

More on the long time stability of Feynman–Kac semigroups

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Abstract

Feynman–Kac semigroups appear in various areas of mathematics: non-linear filtering, large deviations theory, spectral analysis of Schrödinger operators among others. Their long time behavior provides important information, for example in terms of ground state energy of Schrödinger operators, or scaled cumulant generating function in large deviations theory. In this paper, we propose a simple and natural extension of the stability analysis of Markov chains for these non-linear evolutions. As other classical ergodicity results, it relies on two assumptions: a Lyapunov condition that induces some compactness, and a minorization condition ensuring some mixing. We show that these conditions are satisfied in a variety of situations, including stochastic differential equations. Illustrative examples are provided, where the stability of the non-linear semigroup arises either from the underlying dynamics or from the Feynman–Kac weight function. We also use our technique to provide uniform in the time step convergence estimates for discretizations of stochastic differential equations.

Key words : Feynman–Kac dynamics; ergodicity; spectral analysis; large deviations.

1 Introduction

Feynman–Kac semigroups have a long history in physics and mathematics. One of their traditional applications as a probabilistic representation of Schrödinger semigroups [43] is the computation of ground state energies through Diffusion Monte Carlo algorithms [34, 1, 7, 31]. It has also become a significant tool in non-linear filtering and genealogical models [15, 17, 13], as well as in large deviations theory [21, 47, 32, 64]. In all these contexts, the dynamics is evolved and its paths are weighted depending on some cost function. This function is typically a potential energy, a likelihood, or a function whose fluctuations are of interest.

As for Markov chains, the long time behavior of such dynamics is important. However, the long-time analysis is made difficult by the non-linear character of the evolution, so the methods used for the stability of Markov chains [50, 39] cannot be straightforwardly adapted in this context. A series of papers [14, 16, 13] rely on the powerful Dobrushin ergodic coefficient [19, 20]. However, although this tool enables to deal with the nonlinearity and to consider time-inhomogeneous processes, the conditions imposed on the dynamics are not realistic for unbounded domains.

The purpose of this paper is to propose a new scheme of proof for the ergodicity of Feynman–Kac dynamics, suitable for cases where the state space is unbounded. It is based on the principal eigenvalue problem associated to a weighted evolution operator. It then relies on studying a h -transformed version of the dynamics [22], where h is the eigenvector associated to the eigenproblem. This turns the non-linear dynamics into a linear Markov evolution, which can then be studied with standard techniques [50, 39]. However, the spectral properties of the generator fall out of the typical regime of self-adjoint operators, since the dynamics is in general non-reversible. A striking fact of our results is that, under Lyapunov and minorization conditions similar to those of [39] stated for

non-probabilistic kernels, we perform a non self-adjoint spectral analysis that recasts the Feynman–Kac problem into the Markov chain framework studied in [39].

The works of Kontoyannis and Meyn [44, 47] provide elements of answer concerning the spectral properties of the evolution operator, and rely on a nonlinear Lyapunov condition and a regularity in terms of hitting times. If the latter Lyapunov condition is natural in terms of optimal stochastic control [30], we propose instead proofs based on linear conditions. Our generalized linear Lyapunov condition is inspired by [56], and comes together with a minorization condition and a local strong Feller assumption. We will see that these conditions apply to a variety of situations, with natural interpretations. From a broader perspective, it appears as a natural extension of previous works on the stability of Markov chains [39] for evolution kernels that do not conserve probability. To that extent, our work resonates with recent works on Quasi-Stationary Distributions (QSD) [33, 9, 8, 4]. However, our scope and assumptions being different, we leave the comparison for future studies. Let us also mention that our framework applies for both discrete and continuous time processes. This is interesting since one motivation for this work is to understand the behavior of time discretizations of continuous Feynman–Kac dynamics, as in [28].

Let us outline our main results in an informal way. The quantities we are interested in typically correspond to Markov chains $(x_k)_{k \geq 0}$ over a state space \mathcal{X} , whose trajectories are weighted by a function $f : \mathcal{X} \rightarrow \mathbb{R}$. This corresponds to semigroups of the form

$$\Phi_k(\mu)(\varphi) = \frac{\mathbb{E} \left[\varphi(x_k) e^{\sum_{i=0}^{k-1} f(x_i)} \mid x_0 \sim \mu \right]}{\mathbb{E} \left[e^{\sum_{i=0}^{k-1} f(x_i)} \mid x_0 \sim \mu \right]}, \quad (1)$$

where μ is an initial probability distribution, and φ is a test function. We show that, for more general semigroups and under some assumptions on $(x_k)_{k \geq 0}$ and f , there exists a measure μ_f^* such that for any initial measure μ and any φ belonging to a particular class of unbounded test functions,

$$\Phi_k(\mu)(\varphi) \xrightarrow[k \rightarrow +\infty]{} \mu_f^*(\varphi), \quad (2)$$

at an exponential rate. As a corollary of this result, we show that the principal eigenvalue Λ of the generator of the dynamics $(\Phi_k)_{k \geq 1}$ can be obtained as the following limit, for any initial measure μ and suitable functions f :

$$\log(\Lambda) = \lim_{k \rightarrow +\infty} \frac{1}{k} \log \mathbb{E} \left[e^{\sum_{i=0}^{k-1} f(x_i)} \mid x_0 \sim \mu \right],$$

a quantity sometimes called scaled cumulant generating function in large deviations theory [18, 47]. Another natural situation corresponds to continuous semigroups of the form

$$\Theta_t(\mu)(\varphi) = \frac{\mathbb{E} \left[\varphi(X_t) e^{\int_0^t f(X_s) ds} \mid X_0 \sim \mu \right]}{\mathbb{E} \left[e^{\int_0^t f(X_s) ds} \mid X_0 \sim \mu \right]}, \quad (3)$$

where $(X_t)_{t \geq 0}$ is typically a diffusion process. Results similar to the ones obtained in the discrete time setting are then derived for this continuous dynamics. We will see that ergodic properties such as (2) are proved under natural extensions of Lyapunov and minorization conditions, which should be reminiscent of the corresponding theory for Markov chains [39, 56], with additional regularity conditions.

The paper is organized as follows. In Section 2, we present our main results on the stability of Feynman–Kac semigroups. Section 2.2 is devoted to discrete time results, while Section 2.3 is concerned with the continuous time case. Section 3 presents a number of natural applications of the method. In particular, Section 3.3 provides uniform in the time step convergence estimates. Section 4 discusses some links with related works and possible further directions.

2 Results

2.1 Framework

In this section, we present our main convergence results for generalizations of the dynamics (1). The state space \mathcal{X} is assumed to be a Polish space, and for a measurable set $A \subset \mathcal{X}$, we denote by A^c its complement, and $\mathbb{1}_A$ its indicator function. For a Banach space E , we denote by $\mathcal{B}(E)$ the space of bounded linear operators over E , with associated norm $\|T\|_{\mathcal{B}(E)} = \sup\{\|Tu\|_E, \|u\|_E \leq 1\}$. The Banach space of continuous functions is called $C^0(\mathcal{X})$, and the Banach space of measurable functions φ such that

$$\|\varphi\|_{B^\infty} := \sup_{x \in \mathcal{X}} |\varphi(x)| < +\infty$$

is referred to as $B^\infty(\mathcal{X})$. Given a measure μ over \mathcal{X} with finite mass, we use the notation $\mu(\varphi) = \int_{\mathcal{X}} \varphi(x) \mu(dx)$ for $\varphi \in B^\infty(\mathcal{X})$. The spaces of positive measures and probability measures over \mathcal{X} are denoted respectively by $\mathcal{M}(\mathcal{X})$ and $\mathcal{P}(\mathcal{X})$. When we consider Markov chains $(x_k)_{k \in \mathbb{N}}$ over \mathcal{X} , we write \mathbb{E}_μ for the expectation over all the realizations of the Markov chain with initial condition distributed according to the probability measure μ . Appendix A is devoted to reminders on the ergodicity of Markov chains extracted from [39], while Appendix B recalls some useful definitions and theorems used in the proofs of the results of this section.

We consider general kernel operators Q^f over \mathcal{X} , *i.e.* such that for any $x \in \mathcal{X}$, $Q^f(x, \cdot)$ is a positive measure with finite mass (*i.e.* $Q^f \mathbb{1}(x) < +\infty$), and for any measurable set $A \subset \mathcal{X}$, $Q^f(\cdot, A)$ is a measurable function. Such a kernel is referred to as Markov (also probabilistic or conserving) when $Q^f \mathbb{1} = \mathbb{1}$. The notation Q^f instead of Q emphasizes that in general the function $Q^f \mathbb{1} \neq \mathbb{1}$ depends on a measurable function $f : \mathcal{X} \rightarrow \mathbb{R}$. For $\varphi \in B^\infty(\mathcal{X})$, we denote by $Q^f \varphi = \int_{\mathcal{X}} \varphi(y) Q^f(\cdot, dy)$ the action of Q^f on test functions, and by $\mu Q^f = \int_{\mathcal{X}} \mu(dx) Q^f(x, \cdot)$ its action on finite measures μ . We call Feynman–Kac semigroups the dynamics $(\Phi_k)_{k \geq 1}$ defined as follows:

$$\forall k \geq 1, \quad \forall \mu \in \mathcal{P}(\mathcal{X}), \quad \forall \varphi \in B^\infty(\mathcal{X}), \quad \Phi_k(\mu)(\varphi) = \frac{\mu((Q^f)^k \varphi)}{\mu((Q^f)^k \mathbb{1})}. \quad (4)$$

Note that $\Phi_k = \Phi \circ \dots \circ \Phi$, where Φ is the one step evolution operator $\Phi : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X})$:

$$\forall \mu \in \mathcal{P}(\mathcal{X}), \quad \forall \varphi \in B^\infty(\mathcal{X}), \quad \Phi(\mu)(\varphi) = \frac{\mu(Q^f \varphi)}{\mu(Q^f \mathbb{1})}, \quad (5)$$

which is well-defined as soon as $\mu(Q^f \mathbb{1}) > 0$ for any $\mu \in \mathcal{P}(\mathcal{X})$. Lemma 1 below proves that (5) is indeed well-defined under the assumptions presented in Section 2.2.

Although Q^f is not probabilistic, the normalizing factor in (5) ensures that Φ evolves a positive measure of finite mass into a probability measure. An important motivation for studying the general dynamics (5) is that (1) can be written in the form (4) with $Q^f = e^f Q$, where Q is the transition operator of the Markov chain $(x_k)_{k \in \mathbb{N}}$. In this typical setting, $Q^f \mathbb{1} = e^f$. Even when Q^f is not defined in this way (see for instance the continuous time situation (30) considered in Section 2.3), we keep the notation to emphasize that Q^f typically corresponds to a Markov dynamics whose trajectories are weighted by a function f .

2.2 Results in discrete time

We now introduce the assumptions ensuring the well-posedness and ergodicity of the semigroup (4), which should be reminiscent of the ones used in [39, 56] for showing the ergodicity of Markov chains. The first step of the proof is the existence of a principal eigenvector h for Q^f , as shown in Lemma 2. This eigenvector is used in Lemma 3 to study a h -transformed version of Q^f , which leads to our main result, Theorem 1. Note that, in practice, we have in mind the situation $\mathcal{X} = \mathbb{R}^d$ for $d \in \mathbb{N}^*$, but discrete spaces like $\mathcal{X} = \mathbb{Z}^d$ can also be considered, in which case the framework may be simplified.

The first assumption is that a generalized Lyapunov condition holds. We will see in Section 3 that it is satisfied for a large class of processes. In all this section, we consider an increasing sequence of compact sets $(K_n)_{n \geq 1}$ such that, for any compact $K \subset \mathcal{X}$, there exists $m \geq 1$ for which $K \subset K_m$.

Assumption 1 (Lyapunov condition). *There exist a function $W : \mathcal{X} \rightarrow [1, +\infty)$ bounded on compact sets, and positive sequences $(\gamma_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$ with $\gamma_n \rightarrow 0$ as $n \rightarrow +\infty$ such that, for all $n \geq 1$,*

$$Q^f W \leq \gamma_n W + b_n \mathbb{1}_{K_n}. \quad (6)$$

Let us mention that, in many situations, the function W has compact level sets, so that a natural choice of compact sets is $K_n = \{x \in \mathcal{X} \mid W(x) \leq n\}$. When a Lyapunov function W exists, it is natural [39] to consider the following functional space

$$B_W^\infty(\mathcal{X}) = \left\{ \varphi \text{ measurable, } \left\| \frac{\varphi}{W} \right\|_{B^\infty} < +\infty \right\}. \quad (7)$$

In particular, Assumption 1 implies that Q^f is a bounded operator on $B_W^\infty(\mathcal{X})$, since one can show that

$$\forall n \geq 1, \quad \|Q^f\|_{\mathcal{B}(B_W^\infty)} \leq \gamma_n + b_n.$$

We next assume that the following minorization condition holds.

Assumption 2 (Minorization and irreducibility). *For any $n \geq 1$, there exist $\eta_n \in \mathcal{P}(\mathcal{X})$ and $\alpha_n > 0$ such that*

$$\inf_{x \in K_n} Q^f(x, \cdot) \geq \alpha_n \eta_n(\cdot). \quad (8)$$

In addition, for any $n_0 \geq 1$ and any $\varphi \in B_W^\infty(\mathcal{X})$ with $\varphi \geq 0$,

$$\eta_n(\varphi) = 0, \forall n \geq n_0 \implies (Q^f \varphi)(x) = 0, \forall x \in \mathcal{X}. \quad (9)$$

Note that (9) expresses some form of irreducibility with respect to the minorizing measures. It can be reformulated in the following way: for any $n_0 \geq 1$ and any $x \in \mathcal{X}$, $Q^f(x, \cdot)$ is absolutely continuous with respect to the measure

$$\sum_{n \geq n_0} 2^{-n} \eta_n.$$

The typical situation for $\mathcal{X} = \mathbb{R}^d$ is to choose $\eta_n(dx) = \mathbb{1}_{K_n}(x) dx / |K_n|$, where $|K_n|$ denotes the Lebesgue measure of K_n . We also mention that, although we will consider the previous minorization measures η_n in our examples in Section 3, the first part of Assumption 2 can be obtained using irreducibility together with a strong Feller property, see [36], or through the Stroock–Varadhan support theorem [62] with some regularity property, see the discussion in [56]. In our context, we also need some local regularity for the operator Q^f .

Assumption 3 (Local regularity). *The operator Q^f is strong Feller on the compact sets K_n , i.e. for any $n \geq 1$ and any measurable function φ bounded on K_n , $Q^f(\varphi \mathbb{1}_{K_n})$ is continuous over K_n .*

From these assumptions we first state the following preliminary lemma, whose proof can be found in Appendix C.

Lemma 1. *Let Q^f satisfy Assumptions 1 and 2. Then, for any $\mu \in \mathcal{P}(\mathcal{X})$ with $\mu(W) < +\infty$, one has*

$$0 < \mu(Q^f \mathbb{1}) < +\infty. \quad (10)$$

Moreover, for any $n \geq 1$, it holds $1 \leq \eta_n(W) < +\infty$, and there exist infinitely many indices $\bar{n} \geq 1$ such that

$$\eta_{\bar{n}}(K_{\bar{n}}) > 0. \quad (11)$$

The lower bound in (10) implies in particular that the dynamics (4) is well-defined. The inequality (11) means that, for infinitely many minorization conditions, some mass of the minorizing measure remains in the associated compact set. It is used in the proof of Lemma 2 to show that Q^f has a positive spectral radius. Since (11) is satisfied for infinitely many indices, we could consider that it holds for any $n \geq 0$, upon extracting a subsequence and, in the situations considered in Section 3, we can actually check that $\eta_n(K_n) > 0$ for all $n \geq 1$.

We are now in position to state some spectral properties of the operator Q^f , which are a key ingredient for our analysis. Let us recall that the spectral radius of Q^f on $B_W^\infty(\mathcal{X})$, denoted by $\Lambda := \Lambda(Q^f)$, is given by the Gelfand formula [53]:

$$\Lambda = \lim_{k \rightarrow +\infty} \left\| (Q^f)^k \right\|_{\mathcal{B}(B_W^\infty)}^{\frac{1}{k}}, \quad (12)$$

and that the essential spectral radius of Q^f , denoted by $\theta(Q^f)$, reads (see Appendix B):

$$\theta(Q^f) = \lim_{k \rightarrow +\infty} \left(\inf \left\{ \|(Q^f)^k - T\|_{B(B_W^\infty)}, T \text{ compact} \right\} \right)^{\frac{1}{k}}.$$

Lemma 2. *Under Assumptions 1, 2 and 3, the operator Q^f considered on $B_W^\infty(\mathcal{X})$ has a zero essential spectral radius, admits its spectral radius $\Lambda > 0$ as a largest eigenvalue (in modulus), and has an associated eigenfunction $h \in B_W^\infty(\mathcal{X})$, normalized so that $\|h\|_{B_W^\infty} = 1$, and which satisfies*

$$\forall x \in \mathcal{X}, \quad 0 < h(x) < +\infty. \quad (13)$$

In particular, $0 < \eta_n(h) < +\infty$ for all $n \geq 1$.

Note that the eigenspace associated with Λ is a priori not of dimension one. We prove Lemma 2 in Appendix D by using arguments inspired by [56, Theorem 8.9] to show that the essential spectral radius of Q^f is zero, and then relying on the theory of positive operators [12]. Some useful elements of operator theory are reminded in Appendix B for the reader's convenience. Our result is close to those obtained in [47], and the control of the essential spectral radius under Lyapunov and topological conditions has already been studied in [65, 35]. However, our proof uses different techniques based on different assumptions.

Once such a principal eigenvector h is available, the geometric ergodicity of the Feynman–Kac dynamics (4) is derived from the one of a h -transformed kernel, as made clear in the proof of Theorem 1 below. This is the purpose of the next lemma whose proof is postponed to Appendix E.

Lemma 3. *Suppose that Assumptions 1, 2 and 3 hold, and consider an eigenvector h associated with Λ as given by Lemma 2. Since $h > 0$ we can define the corresponding h -transformed operator Q_h as*

$$Q_h \phi = \Lambda^{-1} h^{-1} Q^f (h \phi). \quad (14)$$

Then Q_h is a Markov operator with Lyapunov function $Wh^{-1} : \mathcal{X} \rightarrow [1, +\infty)$. Moreover, there exist a unique $\mu_h \in \mathcal{P}(\mathcal{X})$, which satisfies $\mu_h(Wh^{-1}) < +\infty$, and constants $c > 0$, $\bar{\alpha} \in (0, 1)$ such that, for any $\phi \in B_{Wh^{-1}}^\infty(\mathcal{X})$ and any $k \geq 1$,

$$\|Q_h^k \phi - \mu_h(\phi)\|_{B_{Wh^{-1}}^\infty} \leq c \bar{\alpha}^k \|\phi - \mu_h(\phi)\|_{B_{Wh^{-1}}^\infty}. \quad (15)$$

Although this is not obvious at first glance, the operator Q_h is in fact independent of the choice of h in Lemma 2, and so is the invariant measure μ_h . Actually, Lemma 3 allows to show that the eigenspace associated with h has geometric dimension one, *i.e.* $\text{Ker}(Q^f - \Lambda \text{Id}) = \text{Span}\{h\}$. Indeed, if $\tilde{h} \in B_W^\infty(\mathcal{X})$ is another eigenvector associated with Λ (which may not be of constant sign), it holds, since $h(x) > 0$ for all $x \in \mathcal{X}$ by (13):

$$Q_h \left(\frac{\tilde{h}}{h} \right) = \Lambda^{-1} h^{-1} Q^f \tilde{h} = \frac{\tilde{h}}{h} \in B_{Wh^{-1}}^\infty(\mathcal{X}).$$

From (15), we obtain

$$\frac{\tilde{h}}{h} = \mu_h \left(\frac{\tilde{h}}{h} \right),$$

hence \tilde{h} is proportional to h . It may actually be possible to directly obtain this uniqueness result from stronger Krein–Rutman theorems, like [12, Theorem 19.3], using the irreducibility condition (9) in Assumption 2.

We are now in position to state our main theorem.

Theorem 1. *Consider a kernel operator Q^f satisfying Assumptions 1, 2 and 3 and the associated dynamics (4) with one step evolution operator $\Phi : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X})$. Then Φ admits a unique fixed point $\mu_f^* \in \mathcal{P}(\mathcal{X})$, that is a probability measure such that*

$$\Phi(\mu_f^*) = \mu_f^*, \quad (16)$$

and this measure satisfies $\mu_f^*(W) < +\infty$. Moreover, there exists $\bar{\alpha} \in (0, 1)$ such that, for any $\mu \in \mathcal{P}(\mathcal{X})$ satisfying $\mu(W) < +\infty$, there is $C_\mu > 0$ for which

$$\forall \varphi \in B_W^\infty(\mathcal{X}), \quad \forall k \geq 1, \quad |\Phi_k(\mu)(\varphi) - \mu_f^*(\varphi)| \leq C_\mu \bar{\alpha}^k \|\varphi\|_{B_W^\infty}. \quad (17)$$

We call μ_f^* the invariant measure of Q^f , in analogy with Markov chains. Note that Theorem 1 also implies the convergence of $\Phi_k(\mu)$ towards μ_f^* in the weighted total variation distance (a special type of Wasserstein distance [63, 39]) defined, for $\mu, \nu \in \mathcal{P}(\mathcal{X})$ with $\mu(W) < +\infty, \nu(W) < +\infty$, by

$$\rho_W(\mu, \nu) = \sup_{\|\varphi\|_{B_W^\infty} \leq 1} \int_{\mathcal{X}} \varphi(x) (\mu - \nu)(dx). \quad (18)$$

Proof. The key idea of the proof is to reformulate the dynamics (4) using the h -transformed operator $Q_h = \Lambda^{-1}h^{-1}Q^f h$ of Lemma 3. Using the notation of Lemmas 2 and 3, we rewrite (4) as

$$\Phi_k(\mu)(\varphi) = \frac{\mu((Q^f)^k \varphi) \Lambda^{-k}}{\mu((Q^f)^k \mathbf{1}) \Lambda^{-k}} = \frac{\mu\left(h(\Lambda^{-1}h^{-1}Q^f h)^k(h^{-1}\varphi)\right)}{\mu\left(h(\Lambda^{-1}h^{-1}Q^f h)^k h^{-1}\right)} = \frac{\mu(h(Q_h)^k(h^{-1}\varphi))}{\mu(h(Q_h)^k h^{-1})}.$$

The dynamics (4) is therefore reformulated as the ratio of long time expectations of the Markov chains induced by Q_h , applied to the functions $h^{-1}\varphi$ and h^{-1} . It is then possible to resort to the convergence results given by Lemma 3.

We first construct a probability measure μ_f^* for which (17) is satisfied, namely

$$\mu_f^*(\varphi) = \frac{\mu_h(h^{-1}\varphi)}{\mu_h(h^{-1})}, \quad (19)$$

where μ_h is the probability measure introduced in Lemma 3. Note that μ_f^* is well-defined for $\varphi \in B_W^\infty(\mathcal{X})$. Indeed, for $\varphi \in B_W^\infty(\mathcal{X})$, it holds $h^{-1}\varphi \in B_{Wh^{-1}}^\infty(\mathcal{X})$. Second, we show that

$$\mu_h(h^{-1}) > 0. \quad (20)$$

Indeed, since $\|h\|_{B_W^\infty} = 1$, it holds $h^{-1} \geq W^{-1}$, and since W is upper bounded on any compact set, W^{-1} is lower bounded by a positive constant on any compact set. As $\mu_h \in \mathcal{P}(\mathcal{X})$, we can use Lemma 5 in Appendix B to conclude that $\mu_h(h^{-1}) > 0$. Moreover, μ_f^* does not depend on the choice of normalization for h . Finally, $\mu_f^*(W) < +\infty$ since $\mu_h(Wh^{-1}) < +\infty$.

From Lemma 3, for any $\varphi \in B_W^\infty(\mathcal{X})$, it holds $Q_h^k(h^{-1}\varphi) = \mu_h(h^{-1}\varphi) + a_k$ and $Q_h^k(h^{-1}) = \mu_h(h^{-1}) + b_k$ with $\|a_k\|_{B_{Wh^{-1}}^\infty} \leq c\bar{\alpha}^k \|h^{-1}\varphi - \mu_h(h^{-1}\varphi)\|_{B_{Wh^{-1}}^\infty}$ and $\|b_k\|_{B_{Wh^{-1}}^\infty} \leq c\bar{\alpha}^k \|h^{-1} - \mu_h(h^{-1})\|_{B_{Wh^{-1}}^\infty}$. Since $\varphi \in B_W^\infty(\mathcal{X})$, we have in particular (using also $\|h\|_{B_W^\infty} = 1$),

$$\|h^{-1}\varphi - \mu_h(h^{-1}\varphi)\|_{B_{Wh^{-1}}^\infty} \leq \|h^{-1}\varphi\|_{B_{Wh^{-1}}^\infty} + \mu_h(h^{-1}|\varphi|)\|h\|_{B_W^\infty} \leq (1 + \mu_h(Wh^{-1}))\|\varphi\|_{B_W^\infty} < +\infty.$$

Since $\mu_h(Wh^{-1}) < +\infty$, we can set $c' = 1 + \mu_h(Wh^{-1})$ so that

$$\|a_k\|_{B_{Wh^{-1}}^\infty} \leq c'\bar{\alpha}^k \|\varphi\|_{B_W^\infty}. \quad (21)$$

A similar estimate holds for the sequence $(b_k)_{k \geq 1}$ by taking $\varphi \equiv \mathbf{1}$. This leads to, for any $\varphi \in B_W^\infty(\mathcal{X})$,

$$\begin{aligned} |\Phi_k(\mu)(\varphi) - \mu_f^*(\varphi)| &= \left| \frac{\mu(h(Q_h)^k(h^{-1}\varphi))}{\mu(h(Q_h)^k h^{-1})} - \mu_f^*(\varphi) \right| = \left| \frac{\mu(h(\mu_h(h^{-1}\varphi) + a_k))}{\mu(h(\mu_h(h^{-1}) + b_k))} - \mu_f^*(\varphi) \right| \\ &= \left| \frac{\mu(h)\mu_h(h^{-1}\varphi) + \mu(ha_k)}{\mu(h)\mu_h(h^{-1}) + \mu(hb_k)} - \mu_f^*(\varphi) \right| = \left| \frac{\mu_f^*(\varphi) + c_{\mu,h}\mu(ha_k)}{1 + c_{\mu,h}\mu(hb_k)} - \mu_f^*(\varphi) \right|, \end{aligned}$$

where we introduced

$$c_{\mu,h} = \frac{1}{\mu(h)\mu_h(h^{-1})}. \quad (22)$$

It holds $0 < c_{\mu,h} < +\infty$ because:

- Lemma 2 shows that for any $\mu \in \mathcal{P}(\mathcal{X})$ with $\mu(W) < +\infty$, it holds $0 < \mu(h) < +\infty$;
- we know that $\mu_h(h^{-1}) < +\infty$ from Lemma 3;
- $\mu_h(h^{-1}) > 0$ by (20).

Now, since $|b_k| \leq \|b_k\|_{B_{Wh^{-1}}^\infty} Wh^{-1}$ and (21) holds for b_k with $\varphi \equiv \mathbb{1}$, we have

$$1 + c_{\mu,h}\mu(hb_k) \geq 1 - c_{\mu,h}\mu(h|b_k|) \geq 1 - c_{\mu,h}\mu(W)\|b_k\|_{B_{Wh^{-1}}^\infty} \geq 1 - \bar{\alpha}^k c' c_{\mu,h}\mu(W).$$

Therefore, the choice

$$k \geq -\frac{\log(2c' c_{\mu,h}\mu(W))}{\log(\bar{\alpha})}$$

ensures that

$$1 + c_{\mu,h}\mu(hb_k) \geq \frac{1}{2}.$$

As a result, for k large enough, using $|a_k| \leq \|a_k\|_{B_{Wh^{-1}}^\infty} Wh^{-1}$ and recalling (21),

$$|\Phi_k(\mu)(\varphi) - \mu_f^*(\varphi)| \leq \frac{c_{\mu,h}(\mu(h|a_k|) + \mu_f^*(|\varphi|)\mu(h|b_k|))}{1 + c_{\mu,h}\mu(hb_k)} \leq C_\mu \|\varphi\|_{B_W^\infty} \bar{\alpha}^k, \quad (23)$$

with

$$C_\mu = 2c_{\mu,h}c'\mu(W)(1 + \mu_f^*(W)) = \frac{2}{\mu_h(h^{-1})}(1 + \mu_h(Wh^{-1}))(1 + \mu_f^*(W))\frac{\mu(W)}{\mu(h)}. \quad (24)$$

We therefore obtain (17) from (23) with the constant defined in (24). Note that C_μ depends on the initial measure μ only through the ratio $\mu(W)/\mu(h)$.

Taking the supremum over $\varphi \in B_W^\infty(\mathcal{X})$ such that $\|\varphi\|_{B_W^\infty} \leq 1$, (23) rewrites, with (18):

$$\rho_W(\Phi_k(\mu), \mu_f^*) \leq C_\mu \bar{\alpha}^k.$$

Choosing $\mu = \Phi(\mu_f^*)$ and using the semigroup property we obtain

$$\rho_W(\Phi(\Phi_k(\mu_f^*)), \mu_f^*) \leq C_{\mu_f^*} \bar{\alpha}^k.$$

Taking the limit $k \rightarrow +\infty$ shows that $\Phi(\mu_f^*) = \mu_f^*$, so μ_f^* is a fixed point of Φ .

We have shown the existence of an invariant measure of the form (19), which is a fixed point of Φ and integrates W . We now turn to uniqueness, which follows by a standard fixed point argument. Assume that we have two probability measures μ_1 and μ_2 satisfying (17) and such that $\mu_1(W) < +\infty$, $\mu_2(W) < +\infty$, which are therefore fixed points of Φ . Then, there exists $\bar{\alpha} \in (0, 1)$ such that, for any measure $\mu \in \mathcal{P}(\mathcal{X})$ with $\mu(W) < +\infty$, there is a constant C_μ for which

$$\forall k \geq 1, \quad \rho_W(\Phi_k(\mu), \mu_1) \leq C_\mu \bar{\alpha}^k.$$

Choosing $\mu = \mu_2$ and using the invariance by Φ leads to

$$\rho_W(\mu_2, \mu_1) \leq C_{\mu_2} \bar{\alpha}^k.$$

Taking the limit $k \rightarrow +\infty$ shows that $\mu_1 = \mu_2$, so the invariant measure is unique. \square

Theorem 1 also leads to alternative representations of the spectral radius Λ as a scaled cumulant generating function [47] and as the average rate of creation of probability of the dynamics. This is the purpose of the following result.

Theorem 2. *Let Q^f be as in Theorem 1 and define $\lambda = \log(\Lambda)$. Then, for any $\mu \in \mathcal{P}(\mathcal{X})$ with $\mu(W) < +\infty$,*

$$\lambda = \lim_{k \rightarrow +\infty} \frac{1}{k} \log\left(\mu[(Q^f)^k \mathbb{1}]\right). \quad (25)$$

Moreover,

$$\Lambda = \mu_f^*(Q^f \mathbb{1}). \quad (26)$$

Proof. Considering the operator Q_h introduced in Lemma 3, we have for any $\mu \in \mathcal{P}(\mathcal{X})$ with $\mu(W) < +\infty$,

$$\mu[(Q^f)^k \mathbb{1}] = \mu(\Lambda^k h Q_h^k h^{-1}).$$

Taking the logarithm and dividing by k leads to

$$\frac{1}{k} \log \mu[(Q^f)^k \mathbb{1}] = \log(\Lambda) + \frac{1}{k} \log \mu(h Q_h^k h^{-1}).$$

Lemma 3 shows that $\mu(h Q_h^k h^{-1})$ converges to $c_{\mu,h}^{-1}$, where $c_{\mu,h}$ is defined in (22). Taking the limit $k \rightarrow +\infty$ then leads to (25).

In order to prove (26), we use that μ_f^* is a fixed point of Φ , *i.e.* for any $\varphi \in B_W^\infty(\mathcal{X})$,

$$\mu_f^*(\varphi) = \frac{\mu_f^*(Q^f \varphi)}{\mu_f^*(Q^f \mathbb{1})}.$$

Taking $\varphi = h \in B_W^\infty(\mathcal{X})$ and using $Q^f h = \Lambda h$ we obtain

$$\mu_f^*(h) = \frac{\mu_f^*(\Lambda h)}{\mu_f^*(Q^f \mathbb{1})},$$

so that $\Lambda = \mu_f^*(Q^f \mathbb{1})$, as claimed. \square

Although stated in an abstract setting, Theorem 2 has a natural interpretation. If $Q^f = e^f Q$ where Q is the evolution operator of a Markov chain $(x_k)_{k \in \mathbb{N}}$ with $x_0 \sim \mu$, then (25) rewrites

$$\lambda = \lim_{k \rightarrow +\infty} \frac{1}{k} \log \mathbb{E}_\mu \left[e^{\sum_{i=0}^{k-1} f(x_i)} \right],$$

which is a standard formula for the scaled cumulant generating function (SCGF, or logarithmic spectral radius) in large deviations theory [18, 47]. We remind that \mathbb{E}_μ stands for the expectation with respect to all trajectories with initial condition distributed according to μ . On the other hand, (26) means that this SCGF can be expressed as the average rate of creation of probability of the process under the invariant measure. In particular, if $Q^f = Q$ is the evolution operator of a Markov chain, $\Lambda = 1$ since there is no creation of probability. Formula (26) does not seem typical in the large deviations literature, but was used in [28] to quantify the bias arising from discretizing a continuous Feynman–Kac dynamics.

Remark 1. *It should be clear from the proofs that Assumptions 1 to 3 can be adapted or relaxed depending on the context. In particular, we typically consider situations in which the state space \mathcal{X} is (a subset of) \mathbb{R}^d , and the transition kernel Q^f has a transition density $p^f(x, y) > 0$ jointly continuous in x, y . In this case, Assumptions 2 and 3 are immediately fulfilled by setting $\eta_n(dx) = \mathbb{1}_{K_n}(x) dx / |K_n|$ for each compact K_n , as we will see in Section 2.3. Similarly, the assumption that $W \geq 1$ can be weakened into: W is lower bounded by a positive constant on each compact set.*

Another remark of interest is that the regularity condition (Assumption 3) is not satisfied by Metropolis type kernels [57], which are therefore not covered by our analysis. The obstruction here is that we cannot prove with our techniques that the essential spectral radius of Q^f is 0, because the compactness argument used in the first step of the proof of Lemma 2 in Appendix D fails. However, we believe that a finer spectral analysis can cope with this situation, see for instance [65] for a careful study of the essential spectrum of discrete time Markov chains.

Let us mention that, in Assumption 1, it seems sufficient to suppose that $\gamma_n < \Lambda$ for some $n \geq 1$ in order to obtain that $\theta(Q^f) < \Lambda(Q^f)$ in the proof of Lemma 2. This is sufficient to apply the Krein–Rutman theorem, and to obtain a Lyapunov condition for Q_h (see Remark 4 in Appendix E).

It is also possible to keep track of the constants in the proofs of Lemma 3 and Theorem 1, like in [39], and observe that they depend on the assumptions through the coefficients γ_n, b_n, α_n , the measures η_n and the function W . More precisely, the constants deteriorate when α_n and $\eta_n(h)$ are small, and γ_n, b_n and $\sup_{K_n} W$ are large. Therefore, although the term $\eta_n(h)$ cannot be controlled more explicitly under our assumptions, it seems possible to optimize the final constants in Lemma 3 (and thus in Theorem 1) with respect to the choice of n .

In order to sketch the role of each assumption in the proofs of the results, we display in Figure 1 a schematic representation of the arguments. We hope this will help adapting our framework to situations where our assumptions are not fulfilled as such.

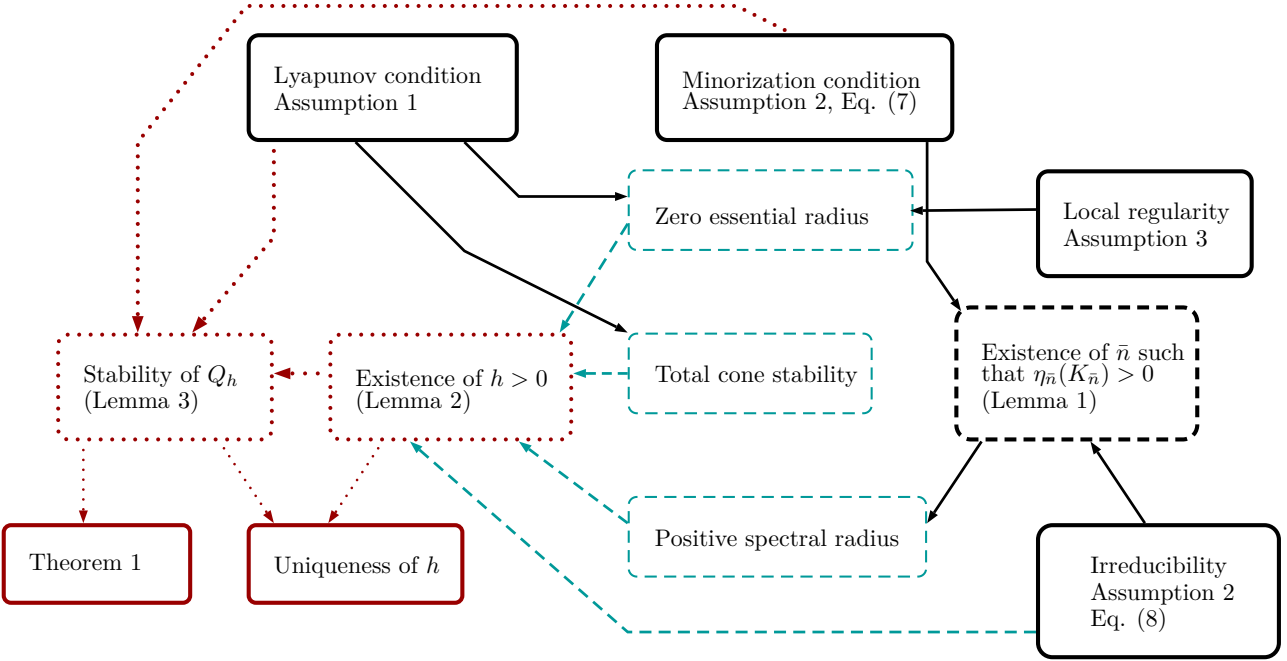


Figure 1: Schematic representation of the arguments used for the proofs of Lemmas 1, 2, 3, and Theorem 1. The plain lines correspond to Assumptions 1-3 pointing towards Lemma 1 and the key ingredients for the proof of Lemma 2. The dashed lines correspond to the actual proof of Lemma 2. The dotted lines correspond to the elements needed for the proof of Lemma 3 and its consequences.

2.3 Results in continuous time

Our analysis carries over to time continuous processes, in particular diffusions. In this case, it is possible to rephrase Assumption 1 in terms of the associated infinitesimal generator. In order to avoid the technical difficulty of dealing with an infinite dimensional process, we consider a diffusion $(X_t)_{t \geq 0}$ over $\mathcal{X} = \mathbb{R}^d$ for some integer $d \geq 1$, satisfying the SDE

$$dX_t = b(X_t) dt + \sigma(X_t) dB_t, \quad (27)$$

where $b : \mathcal{X} \rightarrow \mathbb{R}^d$, $\sigma : \mathcal{X} \rightarrow \mathbb{R}^{d \times m}$ and $(B_t)_{t \geq 0}$ is an m -dimensional Brownian motion (for some integer $m \geq 1$). We assume that b and σ are locally Lipschitz in order for (27) to have at least a local solution [55, Chapter XI, Exercice (2.10)]. The noise in the SDE (27) may be degenerate (*i.e.* $\sigma \sigma^T$ is not of full rank d) provided some technical conditions are met (see Assumptions 4 and 5 below).

The associated infinitesimal generator is given by

$$\mathcal{L} = b \cdot \nabla + \frac{\sigma \sigma^T}{2} : \nabla^2 = \sum_{i=1}^d b_i \partial_{x_i} + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^T)_{ij} \partial_{x_i} \partial_{x_j}. \quad (28)$$

We also consider a measurable function $f : \mathcal{X} \rightarrow \mathbb{R}$ and the corresponding continuous Feynman-Kac semigroup that reads, for all $t > 0$ and all initial distribution $\mu \in \mathcal{P}(\mathcal{X})$,

$$\Theta_t(\mu)(\varphi) = \frac{\mathbb{E}_\mu \left(\varphi(X_t) e^{\int_0^t f(X_s) ds} \right)}{\mathbb{E}_\mu \left(e^{\int_0^t f(X_s) ds} \right)}. \quad (29)$$

In this setting, we define the operator

$$(P_t^f \varphi)(x) = \mathbb{E}_x \left(\varphi(X_t) e^{\int_0^t f(X_s) ds} \right),$$

so that (29) is the natural continuous counterpart of (4) where, for a fixed time $t > 0$, we formally have

$$Q^f := P_t^f = e^{t(\mathcal{L}+f)}. \quad (30)$$

As a result, Θ_t satisfies a semigroup property as the discrete time evolution through (5). In this case, the generator of the weighted evolution operator P_t^f is $\mathcal{L} + f$. As for the discrete semigroup (4), we are interested in the long time behavior of quantities such as (29). When $b = 0$ and $\sigma = \sqrt{2}\text{Id}$, decay estimates of (29) in $L^2(\mathcal{X})$ towards a well-defined limit can be obtained by considering the spectral properties of the Schrödinger type operator $-\Delta - f$, as in [58]. When $\sigma = \sqrt{2}\text{Id}$ and $b = -\nabla U$ is the gradient of a potential energy, the operator $\mathcal{L} + f$ is self-adjoint in $L^2(e^{-U})$ (see for instance [3]), and the unitary transform $\varphi \mapsto \varphi e^{-\frac{U}{2}}$ leads to an analysis similar to the Schrödinger case. More precisely, $\mathcal{L} + f$ is unitarily equivalent to

$$\Delta - \frac{1}{4}|\nabla U|^2 + \frac{1}{2}\Delta U + f,$$

which can be studied by the theory of symmetric operators [41]. In both cases, the operator $\mathcal{L} + f$ is self-adjoint on a suitable Hilbert space, so that the Rayleigh formula can be used. It is also possible to study the spectral properties of P_t^f when $b \neq -\nabla U$ and \mathcal{X} is bounded through the Krein–Rutman theorem (see *e.g.* [28, Proposition 1]). To the best of our knowledge, the case $b \neq -\nabla U$ in an unbounded space \mathcal{X} remains open in general.

Our analysis provides a practical criterion to study the long time behavior of (29) through the Lyapunov function techniques developed in Section 2. The continuous counterpart of Assumption 1 can be stated in the following simple form.

Assumption 4. *Let $(X_t)_{t \geq 0}$ be the dynamics (27) with generator (28). There exists a $C^2(\mathcal{X})$ function $W : \mathcal{X} \rightarrow [1, +\infty)$ going to infinity at infinity such that*

$$\frac{(\mathcal{L} + f)W}{W} \xrightarrow{|x| \rightarrow +\infty} -\infty. \quad (31)$$

In addition, there exist a $C^2(\mathcal{X})$ function $\mathscr{W} : \mathcal{X} \rightarrow [1, +\infty)$ and a constant $c \geq 0$ such that

$$\varepsilon(x) := \frac{\mathscr{W}(x)}{W(x)} \xrightarrow{|x| \rightarrow +\infty} 0, \quad \frac{(\mathcal{L} + f)\mathscr{W}}{\mathscr{W}} \leq c. \quad (32)$$

Condition (31) can be checked by direct computations, as shown on some examples in Section 3.2. Finding a function \mathscr{W} such that (32) holds is usually done by considering Lyapunov functions in an exponential form, *i.e.* $W(x) = e^{aU(x)}$ for some function $U : \mathcal{X} \rightarrow \mathbb{R}$ and $a > 0$, and $\mathscr{W}(x) = e^{a'U(x)}$ for $0 < a' < a$. We refer for instance to [29, Proposition 1] for precise sufficient conditions for (32) to hold in this context. In the proof of Theorem 3, (31)-(32) are used to control P_t^f thanks to a Grönwall lemma. It is also important to remark that, in the case $f = 0$, (31) is a standard condition for the ergodicity of SDEs and compactness of the evolution operator P_t , see [56, Theorem 8.9]. As in Section 2.2, some regularity of the transition kernel is required. A natural condition in the context of diffusions reads as follows [56, Section 7].

Assumption 5. *The function σ is continuous and, for any $t > 0$, the transition kernel P_t^f has a continuous density p_t^f with respect to the Lebesgue measure, that is*

$$\forall x, y \in \mathcal{X}, \quad P_t^f(x, dy) = p_t^f(x, y) dy.$$

Moreover, it holds

$$\forall x, y \in \mathcal{X}, \quad p_t^f(x, y) > 0.$$

This assumption is standard for diffusion processes and, as shown in the proof of Theorem 3, it implies Assumptions 2 and 3 in Section 2.2. It holds true in particular for elliptic diffusions with regular coefficients and additive noise ($b \in C^\infty(\mathcal{X})$ and $\sigma = \text{Id}$). For degenerate diffusions, possibly with multiplicative noise, this result can be obtained through hypoelliptic conditions and controllability [62, 56, 64]. This is explained in detail in [29] in the context of large deviations, which allows to perform a similar spectral analysis as the one performed here.

We now state the continuous version of Theorem 1.

Theorem 3. *Consider the dynamics (29) induced by the SDE (27) and suppose that Assumptions 4 and 5 hold. Then, there exist a unique invariant measure μ_f^* and $\kappa > 0$ such that, for any initial measure $\mu \in \mathcal{P}(\mathcal{X})$ with $\mu(W) < +\infty$, there is $C_\mu > 0$ for which*

$$\forall \varphi \in B_W^\infty(\mathcal{X}), \quad \forall t > 0, \quad |\Theta_t(\mu)(\varphi) - \mu_f^*(\varphi)| \leq C_\mu e^{-\kappa t} \|\varphi\|_{B_W^\infty}. \quad (33)$$

Moreover, the invariant measure satisfies $\mu_f^*(W) < +\infty$ and $\Theta_t(\mu_f^*) = \mu_f^*$ for all $t \geq 0$.

Proof. The idea of the proof is to show that, for any $t > 0$, the evolution operator

$$(P_t^f \varphi)(x) = \mathbb{E}_x \left[\varphi(X_t) e^{\int_0^t f(X_s) ds} \right]$$

satisfies the assumptions of Theorem 1.

Step 1: Minorization and regularity. We first show that, by Assumption 5, P_t^f satisfies Assumptions 2 and 3. A first remark is that, since P_t^f is assumed to have a continuous density with respect to the Lebesgue measure, Assumption 3 immediately holds.

It is enough to prove the minorization condition (Assumption 2) for measurable subsets of $\mathcal{X} = \mathbb{R}^d$. Consider the compact sets $K_n = B(0, n)$, i.e. the balls centered at 0 with radius $n \geq 1$. For a measurable set $S \subset \mathbb{R}^d$ and $n \geq 1$, we have, for all $x \in K_n$,

$$(P_t^f \mathbb{1}_S)(x) = \int_S p_t^f(x, y) dy \geq \int_{S \cap K_n} p_t^f(x, y) dy \geq \left(\inf_{x, y \in K_n} p_t^f(x, y) \right) |S \cap K_n|, \quad (34)$$

where we denote by $|A|$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}^d$. As a result, (8) holds for all $n \geq 1$ with

$$\eta_n(S) = \frac{|S \cap K_n|}{|K_n|}, \quad \alpha_n = |K_n| \left(\inf_{x, y \in K_n} p_t^f(x, y) \right) > 0.$$

Finally, let us check that (9) is satisfied. Take $\varphi \in B_W^\infty(\mathcal{X})$ with $\varphi \geq 0$ such that

$$\eta_n(\varphi) = \frac{1}{|K_n|} \int_{K_n} \varphi(x) dx = 0,$$

for any $n \geq n_0$ for an arbitrary $n_0 \geq 1$. Since for any compact set $K \subset \mathcal{X}$ there exists $m \geq 1$ such that $K \subset K_m$, this implies that $\varphi = 0$ almost everywhere, so $Q^f \varphi = 0$ everywhere since Q^f has a continuous density with respect to the Lebesgue measure. Therefore, Assumption 2 is satisfied.

Step 2: Lyapunov condition. Let us now show that Assumption 1 holds. First, Assumption 4 is equivalent to the existence of positive sequences $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ such that

$$(\mathcal{L} + f)W \leq -a_n W + b_n, \quad (35)$$

with $a_n \rightarrow +\infty$ as $n \rightarrow +\infty$. We then compute, for any $t > 0$ and $n \in \mathbb{N}$,

$$\frac{d}{dt} (e^{a_n t} P_t^f W) = e^{a_n t} P_t^f (a_n W + (\mathcal{L} + f)W) \leq b_n e^{a_n t} P_t^f \mathbb{1}. \quad (36)$$

We can now bound the right hand side of the above expression using (32). Since $\mathscr{W} \geq 1$,

$$(P_t^f \mathbb{1})(x) = \mathbb{E}_x \left[e^{\int_0^t f(X_s) ds} \right] \leq \mathbb{E}_x \left[\mathscr{W}(X_t) e^{\int_0^t f(X_s) ds} \right]. \quad (37)$$

From the second condition in (32), (37) becomes

$$(P_t^f \mathbb{1})(x) \leq e^{ct} \mathbb{E}_x \left[\mathscr{W}(X_t) e^{-\int_0^t \frac{c\mathscr{W}}{\mathscr{W}}(X_s) ds} \right].$$

Inspired by a similar calculation in [64], we see that the right hand side of the above equation is a supermartingale. Indeed, introducing

$$M_t = \mathscr{W}(X_t) e^{-\int_0^t \frac{c\mathscr{W}}{\mathscr{W}}(X_s) ds},$$

Itô formula shows that

$$dM_t = e^{-\int_0^t \frac{c\mathscr{W}}{\mathscr{W}}(X_s) ds} \nabla \mathscr{W}^T(X_t) \sigma(X_t) dB_t,$$

so that M_t is a local martingale (using that σ and b are locally Lipschitz hence continuous and \mathscr{W} has a continuous derivative, see [43, Chapter 3, Proposition 2.24]). Since M_t is nonnegative, it is a supermartingale by Fatou's lemma. As a result, $\mathbb{E}_x[M_t] \leq M_0 = \mathscr{W}(x)$. The inequality (37) then becomes

$$(P_t^f \mathbb{1})(x) \leq e^{ct} \mathbb{E}_x[M_t] \leq e^{ct} \mathscr{W}(x).$$

Coming back to (36), we obtain

$$\frac{d}{dt} (e^{a_n t} P_t^f W) \leq b_n e^{(a_n+c)t} \mathscr{W}.$$

Integrating in time,

$$(e^{a_n t} P_t^f W - W)(x) \leq b_n \frac{e^{(a_n+c)t} - 1}{a_n + c} \mathscr{W}(x).$$

As a result

$$P_t^f W(x) \leq \tilde{\gamma}_n W(x) + c_n \mathscr{W}(x), \quad (38)$$

with

$$\tilde{\gamma}_n = e^{-a_n t}, \quad c_n = \frac{b_n e^{ct}}{a_n + c} \geq 0.$$

At this stage, (6) holds with the indicator function replaced by the function \mathscr{W} . However, using the first condition in (32), we can find a compact set K_n such that $c_n \varepsilon(x) \leq \tilde{\gamma}_n$ outside K_n . Using this set and $\mathscr{W} = \varepsilon W$, (38) becomes

$$\begin{aligned} P_t^f W(x) &\leq \tilde{\gamma}_n W(x) + c_n \mathbb{1}_{K_n}(x) \mathscr{W}(x) + c_n \varepsilon(x) W(x) \mathbb{1}_{K_n^c}(x) \\ &\leq 2\tilde{\gamma}_n W(x) + c_n \left(\sup_{K_n} \mathscr{W} \right) \mathbb{1}_{K_n}(x). \end{aligned}$$

Setting $\gamma_n = 2\tilde{\gamma}_n$ and $b_n = c_n \sup_{K_n} \mathscr{W}$, we see that

$$P_t^f W \leq \gamma_n W + b_n \mathbb{1}_{K_n}, \quad (39)$$

with $\gamma_n \rightarrow 0$ as $n \rightarrow +\infty$. This means that P_t^f satisfies Assumption 1, and hence fullfils all the assumptions of Theorem 1.

Step 3: using Theorem 1. We now use that P_t^f satisfies the assumptions of Theorem 1 to conclude the proof. Fix $t_0 > 0$. There exist a unique measure μ_{f,t_0}^* and a constant $\kappa_{t_0} > 0$ such that for any $\mu \in \mathcal{P}(\mathcal{X})$ with $\mu(W) < +\infty$, it holds (with the constant $C_\mu > 0$ defined in (24))

$$\forall \varphi \in B_W^\infty(\mathcal{X}), \quad \forall k \geq 1, \quad \left| \frac{\mu((P_{t_0}^f)^k \varphi)}{\mu((P_{t_0}^f)^k \mathbb{1})} - \mu_{f,t_0}^*(\varphi) \right| \leq C_\mu e^{-k\kappa_{t_0}} \|\varphi\|_{B_W^\infty}.$$

We next show that (33) can be obtained for any $t > 0$ (and not only multiples of t_0) and that the invariant measure μ_{f,t_0}^* actually does not depend on t_0 . This follows by a standard time decomposition argument [49, 42]. Indeed, for any $t > 0$, we set $t = kt_0 + r$ with $r \in [0, t_0)$, and we use the semigroup property to obtain

$$\Theta_t(\mu)(\varphi) = \Theta_{kt_0} \left(\frac{\mu P_r^f}{\mu(P_r^f \mathbb{1})} \right) (\varphi) = \frac{\mu_r((P_{t_0}^f)^k \varphi)}{\mu_r((P_{t_0}^f)^k \mathbb{1})},$$

where we defined μ_r as

$$\mu_r(\varphi) = \frac{\mu(P_r^f \varphi)}{\mu(P_r^f \mathbb{1})}.$$

We then only need to control the family of initial distributions $(\mu_r)_{r \in [0, t_0]}$. Step 1 in the proof shows that $\mu(P_r^f \mathbb{1}) > 0$ (using (34)). Then, in view of (39), the evolution operator P_r^f maps $B_W^\infty(\mathcal{X})$ to $B_W^\infty(\mathcal{X})$ for any $r > 0$, so $\mu_r(W) < +\infty$ and thus μ_r defines an admissible initial condition in Theorem 1. This leads to:

$$\forall \varphi \in B_W^\infty(\mathcal{X}), \quad \forall t > 0, \quad |\Theta_t(\mu)(\varphi) - \mu_{f, t_0}^*(\varphi)| \leq \left(\sup_{r \in [0, t_0]} C_{\mu_r} \right) e^{-\kappa t_0 \frac{t}{t_0}} \|\varphi\|_{B_W^\infty}, \quad (40)$$

where the constant C_{μ_r} is given in (24). In view of (24), it remains to bound

$$\sup_{r \in [0, t_0]} \frac{\mu_r(W)}{\mu_r(h)} = \sup_{r \in [0, t_0]} \frac{\mu(P_r^f W)}{\mu(P_r^f h_{t_0})}, \quad (41)$$

where h_{t_0} is the principal eigenvector associated to $P_{t_0}^f$ with eigenvalue Λ_{t_0} (using Lemma 2). The numerator in the latter expression is easily bounded uniformly in r using (39). Standard semigroup analysis shows that $h_{t_0} = h$ does not depend on t_0 and $\Lambda_{t_0} = e^{t_0 \alpha}$ for some $\alpha \in \mathbb{R}$. Therefore, for any $r \in [0, t_0]$, $P_r^f h_{t_0} = e^{r \alpha} h$, and the denominator in (41) is bounded away from 0 independently on r .

We finally prove that the invariant measure μ_{f, t_0}^* does not depend on t_0 . Following the same procedure for another time $t_1 > 0$ shows that (40) holds with an invariant measure μ_{f, t_1}^* . Then, for any $\varphi \in B_W^\infty(\mathcal{X})$, $\mu \in \mathcal{P}(\mathcal{X})$ with $\mu(W) < +\infty$ and $t > 0$ we have

$$\begin{aligned} |\mu_{f, t_0}^*(\varphi) - \mu_{f, t_1}^*(\varphi)| &\leq |\Theta_t(\mu)(\varphi) - \mu_{f, t_0}^*(\varphi)| + |\Theta_t(\mu)(\varphi) - \mu_{f, t_1}^*(\varphi)| \\ &\leq \left(\sup_{r \in [0, t_0]} C_{\mu_r} \right) e^{-\kappa t_0 \frac{t}{t_0}} \|\varphi\|_{B_W^\infty} + \left(\sup_{r \in [0, t_1]} C_{\mu_r} \right) e^{-\kappa t_1 \frac{t}{t_1}} \|\varphi\|_{B_W^\infty}. \end{aligned}$$

Taking the limit $t \rightarrow +\infty$ on the right hand side shows that $\mu_{f, t_0}^* = \mu_{f, t_1}^*$, so the invariant measure is independent of the arbitrary time t_0 . This concludes the proof of Theorem 3. \square

We close this section by mentioning that, under the assumptions of Theorem 3, it is also possible to define the logarithmic spectral radius of the dynamics as in Theorem 2, which reads in this case

$$\lambda = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{E}_\mu \left[e^{\int_0^t f(X_s) ds} \right],$$

for any initial measure μ that satisfies $\mu(W) < +\infty$. We do not reproduce the proof of this result which is similar to that of Theorem 2, and refer to [29] for more results on the cumulant function λ .

3 Applications

Since our study was first motivated by practical situations, we provide in this section a number of finite dimensional examples where our framework provides simple criteria for proving convergence of the Feynman–Kac semigroup towards an invariant measure. Sections 3.1 and 3.2 are concerned with discrete and continuous time applications respectively. Section 3.3 presents a convergence result for numerical discretizations of (29), where convergence rates are uniform in the time step.

3.1 Examples in discrete time

In this section, we provide two typical examples of Markov chains for which our results apply. First of all, let us consider the Diffusion Monte Carlo case where $f = -V$ and V stands for a Schrödinger potential.

Proposition 1. Consider a weighted evolution operator $Q^V = e^{-V}Q$ in $\mathcal{X} = \mathbb{R}^d$ with Gaussian increments $Q(x, dy) = (2\pi\sigma^2)^{-\frac{d}{2}} e^{-\frac{(x-y)^2}{2\sigma^2}} dy$, and where V is a continuous function. Then, if $V(x) \rightarrow +\infty$ when $|x| \rightarrow +\infty$, $W(x) = \mathbb{1}$ is a Lyapunov function for Q^V in the sense of Assumption 1. Moreover, if there exist constants $a > 0$ and $c \in \mathbb{R}$ such that

$$V(x) \geq a|x|^2 - c, \quad (42)$$

then $W(x) = e^{\beta x^2}$ is a Lyapunov function for

$$0 < \beta < \frac{a}{2} \left(\sqrt{1 + \frac{2}{a\sigma^2}} - 1 \right).$$

Finally, Assumptions 2 and 3 hold true, so that Theorem 1 applies for these choices of Lyapunov function.

The interpretation of this result is the following. In the Diffusion Monte Carlo setting, the confinement cannot be provided by the dynamics, since it is a Gaussian random walk over \mathbb{R}^d . However, the external potential V gives a small weight to the trajectories going to infinity, which makes the dynamics stable. If more information is available on the growth of V , we obtain better integrability results for the invariant measure μ_V^* through Lyapunov functions growing faster at infinity.

Proof. Let us first check that $W = \mathbb{1}$ is a Lyapunov function when V goes to infinity at infinity. Note that, for any compact set $K \subset \mathbb{R}^d$,

$$(Q^V \mathbb{1})(x) = e^{-V(x)} = \mathbb{1}_{K^c}(x) e^{-V(x)} + \mathbb{1}_K(x) e^{-V(x)}.$$

Taking an increasing sequence of compact sets K_n (in the sense of inclusion) and setting $\gamma_n = \sup_{K_n^c} e^{-V}$, $b_n = \sup_{K_n} e^{-V} < +\infty$, we obtain

$$Q^V \mathbb{1} \leq \gamma_n \mathbb{1} + b_n \mathbb{1}_{K_n},$$

which proves the first assertion since $\gamma_n \rightarrow 0$ as $n \rightarrow +\infty$.

Let us now assume that (42) holds. Setting $W(x) = e^{\beta x^2}$, under the condition

$$\beta < \frac{1}{2\sigma^2}, \quad (43)$$

an easy computation shows that

$$QW(x) = \frac{e^{\frac{\beta}{1-2\beta\sigma^2}x^2}}{(1-2\beta\sigma^2)^{\frac{d}{2}}}.$$

We remark that W is not a Lyapunov function for Q since $1 - 2\beta\sigma^2 < 1$. However, setting

$$C_d = (1 - 2\beta\sigma^2)^{-\frac{d}{2}},$$

we have

$$Q^V W(x) = C_d e^{-V(x) + \frac{\beta}{1-2\beta\sigma^2}x^2} \leq C_d e^{c - ax^2 + \frac{\beta}{1-2\beta\sigma^2}x^2 - \beta x^2} W(x) = C'_d e^{-ax^2 + \frac{2\beta^2\sigma^2}{1-2\beta\sigma^2}x^2} W(x),$$

with $C'_d = C_d e^c$. One can then check that the choice

$$0 < \beta < \frac{a}{2} \left(\sqrt{1 + \frac{2}{a\sigma^2}} - 1 \right) \quad (44)$$

leads to

$$-a + \frac{2\beta^2\sigma^2}{1-2\beta\sigma^2} < 0.$$

Note that, since

$$\frac{a}{2} \left(\sqrt{1 + \frac{2}{a\sigma^2}} - 1 \right) < \frac{1}{2\sigma^2},$$

the condition (43) is automatically satisfied when β is chosen according to (44). Next, when β satisfies (44), the function

$$\varepsilon(x) = e^{-ax^2 + \frac{\beta^2\sigma^2}{1-2\beta\sigma^2}x^2}$$

tends to zero at infinity. Therefore, taking increasing compact sets K_n (such as balls of increasing radii),

$$(Q^V W)(x) = \mathbf{1}_{K_n^c}(x)\varepsilon(x)W(x) + \mathbf{1}_{K_n}(x)\varepsilon(x)W(x) \leq \gamma_n W(x) + b_n \mathbf{1}_{K_n}(x),$$

with $\gamma_n = \sup_{K_n^c} \varepsilon \rightarrow 0$ as $n \rightarrow +\infty$ and $b_n = \sup_{K_n} \varepsilon W < +\infty$. Hence W is a Lyapunov function for Q^V for this choice of β , *i.e.* Assumption 1 is satisfied.

Assumption 3 is easily seen to hold. It therefore suffices to prove the minorization condition (Assumption 2). Take a compact set K with non zero Lebesgue measure, and let us first show that the condition of Assumption 2 holds for Q . It is enough to prove the condition for the indicator function of any borel set $S \subset \mathcal{X}$. Denoting by $D_K = \sup\{|x - y|, x \in K, y \in K\}$ the diameter of K , we compute for any $x \in K$

$$\begin{aligned} (Q\mathbf{1}_S)(x) &= Q(x, S) = \int_S e^{-\frac{(x-y)^2}{2\sigma^2}} dy \geq \int_{S \cap K} e^{-\frac{(x-y)^2}{2\sigma^2}} dy \geq \inf_{x \in K} \int_{S \cap K} e^{-\frac{(x-y)^2}{2\sigma^2}} dy \\ &\geq e^{-\frac{D_K^2}{2\sigma^2}} \int_{S \cap K} dy \geq e^{-\frac{D_K^2}{2\sigma^2}} |S \cap K|, \end{aligned}$$

where we denote again by $|A|$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}^d$. This motivates defining

$$\alpha_K = e^{-\frac{D_K^2}{2\sigma^2}} |K| > 0, \quad \eta_K(S) = \frac{|S \cap K|}{|K|}.$$

Note also that, since $|K| \in (0, +\infty)$, η_K is a probability measure. Finally, since V is continuous,

$$\forall x \in K, \quad Q^V(x, \cdot) \geq \alpha_V \eta_K(\cdot),$$

with $\alpha_V = \alpha_K e^{-\sup_K V} > 0$. Choosing $K_n = B(0, n)$ the centered balls of radius n , we see that (9) holds using arguments similar to the ones used for the proof of Theorem 3, hence Q^V satisfies Assumption 2. \square

We now provide an example where the dynamics Q admits a Lyapunov function W in the sense of the condition (59) recalled in Appendix A, and this function is also a Lyapunov function for Q^f when f does not grow too fast.

Proposition 2. *Consider the dynamics corresponding to a discrete Ornstein–Uhlenbeck process in \mathbb{R}^d , namely*

$$x_{k+1} = \rho x_k + \sigma G_k,$$

where $\rho \in (-1, 1)$, $\sigma \in \mathbb{R}$ and $(G_k)_{k \geq 1}$ is a family of independent standard d -dimensional Gaussian random variables. Define the operator $Q^f = e^f Q$ with f a continuous function such that there exist constants $a > 0$, $c \geq 0$, $0 \leq p < 2$ for which $f(x) \leq a|x|^p + c$.

Then, the Feynman–Kac dynamics associated to Q^f satisfies the assumptions of Theorem 1 with Lyapunov function $W(x) = e^{\beta x^2}$ when

$$0 < \beta < \frac{1 - \rho^2}{2\sigma^2}.$$

The interpretation of this result is quite different from the interpretation of Proposition 1. Here, the confinement is provided by the dynamics itself, and the weight f has to be controlled by the Lyapunov function of the dynamics. In that case it is important to find a «strong enough» Lyapunov function in order for this control to be possible. Quite typically, if f is unbounded, $W(x) = x^2$ is a Lyapunov function for Q , but not for Q^f . On the other hand, if f is bounded above, the result is straightforward.

Proof. We set $W(x) = e^{\beta x^2}$ and first compute

$$QW(x) = \mathbb{E} [W(x_{k+1}) \mid x_k = x] = \mathbb{E}_G [e^{\beta|\rho x + \sigma G|^2}] = e^{\beta\rho^2 x^2} \mathbb{E}_G [e^{\beta(2\sigma\rho x G + \sigma^2 G^2)}].$$

For $\beta < 1/(2\sigma^2)$, an easy computation similar to that of Proposition 1 shows that

$$QW(x) = \frac{1}{(1 - 2\beta\sigma^2)^{\frac{d}{2}}} e^{\frac{\rho^2}{1 - 2\beta\sigma^2} \beta x^2}.$$

Define now

$$\delta_\beta = \frac{\rho^2}{1 - 2\beta\sigma^2}.$$

Then $\delta_\beta \in (0, 1)$ and $1 - 2\beta\sigma^2 > 0$ when

$$\beta \in \left(0, \frac{1 - \rho^2}{2\sigma^2}\right).$$

This leads to

$$e^{f(x)} QW(x) = \frac{1}{(1 - 2\beta\sigma^2)^{\frac{d}{2}}} e^{f(x) + (\delta_\beta - 1)x^2} W(x) \leq \frac{1}{(1 - 2\beta\sigma^2)^{\frac{d}{2}}} e^{a|x|^p + c + (\delta_\beta - 1)x^2} W(x) = \varepsilon(x)W(x),$$

with $\varepsilon(x) \rightarrow 0$ as $|x| \rightarrow +\infty$. Therefore, by considering again $K_n = B(0, n)$, we see that

$$(Q^f W)(x) = \mathbf{1}_{K_n^c}(x) \varepsilon(x) W(x) + \mathbf{1}_{K_n}(x) \varepsilon(x) W(x) \leq \gamma_n W(x) + \mathbf{1}_{K_n}(x) b_n,$$

where $\gamma_n = \sup_{K_n^c} \varepsilon \rightarrow 0$ as $n \rightarrow +\infty$, and $b_n = \sup_{K_n} \varepsilon W < +\infty$. This shows that Assumption 1 is satisfied. Assumptions 2 and 3 follow by arguments similar to those used in the proof of Proposition 1. \square

The latter examples do not intend to form a complete overview of the possible practical cases. However, they seem characteristic of two typical situations: one where the confinement comes from the potential $V = -f$, and another one where it arises from the dynamics. These two strategies correspond respectively to a Diffusion Monte Carlo context [40] and a Large Deviations context [47]. They are both encoded in the condition (6).

3.2 Applications to diffusion processes

We now provide some examples where the conditions of Section 2.3 are met. Our main concern is the Lyapunov condition, Assumption 4, so we assume f and the coefficients of the SDE (27) to be regular enough for Assumption 5 to be satisfied. Let us start with a reversible diffusion.

Proposition 3. *Consider a diffusion process $(X_t)_{t \geq 0}$ over \mathbb{R}^d satisfying (27) with $\sigma = \sqrt{2} \text{Id}$, and assume that the drift is given by $b = -\nabla U$, where $U : \mathcal{X} \rightarrow \mathbb{R}$ is a smooth potential such that $U(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$. Assume moreover that U satisfies*

$$\lim_{|x| \rightarrow +\infty} \frac{|\nabla U(x)|^2}{|\Delta U(x)|} = +\infty, \quad (45)$$

and there exists $1/2 < \beta < 1$ such that

$$\lim_{|x| \rightarrow +\infty} \left(-\beta(1 - \beta)|\nabla U|^2 + \beta\Delta U + f \right) = -\infty. \quad (46)$$

Then Assumption 4 holds for the Lyapunov function $W(x) = e^{\beta U(x)}$.

The conditions (45) and (46) are satisfied for instance for potentials $x \mapsto U(x)$ behaving at infinity as $|x|^q$ with $q > 1$, and weight functions f such that $f(x)/|x|^{2(q-1)} \rightarrow 0$ as $|x| \rightarrow +\infty$.

Proof. The proof follows by simple computations. Indeed, it holds

$$\mathcal{L}W = -\beta\nabla U \cdot (\nabla U)W + \beta\nabla \cdot [(\nabla U)W] = -\beta|\nabla U|^2W + \beta W\Delta U + \beta^2|\nabla U|^2W,$$

so that

$$(\mathcal{L} + f)W = \left(-\beta(1 - \beta)|\nabla U|^2 + \beta\Delta U + f \right)W, \quad (47)$$

hence (31) in Assumption 4 is satisfied. The conditions in (32) are obtained setting

$$\mathscr{W}(x) = e^{\theta U(x)},$$

for some $\theta \in (1/2, \beta)$. It is clear that \mathscr{W}/W goes to zero at infinity, so the first condition in (32) holds true. The key remark is then to note that for our choice of θ, β , we have

$$\beta(1 - \beta) \leq \theta(1 - \theta).$$

Therefore, (45) and (46) show that there exist $c, c' \geq 0$ such that

$$f \leq \beta(1 - \beta)|\nabla U|^2 - \beta\Delta U + c \leq \theta(1 - \theta)|\nabla U|^2 - \theta\Delta U + c' = -\frac{\mathcal{L}\mathscr{W}}{\mathscr{W}} + c'.$$

This proves that the second condition in (32) holds, which concludes the proof. \square

Let us mention that the conditions in Proposition 3 are similar to conditions appearing in works on Poincaré inequalities (see [2] and references therein), and correspond to the case where the confinement comes from the potential U , f being a perturbation that should not go too fast to $+\infty$ with respect to U .

Remark 2. Proposition 3 is also related to confinement conditions for Schrödinger operators. Indeed, using the parameters of Proposition 3, the dynamics is reversible with respect to the measure e^{-U} and, as noted in Section 2.3, it is possible to turn the diffusion operator \mathcal{L} into a Schrödinger operator using the unitary transform:

$$\mathcal{L} \rightarrow e^{-\frac{U}{2}} \mathcal{L} e^{\frac{U}{2}}.$$

Using this transformation, $\mathcal{L} + f$ is unitarily equivalent [49] to the following Schrödinger operator:

$$\Delta - \frac{1}{4}|\nabla U|^2 + \frac{1}{2}\Delta U + f.$$

We then notice that the confinement condition for this Schrödinger operator is precisely (46) for the limit value $\beta = 1/2$. This shows that our Lyapunov condition (31) is a natural extension of this condition for non-reversible dynamics. As a side product, it shows that a slightly modified confinement condition for a Schrödinger operator does not only provide convergence in L^2 -norm, but also in a weighted uniform norm, which does not seem to be a standard result. Let us however conclude this remark by pointing out that we do not claim here that the conditions (45)-(46) are optimal in any way, but they are for sure reminiscent of (admittedly restrictive yet typical) sufficient conditions for Poincaré inequalities to hold.

In the non-reversible setting one cannot hope for a Schrödinger representation, and the Lyapunov function framework shows its usefulness. Let us present such an application, drawn from [25], where the drift behaves polynomially at infinity.

Proposition 4. Let $(X_t)_{t \geq 0}$ satisfy the SDE (27) with $\sigma = \sqrt{2}\text{Id}$ and where the drift b is such that there exist $q > 1$, $\delta > 0$, $R > 0$ for which

$$\forall |x| \geq R, \quad b(x) \cdot x \leq -\delta|x|^q. \quad (48)$$

Assume also that f is smooth and satisfies $f(x) \leq a|x|^p$ for $|x| \geq R$ and some $p < 2q - 2$. Then, Assumption 4 holds for the Lyapunov function

$$W(x) = e^{\beta|x|^q}, \quad \text{with } 0 < \beta < \frac{\delta}{q}. \quad (49)$$

Proof. Setting $W(x) = e^{\beta|x|^q}$, a simple computation shows that

$$\begin{aligned}\mathcal{L}W(x) &= \beta qb(x) \cdot x|x|^{q-2}W(x) + \beta q \nabla \cdot (x|x|^{q-2}W(x)) \\ &= \beta qb(x) \cdot x|x|^{q-2}W(x) + \beta q d|x|^{q-2}W(x) + \beta q(q-2)|x|^{q-2}W(x) + \beta^2 q^2|x|^{2q-2}W(x),\end{aligned}\tag{50}$$

so

$$\frac{\mathcal{L}W}{W}(x) = \beta qb(x) \cdot x|x|^{q-2} + \beta q(q+d-2)|x|^{q-2} + \beta^2 q^2|x|^{2q-2}.$$

Using (48) and the bound on f leads to, for $|x| \geq R$,

$$\frac{\mathcal{L}W}{W}(x) + f(x) \leq -\beta q(\delta - \beta q)|x|^{2q-2} + \beta q(q+d-2)|x|^{q-2} + a|x|^p.\tag{51}$$

Since $p < 2q - 2$, (31) is readily satisfied when $0 < \beta < \delta/q$.

We end the proof by showing that (32) holds. Similarly to the proof of Proposition 3, we consider

$$\mathcal{W}(x) = e^{\theta|x|^q}, \quad \text{with } 0 < \theta < \beta,$$

which satisfies the first condition in (32). Repeating the calculations leading to (51), since $\theta < \delta/q$ and $p < 2q - 2$, we obtain the existence of a constant $c \geq 0$ such that

$$\frac{\mathcal{L}\mathcal{W}}{\mathcal{W}}(x) + f(x) \leq -\theta q(\delta - \theta q)|x|^{2q-2} + \theta q(q-1)|x|^{q-2} + a|x|^p \leq c,$$

so the second condition in (32) holds true, and Assumption 4 is satisfied. \square

3.3 Convergence results uniform with respect to the time step

When one considers continuous semigroups as in Section 2.3, it is natural in practical applications to discretize (29) for example with

$$\Phi_k(\mu)(\varphi) = \frac{\mathbb{E}_\mu \left[\varphi(x_k) e^{\Delta t \sum_{i=0}^{k-1} f(x_i)} \right]}{\mathbb{E}_\mu \left[e^{\Delta t \sum_{i=0}^{k-1} f(x_i)} \right]},\tag{52}$$

where $(x_k)_{k \in \mathbb{N}}$ is a discretization of the SDE (27) with time step $\Delta t > 0$, *i.e.* x_k is an approximation of $X_{k\Delta t}$. First, as mentioned in [28], the stability of the discretization schemes for unbounded state spaces was an open question. Our framework covers this situation, as shown by the examples provided in Section 3.1.

Another interesting consequence of our analysis is that we are able to obtain convergence estimates uniform in the time step Δt , in the sense that the rate of decay towards the invariant measure in fact depends on $k\Delta t$, the physical time of the system, with a prefactor independent of Δt . It has been the purpose of several works to develop such uniform in Δt estimates for long time convergence, in particular in the context of Metropolisized discretizations of overdamped Langevin dynamics [5, 26], discretization of the Langevin dynamics [49, 48], and other discretizations of SDEs [11, 45, 46]. Our goal is to show that similar results can be obtained for Feynman–Kac semigroups. For the remainder of this section, we assume that

$$\mathcal{X} = \mathbb{T}^d$$

is the d -dimensional torus, the function σ in (27) is the identity matrix, and we denote by $[a]$ the upper integer part of a for $a \in \mathbb{R}$. Considering an unbounded state space \mathcal{X} is also possible but, as noted in [28], this leads to serious technical difficulties – we therefore postpone this case to future works.

We consider here a simplified version of the framework extensively developed in [28]. We say that a kernel operator $Q_{\Delta t}^f$ defines a weakly consistent discretization of the semigroup (29) if it satisfies Assumption 3 and there exist $\Delta t^* > 0$, $C > 0$, $p \in \mathbb{N}$, and an operator $\mathcal{R}_{\Delta t} : C^\infty(\mathcal{X}) \rightarrow C^\infty(\mathcal{X})$ (which encodes remainder terms) such that, for any $\varphi \in C^\infty(\mathcal{X})$,

$$Q_{\Delta t}^f \varphi = \varphi + \Delta t(\mathcal{L} + f)\varphi + \Delta t^2 \mathcal{R}_{\Delta t} \varphi,\tag{53}$$

where, for all $\Delta t \in (0, \Delta t^*]$,

$$\|\mathcal{R}_{\Delta t} \varphi\|_{B^\infty} \leq C \sup_{\substack{m \in \mathbb{N}^d \\ |m| \leq p}} \|\partial^m \varphi\|_{B^\infty},$$

using the notation $\partial^m = \partial_{x_1}^{m_1} \dots \partial_{x_d}^{m_d}$ for $m = (m_1, \dots, m_d) \in \mathbb{N}^d$. The dynamics (29) is then approximated by the discrete semigroup

$$\forall k \geq 1, \quad \forall \mu \in \mathcal{P}(\mathcal{X}), \quad \forall \varphi \in B^\infty(\mathcal{X}), \quad \Phi_k(\mu)(\varphi) = \frac{\mu \left((Q_{\Delta t}^f)^k \varphi \right)}{\mu \left((Q_{\Delta t}^f)^k \mathbb{1} \right)}. \quad (54)$$

The latter definition encompasses many numerical schemes – we refer the interested reader to [28] for a justification of this framework and the subsequent numerical analysis. Although the framework is rather abstract, one can think for concreteness to the Euler–Maruyama scheme for discretizing (27), in which case the evolution operation $Q_{\Delta t}$ is defined as

$$\forall x \in \mathcal{X}, \quad (Q_{\Delta t} \varphi)(x) = \mathbb{E}[\varphi(x_{n+1}) \mid x_n = x],$$

for the numerical scheme

$$x_{n+1} = x_n + b(x_n) \Delta t + \sqrt{\Delta t} G_n,$$

where $(G_n)_{n \geq 1}$ is a family of independent standard Gaussian variables. It is then possible to check that, for instance, the left point integration $Q_{\Delta t}^f = e^{\Delta t f} Q_{\Delta t}$ satisfies the expansion (53). The specific numerical scheme to be considered is however not crucial at all, as long as it is a weak approximation of the underlying SDE (as made precise in (53)) with a regularizing evolution operator $Q_{\Delta t}$.

In order to obtain uniform in the time step estimates, we now assume a uniform minorization and boundedness condition of the following form.

Assumption 6. *Fix a time $T > 0$. There exist $\Delta t^* > 0$, $\eta \in \mathcal{P}(\mathcal{X})$ and $\alpha \in (0, 1)$ such that, for any $\Delta t \in (0, \Delta t^*]$, the operator $Q_{\Delta t}^f$ is strong Feller and for any $\varphi \in B^\infty(\mathcal{X})$ with $\varphi \geq 0$, it holds*

$$\forall x \in \mathcal{X}, \quad \alpha \eta(\varphi) \leq \left((Q_{\Delta t}^f)^{\lceil \frac{T}{\Delta t} \rceil} \varphi \right)(x) \leq \frac{1}{\alpha} \eta(\varphi). \quad (55)$$

The lower bound in (55) corresponds to a minorization condition with respect to a physical time $T > 0$, see [49, Section 3]. The upper bound is a standard ingredient for studying Feynman–Kac semigroups, see for instance [14, 13]. We will see in Proposition 5 that Assumption 6 is naturally satisfied if a similar condition holds for $Q_{\Delta t}$ and the evolution operator reads $Q_{\Delta t}^f = e^{\Delta t f} Q_{\Delta t}$ (which corresponds to the discretization (52)).

Remark 3. *Although Assumption 6 holds in many situations when \mathcal{X} is compact, the requirement that the upper bound in (55) holds may not seem natural in view of the results of Section 2.2. Indeed, our framework shows that this upper bound is not necessary to prove the ergodicity of Feynman–Kac semigroups, as opposed to previous works [14, 16, 13, 28]. A careful look at the proof of Theorem 4 shows that this upper bound is only used to show the uniform boundedness of the approximate eigenvector $h_{\Delta t}$ in (57). However, controlling $h_{\Delta t}$ as $\Delta t \rightarrow 0$ does not seem to be an easy task without the upper bound in (55). We therefore stick to this assumption here.*

Before stating our uniform in Δt convergence result, we need the following estimate deduced from [28, Lemma 5], whose proof can be found in Appendix F.

Lemma 4. *Consider the process $(X_t)_{t \geq 0}$ solution to (27) with $\sigma = \text{Id}$, $b \in C^\infty(\mathcal{X})$, and a function $f \in C^\infty(\mathcal{X})$. Then the operator $\mathcal{L} + f$ admits a real isolated largest (in modulus) eigenvalue λ with eigenvector $h \in C^\infty(\mathcal{X})$ and associated eigenspace of dimension one, which satisfies*

$$(\mathcal{L} + f)h = \lambda h, \quad \text{and} \quad P_t^f h = e^{t\lambda} h, \quad \forall t \geq 0.$$

If $Q_{\Delta t}^f$ corresponds to a weakly consistent discretization of (29) (i.e. (53) holds) satisfying Assumption 6, then for any $\Delta t > 0$, the operator $Q_{\Delta t}^f$ has a largest (in modulus) eigenvalue $\Lambda_{\Delta t} \in \mathbb{R}$, which is non-degenerate. Denoting the associated eigenvector by $h_{\Delta t}$,

$$Q_{\Delta t}^f h_{\Delta t} = \Lambda_{\Delta t} h_{\Delta t},$$

with the normalization $\eta(h_{\Delta t}) = 1$, there exist $\Delta t^* > 0$, $C > 0$, $\varepsilon > 0$ such that for all $\Delta t \in (0, \Delta t^*]$, there is $c_{\Delta t} \in \mathbb{R}$ for which

$$\Lambda_{\Delta t} = e^{\Delta t \lambda + \Delta t^2 c_{\Delta t}}, \quad (56)$$

with $|c_{\Delta t}| \leq C$ and

$$\forall x \in \mathcal{X}, \quad \forall \Delta t \in (0, \Delta t^*], \quad \varepsilon \leq h_{\Delta t}(x) \leq \varepsilon^{-1}. \quad (57)$$

Lemma 4 means that the evolution operator associated with a weakly consistent discretization has a principal eigenvalue approximating the principal eigenvalue of the continuous dynamics, and that its associated principal eigenvector remains uniformly bounded from below and above if Δt is sufficiently small. Let us now state the uniform in Δt version of Theorem 1.

Theorem 4. *Consider a consistent discretization $Q_{\Delta t}^f$ of the dynamics (29) satisfying Assumption 6. Then, there exists $\Delta t^* > 0$ such that, for any $\Delta t \in (0, \Delta t^*]$, the dynamics (54) admits a unique invariant measure $\mu_{f, \Delta t}^* \in \mathcal{P}(\mathcal{X})$. Moreover, there exist $\kappa > 0$, $C > 0$ such that for any $\varphi \in B^\infty(\mathcal{X})$, $\mu \in \mathcal{P}(\mathcal{X})$, and $\Delta t \in (0, \Delta t^*]$, it holds*

$$\forall k \geq 0, \quad |\Phi_k(\mu)(\varphi) - \mu_{f, \Delta t}^*(\varphi)| \leq C e^{-\kappa k \Delta t} \|\varphi\|_{B^\infty}.$$

Let us note that the uniformity of the prefactor C in the initial condition is a consequence of the boundedness of \mathcal{X} . Indeed, in this case, we can choose $W \equiv 1$ as a Lyapunov function, so the constant C_μ in (24) can be uniformly bounded using (57). Such a uniformity does not hold for Theorem 1 since in that case \mathcal{X} was not assumed to be bounded. The important part of the theorem is the control of C and κ with respect to the time step, which provides convergence with respect to the physical time $k\Delta t$.

Proof. The proof essentially relies on the fact that if $Q_{\Delta t}^f$ satisfies Assumption 6, then $Q_{h, \Delta t}$ defined as in Lemma 3 satisfies a uniform minorization condition. For controlling the dependencies in the time step, we rely on Lemma 4, and use the same notation.

We want to prove a uniform minorization condition (in the sense of [49, Lemma 3.4]) for the operator defined by

$$Q_{h, \Delta t} = \Lambda_{\Delta t}^{-1} h_{\Delta t}^{-1} Q_{\Delta t}^f h_{\Delta t},$$

and apply [49, Corollary 3.5]. Fix $T > 0$. From (55) and (57) we have, for any $\varphi \geq 0$ and $x \in \mathcal{X}$,

$$Q_{h, \Delta t}^{\lceil \frac{T}{\Delta t} \rceil} \varphi(x) = \Lambda_{\Delta t}^{-\lceil \frac{T}{\Delta t} \rceil} h_{\Delta t}^{-1} (Q_{\Delta t}^f)^{\lceil \frac{T}{\Delta t} \rceil} (h_{\Delta t} \varphi)(x) \geq \Lambda_{\Delta t}^{-\lceil \frac{T}{\Delta t} \rceil} \varepsilon^2 \alpha \eta(\varphi). \quad (58)$$

Moreover, from (56),

$$\Lambda_{\Delta t}^{-\lceil \frac{T}{\Delta t} \rceil} = e^{-\Delta t (\lambda + \Delta t c_{\Delta t}) \lceil \frac{T}{\Delta t} \rceil} \geq e^{-2|\lambda|T} > 0,$$

upon possibly reducing Δt^* . Then, (58) becomes

$$\forall x \in \mathcal{X}, \quad Q_{h, \Delta t}^{\lceil \frac{T}{\Delta t} \rceil} (x, \cdot) \geq \alpha \varepsilon^2 e^{-2|\lambda|T} \eta(\cdot).$$

As a result, $Q_{h, \Delta t}$ satisfies the assumptions of [49, Corollary 3.5]: there exist a unique measure $\mu_{h, \Delta t} \in \mathcal{P}(\mathcal{X})$, $C > 0$, $\kappa > 0$ such that, for any $\phi \in B^\infty(\mathcal{X})$, $k \in \mathbb{N}$ and $\Delta t \in (0, \Delta t^*]$,

$$\|Q_{h, \Delta t}^k \phi - \mu_{h, \Delta t}(\phi)\|_{B^\infty} \leq C e^{-\kappa k \Delta t} \|\phi\|_{B^\infty}.$$

This is a version of Lemma 3 uniform with respect to Δt . The result then follows by rewriting the proof of Theorem 1, with $\bar{\alpha}^k$ replaced by $e^{-\kappa k \Delta t}$.

It only remains to study the constant $C_{\mu, \Delta t}$ arising in Theorem 1 (see (24)), which now also depends on Δt through the eigenvector $h_{\Delta t}$ and the invariant measure $\mu_{h, \Delta t}$. Since \mathcal{X} is bounded, we can actually choose a constant Lyapunov function, *i.e.* $W = \mathbb{1}$. Next, using (57) we obtain that for any $\Delta t \in (0, \Delta t^*]$ and any $\mu \in \mathcal{P}(\mathcal{X})$, it holds

$$C_{\mu, \Delta t} = \frac{4}{\mu_{h, \Delta t}(h_{\Delta t}^{-1})} \left(1 + \mu_{h, \Delta t}(h_{\Delta t}^{-1})\right) \frac{1}{\mu(h_{\Delta t})} \leq 4\varepsilon^{-2} (1 + \varepsilon^{-1}).$$

This provides a uniform bound on $C_{\mu, \Delta t}$, which concludes the proof. \square

We now show that the setting of Theorem 4 is natural, since Assumption 6 can be deduced from a similar assumption on the Markov dynamics $Q_{\Delta t}$ when the evolution operator is $Q_{\Delta t}^f = e^{\Delta t f} Q_{\Delta t}$, which corresponds to the discretization (52). For proving the condition on $Q_{\Delta t}$, we refer to [49] and the references therein.

Proposition 5. *Assume that \mathcal{X} is bounded, $f \in C^0(\mathcal{X})$, and the SDE (27) is discretized for a given time step $\Delta t > 0$ with a Markov chain $(x_k)_{k \in \mathbb{N}}$ whose evolution operator $Q_{\Delta t}$ is strong Feller and satisfies the following uniform minorization and boundedness condition: for a fixed $T > 0$, there exist $\Delta t^* > 0$, $\eta \in \mathcal{P}(\mathcal{X})$ and $\alpha \in (0, 1)$ such that, for any $\Delta t \in (0, \Delta t^*]$ and $\varphi \in B^\infty(\mathcal{X})$ with $\varphi \geq 0$,*

$$\forall x \in \mathcal{X}, \quad \alpha \eta(\varphi) \leq (Q_{\Delta t})^{\lceil \frac{T}{\Delta t} \rceil} \varphi(x) \leq \frac{1}{\alpha} \eta(\varphi).$$

Then, the transition operator $Q_{\Delta t}^f$ defined as $Q_{\Delta t}^f = e^{\Delta t f} Q_{\Delta t}$ satisfies Assumption 6.

Proof. Since $Q_{\Delta t}$ is strong Feller and f is continuous, $Q_{\Delta t}^f$ is strong Feller. Then, for any $k \in \mathbb{N}$ and $\varphi \in B^\infty(\mathcal{X})$,

$$(Q_{\Delta t}^f)^k \varphi(x) = \mathbb{E}_x \left[\varphi(x_k) e^{\Delta t \sum_{i=0}^{k-1} f(x_i)} \right] \geq e^{-k \Delta t \|f\|_{B^\infty}} \mathbb{E}_x [\varphi(x_k)] = e^{-k \Delta t \|f\|_{B^\infty}} ((Q_{\Delta t})^k \varphi)(x).$$

Taking $k = \lceil T/\Delta t \rceil$ with $0 < \Delta t \leq \Delta t^*$ then shows that

$$(Q_{\Delta t}^f)^{\lceil \frac{T}{\Delta t} \rceil} \varphi(x) \geq e^{-2T \|f\|_{B^\infty}} (Q^{\lceil \frac{T}{\Delta t} \rceil} \varphi)(x) \geq e^{-2T \|f\|_{B^\infty}} \alpha \eta(\varphi).$$

A similar computation for the upper bound allows to conclude the proof. \square

4 Discussion

The ideas developed in this work concerning the ergodicity of Feynman–Kac semigroups solve several problems for which, to the best of our knowledge, no solution was available. They are closely related to previous works and we want to highlight two important connections.

First, as we mentioned in the introduction, our framework can be considered as an extension of ergodic theory for Markov chains [50], when the evolution operator of the dynamics does not conserve probability. For this reason, we tried to formulate our assumptions in the flavour of [39]. However, the spectral theory on which we crucially rely in our study requires stronger conditions. This leaves open a few questions, as the converge of Feynman–Kac dynamics based on Metropolis type kernels, which lack regularity, or the case of non-Polish spaces, which may arise for stochastic partial differential equations. Another interesting feature of our framework is that we can prove ergodicity for Feynman–Kac dynamics for which the underlying Markov chain is not ergodic – a case we called Diffusion Monte Carlo (DMC) in analogy with quantum physics models (see Proposition 1). Finally, we mention that it would be interesting to extend our spectral approach to situations where the expected rate of convergence is sub-geometric, see for instance [24, 23] in the context of Markov chains.

The other clear connection concerns Large Deviations theory. Indeed, one motivation for studying Feynman–Kac dynamics is to prove large deviations principles for additive functionals of Markov chains [21, 18, 64, 47], which can be achieved by proving the existence of formulas such as (25). It is then no surprise that the spectral theory we develop, although based on [56], is reminiscent of [47], and requires stronger assumptions than the ones needed for proving ergodicity in [39]. However, the tools we use seem new in this context, and more adapted to the situation at hand, for instance the Krein–Rutman theorem based on the minorization condition. In particular, [47] (like [27]) makes use of nonlinear generators related to an optimal control problem, and we show in [29] that our linear spectral strategy can indeed be used for obtaining large deviations results in weighted topologies, for possibly degenerate diffusions.

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A Stability of Markov chains

In this section, we recall the results presented in [39]. We consider a measurable space \mathcal{X} and Markov chain $(x_k)_{k \geq 0}$ with transition kernel Q on \mathcal{X} . By transition kernel, we mean that (i) for all $x \in \mathcal{X}$, $Q(x, \cdot)$ is a positive measure on \mathcal{X} , (ii) for any measurable set $A \subset \mathcal{X}$, $Q(\cdot, A)$ is measurable, and (iii) $Q\mathbb{1} = \mathbb{1}$. In the notation of Section 2, Q is a kernel operator (*i.e.* (i) and (ii) are satisfied) such that $Q\mathbb{1} = \mathbb{1}$.

The stability of Markov dynamics can be obtained from minorization and Lyapunov conditions [50, 56, 39].

Assumption 7. *There exist a function $\mathcal{W} : \mathcal{X} \rightarrow [1, +\infty)$ and constants $C \geq 0$, $\gamma \in (0, 1)$ such that*

$$\forall x \in \mathcal{X}, \quad (Q\mathcal{W})(x) \leq \gamma\mathcal{W}(x) + C. \quad (59)$$

Given such a Lyapunov function, we consider the associated functional space as in (7).¹ A second key ingredient in the ergodicity of Q is the following minorization condition.

Assumption 8. *There exist $\alpha \in (0, 1)$ and $\eta \in \mathcal{P}(\mathcal{X})$ such that*

$$\inf_{x \in \mathcal{C}} Q(x, \cdot) \geq \alpha\eta(\cdot), \quad (60)$$

where $\mathcal{C} = \{x \in \mathcal{X} \mid \mathcal{W}(x) \leq R + 1\}$ for some $R > 2C/(1 - \gamma)$, and γ, C are the constants from Assumption 7.

The following result holds under these conditions (see [39, Theorem 1.2]).

Theorem 5. *Let Assumptions 7 and 8 hold. Then, Q has a unique invariant measure μ^* , which is such that $\mu^*(\mathcal{W}) < +\infty$. Moreover, there exist $C > 0$ and $\bar{\alpha} \in (0, 1)$ such that, for any $\varphi \in B_{\mathcal{W}}^\infty(\mathcal{X})$,*

$$\forall k \geq 0, \quad \|Q^k\varphi - \mu^*(\varphi)\|_{B_{\mathcal{W}}^\infty} \leq C\bar{\alpha}^k \|\varphi - \mu^*(\varphi)\|_{B_{\mathcal{W}}^\infty}.$$

B Useful theorems

We remind here some definitions and results around the Krein–Rutman theorem, as well as some basic results from analysis. Let us start with some operator theoretic definitions from [52, 54, 12, 56].

Definition 1. *For a Banach space E and an operator $T \in \mathcal{B}(E)$, we denote by $\Lambda(T)$ its spectral radius defined by:*

$$\Lambda(T) = \lim_{k \rightarrow +\infty} \|T^k\|_{\mathcal{B}(E)}^{\frac{1}{k}} = \inf_{k \geq 1} \|T^k\|_{\mathcal{B}(E)}^{\frac{1}{k}}.$$

We denote by $\theta(T)$ the essential spectral radius of T defined by (see [53, Eq. (1.14)] and [52, Theorem 1]):

$$\theta(T) = \lim_{k \rightarrow +\infty} \left(\inf_{\{ \|T^k - Q\|_{\mathcal{B}(E)}, Q \text{ compact} \}} \right)^{\frac{1}{k}} = \inf_{k \geq 1} \left(\inf_{\{ \|T^k - Q\|_{\mathcal{B}(E)}, Q \text{ compact} \}} \right)^{\frac{1}{k}}.$$

An operator $T \in \mathcal{B}(E)$ is said to be compact if it maps bounded sets into precompact sets. In other words, T is compact if, for any bounded sequence $(u_n)_{n \in \mathbb{N}}$ in E , there is a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $(Tu_{n_k})_{k \in \mathbb{N}}$ converges in E , see [54].

¹Compared to [39], we replace \mathcal{W} by $\mathcal{W} + 1$; this is for notational convenience only.

In order to recall the Krein–Rutman theorem, let us first give some definitions for cones in Banach spaces.

Definition 2. Let E be a Banach space. A closed convex set $\mathbb{K} \subset E$ is said to be a cone if $\mathbb{K} \cap -\mathbb{K} = \{0\}$ and for all $u \in \mathbb{K}$ and $\alpha \in \mathbb{R}_+$, it holds $\alpha u \in \mathbb{K}$. A cone is total if the norm closure of $\mathbb{K} - \mathbb{K}$ is equal to E .

We now recall a weak version of the Krein–Rutman theorem, which can be found in [53, Theorem 1.1]. Interesting remarks and comments are also available in [12, Section 19.8].

Theorem 6. Let E be a Banach space, $\mathbb{K} \subset E$ a total cone, and $T \in \mathcal{B}(E)$ be such that $\theta(T) < \Lambda(T)$ and $T\mathbb{K} \subset \mathbb{K}$. Then $\Lambda(T)$ is an eigenvalue of T with an eigenvector in \mathbb{K} .

In Theorem 6, there is no uniqueness of the eigenvector. The non degeneracy can be obtained under stronger positivity conditions on the operator T , as made precise in [12, Theorems 19.3 and 19.5]. In order to control the essential spectral radius and apply the Krein–Rutman theorem, we will need the following classical results, see [60, Theorem 11.28] and [61, Theorem 2.7.19].

Theorem 7 (Ascoli). Let $(\mathcal{Y}, d_{\mathcal{Y}})$ be a compact metric space and $C^0(\mathcal{Y})$ be the space of continuous functions over \mathcal{Y} endowed with the uniform norm $\|f\|_{C^0} = \sup_{y \in \mathcal{Y}} |f(y)|$. Consider a uniformly bounded and equicontinuous sequence $(f_n)_{n \in \mathbb{N}}$, i.e. a sequence for which there exists $M > 0$ such that $\|f_n\|_{C^0} \leq M$ for all $n \geq 1$, and for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_{\mathcal{Y}}(x, y) \leq \delta$ implies $|f(x) - f(y)| \leq \varepsilon$. Then $(f_n)_{n \in \mathbb{N}}$ converges in the uniform norm to some limit f up to extraction.

Theorem 8 (Heine–Cantor). Consider $f : E \rightarrow F$ where (E, d_E) and (F, d_F) are two metric spaces and E is compact. Then, if f is continuous, it is uniformly continuous: for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $x, x' \in E$ with $d_E(x, x') \leq \delta$, it holds $d_F(f(x), f(x')) \leq \varepsilon$.

We close this section with some results in probability theory. The next lemma can be found in [37, Lemma 4.14].

Lemma 5. If \mathcal{X} is a Polish space and $\mu \in \mathcal{P}(\mathcal{X})$, then the family constituted of the single measure μ is tight, i.e. for any $\varepsilon > 0$, there exists a compact set $K \subset \mathcal{X}$ such that $\mu(K) \geq 1 - \varepsilon$.

We finally present results concerning ultra-Feller operators, extending the ones of [38, Appendix A] for transition kernels that are not normalized. Recall that the total variation distance between two positive measures $\mu, \nu \in \mathcal{M}(\mathcal{X})$ is defined by:

$$\|\mu - \nu\|_{\text{TV}} = \sup_{\substack{\varphi \in B^\infty(\mathcal{X}) \\ \|\varphi\|_{B^\infty} \leq 1}} \int_{\mathcal{X}} \varphi d\mu - \int_{\mathcal{X}} \varphi d\nu. \quad (61)$$

Definition 3 (Ultra-Feller). A kernel operator Q is ultra-Feller if the mapping $x \mapsto Q(x, \cdot) \in \mathcal{M}(\mathcal{X})$ is continuous in the total variation distance (61).

The next lemma, used to show that an operator is ultra-Feller, is adapted from [38, Appendix A].

Lemma 6. Suppose that P and Q are two kernel operators over a Polish space \mathcal{X} that satisfy the following properties:

- for all $\varphi \in B^\infty(\mathcal{X})$, $Q\varphi$ is continuous and finite;
- for all ψ such that $|\psi| \leq Q\mathbf{1}$, $P\psi$ is continuous and finite.

Then PQ is ultra-Feller.

We remind some elements of the proof from [38, Theorem 1.6.6], which is based on the Banach–Alaoglu theorem. The details are left to the reader.

Proof. A first element to prove Lemma 6 is to show that, if Q is strong Feller, then there exists a reference probability measure $\zeta \in \mathcal{P}(\mathcal{X})$ such that for any $x \in \mathcal{X}$, $Q(x, \cdot)$ is absolutely continuous with respect to ζ . This is shown in [38, Lemma 1.6.4] for operators Q such that $Q\mathbf{1} = \mathbf{1}$. Even for a non-probabilistic Q , we can consider the normalized probabilities

$$\frac{Q(x, \cdot)}{Q\mathbf{1}(x)},$$

for x in the open set $\tilde{X} := \{x \in \mathcal{X} \mid Q\mathbf{1}(x) > 0\}$. We can apply [38, Lemma 1.6.4] to these probabilities defined over the set \tilde{X} , so there exists a measure ζ such that, for any $x \in \tilde{X}$, $Q(x, \cdot)$ is absolutely continuous with respect to ζ . If $x \in \mathcal{X} \setminus \tilde{X}$, $Q(x, \cdot) = 0$, which is also absolutely continuous with respect to ζ , so that $Q(x, \cdot)$ is absolutely continuous with respect to ζ for any $x \in \mathcal{X}$.

Once this is done, one can write the kernel Q as $Q(y, dz) = k(y, z)\zeta(dz)$ with $k(y, \cdot) \in L^1(\mathcal{X}, \zeta)$ for all $x \in \mathcal{X}$. If one supposes by contradiction that PQ is not ultra-Feller, then Definition 3 shows that there exist a sequence of functions $(g_n)_{n \in \mathbb{N}}$ with $\|g_n\|_{B^\infty} \leq 1$ and a sequence $(x_n)_{n \in \mathbb{N}}$ converging to an element $x \in \mathcal{X}$ such that for some $\delta > 0$ it holds

$$\forall n \in \mathbb{N}, \quad PQg_n(x_n) - PQg_n(x) > \delta. \quad (62)$$

Since the sequence $(g_n)_{n \in \mathbb{N}}$ is bounded, it possesses a weak-* converging subsequence in $L^\infty(\mathcal{X}, \zeta)$ (the space of ζ -essentially bounded functions) to an element $g \in B^\infty(\mathcal{X}, \zeta)$. In particular it holds (upon extracting a subsequence), for any $y \in \mathcal{X}$,

$$\lim_{n \rightarrow +\infty} Qg_n(y) = \lim_{n \rightarrow +\infty} \int_{\mathcal{X}} k(y, z)g_n(z)\zeta(dz) = \int_{\mathcal{X}} k(y, z)g(z)\zeta(dz) = Qg(y).$$

Defining, $f_n = Qg_n$, the latter limit shows that f_n converges pointwise to $f = Qg$. Since $(g_n)_{n \in \mathbb{N}}$ is bounded in $B^\infty(\mathcal{X})$, the second condition in Lemma 6 ensures that $Pf_n(x) \rightarrow Pf(x)$ for all $x \in \mathcal{X}$, by the dominated convergence theorem. This is the main difference compared to the proof in [38, Theorem 1.6.6]. The contradiction follows similarly. Indeed, defining the positive decreasing function $h_n = \sup_{m \geq n} |f_m - f|$ we have, for any $m \in \mathbb{N}$,

$$\lim_{n \rightarrow +\infty} Ph_n(x_n) \leq \lim_{n \rightarrow +\infty} Ph_m(x_n) = Ph_m(x),$$

so that $Ph_n(x_n) \rightarrow 0$ as $n \rightarrow +\infty$. In the end,

$$\lim_{n \rightarrow +\infty} Pf_n(x_n) - Pf(x) \leq \lim_{n \rightarrow +\infty} |Pf_n(x_n) - Pf(x_n)| + \lim_{n \rightarrow +\infty} |Pf(x_n) - Pf(x)| = 0,$$

which comes in contradiction with (62) and concludes the proof. \square

C Proof of Lemma 1

Let us show that $\mu(Q^f \mathbf{1}) > 0$ for any $\mu \in \mathcal{P}(\mathcal{X})$. First, Lemma 5 in Appendix B ensures that, for any $\varepsilon > 0$, there exists a compact $K \subset \mathcal{X}$ such that $\mu(K) \geq 1 - \varepsilon$. Consider next a compact set K_n of Assumption 1 such that $K \subset K_n$. Then, with the corresponding $\alpha_n > 0$ and $\eta_n \in \mathcal{P}(\mathcal{X})$ defined in Assumption 2, we have

$$\forall x \in K_n, \quad (Q^f \mathbf{1})(x) \geq \alpha_n \eta_n(\mathbf{1}) \geq \alpha_n > 0.$$

Integrating with respect to μ leads to

$$\int_{\mathcal{X}} (Q^f \mathbf{1})(x)\mu(dx) \geq \int_{K_n} (Q^f \mathbf{1})(x)\mu(dx) \geq \alpha_n \int_{K_n} \mu(dx) = \alpha_n \mu(K_n) \geq \alpha_n(1 - \varepsilon) > 0,$$

since $K \subset K_n$, which proves the statement. Moreover $W \geq 1$, so (6) implies that $\mu(Q^f \mathbf{1}) \leq \mu(Q^f W) < +\infty$ if $\mu(W) < +\infty$.

Since $W \geq 1$, we immediately have that $\eta_n(W) \geq 1 > 0$ for any $n \geq 1$. Now, for any $n \geq 1$ and $x \in K_n$, Assumptions 1 and 2 lead to

$$\alpha_n \eta_n(W) \leq Q^f W(x) \leq \gamma_n W(x) + b_n \mathbf{1}_{K_n}(x) < +\infty,$$

since W is finite. Moreover, $\alpha_n > 0$, so that $\eta_n(W) < +\infty$ for any $n \geq 1$.

Let us conclude with the proof of (11). We proceed by contradiction and assume that, for any $n \geq n_0$, we have $\eta_n(K_n) = 0$, with $n_0 \geq 1$ an arbitrary integer. Consider $m \geq n_0$ and $\varphi_m = \mathbf{1}_{K_m} \geq 0$. Then, using (8) with $n = n_0$,

$$\forall x \in K_{n_0}, \quad (Q^f \varphi_m)(x) \geq \alpha_{n_0} \eta_{n_0}(K_m).$$

Using again Lemma 5 in Appendix B, we see that for m large enough, $(Q^f \varphi_m)(x) > 0$ for $x \in K_{n_0}$ and so $Q^f \varphi_m \neq 0$. However, for $n \geq m$, we have, using that $K_m \subset K_n$ (since the sets are increasing):

$$0 \leq \eta_n(\varphi_m) = \eta_n(K_m) \leq \eta_n(K_n) = 0,$$

since we assumed $\eta_n(K_n) = 0$ for $n \geq n_0$. The contradiction with (9) shows that there exists $\bar{n} \geq n_0$ such that $\eta_{\bar{n}}(K_{\bar{n}}) > 0$. Since n_0 is arbitrary, \bar{n} can be chosen arbitrarily large, and this concludes the proof of Lemma 1.

D Proof of Lemma 2

The proof is decomposed into three steps. First we show that the essential spectral radius of the operator Q^f considered over $B_W^\infty(\mathcal{X})$ is zero. We next prove that the spectral radius Λ of Q^f is positive. Finally, we use the Krein-Rutman theorem to obtain that Λ is an eigenvalue of Q^f with largest modulus, and that the associated eigenvector is positive.

Step 1: Q^f has zero essential spectral radius

We first perform the following decomposition, for any $n \geq 1$:

$$(Q^f)^2 = \mathbb{1}_{K_n} Q^f \mathbb{1}_{K_n} Q^f + \mathbb{1}_{K_n^c} (Q^f)^2 + \mathbb{1}_{K_n} Q^f \mathbb{1}_{K_n^c} Q^f,$$

where $K_n \subset \mathcal{X}$ are the compact sets from Section 2.2. Applying again Q^f leads to

$$(Q^f)^3 = (\mathbb{1}_{K_n} Q^f \mathbb{1}_{K_n})^2 Q^f + \mathbb{1}_{K_n^c} Q^f (\mathbb{1}_{K_n} Q^f)^2 + Q^f \mathbb{1}_{K_n^c} (Q^f)^2 + Q^f \mathbb{1}_{K_n} Q^f \mathbb{1}_{K_n^c} Q^f. \quad (63)$$

We will show that $Q_n^f := \mathbb{1}_{K_n} Q^f \mathbb{1}_{K_n}$ is such that $(Q_n^f)^2$ is compact on $B_W^\infty(\mathcal{X})$, while $\mathbb{1}_{K_n^c} Q^f$ tends to zero in norm. This will prove that $(Q^f)^3$ is compact as limit of compact operators in operator norm, so the essential spectral radius of Q^f in $B_W^\infty(\mathcal{X})$, denoted by $\theta(Q^f)$, is equal to zero.

Let us first prove that $(Q_n^f)^2$ is compact on $B_W^\infty(\mathcal{X})$ for any $n \in \mathbb{N}$. For this, we use the ultra-Feller property proved in Lemma 6 (see Appendix B) to apply the Ascoli theorem. Consider a sequence $(\varphi_k)_{k \in \mathbb{N}}$ in $B_W^\infty(\mathcal{X})$ such that $\|\varphi_k\|_{B_W^\infty} \leq M$ for some $M \geq 0$. By Assumption 3, the operator Q_n^f is strong Feller over the compact set K_n . In particular, for $\varphi \in B_W^\infty(\mathcal{X})$, $\varphi \mathbb{1}_{K_n} \in B^\infty(\mathcal{X})$, so $Q_n^f \varphi$ is continuous over K_n and finite, so that Lemma 6 in Appendix B applies. Indeed, the second condition in the lemma is easy to check since $Q_n \mathbb{1}$ is equal to zero outside the compact K_n . Therefore, $(Q_n^f)^2$ is ultra-Feller by Lemma 6. By Definition 3, the application $x \in K_n \mapsto (Q_n^f)^2(x, \cdot) \in \mathcal{M}(\mathcal{X})$ is continuous in total variation norm. Since K_n is compact in the metric space \mathcal{X} and $\mathcal{P}(\mathcal{X})$ is a metric space, the Heine-Cantor theorem (Theorem 8 in Appendix B) ensures that this application is continuous over K_n . This means that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x, x' \in K_n$ with $|x - x'| \leq \delta$, it holds

$$\sup_{\|\varphi\|_{B^\infty} \leq 1} \left| ((Q_n^f)^2 \varphi)(x) - ((Q_n^f)^2 \varphi)(x') \right| \leq \varepsilon. \quad (64)$$

Noting that Assumption 1 implies that $1 \leq \sup_{K_n} W < +\infty$, it holds $M_n = (\sup_{K_n} W)^{-1} \in (0, 1]$ for any $n \geq 1$, so

$$\{\varphi \text{ measurable} \mid \|\mathbb{1}_{K_n} \varphi\|_{B^\infty} \leq 1\} \supset \{\varphi \text{ measurable} \mid \|\mathbb{1}_{K_n} \varphi\|_{B_W^\infty} \leq M_n\}. \quad (65)$$

Since $Q_n^f = \mathbb{1}_{K_n} Q^f \mathbb{1}_{K_n}$, (65) shows that (64) implies

$$\sup_{\|\varphi\|_{B_W^\infty} \leq M_n} \left| ((Q_n^f)^2 \varphi)(x) - ((Q_n^f)^2 \varphi)(x') \right| \leq \varepsilon.$$

As a consequence, if $(\varphi_k)_{k \in \mathbb{N}}$ is such that $\|\varphi_k\|_{B_W^\infty} \leq M$, we see that $((Q_n^f)^2 \varphi_k)_{k \in \mathbb{N}}$ is equicontinuous. By the Ascoli theorem, it therefore converges uniformly to a continuous limit on K_n (since the function is supported on K_n , we extend it by 0 on \mathcal{X} outside K_n). Since $W \geq 1$, it also converges as a function in $B_W^\infty(\mathcal{X})$, showing that $(Q_n^f)^2$ is a compact operator on $B_W^\infty(\mathcal{X})$. Since Q^f is bounded over $B_W^\infty(\mathcal{X})$ and the space of compact operators is stable by composition with bounded operators [54], $(Q_n^f)^2 Q^f$ is also compact.

We now show that the second, third and fourth operators on the right hand side of (63) tend to 0 in the operator norm of $B_W^\infty(\mathcal{X})$. For any $\varphi \in B_W^\infty(\mathcal{X})$,

$$\|\mathbb{1}_{K_n^c} Q^f \varphi\|_{B_W^\infty} = \left\| \frac{\mathbb{1}_{K_n^c} Q^f \varphi}{W} \right\|_{B^\infty} \leq \|\varphi\|_{B_W^\infty} \left\| \mathbb{1}_{K_n^c} \frac{Q^f W}{W} \right\|_{B^\infty} \leq \gamma_n \|\varphi\|_{B_W^\infty}.$$

Taking the supremum over $\varphi \in B_W^\infty(\mathcal{X})$ and using $\gamma_n \rightarrow 0$ as $n \rightarrow +\infty$, we obtain:

$$\left\| \mathbb{1}_{K_n^c} Q^f \right\|_{\mathcal{B}(B_W^\infty)} \xrightarrow{n \rightarrow +\infty} 0. \quad (66)$$

Since Q^f is bounded on $\mathcal{B}(B_W^\infty)$, the second, third and fourth operators on the right hand side of (63) vanish in norm as $n \rightarrow +\infty$. As a result, $(Q^f)^3$ is the norm-limit of the compact operators $(Q_n^f)^2 Q^f$ as $n \rightarrow +\infty$ in $\mathcal{B}(B_W^\infty)$. Since the set of compact operators over $B_W^\infty(\mathcal{X})$ is closed in the Banach space $\mathcal{B}(B_W^\infty)$, $(Q^f)^3$ is compact, see *e.g.* [54, Theorem VI.12]. Using Definition 1, we conclude that $\theta(Q^f) = 0$. In this procedure, we see that working in the weighted space $B_W^\infty(\mathcal{X})$ as opposed to $B^\infty(\mathcal{X})$ is crucial in order to obtain the compactness of $(Q^f)^3$ from the control (66) provided by the Lyapunov condition (6).

Step 2: The spectral radius is positive

We now show that the spectral radius Λ of Q^f defined in (12) is positive, in order to use Theorem 6. Given the definition of the operator norm, choosing some arbitrary non negative function $\phi \in B_W^\infty(\mathcal{X})$ with $\|\phi\|_{B_W^\infty} \leq 1$ leads to

$$\|Q^f\|_{\mathcal{B}(B_W^\infty)} \geq \left\| \frac{Q^f \phi}{W} \right\|_{B^\infty} \geq \frac{(Q^f \phi)(x_0)}{W(x_0)},$$

where $x_0 \in \mathcal{X}$ is arbitrary. We now consider a compact set corresponding to some $n = \bar{n}$ as defined in Lemma 1, which satisfies $\eta_{\bar{n}}(K_{\bar{n}}) > 0$, and take $x_0 \in K_{\bar{n}}$. For any non negative function $\phi \in B_W^\infty(\mathcal{X})$ with $\|\phi\|_{B_W^\infty} \leq 1$,

$$\eta_{\bar{n}}(Q^f \phi) = \left(\int_{K_{\bar{n}}} (Q^f \phi)(x) \eta_{\bar{n}}(dx) + \int_{\mathcal{X} \setminus K_{\bar{n}}} (Q^f \phi)(x) \eta_{\bar{n}}(dx) \right) \geq \int_{K_{\bar{n}}} \alpha_{\bar{n}} \eta_{\bar{n}}(\phi) \eta_{\bar{n}}(dx) \geq \alpha_{\bar{n}} \eta_{\bar{n}}(\phi) \eta_{\bar{n}}(K_{\bar{n}}), \quad (67)$$

where we used (8) with $n = \bar{n}$. Iterating the inequality shows that

$$\forall k \geq 1, \quad \eta_{\bar{n}}((Q^f)^k \phi) \geq \alpha_{\bar{n}}^k \eta_{\bar{n}}(K_{\bar{n}})^k \eta_{\bar{n}}(\phi).$$

This leads to the following lower bound on the operator norm of $(Q^f)^k$:

$$\begin{aligned} \|(Q^f)^k\|_{\mathcal{B}(B_W^\infty)} &\geq \frac{((Q^f)^k \phi)(x_0)}{W(x_0)} = \frac{(Q^f((Q^f)^{k-1} \phi))(x_0)}{W(x_0)} \\ &\geq \alpha_{\bar{n}} \frac{\eta_{\bar{n}}((Q^f)^{k-1} \phi)}{W(x_0)} \geq \frac{\alpha_{\bar{n}}^k \eta_{\bar{n}}(K_{\bar{n}})^{k-1}}{W(x_0)} \eta_{\bar{n}}(\phi). \end{aligned}$$

Taking the power $1/k$ and the limit $k \rightarrow +\infty$, together with the choice $\phi = \mathbb{1} \in B_W^\infty(\mathcal{X})$, leads to

$$\Lambda \geq \alpha_{\bar{n}} \eta_{\bar{n}}(K_{\bar{n}}).$$

From Lemma 1, it holds $\eta_{\bar{n}}(K_{\bar{n}}) > 0$, hence $\Lambda > 0$ and Q^f has a positive spectral radius. Note that the existence of $\bar{n} \geq 1$ such that $\eta_{\bar{n}}(K_{\bar{n}}) > 0$ is crucial for this step.

Step 3: Existence of a principal eigenvector

In order to use Theorem 6, we introduce the closed cone:

$$\mathbb{K}_W = \{u \in B_W^\infty(\mathcal{X}) \mid u \geq 0\}.$$

This cone is total, and the positivity of $Q^f \in B_W^\infty(\mathcal{X})$ shows that $Q^f \mathbb{K} \subset \mathbb{K}$. At this stage, Theorem 6 in Appendix B ensures that the spectral radius Λ is an eigenvalue of Q^f of largest modulus with an associated eigenvector $h \in \mathbb{K}_W \setminus \{0\}$.

Step 4: Positivity

We now use the irreducibility condition (9) to show that, for the eigenvector h obtained in Step 3, it holds $h(x) > 0$ for all $x \in \mathcal{X}$ and hence $\eta_n(h) > 0$ all $n \geq 1$.

Let us show the first property by contradiction. Assume that there exists $x_0 \in \mathcal{X}$ such that $h(x_0) = 0$. Since the sets K_n are increasing, there exists n_0 such that for all $n \geq n_0$ it holds $x_0 \in K_n$ so that, by (8),

$$\forall n \geq n_0, \quad (Q^f h)(x_0) \geq \alpha_n \eta_n(h).$$

Since $Q^f h = \Lambda h$ with $\Lambda > 0$, this leads to

$$0 \geq \eta_n(h),$$

and so $\eta_n(h) = 0$ for $n \geq n_0$. By the irreducibility assumption (9), we therefore have $(Q^f h)(x) = 0$ for all $x \in \mathcal{X}$. Using again $Q^f h = \Lambda h$, this shows that $h = 0$, which is in contradiction with the fact that h is an eigenvector associated with Λ .

The second property follows from $h(x) > 0$ for all $x \in \mathcal{X}$ and $\eta_n \in \mathcal{P}(\mathcal{X})$ for all $n \geq 1$. Indeed,

$$\mathcal{X} = \bigcup_{k \geq 1} h^{-1} \left[\frac{1}{k}, +\infty \right), \quad (68)$$

where h^{-1} denotes here the pre-image of h . Therefore, for a given $n \geq 1$,

$$\eta_n(\mathcal{X}) = \eta_n(h^{-1}[1, +\infty)) + \sum_{k \geq 1} \eta_n \left(h^{-1} \left[\frac{1}{k+1}, \frac{1}{k} \right) \right) = 1.$$

Thus, there exists $N \geq 1$ such that

$$\eta_n \left(h^{-1} \left[\frac{1}{N}, +\infty \right) \right) \geq \frac{1}{2},$$

so

$$\eta_n(h) \geq \eta_n(h \mathbf{1}_{h \geq \frac{1}{N}}) \geq \frac{1}{N} \eta_n(\mathbf{1}_{h \geq \frac{1}{N}}) \geq \frac{1}{2N}.$$

Since $n \geq 1$ is arbitrary, this shows that $\eta_n(h) > 0$ for all $n \geq 1$.

E Proof of Lemma 3

A first important remark is that Q_h is a Markov operator. Indeed, it is a well-defined kernel operator (since $0 < h(x) < +\infty$ for all $x \in \mathcal{X}$), and $Q_h \mathbf{1} = \Lambda^{-1} h^{-1} Q^f h = \Lambda^{-1} h^{-1} \Lambda h = \mathbf{1}$. Our goal is therefore to show that the Markov operator Q_h fits the framework reminded in Appendix A, in particular that it satisfies Assumptions 7 and 8.

Let us show that this operator satisfies Assumption 7 in Appendix A with Lyapunov function Wh^{-1} . We first note that the normalization $\|h\|_{B_W^\infty} = 1$ implies that $Wh^{-1} \geq 1$. Using Assumption 1, we obtain

$$Q_h(Wh^{-1}) = \Lambda^{-1} h^{-1} Q^f W \leq \Lambda^{-1} h^{-1} (\gamma_n W + b_n \mathbf{1}_{K_n}) \leq \frac{\gamma_n}{\Lambda} Wh^{-1} + \frac{b_n}{\Lambda h} \mathbf{1}_{K_n}.$$

Noting that, for all $x \in K_n$,

$$\Lambda h(x) = (Q^f h)(x) \geq \alpha_n \eta_n(h),$$

with $\eta_n(h) > 0$ from Lemma 2, the above inequality becomes

$$Q_h(Wh^{-1}) \leq \frac{\gamma_n}{\Lambda} Wh^{-1} + \frac{b_n}{\alpha_n \eta_n(h)} \mathbf{1}_{K_n}. \quad (69)$$

Since γ_n can be taken arbitrarily small and $\eta_n(h) > 0$ for any $n \geq 1$, we deduce that Wh^{-1} is a Lyapunov function for Q_h in the sense of Assumption 7 in Appendix A.

Remark 4. Let us mention that, in order for (69) to define a Lyapunov condition in the sense of Assumption 7, it is not necessary to have $\gamma_n \rightarrow 0$ as $n \rightarrow +\infty$. The existence of $n \geq 1$ such that $\gamma_n < \Lambda$ is sufficient.

We will now prove that: (i) Wh^{-1} has compact level sets, and (ii) Q_h satisfies Assumption 7 in Appendix A on any compact set K_n , that is $\inf_{K_n} Q_h$ is lower bounded by some probability measure. First, choosing $x_n \notin K_n$ in Assumption 1 leads to

$$\Lambda h(x_n) = (Q^f h)(x_n) \leq \gamma_n W(x_n),$$

so that

$$\frac{W(x_n)}{h(x_n)} \geq \frac{\Lambda}{\gamma_n}. \quad (70)$$

Since $\gamma_n \rightarrow 0$ as $n \rightarrow +\infty$, the function Wh^{-1} diverges outside the compact sets K_n defined in Assumption 1. In other words, Wh^{-1} has compact level sets, which shows (i).

Next, for $n \geq 1$, consider $\alpha_n > 0$ and $\eta_n \in \mathcal{P}(\mathcal{X})$ as in Assumption 2, so that, for any bounded measurable function $\varphi \geq 0$ and $x \in K_n$,

$$Q_h \varphi(x) = \Lambda^{-1} \frac{Q^f(h\varphi)(x)}{h(x)} \geq \frac{1}{\Lambda \sup_{K_n} h} \alpha_n \eta_n(h\varphi) \geq \tilde{\alpha}_n \tilde{\eta}_n(\varphi),$$

with

$$\tilde{\alpha}_n = \alpha_n \frac{\eta_n(h)}{\Lambda \sup_{K_n} h} > 0, \quad \tilde{\eta}_n(\varphi) = \frac{\eta_n(h\varphi)}{\eta_n(h)} \in \mathcal{P}(\mathcal{X}).$$

The latter expression is well-defined because, from Lemma 2, we know that $0 < \eta_n(h) < +\infty$ for any $n \geq 1$. Moreover, $0 < \sup_{K_n} h < +\infty$ (since $h \in B_W^\infty(\mathcal{X})$ and $\sup_{K_n} W < +\infty$ by Assumption 1), and this yields precisely (ii). Finally, (i) and (ii) show that Q_h satisfies Assumption 8, so that Q_h satisfies the assumptions of Theorem 5. As a result there exist a unique $\mu_h \in \mathcal{P}(\mathcal{X})$ and constants $c > 0$, $\bar{\alpha} \in (0, 1)$ such that for any $\phi \in B_{Wh^{-1}}^\infty(\mathcal{X})$,

$$\forall k \geq 0, \quad \left\| Q_h^k \phi - \mu_h(\phi) \right\|_{B_{Wh^{-1}}^\infty} \leq c \bar{\alpha}^k \left\| \phi - \mu_h(\phi) \right\|_{B_{Wh^{-1}}^\infty}.$$

Moreover, the measure μ_h satisfies $\mu_h(Wh^{-1}) < +\infty$.

F Proof of Lemma 4

From [28, Proposition 1], we obtain that $\mathcal{L} + f$ has a largest (in modulus) eigenvalue λ with associated smooth eigenvector h . Similarly, Lemma 2 shows that for any $\Delta t \in (0, \Delta t^*]$ the operator $Q_{\Delta t}^f$ has a largest (in modulus) eigenvalue $\Lambda_{\Delta t}$ with continuous eigenvector $h_{\Delta t}$ (since $Q_{\Delta t}^f$ is assumed to be strong Feller). Moreover, there is no restriction of generality in normalizing $h_{\Delta t}$ so that $\eta(h_{\Delta t}) = 1$.

We now turn to the estimate (56) on the spectral radius. In the notation of [28], we have $\Lambda_{\Delta t} = e^{\Delta t \lambda_{\Delta t}}$. A direct application of [28, Theorem 3] then shows that there exist $\Delta t^* > 0$ and $C > 0$ such that $\lambda_{\Delta t} = \lambda + \Delta t c_{\Delta t}$ with $|c_{\Delta t}| \leq C$ for $\Delta t \in (0, \Delta t^*]$, which is the desired result.

Finally, since $Q_{\Delta t}^f h_{\Delta t} = \Lambda_{\Delta t} h_{\Delta t}$, the lower bound (55) applied to $\varphi = h_{\Delta t} \geq 0$ leads to

$$\forall x \in \mathcal{X}, \quad (Q_{\Delta t}^f)^{\lceil \frac{T}{\Delta t} \rceil} h_{\Delta t}(x) = \Lambda_{\Delta t}^{\lceil \frac{T}{\Delta t} \rceil} h_{\Delta t}(x) \geq \alpha \eta(h_{\Delta t}).$$

Using the estimate on $\Lambda_{\Delta t}$ and the normalization $\eta(h_{\Delta t}) = 1$ we obtain, for $\Delta t \in (0, \Delta t^*]$ (possibly upon decreasing Δt^*) and $x \in \mathcal{X}$,

$$h_{\Delta t}(x) \geq \Lambda_{\Delta t}^{-\lceil \frac{T}{\Delta t} \rceil} \alpha \eta(h_{\Delta t}) \geq \alpha e^{-\Delta t(\lambda + \Delta t c_{\Delta t}) \lceil \frac{T}{\Delta t} \rceil} \geq \alpha e^{-2T|\lambda|}.$$

A similar computation leads to an analogous upper bound, which shows (57).

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