

APPROXIMATING REAL-ROOTED AND STABLE POLYNOMIALS, WITH COMBINATORIAL APPLICATIONS

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ABSTRACT. Let $p(x) = a_0 + a_1x + \dots + a_nx^n$ be a polynomial with all roots real and satisfying $x \leq -\delta$ for some $0 < \delta < 1$. We show that for any $0 < \epsilon < 1$, the value of $p(1)$ is determined within relative error ϵ by the coefficients a_k with $k \leq \frac{c}{\sqrt{\delta}} \ln \frac{n}{\epsilon\sqrt{\delta}}$ for some absolute constant $c > 0$. Consequently, if $m_k(G)$ is the number of matchings with k edges in a graph G , then for any $0 < \epsilon < 1$, the total number $M(G) = m_0(G) + m_1(G) + \dots$ of matchings is determined within relative error ϵ by the numbers $m_k(G)$ with $k \leq c\sqrt{\Delta} \ln(v/\epsilon)$, where Δ is the largest degree of a vertex, v is the number of vertices of G and $c > 0$ is an absolute constant. We prove a similar result for polynomials with complex roots satisfying $\Re z \leq -\delta$ and apply it to estimate the number of unbranched subgraphs of G .

1. INTRODUCTION AND MAIN RESULTS

Our main motivation comes from the observation that in some cases, the total number of combinatorial structures of a particular type is determined with high accuracy by the exact number of the structures of the same type but of a small (sometimes, very small) size. For example, if G is a graph and we are interested in the number $M(G)$ of *matchings* in G , that is, collections of vertex-disjoint edges, then within a given relative error $0 < \epsilon < 1$, the number $M(G)$ is determined by the numbers $m_k(G)$ of matchings with exactly k edges for $k \leq c\sqrt{\Delta(G)} \ln \frac{v(G)}{\epsilon}$, where $\Delta(G)$ is the largest degree of a vertex of the graph, $v(G)$ is the number of vertices of G and $c > 0$ is an absolute constant. We deduce this, and some related results, from some general observation about real-rooted polynomials.

Below we talk about approximating some real and complex values up to “relative error ϵ ”. Given a complex number $a \neq 0$, we say that a complex number $b \neq 0$

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approximates a up to (or within) relative error $\epsilon > 0$ if we can write $a = e^z$ and $b = e^w$ for some complex numbers z and w such that $|z - w| \leq \epsilon$.

We prove the following main result.

(1.1) Theorem. *Suppose that $p(x) = a_0 + a_1x + \dots + a_nx^n$ is a polynomial and all roots x of p are real and satisfy $x \leq -\delta$ for some $0 < \delta < 1$. Then, for any $0 < \epsilon < 1$, the value of $p(1)$, up to relative error ϵ , is determined by the coefficients a_k with*

$$k \leq \frac{c}{\sqrt{\delta}} \ln \frac{n}{\epsilon\sqrt{\delta}},$$

where $c > 0$ is an absolute constant.

In fact, we present a polynomial time algorithm, which, given a_0, \dots, a_k with k as in the theorem, computes $p(1)$ within relative error ϵ . We note that we need to know a_0, \dots, a_k exactly (or, at the very least, with very high precision). Theorem 1.1 is a much improved version of the “personal communication” of the author that was referred to in [PR17].

Using the Heilmann-Lieb Theorem [HL72], see also [GG81], we immediately deduce the result about matchings in graphs. We consider undirected graphs, without loops or multiple edges.

(1.2) Theorem. *For a graph G , let $m_k(G)$ be the number of matchings that contain exactly k edges and let $M(G) = m_0(G) + m_1(G) + \dots$ be the total number of matchings. Then, for any $0 < \epsilon < 1$, up to relative error ϵ , the number $M(G)$ is determined by the numbers $m_k(G)$ with*

$$k \leq c\sqrt{\Delta(G)} \ln \frac{v(G)}{\epsilon},$$

where $\Delta(G)$ is the largest degree of a vertex of the graph, $v(G)$ is the number of vertices of G and $c > 0$ is an absolute constant.

Again, we have a polynomial time algorithm, which, given $m_k(G)$, with k as in the theorem, produces an estimate of $M(G)$ within relative error ϵ . We note that Patel and Regts [PR17] constructed a polynomial time algorithm for computing $m_k(G)$ with $k = O(\ln v(G))$ provided the largest degree $\Delta(G)$ is fixed in advance (a straightforward enumeration gives only a quasi-polynomial algorithm of $v(G)^{O(\ln v(G))}$ complexity). For general graphs, the complexity of the algorithm roughly matches that of Bayati et al. [B+07], which estimates $M(G)$ using the correlation decay approach. Although our approach and that of [B+07] look completely different, they are both inspired by the concept of “phase transition” coming from statistical physics; more precisely, two related, but different concepts: ours has to do with the Lee-Yang approach via complex zeros of the “partition function” [YL52], [LY52] while that of [B+07] has to do with correlation decay, cf. [DS87] and [KK98]. Hence the fact that the complexity appears to be roughly the same is not entirely accidental.

Using the Chudnovsky-Seymour extension [CS07] of the Heilmann-Lieb Theorem and the Dobrushin-Shearer bound on the roots of the independence polynomial, see, for example, [SS05], we get another combinatorial application. Recall that a subset of vertices of a graph is called an *independent set* if no two vertices of the subset span an edge of the graph. A graph is called *claw-free* if it does not contain an induced subgraph consisting of a vertex connected to some other three vertices that are pairwise unconnected. We obtain the following result.

(1.3) Theorem. *For a graph G , let $i_k(G)$ be the number of independent sets with exactly k vertices and let $I(G) = i_0(G) + i_1(G) + \dots$ be the total number of independent sets. Then, for any $0 < \epsilon < 1$, up to relative error ϵ , the number $I(G)$ of a claw-free graph is determined by the numbers $i_k(G)$ with*

$$k \leq c\sqrt{\Delta(G)} \ln \frac{v(G)}{\epsilon},$$

where $\Delta(G)$ is the largest degree of a vertex of the graph, $v(G)$ is the number of vertices of G and $c > 0$ is an absolute constant.

We have a polynomial time algorithm, which, given $i_k(G)$, with k as in the theorem, produces an estimate of $I(G)$ within relative error ϵ . Curiously, while the correlation decay approach of [B+07] is essentially harder in the case of independent sets in claw-free graphs than it is in the case of matchings, our approach is the same in both cases (assuming, of course, the hard work done in [HL72] and [CS07]).

Theorems 1.2 and 1.3 vaguely resemble the “approximate inclusion-exclusion” of [LN90] and [K+96]. The methods, however, look completely different. It would be interesting to find out if there is indeed any connection between our Theorem 1.1 and the results of [LN90] and [K+96].

Next, we consider polynomials $p(z) = a_0 + a_1z + \dots + a_nz^n$ with complex roots satisfying $\Re z \leq -\delta$ for some $0 < \delta < 1$ (we call such polynomials “stable”). We allow complex coefficients a_k . We obtain the following result.

(1.4) Theorem. *Suppose that $p(z) = a_0 + a_1z + \dots + a_nz^n$ is a complex polynomial and all roots z of p satisfy $\Re z \leq -\delta$ for some $0 < \delta < 1$. Then, for any $0 < \epsilon < 1$, the value of $p(1)$, up to relative error ϵ , is determined by the coefficients a_k with*

$$k \leq \frac{c}{\delta} \ln \frac{n}{\epsilon\delta},$$

where $c > 0$ is an absolute constant.

We apply Theorem 1.4 to count *unbranched subgraphs*, that is, collections of edges of the graph such that every vertex of the graph is incident to at most two edges from the collection. From Ruelle’s Theorem [R99a], [R99b], see also [Wa09], we deduce the following result.

(1.5) Theorem. For a graph G , let $u_k(G)$ be the number of unbranched subgraphs with exactly k edges and let $U(G) = u_0(G) + u_1(G) + \dots$ be the total number of unbranched subgraphs. Then, for any $0 < \epsilon < 1$, up to relative error ϵ , the number $U(G)$ is determined by the numbers $u_k(G)$ with

$$k \leq c(\Delta(G))^3 \ln \frac{v(G)}{\epsilon},$$

where $\Delta(G)$ is the largest degree of a vertex of the graph, $v(G)$ is the number of vertices of G and $c > 0$ is an absolute constant.

One can easily see that if a non-constant polynomial p satisfies the conditions of Theorems 1.1 or 1.4 then so does its derivative p' . Therefore, in Theorems 1.2, 1.3 and 1.5, we can not only estimate the number of structures of a given type (matchings, independent sets or unbranched subgraphs) by counting structures up to some small size, but also estimate the average size of a structure, the second moment, etc.

Finally, we mention the following result implicit in Section 2.2 of [Ba16].

(1.6) Theorem. Let $p(z) = a_0 + a_1z + \dots + a_nz^n$ be a polynomial and suppose that for some $0 < \delta < 1$ we have $p(z) \neq 0$ for all z in the δ -neighborhood of the interval $[0, 1] \subset \mathbb{C}$ (we measure distances by identifying $\mathbb{C} = \mathbb{R}^2$). Then, for any $0 < \epsilon < 1$, the value of $p(1)$, up to relative error ϵ , is determined by the coefficients a_k with

$$k \leq \exp \left\{ O \left(\frac{1}{\delta} \right) \right\} \ln \frac{n \exp \left\{ O \left(\frac{1}{\delta} \right) \right\}}{\epsilon}.$$

For applications of Theorem 1.6 to computing partition functions, see [Ba16].

We can replace the exponential dependence on $1/\delta$ in Theorem 1.6 by a polynomial dependence if we assume that $p(z) \neq 0$ for z in the δ -neighborhood of the sector $|\arg z| < \alpha$ for some fixed $\alpha > 0$ and some $\delta > 0$. We briefly discuss this in Section 2 and applications to counting subgraphs with prescribed degrees in Section 3.

We prove Theorems 1.1 and 1.4 in Section 2 and Theorems 1.2, 1.3 and 1.5 in Section 3.

2. PROOFS OF THEOREMS 1.1 AND 1.4

We denote the complex plane by \mathbb{C} , the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by $\widehat{\mathbb{C}}$ and the open unit disc by \mathbb{D} , so that

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

By c we denote a positive absolute constant, whose value may change from line to line.

We start with a couple of lemmas.

(2.1) Lemma. Let $h_1, h_2 : \mathbb{C} \rightarrow \mathbb{C}$ be polynomials of degrees n_1 and n_2 respectively and let

$$g(z) = \frac{h_1(z)}{h_2(z)}, \quad g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}},$$

be a rational function. Let $\beta > 1$ be a real number and suppose that

$$h_1(z) \neq 0 \quad \text{and} \quad h_2(z) \neq 0 \quad \text{provided} \quad |z| < \beta,$$

so g has neither zeros nor poles in the disc $\beta\mathbb{D} = \{z \in \mathbb{C} : |z| < \beta\}$.

Let us choose a branch of

$$f(z) = \ln g(z) \quad \text{where} \quad |z| < \beta$$

and let

$$T_m(z) = f(0) + \sum_{k=1}^m \frac{f^{(k)}(0)}{k!} z^k$$

be the Taylor polynomial of degree m of $f(z)$ computed at $z = 0$. Then

$$|f(1) - T_m(1)| \leq \frac{n_1 + n_2}{\beta^m (\beta - 1)(m + 1)}.$$

Proof. In the case when $g(z)$ is a polynomial (that is, when $h_2(z) \equiv 1$), this is Lemma 2.2.1 of [Ba16]. The proof below in the case of a rational function is very similar.

Let $\alpha_{11}, \dots, \alpha_{1n_1}$ be the roots of h_1 and let $\alpha_{21}, \dots, \alpha_{2n_2}$ be the roots of h_2 , counting multiplicity. Hence

$$h_1(z) = h_1(0) \prod_{i=1}^{n_1} \left(1 - \frac{z}{\alpha_{1i}}\right) \quad \text{and} \quad h_2(z) = h_2(0) \prod_{j=1}^{n_2} \left(1 - \frac{z}{\alpha_{2j}}\right),$$

where

$$|\alpha_{1i}| \geq \beta \quad \text{for} \quad i = 1, \dots, n_1 \quad \text{and} \quad |\alpha_{2j}| \geq \beta \quad \text{for} \quad j = 1, \dots, n_2.$$

Then

$$f(z) = f(0) + \sum_{i=1}^{n_1} \ln \left(1 - \frac{z}{\alpha_{1i}}\right) - \sum_{j=1}^{n_2} \ln \left(1 - \frac{z}{\alpha_{2j}}\right),$$

where we choose the branch of the logarithm so that $\ln 1 = 0$.

Approximating the logarithms by their Taylor polynomials, we obtain

$$\ln \left(1 - \frac{1}{\alpha_{1i}}\right) = - \sum_{k=1}^m \frac{1}{k \alpha_{1i}^k} + \eta_{1i} \quad \text{and} \quad \ln \left(1 - \frac{1}{\alpha_{2j}}\right) = - \sum_{k=1}^m \frac{1}{k \alpha_{2j}^k} + \eta_{2j},$$

where

$$|\eta_{1i}| = \left| \sum_{k=m+1}^{\infty} \frac{1}{k\alpha_{1i}^k} \right| \leq \frac{1}{m+1} \sum_{k=m+1}^{\infty} \frac{1}{\beta^k} = \frac{1}{(m+1)\beta^m(\beta-1)}$$

for $i = 1, \dots, n_1$ and, similarly,

$$|\eta_{2j}| = \left| \sum_{k=m+1}^{\infty} \frac{1}{k\alpha_{2j}^k} \right| \leq \frac{1}{m+1} \sum_{k=m+1}^{\infty} \frac{1}{\beta^k} = \frac{1}{(m+1)\beta^m(\beta-1)}$$

for $j = 1, \dots, n_2$.

Since

$$T_m(1) = - \sum_{i=1}^{n_1} \sum_{k=1}^m \frac{1}{k\alpha_{1i}^k} + \sum_{j=1}^{n_2} \sum_{k=1}^m \frac{1}{k\alpha_{2j}^k},$$

the proof follows. □

(2.2) Corollary. For $0 < \epsilon < 1$, under the conditions of Lemma 2.1, we have

$$|f(1) - T_m(1)| \leq \epsilon$$

provided

$$m \geq \frac{c}{\beta-1} \ln \frac{n_1 + n_2}{\epsilon(\beta-1)}$$

where $c > 0$ is an absolute constant.

Proof. Follows by Lemma 2.1. □

To compute the value of $T_m(1)$ in Lemma 2.1 and Corollary 2.2, we need to compute the derivatives $f^{(k)}(0)$ for $k = 0, 1, \dots, m$. This, in turn, reduces to computing the derivatives $g^{(k)}(0)$ for $k = 0, 1, \dots, m$, as explained in Section 2.2.2 of [Ba16]. For completeness, we describe the procedure here.

(2.3) Computing $f^{(k)}(0)$ from $g^{(k)}(0)$. We have

$$f'(z) = \frac{g'(z)}{g(z)} \quad \text{from which} \quad g'(z) = f'(z)g(z)$$

and hence

$$(2.3.1) \quad g^{(k)}(0) = \sum_{j=0}^{k-1} \binom{k-1}{j} f^{(k-j)}(0)g^{(j)}(0) \quad \text{for } k = 1, \dots, m.$$

Now, (2.3.1) is a triangular system of linear equations in the variables $f^{(k)}(0)$ for $k = 1, \dots, m$ with diagonal coefficients $g^{(0)}(0) = g(0) \neq 0$, so the matrix of the system is invertible. Given the values of $g(0)$ and $g^{(k)}(0)$ for $k = 1, \dots, m$, one can compute the values of $f^{(k)}(0)$ for $k = 1, \dots, m$ in $O(m^2)$ time. This is, of course, akin to computing cumulants of a probability distribution from its moments.

Finally, we employ a rational transformation.

(2.4) Lemma. For real $0 < \rho < 1$, let

$$\xi = \xi_\rho = 1 - \sqrt{\frac{\rho}{1+\rho}}, \quad \beta = \beta_\rho = \xi^{-1} \geq 1 + \sqrt{\rho}.$$

and let

$$\psi = \psi_\rho(z) = \frac{\rho}{(1 - \xi z)^2} - \rho, \quad \psi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}},$$

be a rational function. Then $\psi(0) = 0$, $\psi(1) = 1$ and the image of the disc

$$\beta\mathbb{D} = \{z \in \mathbb{C} : |z| < \beta\}$$

under ψ does not intersect the ray

$$\left\{ z \in \mathbb{C} : \Im z = 0 \quad \text{and} \quad \Re z \leq -\frac{3\rho}{4} \right\}.$$

Proof. Clearly, $\psi(0) = 0$ and $\psi(1) = 1$. For $z \in \beta\mathbb{D}$, we have $|\xi z| < 1$ and hence

$$\arg \frac{1}{1 - \xi z} < \frac{\pi}{2}.$$

Therefore the image of $\beta\mathbb{D}$ under the map

$$(2.4.1) \quad z \longmapsto \frac{\rho}{(1 - \xi z)^2}$$

does not contain the non-positive real ray

$$R_- = \{z \in \mathbb{C} : \Im z = 0, \Re z \leq 0\}.$$

The real values of the map (2.4.1) on the disc $\beta\mathbb{D}$ are attained when z is real, and are larger than $\rho/4$, which is attained when $z = -\beta$.

The proof now follows. □

Now we are ready to prove Theorem 1.1.

(2.5) Proof of Theorem 1.1. Let $\rho = 4\delta/3$ and let $\psi = \psi_\rho : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be the corresponding rational transformation of Lemma 2.4. We consider the composition

$$g(z) = p(\psi(z)).$$

Clearly, $g(0) = p(0)$ and $g(1) = p(1)$. Let

$$\xi = 1 - \sqrt{\frac{\rho}{1+\rho}} \quad \text{and} \quad \beta = \xi^{-1} \geq 1 + \sqrt{\frac{4\delta}{3}},$$

as in Lemma 2.4. Since the image $\psi(\beta\mathbb{D})$ does not intersect the ray

$$R = \{z \in \mathbb{C} : \Im z = 0 \text{ and } \Re z \leq -\delta\},$$

we conclude that

$$g(z) \neq 0 \text{ provided } |z| < \beta.$$

For some polynomials $h_1(z)$ and $h_2(z)$, we can write

$$g(z) = \frac{h_1(z)}{h_2(z)} \text{ where } h_2(z) = (1 - \xi z)^{2n} \text{ and } \deg h_2(z) \leq 2n.$$

Let us choose a branch of

$$f(z) = \ln g(z) \text{ for } |z| < \beta$$

and let $T_m(z)$ be the Taylor polynomial of f of degree m , computed at $z = 0$. From Corollary 2.2, we have

$$|T_m(1) - f(1)| = |T_m(1) - \ln p(1)| \leq \epsilon,$$

as long as

$$m \geq \frac{c}{\sqrt{\delta}} \ln \frac{n}{\epsilon\sqrt{\delta}}$$

for some absolute constant $c > 0$.

It remains to show how to compute the values $f^{(k)}(0)$ for $k = 0, \dots, m$ from the coefficients a_k , $k = 0, 1, \dots, m$, of the polynomial p . First, we compute the values $g^{(k)}(0)$ for $k = 0, \dots, m$. To that effect, let

$$p_{[m]}(z) = \sum_{k=0}^m a_k z^k$$

be the truncation of the polynomial p and let

$$\psi_{[m]}(z) = \rho \sum_{k=1}^m (k+1) \xi^k z^k$$

be the truncation of the Taylor series expansion of $\psi(z)$ in the disc $\beta\mathbb{D}$. Note that since $\psi(0) = 0$, the constant term of the expansion is 0. We then compute the composition

$$p_{[m]}(\psi_{[m]}(z))$$

and discard the terms of degree higher than m . A fast way to do it is by Horner's method, successively computing

$$(\dots((a_m \psi_{[m]}(z) + a_{m-1}) \psi_{[m]}(z)) + \dots) \psi_{[m]}(z) + a_0$$

and discarding monomials of degree higher than m on each step. This gives us the Taylor polynomial of degree m of $g(z)$, computed at $z = 0$. We then compute the derivatives $f^{(k)}(0)$ as in Section 2.3. \square

To prove Theorem 1.4, we use a different (simpler) rational transformation.

(2.6) Lemma. For real $0 < \rho < 1$, let

$$\xi = \xi_\rho = \frac{1}{1 + \rho}, \quad \beta = \beta_\rho = \xi^{-1} = 1 + \rho.$$

and let

$$\psi = \psi_\rho(z) = \frac{\rho}{1 - \xi z} - \rho, \quad \psi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}},$$

be a rational function. Then $\psi(0) = 0$, $\psi(1) = 1$ and the image of the disc

$$\beta\mathbb{D} = \{z \in \mathbb{C} : |z| < \beta\}$$

under ψ does not intersect the half-plane

$$\{z \in \mathbb{C} : \Re z \leq -\rho\}.$$

Proof. Clearly, $\psi(0) = 0$ and $\psi(1) = 1$. For $z \in \beta\mathbb{D}$, we have $|\xi z| < 1$ and hence

$$\arg \frac{1}{1 - \xi z} < \frac{\pi}{2}.$$

Therefore, the image of $\beta\mathbb{D}$ under the map

$$z \longmapsto \frac{1}{1 - \xi z}$$

does not intersect the half-plane $\Re z \leq 0$.

The proof now follows. □

(2.7) Proof of Theorem 1.4. We define the transformation $\psi = \psi_\delta$ as in Lemma 2.6, consider the composition $g(z) = p(\psi(z))$ and proceed as in the proof on Theorem 1.1 in Section 2.5 with straightforward modifications. □

(2.8) Remark: approximating $p'(1)$. It follows from Rolle's Theorem that if $p(x) = a_0 + a_1x + \dots + a_nx^n$ is a non-constant polynomial satisfying the conditions of Theorem 1.1 then $p'(x) = a_1 + 2a_2x + \dots + na_nx^{n-1}$ also satisfies the conditions of Theorem 1.1. Similarly, it follows from the Gauss-Lucas Theorem that if a non-constant polynomial $p(z)$ satisfies the conditions of Theorem 1.4, then so does $p'(z)$.

(2.9) Possible ramifications. To prove Theorem 1.6, for $0 < \rho < 1$, we define

$$\xi = \xi_\rho = 1 - e^{-\frac{1}{\rho}}, \quad \beta = \frac{1 - e^{-1 - \frac{1}{\rho}}}{1 - e^{-\frac{1}{\rho}}} > 1$$

and consider the map

$$\psi = \psi_\rho(z) = \rho \ln \frac{1}{1 - \xi z} \quad \text{where } |z| < \beta.$$

Then $\psi(0) = 0$, $\psi(1) = 1$ and if $\rho = \rho(\delta) > 0$ is chosen small enough, the image $\psi(\beta\mathbb{D})$ lies in the $\delta/2$ -neighborhood of $[0, 1] \subset \mathbb{C}$. However, ψ is no longer a rational transformation. Instead of ψ , we use its scaled Taylor polynomial approximation $\tilde{\psi}$ such that we still have $\tilde{\psi}(0) = 0$, $\tilde{\psi}(1) = 1$ and $\tilde{\psi}$ maps the disc $\beta\mathbb{D}$ inside the δ -neighborhood of $[0, 1] \subset \mathbb{C}$, see Section 2.2 of [Ba16] for details. Then the proof proceeds as in Section 2.5.

Suppose now that $p(z)$ is a polynomial of degree n such that $p(z) \neq 0$ whenever z lies in the δ -neighborhood of the sector

$$S_\alpha = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \alpha\}$$

for some fixed $\alpha > 0$ and some $\delta > 0$. In this case, for $0 < \rho < 1$ define

$$\xi = \xi_\rho = 1 - \left(\frac{\rho}{1+\rho}\right)^{\pi/2\alpha}, \quad \beta = \beta_\rho = \xi^{-1} \left(1 - \frac{1}{2} \left(\frac{\rho}{1+\rho}\right)^{\pi/2\alpha}\right) > 1$$

and consider the (well-defined) map

$$\psi(z) = \psi_\rho(z) = \rho(1 - \xi z)^{-2\alpha/\pi} - \rho \quad \text{for } |z| < \beta.$$

We observe that $\psi(0) = 0$, $\psi(1) = 1$ and one can show that by choosing ρ small enough, we can make sure that the image of the disc $\beta\mathbb{D}$ lies in a prescribed neighborhood of the sector $|\arg z| \leq \alpha$. We then can use a sufficiently accurate polynomial approximation $\tilde{\psi}$ to ψ to show that the value of $p(1)$, up to relative error ϵ , is determined by the lowest

$$\frac{c}{\delta^{\pi/2\alpha}} \ln \frac{n}{\epsilon \delta^{\pi/2\alpha}}$$

coefficients of p , where $c > 0$ is an absolute constant.

3. PROOFS OF THEOREMS 1.2, 1.3 AND 1.5

(3.1) Proof of Theorem 1.2. Given a graph G , we define its *matching polynomial* by

$$p(x) = 1 + \sum_{k=1}^{v(G)} m_k(G) x^k.$$

The Heilmann-Lieb Theorem [HL72], see also [GG81], asserts that the roots x of $p(x)$ are real and satisfy $x \leq -\delta$ for

$$\delta = \frac{1}{4(\Delta - 1)} \quad \text{for } \Delta = \max\{\Delta(G), 2\}.$$

The proof now follows from Theorem 1.1. □

(3.2) Proof of Theorem 1.3. Given a graph G , we define its *independence polynomial* by

$$p(x) = 1 + \sum_{k=1}^{v(G)} i_k(G)x^k.$$

Chudnovsky and Seymour proved [CS07] that if G is claw-free, then the roots of $p(x)$ are necessarily non-positive real. On the other hand, the Dobrushin-Shearer bound, cf. [SS05], states that the roots z of the independence polynomial of any graph G satisfy

$$|z| \geq \frac{(\Delta - 1)^{\Delta-1}}{\Delta^\Delta} = \frac{1}{\Delta e} \left(1 + O\left(\frac{1}{\Delta}\right) \right) \quad \text{as } \Delta \rightarrow \infty,$$

where $\Delta = \max\{2, \Delta(G)\}$. The proof now follows from Theorem 1.1. \square

(3.3) Proof of Theorem 1.5. Given a graph G , we define its *unbranched subgraph polynomial* by

$$p(z) = 1 + \sum_{k=1}^{v(G)} u_k(G)z^k.$$

Ruelle proved [R99a], [R99b], see also [Wa09], that all roots z of $p(z)$ satisfy

$$\Re z \leq -\frac{2}{\Delta(\Delta - 1)^2} \quad \text{for } \Delta = \max\{2, \Delta(G)\}.$$

The proof now follows from Theorem 1.4. \square

(3.4) Estimating averages. We note that the value $\frac{p'(1)}{p(1)}$ is interpreted the average number of edges in a matching in Theorem 1.2, the average number of vertices in an independent set in Theorem 1.3 and the average number of edges in a unbranched subgraph in Theorem 1.5. It follows from Remark 2.8 that we can estimate the averages within relative error $\epsilon > 0$ by inspecting the matchings, independent sets and unbranched subgraphs of pretty much the same size as prescribed by Theorems 1.2, 1.3 and 1.5, though with different absolute constants. Similarly, by computing $\frac{p''(1)}{p(1)}$ we can estimate the second moment, etc.

(3.5) Possible ramifications. For each vertex w of a graph G , let us choose a set A_w of allowable degrees of subgraphs. Wagner proved [Wa09] that if $0 \in A_w \subset \{0, 1, 2\}$ for all vertices w , then the corresponding subgraph counting polynomial is non-zero in the sector

$$S_{\pi/3} = \left\{ z \in \mathbb{C} : |\arg z| < \frac{\pi}{3} \right\}$$

and is also non-zero in a δ -neighborhood of $z = 0$ for some $\delta = \Omega(1/\Delta(G))$. Using the approach sketched in Section 2.8, one can show that within relative error $\epsilon > 0$,

the total number of such subgraphs is determined by the numbers of subgraphs with

$$k = (\Delta(G))^{O(1)} \ln \frac{v(G)}{\epsilon}$$

edges.

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