

BESOV CONTINUITY FOR GLOBAL OPERATORS ON COMPACT LIE GROUPS: THE CRITICAL CASE $p = q = \infty$.

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ABSTRACT. In this note, we study the mapping properties of global pseudo-differential operators with symbols in Ruzhansky-Turunen classes on Besov spaces $B_{\infty,\infty}^s(G)$. The considered classes satisfy Fefferman type conditions of limited regularity.

CONTENTS

1. Introduction	1
2. Pseudo-differential operators on compact Lie groups	3
2.1. Fourier analysis and Sobolev spaces on compact Lie groups	3
2.2. Differential and difference operators	5
2.3. Besov spaces	5
3. Global operators on $B_{\infty,\infty}^s(G)$ -spaces	5
References	8

1. INTRODUCTION

This note on the Besov boundedness of pseudo-differential operators on compact Lie groups in $B_{\infty,\infty}^s$ is based in the matrix-valued quantization procedure developed by M. Ruzhansky and V. Turunen in [20].

The Besov spaces $B_{p,q}^s$ arose from attempts to unify the various definitions of several fractional-order Sobolev spaces. By following the historical note of Grafakos [14, p. 113], we recall that Taibleson studied the generalized Hölder-Lipchitz spaces $\Lambda_{p,q}^s$ on \mathbb{R}^n , and these spaces were named after Besov spaces in honor to O. V. Besov who obtained a trace theorem and important embedding properties for them (see Besov [2, 3]). Dyadic decompositions for Besov spaces on \mathbb{R}^n were introduced by J. Peetre as well as other embedding properties (see Peetre [18, 19]). We will use the formulation of Besov spaces $B_{p,q}^s(G)$, trough of the representation theory of compact Lie groups G introduced and consistently developed by E. Nursultanov, M. Ruzhansky, and S. Tikhonov in [16, 17].

The present paper is a continuation of a series of our previous papers [4, 5, 6, 7], where were investigated the mapping properties of global operators (i.e., global pseudo-differential operators on compact Lie groups) on Besov spaces $B_{p,q}^s$, $-\infty <$

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$s < \infty$, $1 < p < \infty$ and $0 < q \leq \infty$. So, in this note we study the mapping properties for global operators in the case of Besov spaces $B_{\infty,\infty}^s$, $-\infty < s < \infty$.

The main tool in the proof of the Besov boundedness results presented in [4, 5, 6, 7], are the L^p -multipliers theorems proved in Ruzhansky and Wirth [21, 22], Delgado and Ruzhansky [10], Fischer [13] and Akylzhanov and Ruzhansky [1]. In general, such L^p -estimates, $1 < p < \infty$, cannot be extended to $L^\infty(G)$ and consequently we need to consider other techniques for the formulation of a boundedness result on $B_{\infty,\infty}^s$. Let us recall some L^p and Besov estimates for global operators, in order to announce our main theorem. First of all, we recall some notions on the global analysis of pseudo-differential operators.

If G is a compact Lie group and \widehat{G} is its unitary dual, that is the set of equivalence classes of continuous irreducible unitary representations of G , Ruzhansky-Turunen's approach associates to every bounded linear operator A on $C^\infty(G)$, a matrix valued symbol $\sigma_A(x, \xi)$ given by $\sigma_A(x, \xi) := \xi(x)^*(A\xi)(x)$, $x \in G$ and $\xi \in [\xi] \in \widehat{G}$. This allows us to write the operator A in terms of representations in G as

$$Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}(\xi(x)\sigma_A(x, \xi)\widehat{f}(\xi)), \quad (1.1)$$

for all $f \in C^\infty(G)$, where $\mathcal{F}(f) := \widehat{f}$ is the Fourier transform on the group G .

The Hörmander classes $\Psi_{\rho,\delta}^m(G)$, $m \in \mathbb{R}$, $\rho > \max\{\delta, 1-\delta\}$, where characterized in [20, 12] by the following condition: $A \in \Psi_{\rho,\delta}^m(G)$ if and only if its matrix-valued symbol $\sigma_A(x, \xi)$ satisfies the inequalities

$$\|\partial_x^\alpha \Delta^\beta \sigma_A(x, \xi)\|_{op} \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}, \quad (1.2)$$

for every $\alpha, \beta \in \mathbb{N}^n$. The discrete differential operator Δ^β (called difference operator of first order) is the main tool in this theory (see [20, 12]).

The L^p -mapping properties for global operators on compact Lie groups can be summarized as follows: if G is a compact Lie group and n is its dimension, \varkappa is the less integer larger than $\frac{n}{2}$ and $l := [n/p] + 1$, under one of the following conditions

- $\|\partial_x^\beta \mathbb{D}_\xi^\alpha \sigma_A(x, \xi)\|_{op} \leq C_\alpha \langle \xi \rangle^{-|\alpha|}$, for all $|\alpha| \leq \varkappa$, $|\beta| \leq l$ y $[\xi] \in \widehat{G}$, (Ruzhansky and Wirth [21, 22]),
- $\|\mathbb{D}_\xi^\alpha \partial_x^\beta \sigma_A(x, \xi)\|_{op} \leq C_{\alpha,\beta} \langle \xi \rangle^{-m-\rho|\alpha|+\delta|\beta|}$, $|\alpha| \leq \varkappa$, $|\beta| \leq l$, $m \geq \varkappa(1 - \rho)|\frac{1}{p} - \frac{1}{2}| + \delta l$, (Delgado and Ruzhansky [10]),
- $\|\mathbb{D}_\xi^\alpha \partial_x^\beta \sigma_A(x, \xi)\|_{op} \leq C_{\alpha,\beta} \langle \xi \rangle^{-\nu-\rho|\alpha|+\delta|\beta|}$, $\alpha \in \mathbb{N}^n$, $|\beta| \leq l$, $0 \leq \nu < \frac{n}{2}(1 - \rho)$, $|\frac{1}{p} - \frac{1}{2}| \leq \frac{\nu}{n}(1 - \rho)^{-1}$, (Delgado and Ruzhansky [10]),
- $\|\sigma_A\|_{\Sigma_s} := \sup_{[\xi] \in \widehat{G}} [\|\sigma_A(\xi)\|_{op} + \|\sigma_A(\xi)\eta(r^{-2}\mathcal{L}_G)\|_{\dot{H}^s(\widehat{G})}] < \infty$, (Fischer [13]),

the global operator $A \equiv T_a$ is bounded on $L^p(G)$. On the other hand, if $A : C^\infty(G) \rightarrow C^\infty(G)$ is a linear and bounded operator, then under any one of the following conditions

- $\|\mathbb{D}_\xi^\alpha \partial_x^\beta \sigma_A(x, \xi)\|_{op} \leq C_\alpha \langle \xi \rangle^{-|\alpha|}$, for all $|\alpha| \leq \varkappa$, $|\beta| \leq l$ and $[\xi] \in \widehat{G}$,
- $\|\mathbb{D}_\xi^\alpha \partial_x^\beta \sigma_A(x, \xi)\|_{op} \leq C_{\alpha,\beta} \langle \xi \rangle^{-\nu-\rho|\alpha|}$, $\alpha \in \mathbb{N}^n$, $|\beta| \leq l$, $|\frac{1}{p} - \frac{1}{2}| \leq \frac{\nu}{n}(1 - \rho)^{-1}$, $0 \leq \nu < \frac{n}{2}(1 - \rho)$,

- $\|\mathbb{D}_\xi^\alpha \partial_x^\beta \sigma_A(x, \xi)\|_{op} \leq C_{\alpha, \beta} \langle \xi \rangle^{-m - \rho|\alpha| + \delta|\beta|}$, $|\alpha| \leq \varkappa$, $|\beta| \leq l$, $m \geq \varkappa(1 - \rho)|\frac{1}{p} - \frac{1}{2}| + \delta l$,

the corresponding pseudo-differential operator A extends to a bounded operator from $B_{p,q}^r(G)$ into $B_{p,q}^r(G)$ for all $r \in \mathbb{R}$, $1 < p < \infty$ and $0 < q \leq \infty$, (see Cardona [4, 5, 6]). For $p, q = \infty$ we have the following theorem which is our main result in this paper.

Theorem 1.1. *Let G be a compact Lie group of dimension n . Let $0 < \rho \leq 1$, $k := [\frac{n}{2}] + 1$, and let $A : C^\infty(G) \rightarrow C^\infty(G)$ be a pseudo-differential operator with symbol σ satisfying*

$$\|\mathbb{D}_\xi^\alpha \sigma(x, \xi)\|_{op} \leq C_\alpha \langle \xi \rangle^{-\frac{n}{2}(1-\rho) - \rho|\alpha|} \quad (1.3)$$

for all $|\alpha| \leq k$. Then $A : B_{\infty, \infty}^s(G) \rightarrow B_{\infty, \infty}^s(G)$ extends to a bounded linear operator for all $-\infty < s < \infty$. Moreover,

$$\|A\|_{\mathcal{B}(B_{\infty, \infty}^s)} \leq C \sup\{C_\alpha : |\alpha| \leq k\}. \quad (1.4)$$

Theorem 1.1 implies the following result.

Corollary 1.2. *Let G be a compact Lie group of dimension n . Let $0 < \rho \leq 1$, $0 \leq \delta \leq 1$, $\ell \in \mathbb{N}$, $k := [\frac{n}{2}] + 1$, and let $A : C^\infty(G) \rightarrow C^\infty(G)$ be a pseudo-differential operator with symbol σ satisfying*

$$\|\partial_x^\beta \mathbb{D}_\xi^\alpha \sigma(x, \xi)\|_{op} \leq C_\alpha \langle \xi \rangle^{-m - \rho|\alpha| + \delta|\beta|} \quad (1.5)$$

for all $|\alpha| \leq k$, $|\beta| \leq \ell$. Then $A : B_{\infty, \infty}^s(G) \rightarrow B_{\infty, \infty}^s(G)$ extends to a bounded linear operator for all $-\infty < s < \infty$ provided that $m \geq \delta \ell + \frac{n}{2}(1 - \rho)$.

Besov spaces on graded Lie groups, as well as the action of Fourier multipliers and spectral multipliers on these spaces can be found in Cardona and Ruzhansky [8, 9]. We refer the reader to the references [11, 23] for boundedness properties of pseudo-differential operators in Besov spaces on \mathbb{R}^n . This paper is organized as follows. In the next section we present some basics on the calculus of global operators. Finally, in Section 3 we prove our Besov estimates.

2. PSEUDO-DIFFERENTIAL OPERATORS ON COMPACT LIE GROUPS

2.1. Fourier analysis and Sobolev spaces on compact Lie groups. In this section we will introduce some preliminaries on pseudo-differential operators on compact Lie groups and some of its properties on L^p -spaces. There are two notions of pseudo-differential operators on compact Lie groups. The first notion in the case of general manifolds (based on the idea of *local symbols* as in Hörmander [15]) and, in a much more recent context, the one of global pseudo-differential operators on compact Lie groups as defined by Ruzhansky and Turunen [20]. We adopt this last notion for our work, because we will use a description of the Besov spaces $B_{\infty, \infty}^s$ through representation theory. We will always equip a compact Lie group with the Haar measure μ_G . For simplicity, we will write $L^\infty(G)$ for $L^\infty(G, \mu_G)$. The following assumptions are respectively the Fourier transform and the Fourier

inversion formula for smooth functions,

$$\widehat{\varphi}(\xi) = \int_G \varphi(x) \xi(x)^* dx, \quad \varphi(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}(\xi(x) \widehat{\varphi}(\xi)).$$

The Peter-Weyl Theorem on G implies the Plancherel identity on $L^2(G)$,

$$\|\varphi\|_{L^2(G)} = \left(\sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}(\widehat{\varphi}(\xi) \widehat{\varphi}(\xi)^*) \right)^{\frac{1}{2}} = \|\widehat{\varphi}\|_{L^2(\widehat{G})}.$$

Here

$$\|A\|_{HS}^2 = \text{Tr}(AA^*),$$

denotes the Hilbert-Schmidt norm of matrices. Now, we introduce global pseudo-differential operators in the sense of Ruzhansky and Turunen. Any continuous linear operator A on G mapping $C^\infty(G)$ into $\mathcal{D}'(G)$ gives rise to a *matrix-valued global (or full) symbol* $\sigma_A(x, \xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ given by

$$\sigma_A(x, \xi) = \xi(x)^*(A\xi)(x), \quad (2.1)$$

which can be understood from the distributional viewpoint. Then it can be shown that the operator A can be expressed in terms of such a symbol as [20]

$$Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}[\xi(x) \sigma_A(x, \xi) \widehat{f}(\xi)]. \quad (2.2)$$

In this paper we use the notation $\text{Op}(\sigma_A) = A$. $L^p(\widehat{G})$ spaces on the unitary dual can be well defined. If $p = 2$, $L^2(\widehat{G})$ is defined by the norm

$$\|\Gamma\|_{L^2(\widehat{G})}^2 = \sum_{[\xi] \in \widehat{G}} d_\xi \|\Gamma(\xi)\|_{HS}^2.$$

Now, we want to introduce Sobolev spaces and, for this, we give some basic tools. Let $\xi \in \text{Rep}(G) := \cup \widehat{G} = \{\xi : [\xi] \in \widehat{G}\}$, if $x \in G$ is fixed, $\xi(x) : H_\xi \rightarrow H_\xi$ is a unitary operator and $d_\xi := \dim H_\xi < \infty$. There exists a non-negative real number $\lambda_{[\xi]}$ depending only on the equivalence class $[\xi] \in \widehat{G}$, but not on the representation ξ , such that $-\mathcal{L}_G \xi(x) = \lambda_{[\xi]} \xi(x)$; here \mathcal{L}_G is the Laplacian on the group G (in this case, defined as the Casimir element on G). Let $\langle \xi \rangle$ denote the function $\langle \xi \rangle = (1 + \lambda_{[\xi]})^{\frac{1}{2}}$.

Definition 2.1. For every $s \in \mathbb{R}$, the *Sobolev space* $H^s(G)$ on the Lie group G is defined by the condition: $f \in H^s(G)$ if only if $\langle \xi \rangle^s \widehat{f} \in L^2(\widehat{G})$. The Sobolev space $H^s(G)$ is a Hilbert space endowed with the inner product $\langle f, g \rangle_s = \langle \Lambda_s f, \Lambda_s g \rangle_{L^2(G)}$, where, for every $r \in \mathbb{R}$, $\Lambda_s : H^r \rightarrow H^{r-s}$ is the bounded pseudo-differential operator with symbol $\langle \xi \rangle^s I_\xi$. In L^p spaces, the p -Sobolev space of order s , $H^{s,p}(G)$, is defined by functions satisfying

$$\|f\|_{H^{s,p}(G)} := \|\Lambda_s f\|_{L^p(G)} < \infty. \quad (2.3)$$

2.2. Differential and difference operators. In order to classify symbols by its regularity we present the usual definition of differential operators and the difference operators used introduced in [20].

Definition 2.2. Let $(Y_j)_{j=1}^{\dim(G)}$ be a basis for the Lie algebra \mathfrak{g} of G , and let ∂_j be the left-invariant vector fields corresponding to Y_j . We define the differential operator associated to such a basis by $D_{Y_j} = \partial_j$ and, for every $\alpha \in \mathbb{N}^n$, the *differential operator* ∂_x^α is the one given by $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Now, if ξ_0 is a fixed irreducible representation, the matrix-valued *difference operator* is the given by $\mathbb{D}_{\xi_0} = (\mathbb{D}_{\xi_0, i, j})_{i, j=1}^{d_{\xi_0}} = \xi_0(\cdot) - I_{d_{\xi_0}}$. If the representation is fixed we omit the index ξ_0 so that, from a sequence $\mathbb{D}_1 = \mathbb{D}_{\xi_0, j_1, i_1}, \dots, \mathbb{D}_n = \mathbb{D}_{\xi_0, j_n, i_n}$ of operators of this type we define $\mathbb{D}_\xi^\alpha = \mathbb{D}_1^{\alpha_1} \cdots \mathbb{D}_n^{\alpha_n}$, where $\alpha \in \mathbb{N}^n$.

2.3. Besov spaces. We introduce the Besov spaces on compact Lie groups using the Fourier transform on the group G as follow.

Definition 2.3. Let $r \in \mathbb{R}$, $0 \leq q < \infty$ and $0 < p \leq \infty$. If f is a measurable function on G , we say that $f \in B_{p, q}^r(G)$ if f satisfies

$$\|f\|_{B_{p, q}^r} := \left(\sum_{m=0}^{\infty} 2^{mrq} \left\| \sum_{2^m \leq \langle \xi \rangle < 2^{m+1}} d_\xi \text{Tr}[\xi(x) \widehat{f}(\xi)] \right\|_{L^p(G)}^q \right)^{\frac{1}{q}} < \infty. \quad (2.4)$$

If $q = \infty$, $B_{p, \infty}^r(G)$ consists of those functions f satisfying

$$\|f\|_{B_{p, \infty}^r} := \sup_{m \in \mathbb{N}} 2^{mr} \left\| \sum_{2^m \leq \langle \xi \rangle < 2^{m+1}} d_\xi \text{Tr}[\xi(x) \widehat{f}(\xi)] \right\|_{L^p(G)} < \infty. \quad (2.5)$$

If we denote by $\text{Op}(\chi_m)$ the Fourier multiplier associated to the symbol

$$\chi_m(\eta) = 1_{\{[\xi]: 2^m \leq \langle \xi \rangle < 2^{m+1}\}}(\eta),$$

we also write,

$$\|f\|_{B_{p, q}^r} = \|\{2^{mr} \|\text{Op}(\chi_m) f\|_{L^p(G)}\}_{m=0}^{\infty}\|_{l^q(\mathbb{N})}, \quad 0 < p, q \leq \infty, r \in \mathbb{R}. \quad (2.6)$$

Remark 2.4. For every $s \in \mathbb{R}$, $H^{s, 2}(G) = H^s(G) = B_{2, 2}^s(G)$. Besov spaces according to Definition (2.3) were introduced in [16] on compact homogeneous manifolds where, in particular, the authors obtained its embedding properties. On compact Lie groups such spaces were characterized, via representation theory, in [17].

3. GLOBAL OPERATORS ON $B_{\infty, \infty}^s(G)$ -SPACES

In this section we prove our Besov estimate for global pseudo-differential operators. Our starting point is the following lemma which is slight variation of one due to J. Delgado and M. Ruzhansky (see Lemma 4.11 of [10]) and whose proof is verbatim to Delgado-Ruzhansky's proof.

Lemma 3.1. *Let G be a compact Lie group of dimension n . Let $0 < \rho \leq 1$, $k := [\frac{n}{2}] + 1$, and let $A : C^\infty(G) \rightarrow C^\infty(G)$ be a pseudo-differential operator with symbol σ satisfying*

$$\|\mathbb{D}_\xi^\alpha \sigma(x, \xi)\|_{op} \leq C_\alpha \langle \xi \rangle^{-\frac{n}{2}(1-\rho) - \rho|\alpha|} \quad (3.1)$$

for all $|\alpha| \leq k$. Let us assume that σ is supported in $\{\xi : R \leq \langle \xi \rangle \leq 2R\}$ for some $R > 0$. Then $A : L^\infty(G) \rightarrow L^\infty(G)$ extends to a bounded linear operator with norm operator independent of R . Moreover,

$$\|A\|_{\mathcal{B}(L^\infty)} \leq C \sup\{C_\alpha : |\alpha| \leq k\}. \quad (3.2)$$

So, we are ready for the proof of our main result.

Proof of Theorem 1.1. Our proof consists of two steps. In the first one, we prove the statement of the theorem for Fourier multipliers, i.e., pseudo-differential operators depending only on the Fourier variables ξ . Later, we extend the result for general global operators.

Step 1. Let us consider $\mathcal{R} := (I - \mathcal{L}_G)^{\frac{1}{2}}$ where \mathcal{L}_G is the Laplacian on G . Let us denote by ψ the characteristic function of the interval $I = [1/2, 1]$. Denote by ψ_l the function $\psi_l(t) = \psi(2^{-l}t)$, $t \in \mathbb{R}$. We will use the following characterization for $B_{\infty,\infty}^s(G) : f \in B_{\infty,\infty}^s(G)$, if and only if,

$$\|f\|_{B_{\infty,\infty}^s(G)} := \sup_{l \geq 0} 2^{ls} \|\psi_l(\mathcal{R})f\|_{L^\infty(G)} \quad (3.3)$$

where $\psi_l(\mathcal{R})$ is defined by the functional calculus associated to the self-adjoint operator \mathcal{R} . If $A \equiv \sigma(D_x)$ has a symbol depending only on the Fourier variables ξ , then

$$\|\sigma(D_x)f\|_{B_{\infty,\infty}^s(G)} := \sup_{l \geq 0} 2^{ls} \|\psi_l(\mathcal{R})\sigma(D_x)f\|_{L^\infty(G)}. \quad (3.4)$$

Taking into account that the operator $\sigma(D_x)$ commutes with $\psi_l(\mathcal{R})$ for every l , that

$$\psi_l(\mathcal{R})\sigma(D_x) = \sigma(D_x)\psi_l(\mathcal{R}) = \sigma_l(D_x)\psi_l(\mathcal{R}) \quad (3.5)$$

where $\sigma_l(D_x)$ is the pseudo-differential operator with matrix-valued symbol

$$\sigma_l(\xi) = \sigma(\xi) \cdot 1_{\{\xi : 2^{l-1} \leq \langle \xi \rangle \leq 2^{l+1}\}},$$

and that $\sigma_l(D_x)$ has a symbol supported in $\{\xi : 2^{l-1} \leq \langle \xi \rangle \leq 2^{l+1}\}$, by Lemma 3.1 we deduce that $\sigma_l(D_x)$ is a bounded operator on $L^\infty(G)$ with operator norm independent on l . In fact, σ_l satisfies the symbol inequalities

$$\|\mathbb{D}_\xi^\alpha \sigma_l(\xi)\|_{op} \leq C_\alpha \langle \xi \rangle^{-\frac{n}{2}(1-\rho) - \varepsilon|\alpha|},$$

for all $|\alpha| \leq k$, and consequently

$$\|\sigma_l(D_x)\|_{\mathcal{B}(L^\infty)} \leq C \sup\{C_\alpha : |\alpha| \leq k\}. \quad (3.6)$$

So, we have

$$\begin{aligned} & \|\psi_l(\mathcal{R})\sigma(D_x)f\|_{L^\infty(G)} \\ &= \|\sigma_l(D_x)\psi_l(\mathcal{R})f\|_{L^\infty(G)} \leq \|\sigma_l(D_x)\|_{\mathcal{B}(L^\infty)} \|\psi_l(\mathcal{R})f\|_{L^\infty(G)} \\ &\lesssim \sup\{C_\alpha : |\alpha| \leq k\} \|\psi_l(\mathcal{R})f\|_{L^\infty(G)}. \end{aligned}$$

As a consequence, we obtain

$$\begin{aligned} \|\sigma(D_x)f\|_{B_{\infty,\infty}^s(G)} &= \sup_{l \geq 0} 2^{ls} \|\psi_l(\mathcal{R})\sigma(D_x)f\|_{L^\infty(G)} \\ &\lesssim \sup\{C_\alpha : |\alpha| \leq k\} \sup_{l \geq 0} 2^{ls} \|\psi_l(\mathcal{R})f\|_{L^\infty(G)} \\ &\asymp \sup\{C_\alpha : |\alpha| \leq k\} \|f\|_{B_{\infty,\infty}^s(G)}. \end{aligned}$$

Step 2. Now, we extend the estimate from multipliers to pseudo-differential operators. So, let us define for every $z \in G$, the multiplier

$$\sigma_z(D_x)f(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}[\xi(x)\sigma(z, \xi)\widehat{f}(\xi)].$$

For every $x \in G$ we have the equality,

$$\sigma_x(D_x)f(x) = Af(x),$$

and we can estimate the Besov norm of the function $\sigma(x, D_x)f$, as follows

$$\begin{aligned} \|\sigma_x(D_x)f(x)\|_{B_{\infty,\infty}^s} &\asymp \sup_{l \geq 0} 2^{ls} \text{ess sup}_{x \in G} |\psi_l(\mathcal{R})\sigma(x, D_x)f(x)| \\ &= \sup_{l \geq 0} 2^{ls} \text{ess sup}_{x \in G} \left| \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}[\xi(x)\psi_l(\xi)\sigma(x, \xi)\widehat{f}(\xi)] \right| \\ &\leq \sup_{l \geq 0} 2^{ls} \text{ess sup}_{x \in G} \sup_{z \in G} \left| \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}[\xi(x)\psi_l(\xi)\sigma(z, \xi)\widehat{f}(\xi)] \right| \\ &= \sup_{l \geq 0} 2^{ls} \text{ess sup}_{x \in G} \sup_{z \in G} |\psi_l(\mathcal{R})\sigma_z(D_x)f(x)| \\ &= \sup_{l \geq 0} 2^{ls} \text{ess sup}_{x \in G} \sup_{z \in G} |\sigma_z(D_x)\psi_l(\mathcal{R})f(x)| \\ &\leq \sup_{l \geq 0} 2^{ls} \text{ess sup}_{x \in G} \sup_{z \in G} \text{ess sup}_{\varkappa \in G} |\sigma_z(D_\varkappa)\psi_l(\mathcal{R})f(\varkappa)| \\ &= \sup_{l \geq 0} 2^{ls} \sup_{z \in G} \|\sigma_z(D_\varkappa)\psi_l(\mathcal{R})f(\varkappa)\|_{L^\infty(G)}. \end{aligned}$$

From the estimate for the operator norm of multipliers proved in the first step, we deduce

$$\sup_{z \in G} \|\sigma_z(D_x)\psi_l(\mathcal{R})f\|_{L^\infty(G)} \lesssim \sup\{C_\alpha : |\alpha| \leq k\} \|\psi_l(\mathcal{R})f\|_{L^\infty(G)}.$$

So, we have

$$\|\sigma_x(D_x)f(x)\|_{B_{\infty,\infty}^s} \lesssim \sup\{C_\alpha : |\alpha| \leq k\} \|f\|_{B_{\infty,\infty}^s(G)}. \quad (3.7)$$

Thus, we finish the proof. \square

Now, we present the following result for symbols admitting differentiability in the spatial variables.

Corollary 3.2. *Let G be a compact Lie group of dimension n . Let $0 < \rho \leq 1$, $0 \leq \delta \leq 1$, $\ell \in \mathbb{N}$, $k := \lfloor \frac{n}{2} \rfloor + 1$, and let $A : C^\infty(G) \rightarrow C^\infty(G)$ be a pseudo-differential operator with symbol σ satisfying*

$$\|\partial_x^\beta \mathbb{D}_\xi^\alpha \sigma(x, \xi)\|_{op} \leq C_\alpha \langle \xi \rangle^{-m-\rho|\alpha|+\delta|\beta|} \quad (3.8)$$

for all $|\alpha| \leq k$, $|\beta| \leq \ell$. Then $A : B_{\infty, \infty}^s(G) \rightarrow B_{\infty, \infty}^s(G)$ extends to a bounded linear operator for all $-\infty < s < \infty$ provided that $m \geq \delta\ell + \frac{n}{2}(1 - \rho)$.

Proof. Let us observe that

$$\langle \xi \rangle^{-m-\rho|\alpha|+\delta|\beta|} \leq \langle \xi \rangle^{-\frac{n}{2}(1-\rho)+\rho|\alpha|} \quad (3.9)$$

when $m \geq \delta\ell + \frac{n}{2}(1 - \rho)$. So, we finish the proof if we apply Theorem 1.1. \square

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