Cut-edges and regular factors in regular graphs of odd degree

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June 15, 2018

Abstract

We study $2k$-factors in $(2r+1)$-regular graphs. Hanson, Loten, and Toft proved that every $(2r+1)$-regular graph with at most $2r$ cut-edges has a 2-factor. We generalize their result by proving for $k \le (2r+1)/3$ that every $(2r+1)$-regular graph with at most $2r-3(k-1)$ cut-edges has a $2k$-factor. Both the restriction on k and the restriction on the number of cut-edges are sharp. We characterize the graphs that have exactly $2r-3(k-1)+1$ cut-edges but no $2k$-factor. For $k > (2r+1)/3$, there are graphs without cut-edges that have no $2k$-factor, as studied by Bollobás, Saito, and Wormald.

1 Introduction

An *ℓ-factor* in a graph is an ℓ-regular spanning subgraph. In this paper we study the relationship between cut-edges and $2k$-factors in regular graphs of odd degree. In fact, all our results are for multigraphs, allowing loops and multiedges, so the model we mean by "graph" allows loops and multiedges.

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The relationship between edge-connectivity and 1-factors in regular graphs is well known. Petersen [9] proved that every 3-regular graph with no cut-edge decomposes into a 1-factor and a 2-factor, noting that the conclusion also holds when all cut-edges lie along a path. Schönberger [11] proved that in a 3-regular graph with no cut-edge, every edge lies in some 1-factor. Berge [3] obtained the same conclusion for r-regular $(r-1)$-edge-connected graphs of even order. Finally, a result of Plesník [10] implies most of these statements: If G is an r-regular $(r-1)$-edge-connected multigraph with even order, and G' is obtained from G by discarding at most $r-1$ edges, then G' has a 1-factor. The edge-connectivity condition is sharp: Katerinis [7] determined the minimum number of vertices in an r-regular $(r-2)$-edge-connected graph of even order having no 1-factor. Belck [2] and Bollobás, Saito, and Wormald [4] (independently) determined all (r,t,k) such that every r-regular t-edge-connected graph has a k-factor; Niessen and Randerath [8] further refined this in terms of also the number of vertices.

Petersen was in fact more interested in 2-factors. The result about 3-regular graphs whose cut-edges lie on a path implies that every 3-regular graph with at most two cut-edges has a 2-factor. Also, there are 3-regular graphs with three cut-edges having no 2-factor (communicated to Petersen by Sylvester in 1889). As a tool in a result about interval edge-coloring, Hanson, Loten, and Toft [6] generalized Petersen's result to regular graphs with larger odd degree.

Theorem 1.1 ([6])**.** *For $r \in \mathbb{N}$, every $(2r+1)$-regular graph with at most $2r$ cut-edges has a 2-factor.*

Petersen [9] also proved that every regular graph of even degree has a 2-factor. Thus when $k \le r$ every $2r$-regular graph has a $2k$-factor. As a consequence, regular factors of degree $2k$ become harder to guarantee as k increases. That is, a decomposition of a $(2r+1)$-regular graph into a 2-factor and $(2r-1)$-factor is easiest to find, while decomposition into a $2r$-factor and 1-factor is hardest to find (and implies the others).

In this paper, we generalize Theorem 1.1 to find the corresponding best possible guarantee for $2k$-factors. Limiting the number of cut-edges suffices when k is not too large.

Theorem 1.2. *For $r, k \in \mathbb{N}$ with $k \le (2r+1)/3$, every $(2r+1)$-regular graph with at most $2r-3(k-1)$ cut-edges has a $2k$-factor. Furthermore, both inequalities are sharp.*

Earlier, Xiao and Liu [14] proved a relationship between cut-edges and $2k$-factors, showing that a $(2kr+s)$-regular graph with at most $k(2r-3)+s$ cut-edges has a $2k$-factor avoiding any given edge. Their number of cut-edges in terms of degree and k is similar to ours, since $(2kr+s)-1-3(k-1) = k(2r-3)+s+2$, but their range of validity of k in terms of the degree of the full graph is more restricted than ours.

Our result is sharp in two ways. First, when $k \le (2r+1)/3$ and there are $2r+1-3(k-1)$ cut-edges, there may be no $2k$-factor. Sylvester found examples of such graphs (forbidding 2-factors in a regular graph of odd degree greater than 1 forbids all regular factors). We complete the Petersen–Sylvester investigation by describing all the extremal graphs without $2k$-factors for general k.

Theorem 1.3. *For $r, k \in \mathbb{N}$ with $k \le (2r+1)/3$, a $(2r+1)$-regular graph with exactly $2r+1-3(k-1)$ cut-edges fails to have a $2k$-factor if and only if it satisfies the constructive structural description stated in Theorem 3.2.*

When $k > (2r+1)/3$, the condition in Theorem 1.2 cannot be satisfied, and in fact there are $(2r+1)$-regular graphs that have no $2k$-factor even though they have no cut-edges. A $2k$-factor can instead be guaranteed by edge-connectivity requirements. The result of Berge [3] implies that $(2r+1)$-regular $2r$-edge-connected graphs have 1-factors and hence factors of all even degrees, by the 2-factor theorem of Petersen [9]. Therefore, when $k > (2r+1)/3$ the natural question becomes what edge-connectivity suffices to guarantee a $2k$-factor.

As mentioned earlier, this problem was solved by Bollobás, Saito, and Wormald [4], who determined all triples (r, t, k) such that every r-regular t-edge-connected multigraph has a k-factor (the triples are the same for simple graphs). As noted by Häggkvist [5] and by Niessen and Randerath [8], earlier Belck [2] obtained the result (in 1950). Earlier still, Baebler [1] proved the weaker result that $2k$-edge-connected $(2r+1)$-regular graphs have $2k$-factors.

The special case of the result of [4] that applies here (even-regular factors of odd-regular multigraphs) is that all $(2r+1)$-regular $2t$-edge-connected or $(2t+1)$-edge-connected multigraphs have $2k$-factors if and only if $k \le \frac{t}{2t+1}(2r+1)$. The general construction given in [4], which covers additional cases, is quite complicated. Here we provide a very simple construction that completes our investigation and shows necessity of their condition for even-regular factors of odd-regular graphs. In particular, for $1 \le t < r$ and $k > \frac{t}{2t+1}(2r+1)$ we present an easily described $(2t+1)$-connected simple graph that has no $2k$-factor.

Our results use the necessary and sufficient condition for the existence of ℓ-factors that was initially proved by Belck [2] and is a special case of the f-Factor Theorem of Tutte [12, 13]. When T is a set of vertices in a graph G, let $d_G(T) = \sum_{v \in T} d_G(v)$, where $d_G(v)$ is the degree of v in G. With $|T|$ for the size of a vertex set T, we also write $\|T\|$ for the number of edges induced by T and $\|A, B\|$ for the number of edges having endpoints in both A and B (when $A \cap B = \varnothing$). The characterization is the following.

Theorem 1.4 ([2, 12, 13])**.** *A multigraph G has a ℓ-factor if and only if*

$$q(S,T) - d_{G-S}(T) \le \ell(|S| - |T|) \qquad (1)$$

for all disjoint subsets $S, T \subset V(G)$, where $q(S,T)$ is the number of components Q of $G-S-T$ such that $\|V(Q), T\| + \ell\,|V(Q)|$ is odd.

Since we consider only the situation where $\ell = 2k$, the criterion for a component Q of $G - S - T$ to be counted by $q(S,T)$ simplifies to $\|V(Q), T\|$ being odd.

2 Cut-edges and $2k$-factors

In this section we generalize Theorem 1.1 to $2k$-factors.

Theorem 2.1. *For $r, k \in \mathbb{N}$ with $k \leq (2r+1)/3$, every $(2r+1)$-regular multigraph with at most $2r - 3(k-1)$ cut-edges has a $2k$-factor.*

Proof. Let G be a $(2r+1)$-regular multigraph having no $2k$-factor, and let p be the number of cut-edges in G. We prove $p > 2r - 3(k-1)$. By setting $\ell = 2k$ in Theorem 1.4, lack of a $2k$-factor requires disjoint sets $S, T \subseteq V(G)$ such that $q(S,T) > 2k(|S| - |T|) + d_{G-S}(T)$.

Letting $R = V(G) - S - T$, the quantity $q(S,T)$ becomes the number of components Q of $G[R]$ such that $\|V(Q), T\|$ is odd. Thus $q(S,T)$ has the same parity as $\|R, T\|$. In turn, $\|R, T\|$ has the same parity as $d_{G-S}(T)$, since the latter counts edges from R to T once and edges within T twice. Hence the two sides of the inequality above have the same parity. We conclude

$$q(S,T) \geq d_{G-S}(T) + 2k(|S| - |T|) + 2. \tag{2}$$

Say that a subgraph H of $G - T$ is *T-odd* if $\|V(H), T\|$ is odd. The components of $G - S - T$ that are T-odd are the components counted by $q(S,T)$. Each T-odd component contributes at least 1 to $d_{G-S}(T)$. Hence (2) cannot hold with $|S| \geq |T|$, and we may assume $|T| > |S|$.

Let q_1 be the number of T-odd components having one edge to T and no edges to S; since that edge is a cut-edge, $q_1 \leq p$. Let q_2 be the number of T-odd components having one edge to T and at least one edge to S; note that $q_2 \leq \|R, S\|$. Let q_3 be the number of T-odd components having at least three edges to T; thus $q_1 + q_2 + 3q_3 \leq d_{G-S}(T)$. Note also that $q(S,T) = q_1 + q_2 + q_3$. Summing the last inequality with two copies of the first two yields

$$3q(S,T) = 3(q_1 + q_2 + q_3) \leq 2p + 2\,\|R, S\| + d_{G-S}(T).$$

Combining this inequality with (2) yields

$$2p + 2\,\|R, S\| + d_{G-S}(T) \geq 3d_{G-S}(T) + 6k(|S| - |T|) + 6,$$

which simplifies to

$$\|R, S\| \geq 3 - p + d_{G-S}(T) + 3k(|S| - |T|). \tag{3}$$

On the other hand, since G is $(2r+1)$-regular,

$$d_{G-S}(T) = (2r+1)|T| - \|T, S\| \geq (2r+1)|T| - \left[(2r+1)|S| - \|R, S\|\right].$$

Using this inequality, (3), and $|T| - |S| \geq 1$, the given hypothesis $2r + 1 - 3k \geq 0$ yields

$$\|R, S\| \geq 3 - p + (2r + 1 - 3k)(|T| - |S|) + \|R, S\| \geq 3 - p + (2r + 1 - 3k) + \|R, S\|.$$

This simplifies to $p \geq 2r + 1 - 3(k - 1)$, as claimed. $\square$

3 Fewest cut-edges with no $2k$-factor

To describe the extremal graphs, we begin with a definition. Keep in mind that here "graph" allows loops and multiedges.

Definition 3.1. In a $(2r + 1)$-regular graph G, the result of *blistering* an edge $e \in E(G)$ by a $(2r+1)$-regular graph H having no cut-edge is a graph G' obtained from the disjoint union $G + H$ by deleting e and an edge $e' \in E(H)$ (where e' may be a loop if $r > 1$), followed by adding two disjoint edges to make each endpoint of e adjacent to one endpoint of e'. The resulting graph G' is $(2r + 1)$-regular.

Figure 1 illustrates blistering of one edge joining S and T in a 3-regular graph G with three cut-edges and no 2-factor to obtain a larger such graph G'. The components of $G' - S - T$ labeled Q_i are components counted by q_i, for $i \in \{1, 2, 3\}$.

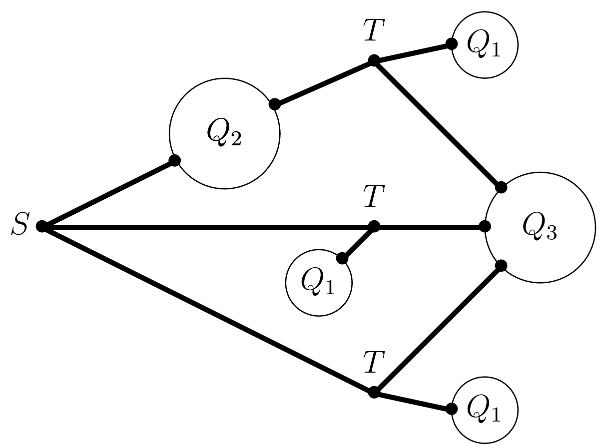

Figure 1: A class of 3-regular graphs with three cut-edges and no 2-factor.

Theorem 3.2. *For $k \leq (2r + 1)/3$, a $(2r + 1)$-regular graph with $2r + 4 - 3k$ cut-edges has no $2k$-factor if and only if the vertex set $V(G)$ has a partition into sets R, S, T such that*
(a) S and T are independent sets with $|T| > |S|$,
(b) all cut-edges join T to distinct components of $G[R]$,

(c) all edges incident to S lead to T (possibly via blisters that are components of $G[R]$),
(d) exactly $k(|T|-|S|)-1$ components of $G[R]$ are joined to T by exactly three edges each,
(e) each remaining component of R is $(2r+1)$-regular, with no cut-edge, and
(f) if $k<(2r+1)/3$, then $|T|-|S|=1$.

Proof. Sufficiency: Let G be a graph G with $2r+4-3k$ cut-edges, and suppose that such a partition $\{R,S,T\}$ of $V(G)$ exists. Let q_2 be the number of components of $G[R]$ that blister edges from S to T. Each cut-edge joins T to a T-odd component, by (b). The $k(|T|-|S|)-1$ components of $G[R]$ joined to T by three edges (according to (d)) are also T-odd, as are the q_2 components of $G[R]$ arising as blisters. Hence $q(S,T)\geq 2r+4-3k+k(|T|-|S|)-1+q_2$. The number of edges joining S and T is $(2r+1)\,|S|-q_2$, by (c). Using also (a), we have $d_{G-S}(T)=(2r+1)(|T|-|S|)+q_2$. We compute

$$\begin{aligned} q(S,T)-d_{G-S}(T) &\geq (2r+1-3k)+2+(k-2r-1)(|T|-|S|) \\ &= -(2r+1-3k)(|T|-|S|-1)+2k(|S|-|T|)+2 = 2k(|S|-|T|)+2, \end{aligned}$$

where the last equality uses (f) and the restriction $k\leq(2r+1)/3$. Hence the given partition R,S,T satisfies (2), and G has no $2k$-factor.

Necessity: Suppose that G has $2r+1-3(k-1)$ cut-edges and no $2k$-factor; we obtain the described partition of $V(G)$. The proof of Theorem 2.1 considers $(2r+1)$-regular graphs with no $2k$-factor and produces $p\geq 2r+4-3k$, where p is the number of cut-edges. To avoid having more cut-edges, we must have equality in all the inequalities used to produce this lower bound.

Recall that $q(S,T)$ counts the components Q of $G[R]$ with $\|V(Q),T\|$ odd. Also $q(S,T)=q_1+q_2+q_3$, where q_1,q_2,q_3 count the components having one edge to T and none to S, one edge to T and at least one to S, and at least three edges to T, respectively. Equality in the computation of Theorem 2.1 requires all of the following.

$$q_1=p \tag{4}$$

$$q_2=\|R,S\| \tag{5}$$

$$q_1+q_2+3q_3=d_{G-S}(T) \tag{6}$$

$$(2r+1)|S|=\|T,S\|+\|R,S\| \tag{7}$$

$$|T|-|S|\geq 1,\text{with equality when } k<(2r+1)/3 \tag{8}$$

By (6), contributions to $d_G(T)$ not in $\|T,S\|$ are counted in $\|T,R\|$, so T is independent. By (7), all edges incident to S lead to T or R, so S is independent, proving (a). The first observation in proving Theorem 2.1 was $|T| > |S|$, and equality in the last step requires $|T| - |S| = 1$ when $2r+1 > 3k$, as stated in (8) and desired in (f). By (4), the cut-edges join T to distinct components of $G[R]$, proving (b).

By (5) and (7), $q_2 = 0$ implies $(2r+1)|S| = \|T,S\|$, making all edges incident to S incident also to T. Since $(2r+1)|S| = \|T,S\| + q_2$, each component of $G[R]$ counted by q_2 generates only one edge from R to S. Thus each such component blisters an edge joining S and T in a smaller such graph. This explains all the edges counted by $\|S,R\|$. Hence we view the edges incident to S as edges to T with possible blisters, proving (c).

We have accounted for $(2r+1)|S|$ edges incident to T leading to S, including through q_2 blisters. There are also p cut-edges leading to components of $G[R]$, where $p = 2r+1-3(k-1)$. This leaves $(2r+1)|T| - (2r+1) + 3(k-1) - (2r+1)|S|$ edges incident to T that are not cut-edges and join T to components of $G[R]$ not counted by q_2.

When $k < (2r+1)/3$ and $|T| - |S| = 1$, this expression simplifies to $3(k-1)$. When $k = (2r+1)/3$, it simplifies to $3[k(|T|-|S|)-1]$, which is valid for both cases. By (6), all remaining edges incident to T connect vertices of T to T-odd components of $G[R]$ counted by q_3, using exactly three edges for each such component. Hence there are exactly $k(|T| - |S|) - 1$ such components of $G[R]$, proving (d). This completes the description of the T-odd components.

Since we have described all edges incident to S and T, any remaining components of $G[R]$ are actually $(2r+1)$-regular components of G without cut-edges, proving (e). They do not affect the number of T-odd components or the existence of a $2k$-factor. □

Theorem 3.2 can be viewed as a constructive procedure for generating all extremal examples from certain base graphs. Given r and k with $k \le (2r+1)/3$, we start with a bipartite graph having parts T and $R \cup S$, where $|T| - |S| \ge 1$, with equality if $k < (2r+1)/3$. Also, vertices in $T \cup S$ have degree $2r+1$, and R has $2r+4-3k$ vertices of degree 1 and $k(|T|-|S|)-1$ vertices of degree 3. We expand the vertices of R to obtain a $(2r+1)$-regular multigraph G. This is a base graph. We can then blister edges from S to T and/or add $(2r+1)$-regular 2-edge-connected components.

The case $|T| = 1$ and $|S| = 0$ gives the graphs found by Sylvester. When $k > (2r+1)/3$, an inequality used in the proof of Theorem 2.1 is not valid. In this range no restriction on cut-edges can guarantee a $2k$-factor; we present a simple general construction. As mentioned earlier, this is a sharpness example for the result of Bollobás, Saito, and Wormald [4] that every $(2r+1)$-regular $2t$-edge-connected or $(2t+1)$-edge-connected multigraph has a $2k$-factor if and only if $k \le \frac{t}{2t+1}(2r+1)$. It is simpler than their more general construction.

Theorem 3.3. *For $1 \leq t < r$ and $k > \frac{t}{2t+1}(2r+1)$, there is a $(2t+1)$-connected $(2r+1)$-regular graph having no $2k$-factor.*

Proof. Let $H_{r,t}$ be the complement of $C_{2t+1} + (r-t+1)K_2$. That is, $H_{r,t}$ is obtained from the complete graph K_{2r+3} by deleting the edges of a $(2t+1)$-cycle and $r-t+1$ other pairwise disjoint edges not incident to the cycle. Note that in $H_{r,t}$ the vertices of the deleted cycle have degree $2r$, while the remaining vertices have degree $2r+1$. Let G be the graph formed from the disjoint union of $2r+1$ copies of $H_{r,t}$ by adding a set T of $2t+1$ vertices and $2r+1$ matchings joining T to the vertices of the deleted cycle in each copy of $H_{r,t}$ (see Figure 2).

Deleting $2t$ vertices cannot separate any copy of $H_{r,t}$ from T, and any two vertices of T are connected by $2r+1$ disjoint paths through the copies of $H_{r,t}$, so G is $(2t+1)$-connected.

Suppose that G has a $2k$-factor F. Every edge cut in an even factor is crossed by an even number of edges, since the factor decomposes into cycles. Hence F has at most $2t$ edges joining T to each copy of $H_{r,t}$. On the other hand, since T is independent, F must have $2k|T|$ edges leaving T. Thus $2k(2t+1) \leq 2t(2r+1)$. $\square$

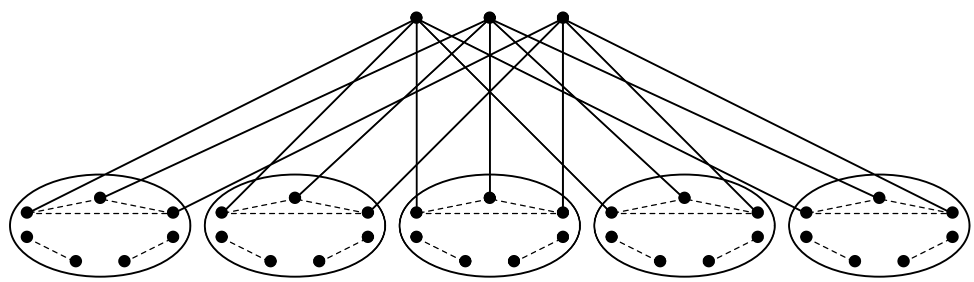

Figure 2: $(2r+1)$-regular, $(2t+1)$-connected, no $2k$-factor ($(r,t,k) = (2,1,2)$ shown).

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