu, R, U)$.

Corollary 1.4. *Under the assumptions of Theorem 1.3, $GRBM(R, \mu, \Gamma, U)$ defined as the strong solution to (1.1), has a unique stationary distribution π, and is uniformly exponentially ergodic. That is, let P^t be the transition kernel of $GRBM(R, \mu, \Gamma, U)$, then there exist*

$$W : \mathbb{R}^d \to [1, \infty) \quad and \quad C(d) > 0,$$

such that

$$||P^t(x, \cdot) - \pi(\cdot)||_{TV} \leq W(x) \exp(-C(d)\, t). \tag{1.13}$$

Moreover, the stationary distribution π satisfies the tail estimate

$$\int_{\mathbb{R}^d} V(x) d\pi(x) < \infty. \tag{1.14}$$

The exponential ergodicity of $GRBM(\Gamma, \mu, R, U)$ holds for $\mu < 0$, which is different from the exponential ergodic condition $R^{-1}\mu < 0$ for $SRBM(\Gamma, \mu, R)$ due to the presence of the potential U. As explained in Section 2, the dependence of $C(d)$ is notoriously difficult. There is no simple way to get the exact rate $C(d)$ from the Lyapunov estimate (1.12). Blanchet and Chen [3] gave an estimate of $C(d)$ for SRBMs. Their result relies on a coupling at the minimal element $\{0\}$ in the positive orthant, which is not available for GRBMs. Eberle, Guillin and Zimmer [17] provided an estimate of $C(d)$ for general diffusions. There a one-sided Lipschitz condition for the drift term is required, but this is not necessarily satisfied in our scenario. In view of (1.12), we make the following conjecture.

Conjecture 1.5. *Under the assumptions of Theorem 1.3, $GRBM(R, \mu, \Gamma, U)$ is uniformly exponentially ergodic with exponent of order $1/d$. That is, there exist*

$$W : \mathbb{R}^d \to [1, \infty) \quad and \quad C > 0$$

such that

$$||P^t(x, \cdot) - \pi(\cdot)||_{TV} \leq W(x) \exp\left(-\frac{Ct}{d}\right). \tag{1.15}$$

Conjecture 1.5 would imply that the relaxation time to global equilibrium is of order d for a class of GRBMs.

Outline of the paper: The rest of the paper is organized as follows. In Section 2, we provide background on stochastic stability theory, and prove Corollary 1.4. In Section 3, we give a proof of Theorem 1.3. In Section 4, we apply these results to the O'Connell-Yor process. There several open questions are raised.

2. Stochastic stability & exponential ergodicity for GRBM

2.1. **Stochastic stability theory.** In this subsection we present the main tool to prove Corollary 1.4: stochastic stability theory for continuous-time Markov processes developed by Meyn and Tweedie [31, 32]. See [30, 31] for background.

Meyn and Tweedie [32, Theorem 4.2] provided criteria for a Markov process to be positive Harris recurrent in terms of its infinitesimal generator.

Theorem 2.1. *Let $(Z_t; t \geq 0)$ be a $\mathbb{R}^d$-valued Markov process with generator $\mathcal{L}$.*

(1) *If there exist $k > 0$, $b < \infty$, a petite set $C \subset \mathbb{R}^d$, and a norm-like function $V : \mathbb{R}^d \to [1, \infty)$ such that*

$$\mathcal{L}V \leq -k + b 1_C,$$

then $(Z_t; t \geq 0)$ is positive Harris recurrent.

(2) *If $(Z_t; t \geq 0)$ is positive Harris recurrent, then it has a unique stationary distribution.*

Meyn and Tweedie [32, Theorem 6.1] also gave a criterion for a Markov process to be uniformly exponentially ergodic.

Theorem 2.2. *Let $(Z_t; t \geq 0)$ be a $\mathbb{R}^d$-valued Markov process with generator $\mathcal{L}$. If all compact sets are petite, and there exist $k > 0$, $b < \infty$, a petite set $C \subset \mathbb{R}^d$, and a norm-like function $V : \mathbb{R}^d \to [1, \infty)$ such that*

$$\mathcal{L}V \leq -kV + b 1_C,$$

then $(Z_t; t \geq 0)$ is uniform exponential ergodic.

It is well known that under the *geometric drift condition*, a Markov chain converges to its equilibrium with rate ρ^n for some $\rho < 1$, see [30, Chapter 15]. Down, Meyn and Tweedie [14] extended this result to the continuous setting. Under the exponential drift condition, a Markov process converges exponentially to its stationary distribution with some exponent $\delta > 0$. But the explicit value or bounds of $\rho < 1$ and $\delta > 0$ were unknown. Efforts have been made to derive bounds of ρ and δ under extra assumptions that

- the Markov chain/process is stochastically ordered, and the state space has a minimal element, see [3, 28, 29].

- the Markov chain/process satisfies a minorisation condition: there exists a Borel set $C \subset \mathbb{R}^d$, $t^* > 0$, $\varepsilon > 0$, and a probability distribution ν on $\mathbb{R}^d$ such that for each Borel set $A \in \mathbb{R}^d$,

$$P^{t^*}(x, A) \geq \varepsilon \nu(A) \quad \text{for all } x \in C.$$

But it is difficult to provide good estimates of (t^*, ε). See [38, 39, 41].

2.2. **Proof of Corollary 1.4.** In this subsection we explain how to use the exponential drift condition (1.12) to derive the uniform exponential ergodicity for $\mathrm{GRBM}(\Gamma, \mu, R, U)$. We begin with two simple lemmas.

Lemma 2.3. *Under Assumption 1.2, the SDE (1.1) has a strong solution which is pathwise unique.*

Proof. As we will see in Section 3, a key step to prove Theorem 1.3 is the following estimate

$$\sum_{i=1}^{d} x_i(\mu_i + U'(x_i) - U'(x_{i-1})) \leq \left(- \min_{1 \leq i \leq d} |\mu_i| + \epsilon \right) ||x||, \tag{2.1}$$

for arbitrary small $\epsilon > 0$ and $||x||$ large enough. Plainly, the Khasminskii non-explosion condition (1.8) is satisfied. Combined with the local Lipschitz property of U', we conclude that the SDE (1.1) has a strong solution which is pathwise unique. $\qquad\qquad\square$

Lemma 2.4. *Under Assumption 1.2, $\mathrm{GRBM}(\Gamma, \mu, R, U)$ defined as the strong solution to the SDE (1.1) is positive Harris recurrent and has a unique stationary distribution.*

Proof. According to the exponential drift condition (1.12), there exist $k > 0$, $b < \infty$, $R > 0$, and a norm-like function $V : \mathbb{R}^d \to [1, \infty)$ such that

$$\mathcal{L}V \leq -k + b1_{\mathcal{B}_R},$$

where $\mathcal{L}$ is defined as in (1.11). According to [45, Theorem 2.3], a GRBM is Lebesgue-irreducible $T-$process. It follows from [31, Theorem 4.1(i)] that for a Lebesgue-irreducible T-process, every compact set is petite. In particular, $\mathcal{B}_R$ as a compact set is petite. By Theorem 2.1, $\mathrm{GRBM}(R, \mu, \Gamma, U)$ is positive Harris recurrent and has a unique stationary distribution.   □

The existence and uniqueness of the stationary distribution of GRBM can also be derived from the exponential drift condition (1.12) in a purely analytical way. By [4, Corollary 1.3], there exists a stationary distribution which is absolute continuous relative to Lebesgue measure with density $p \in L^{d/(d-1)}(\mathbb{R}^d)$. The uniqueness follows from [5, Example 5.1].

Proof of Corollary 1.4. Lemma 2.3 and 2.4 guarantees that $\mathrm{GRBM}(R, \mu, \Gamma, U)$ is well-defined, and has a unique stationary distribution. It suffices to apply Theorem 2.2 with Theorem 1.3 to conclude.   □

3. Exponential drift condition for GRBM

In this section we prove Theorem 1.3 by induction on d – the dimension of GRBM. To proceed further, we need the following notations. Let $\epsilon > 0$ chosen to be small enough and $L > 0$ chosen to be large enough. Define

- $\mu_{\min}^{(d)}(\epsilon) := - \min_{1 \leq i \leq d} |\mu_i| + \epsilon < 0$,

- $r_+(d, \epsilon) > 0$ such that for all $1 \leq i \leq n$, $\mu_i + U'(x) \leq \mu_{\min}^{(d)}(\epsilon)$ for $x \geq r_+(d, \epsilon)$,

- $r_-(d, \epsilon, L) > 0$ such that for all $1 \leq i \leq n$, $\mu_i + U'(x) \geq |\mu_{\min}^{(d)}(\epsilon)| + L$ for $x \leq -r_-(d, \epsilon, L)$.

To avoid heavy notations, we abandon the dependance on (d, ϵ, L), and write $\mu_{\min}^{(d)}$, r_+, r_- instead of $\mu_{\min}^{(d)}(\epsilon)$, $r_+(d, \epsilon)$, $r_-(d, \epsilon, L)$.

Proof of Theorem 1.3. Define $V : \mathbb{R}^d \to [1, \infty)$ by

$$V(x) := \exp\left(\lambda\phi(||x||)\right) \quad \text{for } \lambda > 0, \tag{3.1}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing $\mathcal{C}^2$ function such that $\phi(s) = 0$ for $s \leq \frac{1}{2}$, and $\phi(s) = s$ for $s \geq 1$. Let $\psi(x) := \phi(||x||)$. We get

$$DV(x) = \lambda D\psi(x)V(x),$$

$$D^2V(x) = \left(\lambda D^2\psi(x) + \lambda^2 D\psi(x)D\psi(x)^T\right)V(x).$$

Note that for $||x|| \geq 1$, $||D\psi(x)|| = 1$ and

$$||D^2\psi(x)|| = \left|\left|\frac{I_d}{||x||} - \frac{xx^T}{||x||^3}\right|\right| \leq \frac{2}{||x||} \to 0 \quad \text{as } ||x|| \to \infty.$$

So there exists $r_\epsilon \geq 1$ such that for $x \notin \mathcal{B}_{r_\epsilon}$, we have $||D^2\psi(x)|| \leq \epsilon$. In this case, we have

$$\left| \sum_{i,j=1}^{d} \Gamma_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}(x) \right| \leq d||\Gamma|| ||D^2 V(x)|| \leq d||\Gamma||(\lambda\epsilon + \lambda^2)V(x), \qquad (3.2)$$

and

$$\sum_{i=1}^{d} \left(\mu_i + U'(x_i) - U'(x_{i-1}) \right) \frac{\partial\psi}{\partial x_i}(x) = \sum_{i=1}^{d} \left(\mu_i + U'(x_i) - U'(x_{i-1}) \right) \frac{x_i}{||x||}.$$

Consequently, for $||x||$ large enough,

$$\mathcal{L}V(x) \leq \left(\frac{1}{2}d||\Gamma||\lambda^2 + (\beta_d(x) + \mathcal{O}(\epsilon))\lambda \right) V(x), \qquad (3.3)$$

where

$$\beta_d(x) := (\mu_1 + U'(x_1))\frac{x_1}{||x||} + \sum_{k=2}^{d} (\mu_k + U'(x_k) - U'(x_{k-1}))\frac{x_k}{||x||}. \qquad (3.4)$$

The case $d = 1$ is straightforward. We consider $d = 2$ as the base case.

Step 1 ($d = 2$): By (3.3),

$$\mathcal{L}V(x) \leq \left[||\Gamma||\lambda^2 + (\beta_2(x) + \mathcal{O}(\epsilon))\lambda \right] V(x), \qquad (3.5)$$

where

$$\beta_2(x) := (\mu_1 + U'(x_1))\frac{x_1}{||x||} + (\mu_2 + U'(x_2) - U'(x_1))\frac{x_2}{||x||}.$$

Lemma 3.1. $\beta_2(x) \leq \mu_{\min}^{(2)} + \mathcal{O}(\epsilon)$ *for* $||x|| \geq \max(r_+/\epsilon, r_-/\epsilon, r_\epsilon')$, *where* r_ϵ' *is given in the proof.*

Proof. There are four cases according to the signs of (x_1, x_2).

Case 1: $x_1 \geq 0$ and $x_2 \geq 0$.

(1) If $\frac{x_1}{||x||} \geq \epsilon$ and $\frac{x_2}{||x||} \geq \epsilon$, then $x_1 \geq r_+$ and $x_2 \geq r_+$. We have

$$\mu_1 + U'(x_1) \leq \mu_{\min}^{(2)} \quad \text{and} \quad \mu_2 + U'(x_2) - U'(x_1) \leq \mu_2 + U'(x_2) \leq \mu_{\min}^{(2)}.$$

Therefore,

$$\beta_2(x) \leq \mu_{\min}^{(2)} \left(\frac{x_1}{||x||} + \frac{x_2}{||x||} \right) \leq \mu_{\min}^{(2)}.$$

(2) If $\frac{x_1}{||x||} \leq \epsilon$, then $\frac{x_2}{||x||} \geq \sqrt{1 - \epsilon^2}$. We get $x_2 \geq \sqrt{1 - \epsilon^2}r_+/\epsilon \geq r_+$, so

$$\mu_2 + U'(x_2) - U'(x_1) \leq \mu_{\min}^{(2)}.$$

Moreover, $\mu_1 + U'(x_1) = \mathcal{O}(1)$ since $x_1 \geq 0$. Thus,

$$\beta_2(x) \leq \mathcal{O}(\epsilon) + \mu_{\min}^{(2)} \frac{x_2}{||x||} \leq \mu_{\min}^{(2)} + \mathcal{O}(\epsilon).$$

(3) If $\frac{x_2}{||x||} \leq \epsilon$, then $\frac{x_1}{||x||} \geq \sqrt{1 - \epsilon^2}$. By symmetry, we get

$$\beta_2(x) \leq \mu_{\min}^{(2)} \frac{x_1}{||x||} + \mathcal{O}(\epsilon) \leq \mu_{\min}^{(2)} + \mathcal{O}(\epsilon).$$

Case 2: $x_1 \leq 0$ and $x_2 \geq 0$.

(1) If $\frac{x_1}{||x||} \leq -\epsilon$ and $\frac{x_2}{||x||} \geq \epsilon$, then $x_1 \leq -r_-$ and $x_2 \geq r_+$. We have

$$\mu_1 + U'(x_1) \geq |\mu_{\min}^{(2)}| \quad \text{and} \quad \mu_2 + U'(x_2) - U'(x_1) \leq \mu_2 + U'(x_2) \leq \mu_{\min}^{(2)}.$$

Thus,

$$\beta_2(x) \leq \mu_{\min}^{(2)} \left(\frac{-x_1}{||x||} + \frac{x_2}{||x||} \right) \leq \mu_{\min}^{(2)}.$$

(2) If $-\epsilon \leq \frac{x_1}{||x||} \leq 0$, then $\frac{x_2}{||x||} \geq \sqrt{1-\epsilon^2}$. We get $x_2 \geq \sqrt{1-\epsilon^2} r_+/\epsilon \geq r_+$, so

$$\mu_2 + U'(x_2) - U'(x_1) \leq \mu_{\min}^{(2)}.$$

Since $\mu_1 + U'(x_1) \geq \mu_1$, we have

$$\beta_2(x) \leq -\mu_1 \epsilon + \mu_{\min}^{(2)} \sqrt{1-\epsilon^2} = \mu_{\min}^{(2)} + \mathcal{O}(\epsilon).$$

(3) If $0 \leq \frac{x_2}{||x||} \leq \epsilon$, thus $\frac{x_1}{||x||} \leq -\sqrt{1-\epsilon^2}$. We have $x_1 \leq -\sqrt{1-\epsilon^2} r_-/\epsilon \leq -r_-$, so

$$\mu_1 + U'(x_1) \geq |\mu_{\min}^{(2)}|.$$

For L large enough, $\mu_2 + U'(x_2) - U'(x_1) \leq \mu_2 + \sup_{x>0} U'(x) - L \leq 0$. Therefore,

$$\beta_2(x) \leq |\mu_{\min}^{(2)}| \frac{x_1}{||x||} \leq \mu_{\min}^{(2)} \sqrt{1-\epsilon^2} = \mu_{\min}^{(2)} + \mathcal{O}(\epsilon).$$

Case 3: $x_1 \geq 0$ and $x_2 \leq 0$.

(1) If $\frac{x_1}{||x||} \geq \epsilon$ and $\frac{x_2}{||x||} \leq -\epsilon$, then $x_1 \geq r_+$ and $x_2 \leq -r_-$. We have

$$\mu_1 + U'(x_1) \geq |\mu_{\min}^{(2)}|,$$

and

$$\mu_2 + U'(x_2) - U'(x_1) \geq |\mu_{\min}^{(2)}| \text{ for } L \text{ large enough.}$$

Thus,

$$\beta_2(x) \leq \mu_{\min}^{(2)} \left(\frac{x_1}{||x||} + \frac{-x_2}{||x||} \right) \leq \mu_{\min}^{(2)}.$$

(2) If $\frac{x_1}{||x||} \leq \epsilon$, then $\frac{x_2}{||x||} \leq -\sqrt{1-\epsilon^2}$. We get $x_2 \leq -\sqrt{1-\epsilon^2} r_-/\epsilon \leq -r_-$, so

$$\mu_2 + U'(x_2) - U'(x_1) \geq |\mu_{\min}^{(2)}| \text{ for } L \text{ large enough.}$$

Moreover, $\mu_1 + U'(x_1) = \mathcal{O}(1)$ since $x_1 \geq 0$. Thus,

$$\beta_2(x) \leq \mathcal{O}(\epsilon) + \mu_{\min}^{(2)} \sqrt{1-\epsilon^2} = \mu_{\min}^{(2)} + \mathcal{O}(\epsilon).$$

(3) If $-\epsilon \leq \frac{x_2}{||x||} \leq 0$, thus $\frac{x_1}{||x||} \geq \sqrt{1-\epsilon^2}$. We have $x_1 \geq \sqrt{1-\epsilon^2} r_+/\epsilon \geq r_+$, so

$$\mu_1 + U'(x_1) \leq \mu_{\min}^{(2)}.$$

Since $\mu_2 + U'(x_2) - U'(x_1) \geq \mu_2 - \sup_{x>0} U'(x)$, we get

$$\beta_2(x) \leq \mu_{\min}^{(2)} \sqrt{1-\epsilon^2} - \left(\mu_2 - \sup_{x>0} U'(x) \right) \epsilon = \mu_{\min}^{(2)} + \mathcal{O}(\epsilon).$$

Case 4: $x_1 \leq 0$ and $x_2 \leq 0$.

(1) If $\frac{x_1}{||x||} \leq -\epsilon$ and $\frac{x_2}{||x||} \leq -\epsilon$, we have

$$\begin{aligned}
\beta_2(x) &= \mu_1 \frac{x_1}{||x||} + \frac{(x_1 - x_2)U'(x_1)}{||x||} + \frac{x_2 U'(x_2)}{||x||} + \mu_2 \frac{x_2}{||x||} \\
&\leq \mathcal{O}(1) + \frac{(x_1 - x_2)U'(x_2) + x_2 U'(x_2)}{||x||} \qquad\qquad (3.6) \\
&= \mathcal{O}(1) + \frac{x_1 U'(x_2)}{||x||} \\
&\leq \mathcal{O}(1) - \epsilon U'(-\epsilon ||x||) \to -\infty \quad \text{as } ||x|| \to \infty,
\end{aligned}$$

where the inequality (3.6) follows from the fact that $\mu_1 \frac{x_1}{||x||}$, $\mu_2 \frac{x_2}{||x||} = \mathcal{O}(1)$ and $(x_1 - x_2)(U'(x_1) - U'(x_2)) \leq 0$.

(2) If $\frac{x_1}{||x||} \geq -\epsilon$, then $\frac{x_2}{||x||} \leq -\sqrt{1 - \epsilon^2}$. We have

$$\begin{aligned}
\beta_2(x) &= (\mu_1 + U'(x_1)) \frac{x_1}{||x||} + (\mu_2 + U'(x_2) - U'(x_1)) \frac{x_2}{||x||} \\
&\leq -\mu_1 \epsilon - (\mu_2 + U'(-\sqrt{1 - \epsilon^2}||x||) - U'(-\epsilon||x||))\sqrt{1 - \epsilon^2} \\
&\to -\infty \quad \text{as } ||x|| \to \infty.
\end{aligned}$$

(3) If $\frac{x_2}{||x||} \geq -\epsilon$, then $\frac{x_1}{||x||} \leq -\sqrt{1 - \epsilon^2}$. We have

$$\begin{aligned}
\beta_2(x) &= \mu_1 \frac{x_1}{||x||} + \frac{(x_1 - x_2)U'(x_1)}{||x||} + (\mu_2 + U'(x_2)) \frac{x_2}{||x||} \\
&\leq \mathcal{O}(1) + (\epsilon - \sqrt{1 - \epsilon^2})U'(-\sqrt{1 - \epsilon^2}||x||) - \mu_2 \epsilon \\
&\to -\infty \quad \text{as } ||x|| \to \infty.
\end{aligned}$$

It suffices to take $r'_\epsilon > 0$ such that $\beta_2(x) \leq \mu_{\min}^{(2)}$ for $x \leq 0$ and $||x|| \geq r'_\epsilon$. $\qquad\qquad\square$

By Lemma 3.1 and (3.5), we get for $||x||$ large enough,

$$\mathcal{L}V(x) \leq \left(||\Gamma|| \lambda^2 + \mu_{\min}^{(2)} \lambda \right) V(x).$$

By taking $\lambda = -\frac{\mu_{\min}^{(2)}}{2||\Gamma||}$, we have

$$\mathcal{L}V(x) \leq \left(-\frac{1}{4||\Gamma||} (\mu_{\min}^{(2)})^2 \right) V(x).$$

Step 2 $(d - 1 \to d)$**:** We prove the following lemma by induction on d. The case $d = 2$ was proved in Lemma 3.1.

Lemma 3.2. $\beta_d(x) \leq \mu_{\min}^{(d)}$ *for* $||x||$ *large enough, and* $\beta_d(x) \to -\infty$ *as* $x < 0$ *and* $||x|| \to \infty$.

Proof. Let $i^+ := \sup\{i \geq 0; x_i \geq 0\}$, with the convention $i^+ = 0$ if $x < 0$. There are three cases.

Case 1: If $i^+ \geq 2$, then we have

$$\beta_d(x) = \left[(\mu_1 + U'(x_1))\frac{x_1}{||x||} + \sum_{k=2}^{i^+-1} (\mu_k + U'(x_k) - U'(x_{k-1}))\frac{x_k}{||x||} \right] - U'(x_{i^+-1})\frac{x_{i^+}}{||x||}$$

$$+ \left[(\mu_{i^+} + U'(x_{i^+}))\frac{x_{i^+}}{||x||} + \sum_{k=i^++1}^{d} (\mu_k + U'(x_k) - U'(x_{k-1}))\frac{x_k}{||x||} \right]$$

$$\leq \beta_{i^+-1}(x_1, \ldots, x_{i^+-1})\frac{\sqrt{\sum_{k=1}^{i^+-1} x_k^2}}{||x||} + \beta_{d-i^++1}(x_{i^+}, \ldots, x_d)\frac{\sqrt{\sum_{k=i^+}^{d} x_k^2}}{||x||}.$$

By induction hypothesis, $\beta_{i^+-1}(x_1, \ldots, x_{i^+-1}) \leq \mu_{\min}^{(i^+-1)}$ for $||(x_1, \ldots, x_{i^+-1})|| > r_{d-1}$ and $\beta_{d-i^++1}(x_{i^+}, \ldots, x_d) \leq \mu_{\min}^{(d-i^++1)}$ for $||(x_{i^+}, \ldots, x_d)|| > r_{d-1}$. There are three subcases. If $||(x_1, \ldots, x_{i^+-1})|| \leq r_{d-1}$, then

$$\frac{\sqrt{\sum_{k=1}^{i^+-1} x_k^2}}{||x||} \to 0 \quad \text{and} \quad \frac{\sqrt{\sum_{k=i^+}^{d} x_k^2}}{||x||} \to 1 \quad \text{as } ||x|| \to \infty.$$

So we get

$$\beta_d(x) \leq o(1) + \mu_{\min}^{(d-i^++1)} \leq \mu_{\min}^{(d)}.$$

The same result holds if $||(x_{i^+}, \ldots, x_d)|| \leq r_{d-1}$. Assume that $||(x_1, \ldots, x_{i^+-1})|| > r_{d-1}$ and $||(x_{i^+}, \ldots, x_d)|| > r_{d-1}$. We have

$$\beta_d(x) \leq \mu_{\min}^{(i^+-1)}\frac{\sqrt{\sum_{k=1}^{i^+-1} x_k^2}}{||x||} + \mu_{\min}^{(d-i^++1)}\frac{\sqrt{\sum_{k=i^+}^{d} x_k^2}}{||x||} \leq \mu_{\min}^{(d)}.$$

Case 2: If $i^+ = 1$, then we have

$$\beta_d(x) = \mu_1\frac{x_1}{||x||} + \frac{(x_1 - x_2)U'(x_1)}{||x||} + \beta_{d-1}(x_2, \ldots, x_d)\frac{\sqrt{||x||^2 - x_1^2}}{||x||}.$$

(1) If $\frac{x_1}{||x||} \leq \epsilon$, then $\frac{\sqrt{||x||^2-x_1^2}}{||x||} \geq \sqrt{1-\epsilon^2}$. By induction hypothesis, $\beta_{d-1}(x_2, \ldots, x_d) \to -\infty$ as $||x|| \to \infty$. Therefore,

$$\beta_d(x) \leq \mathcal{O}(1) + \sqrt{1-\epsilon^2}\beta_{d-1}(x_2, \ldots, x_d) \to -\infty \quad \text{as } ||x|| \to \infty.$$

(2) If $\frac{x_1}{||x||} \geq \epsilon$, then $U'(x_1) \leq \epsilon$ for $||x||$ large enough. As a consequence,

$$\beta_d(x) \leq \mu_1\frac{x_1}{||x||} + \epsilon + \beta_{d-1}(x_2, \ldots, x_d)\frac{\sqrt{||x||^2 - x_1^2}}{||x||}.$$

Similar as Case 1, we get $\beta_d(x) \leq \mu_{\min}^{(d)}$.

Case 3: If $i^+ = 0$, i.e. $x < 0$, then we write $\beta_d(x) = \sum_{k=1}^{d} \mu_k\frac{x_k}{||x||} + \gamma_d(x)$, where

$$\gamma_d(x) := \sum_{k=1}^{d-1} \frac{(x_k - x_{k+1})U'(x_k)}{||x||} + \frac{x_d U'(x_d)}{||x||}.$$

It suffices to prove that $\gamma_d(x) \to -\infty$ as $||x|| \to \infty$. Let $i_1 := \underset{1 \leq k \leq d}{\operatorname{argmin}}\, x_k$.

If $i_1 \geq 2$, then we have

$$
\gamma_d(x) = \sum_{k=1}^{i_1-1} \frac{(x_k - x_{k+1})U'(x_k)}{||x||} + \frac{(x_{i_1} - x_{i_1+1})U'(x_{i_1})}{||x||}
$$

$$
+ \sum_{k=i_1+1}^{d-1} \frac{(x_k - x_{k+1})U'(x_k)}{||x||} + \frac{x_d U'(x_d)}{||x||}
$$

$$
\leq \sum_{k=1}^{i_1-1} \frac{(x_k - x_{k+1})U'(x_k)}{||x||} + \frac{(x_{i_1} - x_{i_1+1})U'(x_{i_1-1})}{||x||}
$$

$$
+ \sum_{k=i_1+1}^{d-1} \frac{(x_k - x_{k+1})U'(x_k)}{||x||} + \frac{x_d U'(x_d)}{||x||} \tag{3.7}
$$

$$
= \sum_{k=1}^{i_1-2} \frac{(x_k - x_{k+1})U'(x_k)}{||x||} + \frac{(x_{i_1-1} - x_{i_1+1})U'(x_{i_1-1})}{||x||}
$$

$$
+ \sum_{k=i_1+1}^{d-1} \frac{(x_k - x_{k+1})U'(x_k)}{||x||} + \frac{x_d U'(x_d)}{||x||}
$$

$$
= \gamma_{d-1}(x_1, \ldots, x_{i_1-1}, x_{i_1+1}, \ldots, x_d)\frac{\sqrt{||x||^2 - x_{i_1}^2}}{||x||},
$$

where the inequality follows from the fact that $x_{i_1} - x_{i_1+1} \leq 0$ and $U'(x_{i_1}) \geq U'(x_{i_1-1})$ by minimality of x_{i_1}. If $\frac{x_{i_1}}{||x||} \geq -(1-\epsilon)$, then

$$
\gamma_d(x) \leq \gamma_{d-1}(x_1, \ldots, x_{i_1-1}, x_{i_1+1}, \ldots, x_d)\sqrt{2\epsilon - \epsilon^2} \to -\infty \quad \text{as } ||x|| \to \infty.
$$

If $\frac{x_{i_1}}{||x||} \leq -(1-\epsilon)$, then $\frac{x_k}{||x||} \geq -\epsilon$ for all $k \neq i_1$. Consequently,

$$
\gamma_d(x) \leq \mathcal{O}(U'(-\epsilon||x||)) + (-1 + 2\epsilon)U'(-(1-\epsilon)||x||) \to -\infty \quad \text{as } ||x|| \to \infty.
$$

If $i_1 = 1$, let $i_2 := \operatorname{argmin}_{2 \leq k \leq d} x_k$. The same argument shows that $\gamma_d(x) \to -\infty$ as $||x|| \to \infty$ for $i_2 \geq 3$. We continue this algorithm and the only remaining case is $x_1 \leq x_2 \leq \cdots \leq x_d \leq 0$. In this case, we get

$$
\gamma_d(x) \leq \frac{\sum_{k=1}^{d-1}(x_k - x_{k+1}) + x_d}{||x||} U'(x_d)
$$

$$
= \frac{x_1}{||x||}U'(x_d) \leq -\frac{1}{\sqrt{d}}U'(x_d) \to -\infty \quad \text{as } x_d \to -\infty.
$$

Now assume that $x_d \geq -r^*$ for some $r^* > 0$. We have

$$
\gamma_d(x) \leq \frac{x_1 - x_d}{||x||}U'(x_{d-1}) + \frac{x_d}{||x||}U'(x_d)
$$

$$
= \left(-\frac{1}{\sqrt{d}} + o(1)\right)U'(x_{d-1}) + \mathcal{O}(1) \to -\infty \quad \text{as } x_{d-1} \to -\infty.
$$

So it suffices to consider the case $x_d \geq x_{d-1} \geq -r^*$ for some $r^* > 0$. We repeat the procedure until $x_d \geq x_{d-1} \geq \cdots \geq x_2 \geq -r^*$ for some $r^* > 0$. Then we have

$$\frac{x_1}{||x||} \leq -\frac{\sqrt{||x||^2 - dr^{*2}}}{||x||} \to -1 \quad \text{as } ||x|| \to \infty.$$

The above condition implies that

$$\gamma_d(x) \leq (-1 + o(1))U'(-(1 + o(1))||x||) + \mathcal{O}(1) \to -\infty \quad \text{as } ||x|| \to \infty.$$

$\square$

By Lemma 3.2 and (3.3), we get for $||x||$ large enough,

$$\mathcal{L}V(x) \leq \left(\frac{1}{2}d||\Gamma||\lambda^2 + \mu_{\min}^{(d)}\lambda\right)V(x).$$

By taking $\lambda = -\frac{\mu_{\min}^{(d)}}{d||\Gamma||}$, we have

$$\mathcal{L}V(x) \leq \left(-\frac{1}{2d||\Gamma||}(\mu_{\min}^{(d)})^2\right)V(x).$$

$\square$

4. BROWNIAN DIFFUSIONS WITH HARD AND SOFT REFLECTION

In this section we apply Theorem 1.3 and Corollary 1.4 to a class of Brownian diffusions with soft reflection, including the O'Connell-Yor process. We compare the Brownian TASEP to these diffusions with soft reflection, and present several conjectures regarding the rate of convergence as the dimension d is large.

Brownian TASEP

Consider the Brownian TASEP on the real line. There are d particles with positions $Z_1^h, \cdots, Z_d^h$ such that $Z_1^h(t) \leq \cdots \leq Z_d^h(t)$ for all $t \geq 0$. The leftmost particle Z_1^h evolves as Brownian motion with drift μ_1. The second leftmost particle Z_2^h evolves as Brownian motion with drift μ_2 reflected off Z_1^h, and so on.

It is well known that this process is governed by the following SDE:

$$dZ_1^h(t) = \mu_1 dt + dB_1(t),$$

$$dZ_i^h(t) = \mu_i dt + \frac{1}{\sqrt{2}}(dL_{i-1,i}(t) - dL_{i,i+1}(t)) + dB_i(t) \quad \text{for } 2 \leq i \leq d, \tag{4.1}$$

where $B := (B_i(t); t \geq 0)_{1 \leq i \leq d}$ is a d-dimensional Brownian motion with the identity covariance matrix, and $L_{j,j+1}$ is the local time process of the semimartingale $(Z_{j+1}^h - Z_j^h)/\sqrt{2}$, with the convention $L_{d,d+1} := 0$. See Pal and Pitman [36, Section 2].

We consider the gap process $G^h := (Z_{i+1}^h(t) - Z_i^h(t); t \geq 0)_{1 \leq i \leq d-1}$ of the Brownian TASEP. It was proved in [2, Section 4] that the gap process $(G^h(t); t \geq 0)$ is a $(d-1)$-dimensional

SRBM($\Gamma, \tilde{\mu}, R$), with the reflection matrix

$$R := \begin{pmatrix} 1 & -1/2 & 0 & \cdots & 0 \\ -1/2 & 1 & -1/2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1/2 & 1 & -1/2 \\ 0 & \cdots & 0 & -1/2 & 1 \end{pmatrix},$$

the drift $\tilde{\mu} := (\mu_{i+1} - \mu_i)_{1 \le i \le d-1}$, and the covariance matrix $\Gamma := 2R$. Let $\nu_i := \sum_{k=1}^{i} \mu_k - \frac{k}{d} \sum_{k=1}^{d} \mu_k$ for $1 \le i \le d-1$. Sarantsev [43] proved that the gap process G^h of the Brownian TASEP has a unique stationary distribution if and only if $\nu_i < 0$ for all $1 \le i \le d-1$. Further by letting $\mathcal{L}^h$ be the infinitesimal generator of G^h, there exist a norm-like function V and $b < \infty$ such that

$$\mathcal{L}^h V \le -K^h V + b \mathbb{1}_{\mathcal{B}_r} \quad \text{for } r \text{ large enough,}$$

where

$$K^h := \frac{4}{d} \left(1 - \cos \frac{\pi}{d} \right)^3 \left(1 + \cos \frac{\pi}{d} \right)^{-1} \min_{1 \le i \le d-1} \nu_i^2. \tag{4.2}$$

Consequently, the gap process G^h is uniformly exponentially ergodic.

Brownian diffusions with soft reflection

We replace the local time process in (4.1) with soft reflection U'. Precisely, the particle system $Z_1^s, \cdots, Z_d^s$ is governed by the following SDE:

$$dZ_1^s(t) = \mu_1 dt + dB_1(t),$$

$$dZ_i^s(t) = \left(\mu_i + U'(Z_{i+1}^s(t) - Z_i^s(t)) \right) dt + dB_i(t) \quad \text{for } 2 \le i \le d, \tag{4.3}$$

where U is a potential function satisfying Assumption 1.2. This multidimensional diffusion is the O'Connell-Yor process with the choice $U(x) = -e^{-x}$.

Consider the gap process $G^s := (Z_{i+1}^s(t) - Z_i^s(t); t \ge 0)_{1 \le i \le d-1}$ of the Brownian diffusion with soft reflection. We write

$$dG^s(t) = \left(\tilde{\mu} + \sum_{i=1}^{d} U'(G_i^s(t)) r_i \right) dt + d\tilde{B}(t).$$

So the gap process $(G^s(t); t \ge 0)$ is a $(d-1)$-dimensional GRBM($\Gamma, \tilde{\mu}, R, U$), where the reflection matrix R is given by (1.10), the drift $\tilde{\mu}$ and the covariance matrix Γ are the same as those defined for the Brownian TASEP. The following proposition is a consequence of Theorem 1.3.

Proposition 4.1. *Assume that the input data $(R, \tilde{\mu}, \Gamma, U)$ satisfy the assumptions in Theorem 1.3. Then the gap process G^s of the Brownian diffusion with soft reflection has a unique stationary distribution. Let $\mathcal{L}^s$ be the infinitesimal generator of G^s. Then there exist a norm-like function V and $b < \infty$ such that*

$$\mathcal{L}^s V \le -K^s V + b \mathbb{1}_{\mathcal{B}_r} \quad \text{for } r \text{ large enough,}$$

where

$$K^s := \frac{1}{4d(1 + \cos \frac{\pi}{d})} \min_{1 \le i \le d} |\tilde{\mu}_i|^2. \tag{4.4}$$

Consequently, the gap process G^s is uniformly exponentially ergodic.

From the exponential drift conditions (4.2)–(4.4), we see that

$$K^h \sim d^{-7} \quad \text{and} \quad K^s \sim d^{-1} \quad \text{as } d \to \infty. \tag{4.5}$$

This suggests that Brownian diffusions with soft reflection converges faster than those with hard reflection. Blanchet and Chen [3] proved a bound $d^{-4}(\log d)^{-2}$ for the exponent of the gap process G^h. The exact rate of convergence of the gap process G^s (resp. G^h) remains open. We leave these problems for future research.

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