Law of two-sided exit by a spectrally positive strictly stable process[1]

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Abstract

For a spectrally positive strictly stable process with index in $(1, 2)$, the paper obtains i) the density of the time when the process makes first exit from an interval by hitting the interval's lower end point before jumping over its upper end point, and ii) the joint distribution of the time, the undershoot, and the jump of the process when it makes first exit the other way around. For i), the density of the time of first exit is expressed as an infinite sum of functions, each the product of a polynomial and an exponential function, with all coefficients determined by the roots of a Mittag-Leffler function. For ii), conditional on the undershoot, the time and the jump of first exit are independent, and the marginal conditional densities of the time has similar features as i).

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1 Introduction

The so-called exit problems, which concern the random event that a stochastic process gets out of a set for the first time, occupy a prominent place in the study of Lévy processes. For spectrally positive Lévy processes, years of intensive research have revealed many remarkable facts about first exit from a bounded closed interval [2, 8, 9]. An essential tool for many of the results is the scale function. Since the function can be analytically extended to $\mathbb{C}$ ([9], Lemma 8.3), it is amenable to treatments by complex analysis. By combining the scale function and residual calculus, this paper obtains series expressions of the distribution of first exit of a spectrally positive strictly stable process with index in $(1, 2)$.

Henceforth, without loss of generality, let X be a Lévy process with

$$\mathbb{E}[e^{-qX_t}] = e^{tq^\alpha}, \quad t > 0,\ q > 0,\ \alpha \in (1, 2). \tag{1}$$

Given $b, c \in (0, \infty)$, first exit by X from interval $[-b, c]$ consists of two possibilities: either the process makes a continuous downward passage of $-b$ before it makes an upward jump across c or the other way around. The probability of each possibility is well-known (cf. [2], Theorem VII.8). Meanwhile, the scale functions of the first exit have been known [9]. On the other hand, not much is known about the explicit joint probability density function (p.d.f.) for the first exit.

In [6], by using Laplace transform, the distribution of first upward passage of a fixed level by X is obtained. It turns out that the method used there can be extended to first exit from an interval. Section 2 considers the time of first exit from $[-b, c]$ at $-b$. It will be shown that the probability density function (p.d.f.) of the time has an expression in terms of the residuals of a certain function at the roots of a Mittag-Leffler function, and as a result, is of the form $f(t) = \sum_\varsigma p_\varsigma(t) e^{t\varsigma}$, where

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the sum runs over the roots and for each root ς, $p_\varsigma(t)$ is a polynomial in t whose coefficients are determined by ς and several Mittag-Leffler functions. For all but a finite number of ς, $p_\varsigma(t)$ is of order 0. The result provides a connection to some known results on first exit of a standard Brownian motion. It also highlights the importance of gaining more information on the roots of Mittag-Leffler functions [13]. Section 3 considers the joint distribution of the time, the undershoot, and the jump of X when its first exit from $[-b, c]$ occurs at c. It will be shown that conditional on the undershoot, the time and the jump are independent. This allows the joint distribution to be factorized into the marginal p.d.f. of the undershoot, and the marginal conditional p.d.f.'s of the time and the jump, respectively. The expression of the marginal conditional p.d.f. of the time has similar features as the one of first exit at the lower end. This is in contrast to the power series expression of the time of first upward passage of c [1, 6, 12, 16], even though the first passage can be regarded as the limit of first exit from $[-b, c]$ as $b \to \infty$. In both sections, the asymptotics of the time of first exit as $t \to 0$ or ∞ are also considered.

The rest of the section fixes notation and collects some background information for later use.

Integral transforms. For $f \in L^1(\mathbb{R})$, denote its Laplace transform and Fourier transform, respectively, by

$$\widetilde{f}(z) = \int e^{-tz} f(t)\,\mathrm{d}t \quad \text{and} \quad \widehat{f}(\theta) = \widetilde{f}(-\mathrm{i}\theta), \quad \theta \in \mathbb{R}.$$

The domain of $\widetilde{f}$ is $\{z \in \mathbb{C} : e^{-tz} f(t) \in L^1(\mathrm{d}t)\}$. Similarly, for a finite measure μ on $\mathbb{R}$, denote its Laplace transform and Fourier transform, respectively, by

$$\widetilde{\mu}(z) = \int e^{-tz} \mu(\mathrm{d}t) \quad \text{and} \quad \widehat{\mu}(\theta) = \widetilde{\mu}(-\mathrm{i}\theta), \quad \theta \in \mathbb{R}.$$

The domain of $\widetilde{\mu}$ is $\{z \in \mathbb{C} : e^{-tz} \in L^1(\mathrm{d}t)\}$.

Let $z_0 \in \mathbb{C}$. Denote $U_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$. If function g is analytic in $U_r(z_0) \setminus \{z_0\}$ for some $r > 0$ and has z_0 as a pole, possibly removable, then the residual of g at z_0 is

$$\operatorname{Res}(g(z), z_0) = c_1 = \frac{1}{2\pi\mathrm{i}} \oint_\gamma g(z)\,\mathrm{d}z,$$

where γ is any counterclockwise simple closed contour in $U_r(z_0) \setminus \{z_0\}$ ([14], p. 224).

Some properties of a Mittag-Leffler function. A Mittag-Leffler function with parameters $a > 0$ and $b \in \mathbb{C}$ is an entire function defined as

$$E_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an + b)}, \quad z \in \mathbb{C}.$$

Let $\mathscr{Z}_{a,b} = \{z \in \mathbb{C} : E_{a,b}(z) = 0\}$. The focus will be mostly on $E_{\alpha,\alpha}(z)$ with $\alpha \in (1, 2)$. By [13], Theorems 1.2.1, 1.4.2, and 1.5.1,

$$E_{\alpha,\alpha}(z) = \alpha^{-1} z^{1/\alpha - 1} \exp(z^{1/\alpha}) - \frac{(\alpha - 1)\alpha z^{-2}}{\Gamma(2 - \alpha)} + O(|z|^{-3}), \quad |z| \to \infty, \tag{2}$$

where the $O(\cdot)$ term is uniform in $\arg z$ and the principle branch of z is used so that $\arg(e^{\mathrm{i}\theta}) = \theta - 2k\pi$ for any $\theta \in \mathbb{R}$ with k the unique integer satisfying $2k - 1 < \theta/\pi \le 2k + 1$. Furthermore, from Theorems 2.1.1 and 4.2.1, and Chapter 6 in [13],

1) $\mathscr{Z}_{\alpha,\alpha}$ has infinitely many elements and all those with large enough modulus are simple roots of $E_{\alpha,\alpha}(z)$ and can be enumerated as $\varsigma_{\pm n}$, $n \ge N$, for some large N, such that

$$\varsigma_{\pm n} = [1 + o(1)](2\pi n)^{\alpha} e^{\pm \mathrm{i}\alpha\pi/2}, \quad n \to \infty; \tag{3}$$

2) $\mathscr{Z}_{\alpha,\alpha} \subset \{z : |\arg(z)| > \alpha\pi/2\}$; and

3) $E_{\alpha,\alpha}(z)$ has a finite positive number of real roots, all being negative.

First passage time and hitting time. For $c > 0$ and $x \in \mathbb{R}$, denote $T_c = \inf\{t > 0 : X_t > c\}$ and $\tau_x = \inf\{t > 0 : \ X_t = x\}$. Under the law of X, $T_c < \tau_c < \infty$ and $X_{T_c} > c > X_{T_c-}$ a.s. [16], both T_c and τ_x have p.d.f.'s [1, 2, 12, 16], and given $b > 0$, as downward movement is continuous and 0 is regular for $(-\infty, 0)$, $\tau_{-b} = \inf\{t : X_t < -b\}$ ([8], Theorem 5.17), so the time of first exit from $[-b, c]$ is $\min(\tau_{-b}, T_c)$. When $\tau_{-b} < T_c$ (resp. $\tau_{-b} > T_c$), X is said to make first exit from $[-b, c]$ at the lower (resp. upper) end. Denote by g_t the p.d.f. of X_t and by f_x the p.d.f. of τ_x. The distribution of τ_x is classical for $x < 0$ ([2], Theorem VII.1) and is also known for $x > 0$ [16].

We will rely on the scale function $W^{(q)}$ of $-X$ for the derivation ([9], Chapter 8; also cf. [2, 8]). For the spectrally negative strictly stable process $-X$,

$$W^{(q)}(x) = x_+^{\alpha-1} E_{\alpha,\alpha}(q x_+^{\alpha}), \quad q \ge 0, \tag{4}$$

where $x_+ = \max(x, 0)$ ([9], p. 250).

2 Distribution of first exit time at lower end

Given $c > 0$ and $x < c$, denote by

$$k_{x,c}(t) = \frac{\mathbb{P}\{\tau_x \in \mathrm{d}t,\, X_s < c\,\forall s \le t\}}{\mathrm{d}t} = \frac{\mathbb{P}\{\tau_x \in \mathrm{d}t,\, T_c > \tau_x\}}{\mathrm{d}t}.$$

Since τ_x has a p.d.f., $k_{x,c}(t)$ is well-defined, and since its integral over t is $\mathbb{P}\{T_c > \tau_x\} < 1$, it is a sub-p.d.f. rather than a proper one. For $b > 0$ and $c > 0$, it is well-known that ([9], Theorem 8.1)

$$\widetilde{k}_{-b,c}(q) = \frac{W^{(q)}(c)}{W^{(q)}(b+c)} = \frac{c^{\alpha-1}}{(b+c)^{\alpha-1}} \frac{E_{\alpha,\alpha}(c^{\alpha} q)}{E_{\alpha,\alpha}((b+c)^{\alpha} q)}. \tag{5}$$

Proposition 1. *Given $b > 0$ and $c > 0$, put $s = c^{\alpha}/(b+c)^{\alpha}$. Then*

$$k_{-b,c}(t) = \frac{c^{\alpha-1}}{(b+c)^{2\alpha-1}} \psi_s\left(\frac{t}{(b+c)^{\alpha}}\right), \tag{6}$$

where

$$\psi_s(t) = \sum_{\varsigma \in \mathscr{Z}_{\alpha,\alpha}} \operatorname{Res}(H_s(z) e^{tz}, \varsigma), \quad t > 0, \tag{7}$$

is a p.d.f. concentrated on $[0, \infty)$ and for $s \in [0, 1)$

$$H_s(z) = \frac{E_{\alpha,\alpha}(sz)}{E_{\alpha,\alpha}(z)}, \quad z \in \mathbb{C}. \tag{8}$$

Furthermore, $\psi_s \in C^{\infty}(\mathbb{R})$ such that for all $n \ge 1$, $\psi_s^{(n)}(x) \to 0$ as $x \downarrow 0$ or $x \to \infty$.

Remark. Since $\mathbb{P}\{\tau_{-b} < T_c\} = \mathbb{P}\{\tau_{-b} < \tau_c\} = c^{\alpha-1}/(b+c)^{\alpha-1}$ ([2], Theorem VII.8), by (6), conditional on the event that X makes first exit from $[-b, c]$ by hitting $-b$, the scaled exit time $(b+c)^\alpha \tau_{-b}$ has p.d.f. ψ_s with $s = c^\alpha/(b+c)^\alpha$.

The main feature of Proposition 1 is that it expresses the p.d.f. of the time of first exit in terms of the roots of the Mittag-Leffler function $E_{\alpha,\alpha}(z)$. As one may suspect, the expression results from residual calculus for (5) as a meromorphic function on $\mathbb{C}$. However, since currently there is little precise knowledge on the values of the roots of $E_{\alpha,\alpha}$, the contour involved in the calculation has to be chosen carefully. For each term in the sum (7), if $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ has multiplicity n, then for $t > 0$,

$$\operatorname{Res}(H_s(z)e^{tz}, \varsigma) = \frac{1}{(n-1)!} \lim_{z\to\varsigma} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[\frac{E_{\alpha,\alpha}(sz)e^{tz}}{E_{\alpha,\alpha}(z)/(z-\varsigma)^n}\right],$$

which has the form $\sum_{k=0}^{n-1} c_k(\varsigma) t^{n-1-k} e^{t\varsigma}$. Moreover, if ς has large enough modulus, then it is a simple root ([13], Theorem 2.1.1), giving

$$\operatorname{Res}(H_s(z)e^{tz}, \varsigma) = \frac{E_{\alpha,\alpha}(s\varsigma)e^{t\varsigma}}{E'_{\alpha,\alpha}(\varsigma)}.$$

Proposition 1 is an extension of a similar result on first exit of a standard Brownian motion. If $\alpha = 2$, then by $\mathbb{E}[e^{-qX_t}] = e^{tq^2}$ for $q > 0$, $X_t = B_{2t}$ with B_t a standard Brownian motion. By $E_{2,2}(z) = \sinh(\sqrt{z})/\sqrt{z}$, $\mathscr{Z}_{2,2} = \{-k^2\pi^2, k \in \mathbb{N}\}$ and $E'_{2,2}(z) = [\cosh(\sqrt{z}) - E_{2,2}(z)]/(2z)$. Since $E'_{2,2}(-k^2\pi^2) = (-1)^{k-1}/(2k^2\pi^2)$, each root is simple. Then from the above display with $s = c^2/(b+c)^2$ and Proposition 1,

$$\frac{\mathbb{P}\{\tau_{-b} \in \mathrm{d}t,\, \tau_c > \tau_{-b}\}}{\mathrm{d}t} = \frac{2\pi}{(b+c)^2} \sum_{k=1}^\infty (-1)^{k-1} k \sin\left(\frac{k\pi c}{b+c}\right) \exp\left\{-\frac{k^2\pi^2 t}{(b+c)^2}\right\}.$$

The series expansion is different from the one in [3] (p. 212, 3.0.6). However, it can be proved using the heat equation method ([11], section 7.4); see for example [5] for a derivation.

Following a heuristic for a standard Brownian motion (cf. [11], p. 217), one can get a different expression of $k_{-b,c}$ analogous to the one for the standard Brownian motion in [3] (p. 212, 3.0.6):

$$\begin{aligned} k_{-b,c} &= f_{-b} - f_c * f_{-b-c} + f_{-b} * f_{b+c} * f_{-b-c} - f_c * f_{-b-c} * f_{b+c} * f_{-b-c} + \cdots \\ &= \sum_{n=0}^\infty f_{-b} * (\delta - f_c * f_{-c}) * (f_{b+c} * f_{-b-c})^{*n}, \end{aligned} \tag{9}$$

where all the terms involved are taken as p.d.f.'s of time, δ is the degenerate distribution at 0, and $p^{*0} := \delta$ for any p.d.f. p. Indeed, thinking of $f_{-b}(t)$ as the probability that X hits $-b$ at time t for the first time, and $k_{-b,c}(t)$ as the one that X does so before it ever hits c, $f_{-b}(t) - k_{-b,c}(t)$ is the probability that X does so after it hits c, so by strong Markov property,

$$k_{-b,c}(t) = f_{-b}(t) - (k_{c,-b} * f_{-b-c})(t), \tag{10}$$

where we have used the extended definition $k_{x,c}(t) = \mathbb{P}\{\tau_x \in \mathrm{d}t,\, \tau_c > \tau_x\}/\mathrm{d}t$ for any x, $c \in \mathbb{R}$. Likewise, $k_{c,-b}(t) = f_c(t) - (k_{-b,c} * f_{b+c})(t)$. Then iterating the two identities gives (9). A rigorous proof of (9) will be considered in Section 2.3.

2.1 Properties of scaled first exit time at lower end

This subsection considers some properties of $H_s(z)$ as defined in (8). Along the way it proves the smoothness of ψ_s stated at the end of Proposition 1.

From (5) and scaling, it follows that for any $s \in (0,1)$, $H_s(q)$ is the Laplace transform of the probability distribution

$$\mu_s(\mathrm{d}t) = \mathbb{P}\{\tau_{s^{1/\alpha}-1} \in \mathrm{d}t \,|\, \tau_{s^{1/\alpha}-1} < T_{s^{1/\alpha}}\}, \quad t > 0,$$

i.e., for $q > 0$,

$$\widetilde{\mu}_s(q) = H_s(q) = \frac{E_{\alpha,\alpha}(sq)}{E_{\alpha,\alpha}(q)}. \tag{11}$$

It is seen that the identity holds for all $q \in \mathbb{C}$ with $\mathrm{Re}(q) \ge 0$. By (11), given $s \in (0,1)$, $H_s(q)$ is completely monotone in $q \ge 0$, and for $z \in \mathbb{C}$ with $\mathrm{Re}(z) \ge 0$, $|H_s(z)| \le 1$, and so $|E_{\alpha,\alpha}(sz)| \le |E_{\alpha,\alpha}(z)|$. Thus for any $\theta \in [-\pi/2, \pi/2]$, $|E_{\alpha,\alpha}(e^{\mathrm{i}\theta} r)|$ is increasing in $r \ge 0$, so letting $r \to 0+$,

$$|E_{\alpha,\alpha}(z)| \ge E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}, \quad \mathrm{Re}(z) \ge 0.$$

Fix $s \in (0,1)$. For $\theta \in \mathbb{R}$,

$$\widehat{\mu}_s(\theta) = H_s(-\mathrm{i}\theta) = \frac{E_{\alpha,\alpha}(-\mathrm{i}s\theta)}{E_{\alpha,\alpha}(-\mathrm{i}\theta)}. \tag{12}$$

On the other hand, $|e^{(-\mathrm{i}\theta)^{1/\alpha}}| = e^{\lambda|\theta|^{1/\alpha}}$ with $\lambda = \cos(\alpha^{-1}\pi/2) > 0$, so by (2),

$$|E_{\alpha,\alpha}(-\mathrm{i}\theta)| \sim \alpha^{-1}|\theta|^{1/\alpha-1} e^{\lambda|\theta|^{1/\alpha}}, \quad \theta \to \pm\infty \tag{13}$$

where $x \sim y$ means $x/y \to 1$. Then from (12), $|\widehat{\mu}_s(\theta)| \sim s^{1/\alpha-1} e^{\lambda(s^{1/\alpha}-1)|\theta|^{1/\alpha}}$. As a result, $\int |\widehat{\mu}_s(\theta)||\theta|^n \,\mathrm{d}\theta < \infty$ for all $n \ge 0$, so μ_s has a p.d.f. in $C^\infty(\mathbb{R})$ with vanishing derivative of any order at $\pm\infty$ ([15], Proposition 28.1). By (5), the p.d.f. is exactly ψ_s in Proposition 1. Since ψ_s is supported on $[0,\infty)$, it is seen that $\psi_s^{(n)}(x) \to 0$ as $x \to 0+$.

From (12) and the Continuity Theorem of characteristic functions (cf. [4], Theorem 8.28), as $s \to 0+$, μ_s weakly converges to a probability distribution μ_0 with

$$\widehat{\mu}_0(\theta) = H_0(-\mathrm{i}\theta) = \frac{1}{\Gamma(\alpha) E_{\alpha,\alpha}(-\mathrm{i}\theta)}, \quad \theta \in \mathbb{R}.$$

Similar to μ_s with $s \in (0,1)$, μ_0 has a p.d.f. $\psi_0 \in C^\infty(-\infty,\infty)$ with support on $[0,\infty)$ such that all its derivatives $\psi_0^{(n)}(x)$ vanish as $x \to 0+, \infty$. Consequently, for each $s \in [0,1)$, ψ_s cannot be analytically extended to a neighborhood of 0, otherwise, ψ_s would be constant 0. On the other hand, by Fourier inversion ([14], p. 185),

$$\psi_s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\mu}_s(\theta) e^{-\mathrm{i}\theta t} \,\mathrm{d}\theta = \frac{1}{2\pi} \lim_{M\to\infty} \int_{-M}^{M} \widehat{\mu}_s(\theta) e^{-\mathrm{i}\theta t} \,\mathrm{d}\theta, \tag{14}$$

and from (11), μ_s has finite moment of any order, with the n^{th} moment equal to $(-1)^n H_s^{(n)}(0)$.

2.2 Contour integration

This subsection completes the proof of Proposition 1. Because (6) directly follows from (5), it only remains to show (7).

Proof of Eq. (7). Define function

$$\sigma(\theta) = \frac{1}{|\sin(\theta/\alpha)|}.$$

Since $\alpha > 1$, $\sigma(\theta)$ is bounded on $[-\pi, -\pi/2] \cup [\pi/2, \pi]$. Fix $\beta \in (\pi/2, \pi/\alpha)$. For $R > 0$, let C_R be the contour that travels along the curve

$$\{[R\sigma(\theta)]^\alpha e^{\mathrm{i}\theta} : \pi/2 \le |\theta| \le \pi\} \tag{15}$$

from $\mathrm{i}(R\sigma_0)^\alpha$ to $-\mathrm{i}(R\sigma_0)^\alpha$, where $\sigma_0 = \sigma(\pi/2)$. The contour is smooth except at its intersection with $(-\infty, 0)$ and its length is proportional to R^α. Fix $\beta \in (\pi/2, \alpha\pi/2)$. Let $C_{R,1}$ denote the part of C_R in the section $\pi/2 \le |\arg z| \le \beta$, and $C_{R,2}$ the part in the section $\beta \le |\arg z| \le \pi$.

For $z = re^{\mathrm{i}\theta}$ with $\theta = \arg z$, $|\exp(z^{1/\alpha})| = \exp\{r^{1/\alpha}\cos(\theta/\alpha)\}$. If $z \in C_{R,1}$, then $|\theta/\alpha| \le \beta/\alpha < \pi/2$, and so $\cos(\theta/\alpha) \ge \lambda := \cos(\beta/\alpha) > 0$. As a result, for $z \in C_{R,1}$,

$$|\exp(z^{1/\alpha})| \ge \exp(\lambda|z|^{1/\alpha}). \tag{16}$$

Then by (2), given $s \in (0,1)$, as $R \to \infty$,

$$\begin{aligned} H_s(z) = \frac{E_{\alpha,\alpha}(sz)}{E_{\alpha,\alpha}(z)} &= (1+o(1))\frac{(sz)^{1/\alpha-1}\exp((sz)^{1/\alpha})}{z^{1/\alpha-1}\exp(z^{1/\alpha})} \\ &= (1+o(1))s^{1/\alpha-1}\exp\{(s^{1/\alpha}-1)z^{1/\alpha}\}, \quad z \in C_{R,1}, \end{aligned}$$

where the $o(1)$ term is uniform in $z \in C_{R,1}$. Since $|z| \ge R^\alpha$, from (16),

$$\sup_{z \in C_{R,1}} |H_s(z)| = O(\exp\{-\lambda(1-s^{1/\alpha})R\}). \tag{17}$$

We also need a bound for $|H_s(z)| = |E_{\alpha,\alpha}(sz)/E_{\alpha,\alpha}(z)|$ on $C_{R,2}$. However, since $E_{\alpha,\alpha}(z)$ has infinitely many roots in the section $\beta \le |\arg z| \le \pi$, R cannot be any large positive number but has to be selected appropriately. We need the following result.

Lemma 2. *Let $R_n = 2\pi n$, $n = 1, 2, \ldots$. Then given any $A \in \mathbb{R} \setminus \{0\}$,*

$$\varliminf_{n\to\infty} \inf_{z \in C_{R_n,2}} |z^{1/\alpha+1}\exp(z^{1/\alpha}) - A| > 0. \tag{18}$$

Assuming the lemma is true for now, let $A = \alpha^2(\alpha-1)/\Gamma(2-\alpha)$. By (2),

$$E_{\alpha,\alpha}(z) = \alpha^{-1}z^{-2}[z^{1/\alpha+1}\exp(z^{1/\alpha}) - A] + O(|z|^{-3}).$$

Then by Lemma 2, there is $\epsilon > 0$, such that for all large n and $z \in C_{R_n,2}$, $E_{\alpha,\alpha}(z) \ge \epsilon|z|^{-2}$. Let $m_0 = \sup_{\pi/2 \le |\theta| \le \pi} \sigma(\theta)$. Then by $|z| \le (m_0 R_n)^\alpha$,

$$E_{\alpha,\alpha}(z) \ge \epsilon m_0^{-2\alpha} R_n^{-2\alpha}.$$

On the other hand, again by (2),

$$|E_{\alpha,\alpha}(sz)| \le E_{\alpha,\alpha}(|z|) \le E_{\alpha,\alpha}(m_0^\alpha R_n^\alpha) = O(R_n^{1-\alpha}\exp(m_0 R_n)).$$

Combining with the lower bound, this implies

$$\sup_{z\in C_{R_n,2}} |H_s(z)| = O(R_n^{1+\alpha} e^{m_0 R_n}), \quad n\to\infty. \tag{19}$$

Let D_R be the domain bounded by C_R and $\{\mathrm{i}\theta : |\theta| \le (R\sigma_0)^\alpha\}$. Let $t > 0$. If $C_R \cap \mathscr{Z}_{\alpha,\alpha} = \emptyset$, then by (12) and residual theorem,

$$\begin{aligned}\frac{1}{2\pi}\int_{-(R\sigma_0)^\alpha}^{(R\sigma_0)^\alpha} \widehat{\mu}_s(\theta) e^{-\mathrm{i}\theta t}\,\mathrm{d}\theta &= \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}(R\sigma_0)^\alpha}^{\mathrm{i}(R\sigma_0)^\alpha} H_s(z) e^{tz}\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}}\int_{C_R} H_s(z) e^{tz}\,\mathrm{d}z - \sum_{\varsigma\in D_R\cap\mathscr{Z}_{\alpha,\alpha}} \mathrm{Res}(H_s(z)e^{tz},\varsigma). \end{aligned}\tag{20}$$

Consider the contour integral along C_R. For $z = re^{\mathrm{i}\theta} \in C_R$ with $\theta = \arg z$, by $\pi/2 \le |\theta| \le \pi$, $|e^{tz}| = e^{rt\cos\theta} \le 1$. Then by (17),

$$\begin{aligned}\left|\int_{C_{R,1}} H_s(z)e^{tz}\,\mathrm{d}z\right| &\le \mathrm{Length}(C_{R,1}) \times O(e^{-\lambda(1-s^{1/\alpha})R}) \\ &= O(1)R^\alpha e^{-\lambda(1-s^{1/\alpha})R}.\end{aligned}$$

Furthermore, if $z\in C_{R,2}$, then by $\beta\le|\theta|\le\pi$ and $r\ge R^\alpha$, $|e^{tz}| \le e^{-b_0 R^\alpha t}$, where $b_0 = -\cos\beta > 0$. Then by (19),

$$\begin{aligned}\left|\int_{C_{R_n,2}} H_s(z)e^{tz}\,\mathrm{d}z\right| &\le \mathrm{Length}(C_{R_n,2}) \times O(R_n^{1+\alpha} e^{m_0 R_n - b_0 R_n^\alpha t}) \\ &= O(1) R_n^{1+2\alpha} e^{m_0 R_n - b_0 R_n^\alpha t}.\end{aligned}\tag{21}$$

By $\alpha > 1$, combining the above two bounds yields

$$\int_{C_{R_n}} H_s(z) e^{tz}\,\mathrm{d}z \to 0.$$

Then by the Fourier inversion (14) and (20),

$$\psi_s(t) = \lim_{n\to\infty} \sum_{\varsigma\in D_{R_n}\cap\mathscr{Z}_{\alpha,\alpha}} \mathrm{Res}(H_s(z)e^{tz},\varsigma), \quad t>0.$$

The above formula is proved for $s\in(0,1)$. For $s=0$, the formula can be similarly proved. To complete the proof, it only remains to show that the series on the r.h.s. of (7) converges absolutely. Since in any bounded domain there are only a finite number of roots of $E_{\alpha,\alpha}(z)$, it suffices to show that for a large enough $M>0$,

$$\sum_{\varsigma\in\mathscr{Z}_{\alpha,\alpha}\setminus U_M(0)} |\mathrm{Res}(H_s(z)e^{tz},\varsigma)| < \infty.$$

By Theorems 2.1.1 and Chapter 6 of [13], $M > 0$ can be chosen such that all elements in $\mathscr{Z}_{\alpha,\alpha} \setminus U_M(0)$ are not real and are simple roots of $E_{\alpha,\alpha}$. Then for each $\varsigma \in \mathscr{Z}_{\alpha,\alpha} \setminus U_M(0)$,

$$\mathrm{Res}(H_s(z)e^{tz}, \varsigma) = \mathrm{Res}\left(\frac{E_{\alpha,\alpha}(sz)e^{tz}}{E_{\alpha,\alpha}(z)}, \varsigma\right) = \frac{E_{\alpha,\alpha}(s\varsigma)e^{t\varsigma}}{E'_{\alpha,\alpha}(\varsigma)}.$$

Letting $z = \varsigma$ in the following identity ([13], p. 333)

$$\alpha z E'_{\alpha,\alpha}(z) + (\alpha - 1)E_{\alpha,\alpha}(z) = E_{\alpha,\alpha-1}(z),$$

one gets $E'_{\alpha,\alpha}(\varsigma) = E_{\alpha,\alpha-1}(\varsigma)/(\alpha\varsigma)$ and hence

$$\mathrm{Res}(H_s(z)e^{tz}, \varsigma) = \frac{\alpha\varsigma E_{\alpha,\alpha}(s\varsigma)e^{t\varsigma}}{E_{\alpha,\alpha-1}(\varsigma)}. \tag{22}$$

By $E_{\alpha,\alpha}(\varsigma) = 0$ and (2),

$$\alpha^{-1}\varsigma^{1/\alpha-1}\exp(\varsigma^{1/\alpha}) = \frac{\alpha(\alpha-1)\varsigma^{-2}}{\Gamma(2-\alpha)} + O(|\varsigma|^{-3}).$$

On the other hand, by Theorems 1.2.1, 1.4.2, and 1.5.1 in [13],

$$E_{\alpha,\alpha-1}(\varsigma) = \alpha^{-1}\varsigma^{2/\alpha-1}\exp(\varsigma^{1/\alpha}) + \frac{\alpha(\alpha^2-1)\varsigma^{-2}}{\Gamma(2-\alpha)} + O(|\varsigma|^{-3}).$$

Combining the two displays yields $E_{\alpha,\alpha-1}(\varsigma) \asymp \varsigma^{1/\alpha-2}$. Then by (22),

$$|\mathrm{Res}(H_s(z)e^{tz}, \varsigma)| = O(1)|\varsigma^{3-1/\alpha}E_{\alpha,\alpha}(s\varsigma)e^{t\varsigma}|.$$

From (2), as $|\varsigma| \to \infty$,

$$|E_{\alpha,\alpha}(s\varsigma)| \le E_{\alpha,\alpha}(|\varsigma|) = O(1)|\varsigma|^{1/\alpha-1}\exp(|\varsigma|^{1/\alpha}).$$

On the other hand, let $\varsigma = re^{\mathrm{i}\theta}$ with $\theta = \arg\varsigma$. By $\alpha\pi/2 < |\theta| \le \pi$, $|e^{\varsigma t}| = e^{tr\cos\theta} \le \exp(-\lambda|\varsigma|t)$, where $\lambda = -\cos(\alpha\pi/2) > 0$. Then

$$|\mathrm{Res}(H_s(z)e^{tz}, \varsigma)| = O(1)|\varsigma|^2 e^{|\varsigma|^{1/\alpha} - \lambda t|\varsigma|} = O(1)e^{-\eta|\varsigma|}$$

for some $\eta = \eta(t) > 0$. By (3), for $M > 0$ large enough, all $\varsigma \in \mathscr{Z}_{\alpha,\alpha} \setminus U_M(0)$ can be enumerated as $\varsigma_{\pm n}$, $n = N, N+1, \ldots$, with $N \ge 1$ being some large integer, such that $|\varsigma_{\pm n}| \asymp n^\alpha$. This combined with the above display then yields the desired absolute convergence. □

Proof of Lemma 2. For $z = [R\sigma(\theta)]^\alpha e^{\mathrm{i}\theta} \in C_{R,2}$ with $\theta = \arg z$,

$$\begin{aligned} z^{1/\alpha+1}\exp(z^{1/\alpha}) &= [R\sigma(\theta)]^{1+\alpha}e^{\mathrm{i}(1/\alpha+1)\theta}\exp\{R\sigma(\theta)e^{\mathrm{i}\theta/\alpha}\} \\ &= [R\sigma(\theta)]^{1+\alpha}e^{R\sigma(\theta)\cos(\theta/\alpha)}e^{\mathrm{i}[(1/\alpha+1)\theta + R\sigma(\theta)\sin(\theta/\alpha)]} \\ &= [R\sigma(\theta)]^{1+\alpha}e^{R\sigma(\theta)\cos(\theta/\alpha)}e^{\mathrm{i}[(1/\alpha+1)\theta + R\mathrm{sign}(\theta)]}. \end{aligned}$$

Put $a(\theta, R) = [R\sigma(\theta)]^{1+\alpha}e^{R\sigma(\theta)\cos(\theta/\alpha)}$. Then for $z \in C_{R_n,2}$, by $R_n = 2\pi n$, $z^{1/\alpha+1}\exp(z^{1/\alpha}) = a(\theta, R_n)e^{\mathrm{i}(1/\alpha+1)\theta}$. If there were $z_n = [R\sigma(\theta_n)]^\alpha e^{\mathrm{i}\theta_n} \in C_{R_n,2}$ such that $z_n^{1/\alpha+1}\exp(z_n^{1/\alpha}) \to A$, then taking modulus, $a(\theta_n, R_n) = [R_n\sigma(\theta_n)]^{1+\alpha}e^{R_n\sigma(\theta_n)\cos(\theta_n/\alpha)} \to |A| > 0$. By $|R_n\sigma(\theta_n)| \to \infty$, it follows that $\cos(\theta_n/\alpha) \to 0$, as any sequence n with $R_n\sigma(\theta_n)\cos(\theta_n/\alpha) \to \infty$ (resp. $-\infty$) has $a(\theta_n, R_n) \to \infty$ (resp. 0). Because $|\theta_n|/\alpha \in (\pi/(2\alpha), \pi/\alpha]$, this implies $\theta_n/\alpha = k_n\pi/2 + \epsilon_n$ with $k_n = \pm 1$ and $\epsilon_n \to 0$. But then

$$z_n^{1/\alpha+1}\exp(z_n^{1/\alpha}) = a(\theta_n, R_n)e^{\mathrm{i}(1/\alpha+1)\theta_n} = |A|e^{\mathrm{i}(1+\alpha)k_n\pi/2} + o(1) \not\to A,$$

a contradiction. □

2.3 Alternative expression and asymptotics

We first consider (9). Let

$$M(t) = \sum_{n=0}^{\infty} f_{-b} * (\delta + f_c * f_{-c}) * (f_{b+c} * f_{-b-c})^{*n}(t).$$

Given $q > 0$, by Fubini's theorem, $\widetilde{M}(q) = \widetilde{f}_{-b}(q)[1 + \widetilde{f}_c(q)\widetilde{f}_{-c}(q)] \sum_{n=0}^{\infty} \widetilde{f}_{b+c}(q)^n \widetilde{f}_{-b-c}(q)^n$, which is finite due to $\widetilde{f}_{-b-c}(q) < 1$. Then $M(t) < \infty$ a.e. As a result, the r.h.s. of (9) converges a.e. Denote it by $F(t)$ for now. By dominated convergence and the formula for $\widetilde{M}(q)$,

$$\widetilde{F}(q) = \frac{\widetilde{f}_{-b}(q)[1 - \widetilde{f}_c(q)\widetilde{f}_{-c}(q)]}{1 - \widetilde{f}_{b+c}(q)\widetilde{f}_{-b-c}(q)}.$$

On the other hand, it is known that [7] (also cf. [9], p. 253)

$$\widetilde{f}_x(q) = e^{xq^{1/\alpha}} - \alpha x_+^{\alpha-1} q^{1-1/\alpha} E_{\alpha,\alpha}(x_+^\alpha q), \quad x \in \mathbb{R},\ q > 0.$$

Then for $x \geq 0$, $\widetilde{f}_{-x}(q) = e^{-xq^{1/\alpha}}$ and $1 - \widetilde{f}_x(q)\widetilde{f}_{-x}(q) = \alpha x^{\alpha-1} q^{1-1/\alpha} E_{\alpha,\alpha}(x^\alpha q) e^{-xq^{1/\alpha}}$. Plugging the identities into the display, it is seen that $\widetilde{F}(q)$ equals the r.h.s. of (5), giving $F(t) = k_{-b,c}(t)$.

Based on the alternative expression, it is quite easy to get that as $t \downarrow 0$,

$$k_{-b,c}(t) \sim f_{-b}(t), \tag{23}$$

in particular, by Eq. (14.35) in [15], $\ln k_{-b,c}(t) \sim -Cb^{\alpha/(\alpha-1)} t^{-1/(\alpha-1)}$, where $C > 0$ is constant. First, by (10), $0 < f_{-b}(t) - k_{-b,c}(t) = (k_{c,-b} * f_{-b-c})(t) < (f_c * f_{-b-c})(t) = (u * f_{-b})(t)$, where $u = f_c * f_{-c}$ is a p.d.f. and we have used $k_{c,-b} < f_c$ and $f_{-b-c} = f_{-b} * f_{-c}$. Next,

$$(u * f_{-b})(t) = \int_0^t u(s) f_{-b}(t-s)\,\mathrm{d}s \leq \sup_{s \leq t} f_{-b} \times \int_0^t u = o(1) \sup_{s \leq t} f_{-b}, \quad t \downarrow 0.$$

Since f_{-b} is unimodal ([15], p. 416), for all $t > 0$ small enough, $\sup_{s \leq t} f_{-b} = f_{-b}(t)$, implying (23).

Although Proposition 1 gives a series expression of μ_s, it does not provide the radius of convergence of $\widetilde{\mu}_s$, defined as $\sup\{r > 0 : e_z \in L^1(\mu_s)\, \forall z \in U_r(0)\}$, where e_z is the function $t \mapsto e^{-tz}$. This is also related to the tail of $k_{-b,c}(t)$ as $t \to \infty$. We have the following.

Proposition 3. *Let $-\varrho$ be the largest real root of $E_{\alpha,\alpha}(z)$, where $\varrho > 0$. Then given $s \in [0, 1)$, $\widetilde{\mu}_s(x) < \infty$ for $x \in (-\varrho, \infty)$ and $\widetilde{\mu}_s(x) \uparrow \widetilde{\mu}_s(-\varrho) = \infty$ as $x \downarrow -\varrho$. In particular, the radius of convergence of $\widetilde{\mu}_s$ is ϱ and $\mathscr{Z}_{\alpha,\alpha} \subset \{z : |\arg z| > \alpha\pi/2,\ \mathrm{Re}(z) \leq -\varrho\}$.*

Proof. Recall that for a measure ν on $[0, \infty)$ with finite total mass, the domain of $\widetilde{\mu}_s$ contains $\{z : \mathrm{Re}(z) \geq 0\}$, and if $\widetilde{\nu}(z)$ can be analytically extended to $U_r(0) \cap \{z : \mathrm{Re}(z) < 0\}$ for some $r > 0$, then the domain of $\widetilde{\nu}$ contains $\{z : \mathrm{Re}(z) > -r\}$ and $\widetilde{\nu}$ is analytic in the region.

Let ς_0 be a root of $E_{\alpha,\alpha}(z)$ with the largest real part. Put $a = -\mathrm{Re}(\varsigma_0)$. Clearly, $\varrho \geq a$. From $\mathscr{Z}_{\alpha,\alpha} \subset \{z : |\arg(z)| > \alpha\pi/2\}$ ([13], Theorem 4.2.1), $a > 0$. Since $E_{\alpha,\alpha}(z) \neq 0$ for any z with $\mathrm{Re}(z) > -a$, $H_s(z)$ is analytic in $U_a(0)$. Since, by (8) and (11), $\widetilde{\mu}_s(z) = H_s(z)$ for z with $\mathrm{Re}(z) \geq 0$, from the above remark, the domain of $\widetilde{\mu}_s$ contains $\{z : \mathrm{Re}(z) > -a\}$. Let $z \to \varsigma_0$ along the ray from 0 to ς_0. Then $|\widetilde{\mu}_s(z)| = |H_s(z)| \to |H_s(\varsigma_0)| = \infty$. By $|\widetilde{\mu}_s(z)| \leq \widetilde{\mu}_s(\mathrm{Re}(z))$, it follows that if $x \in \mathbb{R}$ and $x \downarrow -a$, then $\widetilde{\mu}_s(x) = H_s(x) \to \infty$. Since $E_{\alpha,\alpha}(x) \to E_{\alpha,\alpha}(-sa) > 0$, then $E_{\alpha,\alpha}(-a) = 0$, and so $a = \varrho$. Thus, $\mathscr{Z}_{\alpha,\alpha} \subset \{z : \mathrm{Re}(z) \leq -\varrho\}$. The proof is complete. ∎

Combining Propositions 1 and 3, if the multiplicity of $-\varrho$ is $n \geq 1$, then $k_{-b,c}(t)$ decreases exponentially fast with

$$\limsup_{t\to\infty} \frac{\ln k_{-b,c}(t)}{t} = -\frac{\varrho}{(b+c)^\alpha}.$$

However, the exact asymptotic of $k_{-b,c}(t)$ depends on the multiplicity of $-\varrho$ as well as other roots of $E_{\alpha,\alpha}(z)$ on the line $\mathrm{Re}(z) = -\varrho$, if there are any.

3 Distribution of first exit at upper end

The main result of this section is Theorem 4 below. It provides a factorization of the joint sub-p.d.f. of the time T_c, the undershoot X_{T_c-}, and the jump $\Delta_{T_c} = X_{T_c} - X_{T_c-}$ when X makes first exit from $[-b, c]$ by jumping upward across c before hitting $-b$. The following function plays an important role. For $x \in (-b, c)$ and $t > 0$, define

$$l_{x,-b,c}(t) = \frac{\mathbb{P}\{X_t \in \mathrm{d}x,\ X_s \in (-b,c)\,\forall s \leq t\}}{\mathrm{d}x}. \tag{24}$$

While the function can be defined for any process that has a p.d.f. at any time point, in the case of a spectrally positive strictly stable process, it has an explicit representation.

To start with, it is known that for $q \geq 0$ ([9], Theorem 8.7)

$$\begin{aligned}\widetilde{l}_{x,-b,c}(q) &= \frac{W^{(q)}(c)W^{(q)}(b+x)}{W^{(q)}(b+c)} - W^{(q)}(x_+)\\ &= \frac{c^{\alpha-1}(b+x)^{\alpha-1}}{(b+c)^{\alpha-1}} \frac{E_{\alpha,\alpha}(c^\alpha q)E_{\alpha,\alpha}((b+x)^\alpha q)}{E_{\alpha,\alpha}((b+c)^\alpha q)} - x_+^{\alpha-1}E_{\alpha,\alpha}(x_+^\alpha q).\end{aligned} \tag{25}$$

Theorem 4. *Fix $b > 0$ and $c > 0$. For $x \in \mathbb{R}$,*

$$\mathbb{P}\{T_c < \tau_{-b},\, X_{T_c-} \in \mathrm{d}x\} = \mathbf{1}\{x \in (-b,c)\}\, \frac{|\sin(\alpha\pi)|}{\pi} \left[\frac{c^{\alpha-1}(b+x)^{\alpha-1}}{(b+c)^{\alpha-1}} - x_+^{\alpha-1}\right] \frac{\mathrm{d}x}{(c-x)^\alpha}, \tag{26}$$

and conditional on $T_c < \tau_{-b}$ and $X_{T_c-} = x \in (-b, c)$, Δ_{T_c} and T_c are independent, such that Δ_{T_c} follows the Pareto distribution with p.d.f.

$$\pi(u) = \alpha(c-x)^\alpha u^{-\alpha-1}\mathbf{1}\{u > c-x\}$$

and T_c has p.d.f.

$$p(t) = \Gamma(\alpha)\left[\frac{c^{\alpha-1}(b+x)^{\alpha-1}}{(b+c)^{\alpha-1}} - x_+^{\alpha-1}\right]^{-1} l_{x,-b,c}(t) \tag{27}$$

with $l_{x,-b,c}(t)$ having the following expression

$$l_{x,-b,c}(t) = \frac{c^{\alpha-1}(b+x)^{\alpha-1}}{(b+c)^{\alpha-1}} \sum_{\varsigma\in\mathscr{Z}_{\alpha,\alpha}} \mathrm{Res}\left(\frac{E_{\alpha,\alpha}(c^\alpha z)E_{\alpha,\alpha}((b+x)^\alpha z)}{E_{\alpha,\alpha}((b+c)^\alpha z)} e^{tz},\ \frac{\varsigma}{(b+c)^\alpha}\right). \tag{28}$$

Furthermore, given $t > 0$, the mapping $x \to l_{x,-b,c}(t)$ has an analytic extension from $(-b, c)$ to $\mathbb{C} \setminus (-\infty, -b]$.

From (25), one may suspect that (28) can be obtained by residual calculus. However, when $x > 0$, the function in the residuals of (28) have the term $x^{\alpha-1}E_{\alpha,\alpha}(x^\alpha q)$ missing. A direct inversion of the Laplace transform (25) when $x > 0$ seems to be involved. Instead, we will first show (28) for $x < 0$ by residual calculus, and then establish (28) for $x \geq 0$ through analytic extension.

3.1 Factorization and conditional independence

The factorization in Theorem 4 follows from the next result, which actually holds under much more general assumptions on a Lévy process.

Lemma 5. *Given $b > 0$ and $c > 0$, for $t > 0$,*

$$\begin{aligned}
&\mathbb{P}\{T_c < \tau_{-b},\, T_c \in \mathrm{d}t,\, X_{T_c-} \in \mathrm{d}x,\, \Delta_{T_c} \in \mathrm{d}u\} \\
&\qquad = \mathbf{1}\{x > -b,\, u > c - x > 0\}\, \mathrm{d}t\, \mathbb{P}\{X_t \in \mathrm{d}x,\, X_s \in (-b, c)\, \forall s \le t\}\, \Pi(\mathrm{d}u). \qquad (29)
\end{aligned}$$

Proof. The proof follows the one on p. 76 of [2]. As already noted, $\mathbb{P}\{X_{T_c} > c > X_{T_c-} > -b\} = 1$. By Rogozin's criterion (cf. [8], Theorem 5.17), under the law of X, 0 is regular for $(0, \infty)$ as well as for $(-\infty, 0)$, so $X_t < c$ for all $t < T_c$ and $X_t > -b$ for all $t < \tau_{-b}$. Therefore, almost surely, for any bounded function $f(t, x, u) \ge 0$,

$$\begin{aligned}
&f(T_c,\, X_{T_c-}, \Delta_{T_c}) \mathbf{1}\{T_c < \tau_{-b}\} \\
&= \sum_t f(t,\, X_{t-},\, \Delta_t) \mathbf{1}\{\Delta_t > c - X_{t-} > 0,\, X_{t-} > -b,\, -b < X_s < c\, \forall s \le t\}\,.
\end{aligned}$$

The sum on the r.h.s. is well-defined as it runs over the set of t's where X has a jump, which is countable. The rest of the proof then applies the compensation formula to show that the expectation of the random sum is an integral of $f(t, x, u)$ with respect to the measure on the r.h.s. of (29). Since the argument has become standard, it is omitted for brevity. ☐

Since the Lévy measure of X is

$$\Pi(\mathrm{d}x) = \frac{\alpha(\alpha - 1)\mathbf{1}\{x > 0\}}{\Gamma(2 - \alpha)} \frac{\mathrm{d}x}{x^{\alpha+1}}.$$

by (24) and Lemma 5,

$$\begin{aligned}
&\mathbb{P}\{T_c < \tau_{-b},\, T_c \in \mathrm{d}t,\, X_{T_c-} \in \mathrm{d}x,\, \Delta_{T_c} \in \mathrm{d}u\} \\
&= \mathbf{1}\{c > x > -b\}\, l_{-x,b,c}(t)\, \mathrm{d}t\, \mathrm{d}x\, \frac{\mathbf{1}\{u > c - x\}\, \alpha(\alpha - 1)\mathrm{d}u}{\Gamma(2 - \alpha) u^{\alpha+1}}. \qquad (30)
\end{aligned}$$

Now let $q = 0$ in (25). Then

$$\int_0^\infty l_{x,-b,c}(t)\, \mathrm{d}t = \widetilde{l}_{x,-b,c}(0) = \frac{1}{\Gamma(\alpha)} \left[\frac{c^{\alpha-1}(b + x)^{\alpha-1}}{(b + c)^{\alpha-1}} - x_+^{\alpha-1} \right].$$

Then by (30), for $x \in (-b, c)$,

$$\begin{aligned}
\mathbb{P}\{T_c < \tau_{-b},\, X_{T_c-} \in \mathrm{d}x\} &= \mathrm{d}x \int_0^\infty l_{x,-b,c}(t)\, \mathrm{d}t \int_{c-x}^\infty \frac{\alpha(\alpha - 1)}{\Gamma(2 - \alpha)} \frac{\mathrm{d}u}{u^{\alpha+1}} \\
&= \left[\frac{c^{\alpha-1}(b + x)^{\alpha-1}}{(b + c)^{\alpha-1}} - x_+^{\alpha-1} \right] \frac{(\alpha - 1)}{\Gamma(\alpha)\Gamma(2 - \alpha)} \frac{\mathrm{d}x}{(c - x)^\alpha}.
\end{aligned}$$

Since $\mathbb{P}\{X_{T_c-} \in (-b, c)\} = 1$ by Lemma 5, then the sub-p.d.f. formula in (26) follows. On the other hand, given $x \in (-b, c)$,

$$\mathbb{P}\{T_c \in \mathrm{d}t,\, \Delta_{T_c} \in \mathrm{d}u \,|\, T_c < \tau_{-b},\, X_{T_c-} \in \mathrm{d}x\} = C l_{-x,b,c}(t)\, \mathrm{d}t \times \frac{\mathbf{1}\{u > c - x\}\, \mathrm{d}u}{u^{\alpha+1}},$$

for some constant $C = C(x)$. It follows that conditional on $T_c < \tau_{-b}$ and $X_{T_c-} = x$, T_c and Δ_{T_c} are independent, with Δ_{T_c} following a Pareto distribution and T_c having a p.d.f. of the form $C' l_{-x,b,c}(t)$ with C' another constant, as claimed in Theorem 4.

3.2 Contour integration

The main step is to get the expression of $l_{x,-b,c}(t)$ for given $x \in (-b, c)$. As marked after Theorem 4, we need to show that $l_{x,-b,c}(t)$ as a function of x can be analytically extended from $x \in (-b, 0)$ to the entire $(-b, c)$. To this end, define

$$h_{x,c}(t) = \frac{\mathbb{P}\{X_t \in \mathrm{d}x,\, X_s < c\,\forall s \le t\}}{\mathrm{d}x}. \tag{31}$$

Given $c > 0$, $h_{x,c}(t)$ can be either regarded as a function of t or a function of x. It already plays a critical role in [6] in the derivation of the distribution of the triple (T_c, X_{T_c-}, X_{T_c}), known as Gerber-Shiu distribution ([9], Chapter 10).

Lemma 6. *Given $b > 0$, $c > 0$, and $x \in (-b, c)$, $l_{x,-b,c}(t) = h_{x,c}(t) - (k_{-b,c} * h_{b+x,b+c})(t)$.*

Proof. Let $f \ge 0$ be a function with support in $(-b, c)$. Then for any $t > 0$,

$$\mathbb{E}[f(X_t)\mathbf{1}\{X_s \in (-b, c)\forall s \le t\}] = \int_{-b}^{c} f(x) l_{x,-b,c}(t)\,\mathrm{d}x.$$

On the other hand, the expectation can be decomposed as

$$\mathbb{E}[f(X_t)\mathbf{1}\{X_s < c\forall s \le t\}] - \mathbb{E}[f(X_t)\mathbf{1}\{\tau_{-b} \le t,\, X_s < c\forall s \le t\}].$$

The first expectation in the display is equal to

$$\int_{-b}^{c} f(x) h_{x,c}(t)\,\mathrm{d}x.$$

By strong Markov property of X, the second expectation is equal to

$$\begin{aligned}
&\int_{-b}^{c}\int_0^t f(x)\mathbb{P}\{X_t \in \mathrm{d}x,\, \tau_{-b} \in \mathrm{d}u,\, X_s < c\forall s \le t\} \\
&= \int_{-b}^{c}\int_0^t f(x)\mathbb{P}\{X_t \in \mathrm{d}x + b,\, X_s < b + c\forall s \le t - u\}\mathbb{P}\{\tau_{-b} \in \mathrm{d}u,\, X_s < c\forall s \le u\} \\
&= \int_{-b}^{c} f(x)\left[\int_0^t h_{x+b,b+c}(t-u) k_{-b,c}(u)\,\mathrm{d}u\right]\mathrm{d}x.
\end{aligned}$$

Comparing the integrals and by f being arbitrary, the claimed identity follows. □

Lemma 7. *Given $b > 0$, $c > 0$, and $t > 0$, the mapping $x \to l_{x,-b,c}(t)$ has an analytic extension from $(-b, c)$ to $\{z - b : z \in \Omega\} \cap \{c - z : z \in \Omega\}$, where $\Omega = \{z \in \mathbb{C} : |\arg z| < \kappa^{-1}\pi/2\}$ with $1/\kappa = 1 - 1/\alpha$.*

Proof. It is shown in [6] that given $c > 0$ and $t > 0$, the mapping $x \to h_{x,c}(t)$ has an analytic extension from $(-\infty, c)$ to $\{c - z : z \in \Omega\}$. Put $a = b + c$. Then by Lemma 6, it suffices to show that $x \to (k_{-b,c} * h_{x,a})(t)$ has an analytic extension from $(0, a)$ to $\Omega \cap \{a - z : z \in \Omega\}$.

For $x < a$, by p. 4/10 of [10],

$$h_{x,a}(t) = g_t(x) - \phi_t(x) > 0,$$

with

$$\phi_t(x) = \int_0^t f_{x-a}(t-s) g_s(a)\,\mathrm{d}s > 0.$$

Then $(k_{-b,c} * h_{x,a})(t) = G(x) - \Phi(x)$, where

$$G(x) = \int_0^t k_{-b,c}(t-s) g_s(x)\,\mathrm{d}s$$

and

$$\Phi(x) = \int_0^t k_{-b,c}(t-s)\phi_s(x)\,\mathrm{d}s.$$

We shall show that $G(x)$ has an analytic extension from $(0,\infty)$ to Ω and $\Phi(x)$ has an analytic extension from $(-\infty, a)$ to $\{a - z : z \in \Omega\}$. This will finish the proof.

First consider $\Phi(x)$. In [6], the proof of its Lemma 9 establishes that ϕ_t has an analytic extension from $(-\infty, a)$ to $\{a - z : z \in \Omega\}$ such that, given $0 < r_1 < r_2 < \infty$ and $0 < \beta_0 < \kappa^{-1}\pi/2$, for $z = re^{\mathrm{i}\beta} \in \Omega$ with $r \in [r_1, r_2]$ and $-\beta_0 \le \beta = \arg z \le \beta_0$, letting $x_0 = r_1 d(\beta_0) > 0$, where $d(\beta) = \cos(\kappa\beta)^{1/\kappa}$,

$$|\phi_t(a - z)| \le (r_2/x_0)^{\alpha/(\alpha-1)} g_t(a - x_0) < \infty.$$

By scaling, $g_t(a - x_0) \le t^{-1/\alpha} \sup g_1$. By (6), $\sup k_{-b,c} < \infty$. Then by

$$\int_0^t k_{-b,c}(t-s)|\phi_s(a-z)|\,\mathrm{d}s \le \sup k_{-b,c} \int_0^t (r_2/x_0)^{\alpha/(\alpha-1)} s^{-1/\alpha} \sup g_1\,\mathrm{d}s,$$

the l.h.s. is uniformly bounded for z in any compact subset of Ω. By dominated convergence, $\Phi(a - z)$ is continuous in Ω. Then by Fubini's theorem followed by Morera's theorem (cf. [14], p. 208), $\Phi(a - z)$ is analytic in Ω, as desired.

Next consider $G(x)$. It is known that $x \to g_t(x)$ has an analytic extension from $\mathbb{R}$ to $\mathbb{C}$ ([15], p. 88). Then, as above, it suffices to show $\int_0^t k_{-b,c}(t-s)|g_s(z)|\,\mathrm{d}s$, or more simply, $\int_0^t |g_s(z)|\,\mathrm{d}s$ is uniformly bounded for z in any compact subset of Ω. By [17], for $x > 0$,

$$g_t(x) = \frac{\alpha\pi}{\alpha - 1}(x/t)^{\kappa/\alpha} \int_{\pi(1/\alpha - 1/2)}^{\pi/2} a(\theta) \exp\{-x^\kappa t^{-\kappa/\alpha} a(\theta)\}\,\mathrm{d}\theta,$$

where $a(\theta) > 0$ is a continuous function in the open interval of the integral. Since $z \to z^\kappa$ maps Ω to $\{z : \mathrm{Re}(z) > 0\}$, the integral on the r.h.s. can be analytically extended to Ω, and hence the identity can be extended to all $z \in \Omega$. If $z = re^{\mathrm{i}\beta}$ with $\beta = \arg z$, then by $|\beta| < \kappa^{-1}\pi/2$, $\mathrm{Re}(z^\kappa) = [rd(\beta)]^\kappa > 0$, where $d(\beta)$ is defined above. Then

$$\begin{aligned}
|g_t(z)| &\le \frac{\alpha\pi}{\alpha - 1}(r/t)^{\kappa/\alpha} \int_{\pi(1/\alpha - 1/2)}^{\pi/2} a(\theta) \exp\{-[rd(\beta)]^\kappa t^{-\kappa/\alpha} a(\theta)\}\,\mathrm{d}\theta \\
&= \frac{g_t(rd(\beta))}{d(\beta)^{\kappa/\alpha}} \le \frac{t^{-1/\alpha} \sup g_1}{d(\beta)^{\kappa/\alpha}},
\end{aligned}$$

where the equality on the second line follows by comparing the integral with the previous display. Since any compact subset of Ω is contained in a section $\{|\arg z| \le \beta_0\}$ for some $\beta_0 < \kappa^{-1}\pi/2$ and since $d(\beta) > 0$ is decreasing in $|\beta|$ in the section, it is then easy that $|g_t(z)| \le t^{-1/\alpha} \sup g_1 / d(\beta_0)^{\kappa/\alpha}$, yielding the desired uniform boundedness. ∎

We finally can prove the expression (28) for $l_{x,-b,c}(t)$ and that it can analytically extended to $\mathbb{C} \setminus (-\infty, b)$, which then finishes the proof of Theorem 4.

Proof. First, suppose $x \in (-b, 0)$. Then

$$\widetilde{l}_{x,-b,c}(q) = \frac{c^{\alpha-1}(b+x)^{\alpha-1}}{(b+c)^{\alpha-1}} \frac{E_{\alpha,\alpha}(c^\alpha q) E_{\alpha,\alpha}((b+x)^\alpha q)}{E_{\alpha,\alpha}((b+c)^\alpha q)}$$

and (28) will follow once it is proved that $\widehat{l}_{x,-b,c}(\theta) = \widetilde{l}_{x,-b,c}(-\mathrm{i}\theta)$ is in $L^1(\mathrm{d}\theta)$ and that

$$\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{E_{\alpha,\alpha}(c^\alpha z) E_{\alpha,\alpha}((b+x)^\alpha z)}{E_{\alpha,\alpha}((b+c)^\alpha z)} e^{zt} \, \mathrm{d}z$$

is equal to the sum on the r.h.s. of (28). The argument is similar to that for Proposition 1. Let

$$s = c^\alpha/(b+c)^\alpha, \quad v = (b+x)^\alpha/(b+c)^\alpha. \tag{32}$$

Then by making change of variables $z' = (b+c)^\alpha z$ and $t' = t/(b+c)^\alpha$, and using the function $H_s(z)$ defined in (8), it boils down to showing that

$$\int_{-\infty}^{\infty} |H_s(-\mathrm{i}\theta) E_{\alpha,\alpha}(-\mathrm{i}v\theta)| \, \mathrm{d}\theta < \infty \tag{33}$$

and for any $t > 0$,

$$\int_{C_{R_n}} H_s(z) E_{\alpha,\alpha}(vz) e^{tz} \, \mathrm{d}z \to 0, \quad n \to \infty, \tag{34}$$

where the contour C_R and the numbers R_n are defined in the proof of Proposition 1.

By (13), with $\lambda = \cos(\alpha^{-1}\pi/2) > 0$,

$$|H_s(-\mathrm{i}\theta) E_{\alpha,\alpha}(-\mathrm{i}v\theta)| \sim \alpha^{-1} |sv\theta|^{1/\alpha - 1} e^{\lambda(s^{1/\alpha} + v^{1/\alpha} - 1)|\theta|^{1/\alpha}}, \quad |\theta| \to \infty.$$

Then (34) easily follows by noting

$$s^{1/\alpha} + v^{1/\alpha} - 1 = x/(b+c) < 0. \tag{35}$$

Next, as in the proof of Proposition 1, divide C_R into $C_{R,1}$ and $C_{R,2}$. As $R \to \infty$, uniformly for $z \in C_{R,1}$, $H_s(z) = O(1) \exp\{(s^{1/\alpha} - 1) z^{1/\alpha}\}$ and $E_{\alpha,\alpha}(vz) = O(1)(vz)^{1/\alpha - 1} \exp\{v^{1/\alpha} z^{1/\alpha}\}$; see the derivation of (17). From (35) again, there is $\eta = \eta(x) > 0$, such that $\sup_{z \in C_{R,1}} |H_s(z) E_{\alpha,\alpha}(vz)| = O(e^{-\eta R})$. Meanwhile, $|e^{tz}| \le 1$ for $z \in C_{R,1}$ and $\mathrm{Length}(C_{R,1}) = O(R^\alpha)$. Then

$$\int_{C_{R,1}} H_s(z) E_{\alpha,\alpha}(vz) e^{tz} \, \mathrm{d}z = O(R^\alpha e^{-\eta R}) \to 0, \quad R \to \infty.$$

On the other hand, according to derivation of (19), for some $m_0 > 0$,

$$\sup_{z \in C_{R_n,2}} |H_s(z) E_{\alpha,\alpha}(vz)| = O(R_n^{1+\alpha} e^{m_0 R_n}) \cdot O(R_n^{1-\alpha} e^{m_0 R}) = O(R_n^2 e^{2 m_0 R_n}).$$

Meanwhile, according to the argument leading to (21), for some $b_0 > 0$, $|e^{tz}| \le e^{-b_0 R^\alpha t}$ for $t \in C_{R,2}$. Then by $\alpha > 1$ and $t > 0$,

$$\int_{C_{R_n,2}} H_s(z) E_{\alpha,\alpha}(vz) e^{tz} \, \mathrm{d}z = O(R_n^{2+\alpha} e^{2 m_0 R_n - b_0 R_n^\alpha t}), \quad n \to \infty.$$

The desired convergence in (34) then follows and hence (28) is proved in the case $x \in (-b, 0)$.

It only remains to show that the r.h.s. of (28) as a function of x has an analytic extension to $\mathbb{C} \setminus (-\infty, -b]$ for given $t > 0$. Once this is done, since by Lemma 7, $x \to l_{x,-b,c}(t)$ has an analytic extension to a domain containing $(-b, c)$ and since it was just proved that the two functions are equal on $(-b, 0)$, then they must be equal on $(-b, c)$ and $x \to l_{x,-b,c}(t)$ can actually be extended to $\mathbb{C} \setminus (-\infty, -b]$, finishing the proof.

Thus, let

$$w_\varsigma(x) = \operatorname{Res}\left(\frac{E_{\alpha,\alpha}(c^\alpha z) E_{\alpha,\alpha}((b+x)^\alpha z)}{E_{\alpha,\alpha}((b+c)^\alpha z)} e^{tz}, \frac{\varsigma}{(b+c)^\alpha}\right).$$

It is easy to see that w_ς has an analytic extension from $(-b, c)$ to $\mathbb{C} \setminus (-\infty, -b]$. All $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ with large enough modulus are simple roots of $E_{\alpha,\alpha}(z)$ and have $|\arg \varsigma|$ arbitrarily close but strictly greater than $\alpha\pi/2$. For each such ς and each $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$w_\varsigma(z - b) = \frac{E_{\alpha,\alpha}(s\varsigma) e^{t'\varsigma}}{(b+c)^\alpha E'_{\alpha,\alpha}(\varsigma)} \times E_{\alpha,\alpha}\left(\frac{z^\alpha \varsigma}{(b+c)^\alpha}\right),$$

where s is defined as in (32) and $t' = t/(b+c)^\alpha$. From the last part of the proof for Proposition 1, there is $c' = c'(t') > 0$, such that

$$\left|\frac{E_{\alpha,\alpha}(s\varsigma) e^{t'\varsigma}}{(b+c)^\alpha E'_{\alpha,\alpha}(\varsigma)}\right| = O(1) e^{-c'|\varsigma|}.$$

On the other hand, by (2), there is a constant $C > 0$ such that

$$\left|E_{\alpha,\alpha}\left(\frac{z^\alpha \varsigma}{(b+c)^\alpha}\right)\right| \le E_{\alpha,\alpha}\left(\frac{(|z| \vee 1)^\alpha |\varsigma|}{(b+c)^\alpha}\right) = O(1) \exp\left\{C(|z| \vee 1)|\varsigma|^{1/\alpha}\right\}.$$

Combining the two bounds then yields a bound on $|w_\varsigma(z - b)|$. Following the last part of the proof of Proposition 1, $\sum_{\varsigma \in \mathscr{Z}_{\alpha,\alpha}} |w_\varsigma(z - b)|$ converges uniformly for z in any compact subset of $\mathbb{C} \setminus (-\infty, 0]$. Then $z \mapsto \sum_{\varsigma \in \mathscr{Z}_{\alpha,\alpha}} w_\varsigma(z - b)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$. Then by Fubini's theorem followed by Morera's theorem, the r.h.s. of (28) as a function of x has an analytic extension to $\mathbb{C} \setminus (-\infty, -b]$. □

3.3 Asymptotics

We consider the asymptotics of $l_{x,-b,c}(t)$ as $t \downarrow 0$ or $\to \infty$. First, we have

Proposition 8. *Given $b > 0$, $c > 0$, and $x \in (-b, c)$, as $t \downarrow 0$, $l_{x,-b,c}(t) \sim g_t(x)$.*

Proof. It is clear that $l_{x,-b,c}(t) < g_t(x)$. On the other hand,

$$\begin{aligned} g_t(x) - l_{x,-b,c}(t) &\le \frac{\mathbb{P}\{X_t \in \mathrm{d}x,\, \tau_{-b} < t\}}{\mathrm{d}x} + \frac{\mathbb{P}\{X_t \in \mathrm{d}x,\, T_c < t\}}{\mathrm{d}x} \\ &= \frac{\mathbb{P}\{X_t \in \mathrm{d}x,\, \tau_{-b} < t\}}{\mathrm{d}x} + \frac{\mathbb{P}\{X_t \in \mathrm{d}x,\, \tau_c < t\}}{\mathrm{d}x} \\ &= \frac{\mathbb{P}\{X_t \in \mathrm{d}x,\, \tau_{-b} < t\}}{\mathrm{d}x} + \frac{\mathbb{P}\{X_t \in \mathrm{d}x,\, \tau_{-(c-x)} < t\}}{\mathrm{d}x}, \end{aligned}$$

where the third line follows by considering $\sup\{s < t : X_s = c\}$ as well as time reversal. Note that both b and $c - x$ are greater than $(-x)_+$. Then it suffices to show that for any $x \in \mathbb{R}$ and $b > (-x)_+$, $j(t) := \mathbb{P}\{X_t \in \mathrm{d}x,\, \tau_{-b} < t\}/\mathrm{d}x = o(g_t(x))$ as $t \to 0$.

By strong Markov property and $\sup g_s = O(s^{-1/\alpha})$,

$$j(t) = \int_0^t f_{-b}(t-s) g_s(x+b)\,\mathrm{d}s = O(1) t^{1-1/\alpha} \sup_{s \le t} f_{-b}(s).$$

Since f_{-b} is unimodal ([15], p. 416), then for small $t > 0$, $j(t) = O(1) f_{-b}(t)$. On the other hand,

$$g_t(x) \asymp \begin{cases} t & \text{if } x > 0, \\ t^{-1/\alpha} & \text{if } x = 0, \\ t f_x(t)/(-x) & \text{if } x < 0 \end{cases} \quad \text{as } t \downarrow 0,$$

where the last line is by Corollary VII.3 in [2]. It is then clear that if $x \ge 0$, then $j(t) = o(g_t(x))$. On the other hand, if $x < 0$, since $b > |x|$, then $f_{-b}(t) = o(f_{-x}(t))$, again yielding $j(t) = o(g_t(x))$. □

Finally, by the same argument for the tail of $k_{-b,c}(t)$, as $t \to \infty$, $l_{x,-b,c}(t)$ decreases exponentially fast with

$$\limsup_{t \to \infty} \frac{\ln l_{x,-b,c}(t)}{t} = -\frac{\varrho}{(b+c)^\alpha},$$

where $-\varrho < 0$ is the largest real root of $E_{\alpha,\alpha}$ and n its multiplicity. Again, the exact asymptotic of $l_{x,-b,c}(t)$ depends on more detail of the roots along the line $\mathrm{Re}(z) = -\varrho$.

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