ON A RESOLUTION OF SINGULARITIES WITH TWO STRATA

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ABSTRACT. Let X be a complex, irreducible, quasi-projective variety, and $\pi : \widetilde{X} \to X$ a resolution of singularities of X. Assume that the singular locus $\mathrm{Sing}(X)$ of X is smooth, that the induced map $\pi^{-1}(\mathrm{Sing}(X)) \to \mathrm{Sing}(X)$ is a smooth fibration admitting a cohomology extension of the fiber, and that $\pi^{-1}(\mathrm{Sing}(X))$ has a negative normal bundle in $\widetilde{X}$. We present a very short and explicit proof of the Decomposition Theorem for π, providing a way to compute the intersection cohomology of X by means of the cohomology of $\widetilde{X}$ and of $\pi^{-1}(\mathrm{Sing}(X))$. Our result applies to special Schubert varieties with two strata, even if π is non-small. And to certain hypersurfaces of $\mathbb{P}^5$ with one-dimensional singular locus.

Keywords: Projective variety, Smooth fibration, Resolution of singularities, Derived category, Intersection cohomology, Decomposition Theorem, Poincaré polynomial, Betti numbers, Schubert varieties.

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1. Introduction

The Decomposition Theorem is a beautiful and very deep result about algebraic maps. In the words of MacPherson "it contains as special cases the deepest homological properties of algebraic maps that we know"[21], [27]. In literature, one can find different approaches to the Decomposition Theorem [1], [6], [7], [23], [27]. Let us say they have in common a fairly heavy formalism, that may discourage the reader to the point that the Decomposition Theorem is often used like a "black box" by many authors [17], [3]. Furthermore, it is often very difficult to calculate the intersection cohomology of a singular algebraic variety, unless either the singular locus if finite, or the variety admits a small resolution with known Betti numbers.

However, there are many special cases for which the Decomposition Theorem admits a simplified approach. One of these is the case of varieties with isolated singularities. This is a key point also in the general case since, as observed in [27, Remark 2.14], the proof of the Decomposition Theorem proceeds by induction on the dimension of the strata of the singular locus.

For instance, in our previous work [10], we reduced the proof of the Decomposition Theorem for varieties with isolated singularities, to the vanishing of certain maps between ordinary cohomology groups [10, Theorem 3.1]. This in turn is related with the existence of a "natural Gysin morphism". By a natural Gysin morphism we mean a topological bivariant class [14, p. 83], [5], [9]:

$$\theta \in T^0(\widetilde{X} \stackrel{\pi}{\to} X) = Hom_{D^b(X)}(R\pi_*\mathbb{Q}_{\widetilde{X}}, \mathbb{Q}_X),$$

commuting with restrictions to the smooth locus of X (here $\pi : \widetilde{X} \to X$ is a resolution of singularities of X with isolated singularities). In [10, Theorem 1.2], we gave a complete characterization of morphisms like π admitting a natural Gysin morphism, providing a relationship between the Decomposition Theorem and Bivariant Theory. In fact, π admits a natural Gysin morphism if and only if X is a $\mathbb{Q}$-intersection cohomology manifold, i.e. $IC^\bullet_X \cong \mathbb{Q}_X[n]$ in $D^b(X)$ ($IC^\bullet_X$ denotes the intersection cohomology complex of X [11, p. 156], [22]). In this case, there is a unique natural Gysin morphism θ, and it arises from the Decomposition Theorem.

Our aim in this work is to develop another case for which the Decomposition Theorem admits a simplified approach. More precisely, we assume X to be a complex, irreducible, quasi-projective variety of dimension $n \geq 1$, and

$$\pi : \widetilde{X} \to X$$

a resolution of singularities of X. Moreover, we assume that the singular locus $\mathrm{Sing}(X)$ of X is smooth, and that the induced fibre square diagram:

$$\begin{array}{ccc} \pi^{-1}(\mathrm{Sing}(X)) & \hookrightarrow & \widetilde{X} \\ \downarrow & & {}^{\pi}\!\downarrow \\ \mathrm{Sing}(X) & \hookrightarrow & X \end{array}$$

is such that $\pi^{-1}(\mathrm{Sing}(X)) \to \mathrm{Sing}(X)$ is a smooth fibration, with negative normal bundle, admitting a cohomology extension of the fiber (see Notations, (*iii*), below, for a precise statement of our assumptions).

Our main result is a very short and explicit proof of the Decomposition Theorem (compare with Theorem 3.1), providing a way to compute the intersection cohomology of X by means of the cohomology of $\widetilde{X}$ and of $\pi^{-1}(\mathrm{Sing}(X))$ (Corollary 3.2). In the last two sections, we apply our main result. First, to *special Schubert varieties with two strata*, for which it is known to exist both a small and a non-small resolution. Comparing our computation of the intersection cohomology by means of Corollary 3.2, with the one given in [4], we find some polynomial identities apparently not known so far (Remark 4.2, (*ii*)). Next, we compute the intersection cohomology of certain hypersurfaces of $\mathbb{P}^5$ with one-dimensional singular locus. As far as we know, Corollary 5.2 is completely new.

2. Notations

(*i*) All cohomology and intersection cohomology groups are with $\mathbb{Q}$-coefficients.

(*ii*) Let Y be a complex, possible reducible, quasi-projective variety. We denote by $H^\alpha(Y)$ and $IH^\alpha(Y)$ its cohomology and intersection cohomology groups ($\alpha \in \mathbb{Z}$). Let $D^b(Y)$ be the bounded derived category of sheaves of $\mathbb{Q}$-vector spaces on Y. Let $\mathcal{F}^\bullet \in D^b(Y)$ be a complex of sheaves. We denote by $\mathcal{H}^\alpha(\mathcal{F}^\bullet)$ its cohomology sheaves, and by $\mathbb{H}^\alpha(\mathcal{F}^\bullet)$ its hypercohomology groups. Let $IC^\bullet_Y$ be the intersection cohomology complex of Y. If Y is irreducible, we have $IH^\alpha(Y) =$

$\mathcal{H}^\alpha(IC^\bullet_Y[-\dim Y])$. If Y is irreducible and nonsingular, and $\mathbb{Q}_Y$ is the constant sheaf $\mathbb{Q}$ on Y, we have $IC^\bullet_Y \cong \mathbb{Q}_Y[\dim_{\mathbb{C}} Y]$.

(*iii*) Let X be a complex, irreducible, quasi-projective variety of dimension $n \geq 1$, and

$$\pi : \widetilde{X} \to X$$

a resolution of singularities of X. This means that π is a projective, surjective, birational morphism, such that $\widetilde{X}$ is irreducible and nonsingular. Fix a closed, nonsingular subvariety Δ of X, of pure dimension m. Consider the induced fibre square commutative diagram:

$$\begin{array}{ccc} \widetilde{\Delta} & \stackrel{\jmath}{\hookrightarrow} & \widetilde{X} \\ {\scriptstyle\rho}\downarrow & & {\scriptstyle\pi}\downarrow \\ \Delta & \stackrel{\imath}{\hookrightarrow} & X, \end{array} \tag{1}$$

where $\widetilde{\Delta} = \pi^{-1}(\Delta)$, $\imath$ and $\jmath$ are the inclusion maps, and ρ the restriction of π. We make the following assumptions $(a1)$, $(a2)$, and $(a3)$:

$(a1)$ $\mathrm{Sing}(X) \subseteq \Delta$, and the induced map $\pi^{-1}(X \backslash \Delta) \to X \backslash \Delta$ is an isomorphism.

$(a2)$ $\widetilde{\Delta}$ is nonsingular, of pure dimension $m + p$, and the map $\rho : \widetilde{\Delta} \to \Delta$ is a smooth fibration, with fiber say G, well-defined up to diffeomorphisms, such that the restriction map $H^\alpha(\widetilde{\Delta}) \to H^\alpha(G)$ is onto for all $\alpha \in \mathbb{Z}$.

In view of previous assumptions, the fiber G is a projective variety, nonsingular, purely dimensional, of dimension p. Let N be the normal bundle of $\widetilde{\Delta}$ in $\widetilde{X}$, and set

$$q := n - m - p$$

its rank. Let $\overline{c} \in H^{2q}(\widetilde{\Delta})$ be the top Chern class of N, and let $c \in H^{2q}(G)$ be the restriction of $\overline{c}$ to G.

$(a3)$ The map $H^\alpha(G) \stackrel{\cdot \cup c}{\longrightarrow} H^{\alpha+2q}(G)$, determined by cup-product with c, is onto for all integers $\alpha \geq p - q$.

Remark 2.1. Combining the Universal Coefficient Theorem with the Poincaré Duality Theorem, it follows that condition $(a3)$ is equivalent to require that the map $H^\alpha(G) \stackrel{\cdot \cup c}{\longrightarrow} H^{\alpha+2q}(G)$ is injective for all integers $\alpha \leq p-q$. Notice that if $p-q < 0$, then condition $(a3)$ is satisfied. In fact, in this case, we have $H^{\alpha+2q}(G) = 0$ for all $\alpha \geq p - q$ (and $H^\alpha(G) = 0$ for all $\alpha \leq p - q$). Moreover, in view of the Hard Lefschetz Theorem for ample bundles, the condition $(a3)$ is satisfied when the restriction of N to G, either of $N^\vee$ to G, is ample [20, p. 69].

(*iv*) For all $\alpha \in \mathbb{Z}$, set $A^\alpha := H^\alpha(G)$, and $a^\alpha := \dim H^\alpha(G)$. When $\alpha \leq p-q$, set $B^{\alpha+2q} := \Im(H^\alpha(G) \stackrel{\cdot \cup c}{\longrightarrow} H^{\alpha+2q}(G)) \cong H^\alpha(G)$. When $\alpha \geq p - q$, choose a subspace $C^\alpha \subseteq H^\alpha(G)$ such that $C^\alpha \cong H^{\alpha+2q}(G)$ via $H^\alpha(G) \stackrel{\cdot \cup c}{\longrightarrow} H^{\alpha+2q}(G)$. Observe that by the Universal Coefficient Theorem and the Poincaré Duality Theorem, it follows that

$$A^{p-q-\alpha} \cong C^{p-q+\alpha} \quad \text{for every } \alpha \geq 0. \tag{2}$$

Remark 2.2. By previous assumption ($a2$), there exists a *cohomology extension* $H^*(G) \stackrel{\theta}{\to} H^*(\widetilde{\Delta})$ *of the fiber* [24, p. 256-258]. By the Leray-Hirsch Theorem [26, p. 182 and p. 195], [8, Lemma 2.5 and proof], it determines a decomposition in $D^b(\Delta)$:

$$\sum_{\alpha=0}^{2p} A^\alpha \otimes \mathbb{Q}_\Delta[-\alpha] \stackrel{\theta}{\cong} R\rho_*\mathbb{Q}_{\widetilde{\Delta}}. \tag{3}$$

(v) We define the following complex $\mathcal{F}^\bullet$ in $D^b(X)$. It will appear in the claim of Theorem 3.1 below. When $p-q\geq 0$, we set:

$$\mathcal{F}^\bullet := \left(\sum_{\alpha=0}^{p-q} A^\alpha \otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[n-2q-\alpha]\right) \oplus \left(\sum_{\alpha=p-q+1}^{2p-2q} C^\alpha \otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[n-2q-\alpha]\right).$$

When $p-q<0$, we set $\mathcal{F}^\bullet := 0$. If $p-q=0$, then we simply have

$$\mathcal{F}^\bullet = H^0(G)\otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[n-2q].$$

(vi) We denote by $IH_X(t)$ the Poincaré polynomial of the intersection cohomology of X, i.e.

$$IH_X(t) := \sum_{\alpha\in\mathbb{Z}} \dim_{\mathbb{Q}} IH^\alpha(X)\, t^\alpha.$$

We denote by $H_{\widetilde{X}}(t)$ and $H_\Delta(t)$ the Poincaré polynomials of the cohomology of $\widetilde{X}$ and Δ, i.e.

$$H_{\widetilde{X}}(t) := \sum_{\alpha\in\mathbb{Z}} \dim_{\mathbb{Q}} H^\alpha(\widetilde{X})\, t^\alpha, \quad \text{and} \quad H_\Delta(t) := \sum_{\alpha\in\mathbb{Z}} \dim_{\mathbb{Q}} H^\alpha(\Delta)\, t^\alpha.$$

(vii) We define the polynomial $g(t)$ as follows. First, when $p-q\geq 0$, we set:

$$r(t) := \frac{1}{2}a^{p-q}t^{p-q} + \sum_{0\leq\alpha\leq p-q-1} a^\alpha t^\alpha.$$

Next we define:

$$g(t) := t^{2q}r(t) + t^{2p}r\left(t^{-1}\right).$$

When $p-q<0$, we set $g(t):=0$. When $p-q=0$ and G is connected, we simply have $g(t)=t^{2p}$. We denote by $f(t)$ the Poincaré polynomial of the complex $\mathcal{F}^\bullet[-n]$, i.e.

$$f(t) := \sum_{\alpha\in\mathbb{Z}} \mathbf{h}^\alpha(\mathcal{F}^\bullet[-n])\, t^\alpha,$$

where $\mathbf{h}^\alpha(\mathcal{F}^\bullet[-n]) := \dim_{\mathbb{Q}} \mathbb{H}^\alpha(\mathcal{F}^\bullet[-n])$ denotes the dimension of the hypercohomology.

($viii$) Let $\mathbb{G}_k(\mathbb{C}^l)$ denote the Grassmann variety of k planes in $\mathbb{C}^l$ (compare with [4, p. 328]). Recall that

$$\dim \mathbb{G}_k(\mathbb{C}^l) = k(l-k),$$

and that the Poincaré polynomial

$$Q_k^l := Q_k^l(t) := \sum_{\alpha\in\mathbb{Z}} \dim H^\alpha(\mathbb{G}_k(\mathbb{C}^l))t^\alpha$$

of $\mathbb{G}_k(\mathbb{C}^l)$ is equal to

$$Q_k^l = \frac{P_l}{P_k P_{l-k}}, \tag{4}$$

where, for every integer $\alpha \geq 0$, we set:

$$P_\alpha := P_\alpha(t) := h_0 \cdot h_1 \cdot \dots \cdot h_{\alpha-1} \quad \text{and} \quad h_\alpha := h_\alpha(t) := 1 + t^2 + t^4 + \dots + t^{2\alpha}$$

(assume that $P_0 = 1$, and notice that $P_1 = 1$).

3. The main results

We are in position to state our main results. We keep the notations stated before, together the assumptions $(a1)$, $(a2)$, and $(a3)$.

Theorem 3.1. *In $D^b(X)$ we have a decomposition:*

$$R\pi_*\mathbb{Q}_{\widetilde{X}}[n] \cong IC_X^\bullet \oplus \mathcal{F}^\bullet.$$

Corollary 3.2.

$$IH_X(t) = H_{\widetilde{X}}(t) - H_\Delta(t)g(t).$$

In order to prove our results, we need the following:

Lemma 3.3. (a) *$\mathcal{F}^\bullet$ is self-dual in $D^b(X)$.*

(b) *$\mathcal{F}^\bullet$ is a direct summand of $R\pi_*\mathbb{Q}_{\widetilde{X}}[n]$ in $D^b(X)$.*

(c) *For every $x \in \Delta$ and $\alpha \geq -m$, one has*

$$\mathcal{H}^\alpha\left(R\pi_*\mathbb{Q}_{\widetilde{X}}[n]\right)_x \cong \mathcal{H}^\alpha\left(\mathcal{F}^\bullet\right)_x \cong A^{\alpha+n}.$$

(d) *$f(t) = H_\Delta(t)g(t)$.*

Proof of the Lemma. (a) If we set $\beta = p - q - \alpha$, we have:

$$\sum_{\alpha=0}^{p-q-1} A^\alpha \otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[n-2q-\alpha] = \sum_{\beta=1}^{p-q} A^{p-q-\beta} \otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[m+\beta].$$

On the other hand, setting $\beta = \alpha - (p-q)$, we have:

$$\sum_{\alpha=p-q+1}^{2p-2q} C^\alpha \otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[n-2q-\alpha] = \sum_{\beta=1}^{p-q} C^{p-q+\beta} \otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[m-\beta].$$

By (2), we deduce:

$$\mathcal{F}^\bullet \cong \left(A^{p-q} \otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[m]\right) \oplus \left(\sum_{\beta=1}^{p-q} A^{p-q-\beta} \otimes_{\mathbb{Q}} R\imath_*\left(\mathbb{Q}_\Delta[m+\beta] \oplus \mathbb{Q}_\Delta[m-\beta]\right)\right).$$

Taking into account that $\mathbb{Q}_\Delta[m]$ is self-dual in $D^b(\Delta)$, previous formula shows $\mathcal{F}^\bullet$ as a direct sum of self-dual complexes (compare with [11, p. 69, Proposition 3.3.7 (ii), and Remark 3.3.6 (i)]).

(b) Consider the commutative diagram (1). The inclusion $\widetilde{\Delta} \overset{\jmath}{\hookrightarrow} \widetilde{X}$ gives rise a pull-back morphism $\mathbb{Q}_{\widetilde{X}} \to R\jmath_*\mathbb{Q}_{\widetilde{\Delta}}$ in $D^b(\widetilde{X})$ [26, p. 176]. On the other hand, since $\widetilde{\Delta}$ is smooth (Notations, (iii), $(a2)$), the inclusion $\widetilde{\Delta} \overset{\jmath}{\hookrightarrow} \widetilde{X}$ induces also a Gysin morphism $R\jmath_*\mathbb{Q}_{\widetilde{\Delta}} \to \mathbb{Q}_{\widetilde{X}}[2q]$ [14, p. 83]. So, we have the following sequence of morphisms:

$$R\jmath_*\mathbb{Q}_{\widetilde{\Delta}} \to \mathbb{Q}_{\widetilde{X}}[2q] \to R\jmath_*\mathbb{Q}_{\widetilde{\Delta}}[2q].$$

Composing with π, and taking into account that diagram (1) commutes, we deduce the sequence in $D^b(X)$:

$$R(\imath\circ\rho)_*\mathbb{Q}_{\widetilde{\Delta}} \to R\pi_*\mathbb{Q}_{\widetilde{X}}[2q] \to R(\imath\circ\rho)_*\mathbb{Q}_{\widetilde{\Delta}}[2q]. \tag{5}$$

The idea of the proof consists in using the Leray-Hirsch decomposition (3), and the self-intersection formula [14, p. 92], [12], [25], in order to identify the image of $\mathcal{F}^\bullet[-n+2q]$ via the composite morphism $R(\imath\circ\rho)_*\mathbb{Q}_{\widetilde{\Delta}} \to R(\imath\circ\rho)_*\mathbb{Q}_{\widetilde{\Delta}}[2q]$ given by (5).

More precisely, consider the complex $\mathcal{F}^\bullet$ (Notations, (v)). We may write:

$$\mathcal{F}^\bullet[-n+2q] = \left(\sum_{\alpha=0}^{p-q} A^\alpha \otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[-\alpha]\right) \oplus \left(\sum_{\alpha=p-q+1}^{2p-2q} C^\alpha \otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[-\alpha]\right).$$

By the Leray-Hirsch Theorem (3), we have:

$$\sum_{\alpha=0}^{2p} A^\alpha \otimes R\imath_*\mathbb{Q}_\Delta[-\alpha] \overset{\theta}{\cong} R(\imath\circ\rho)_*\mathbb{Q}_{\widetilde{\Delta}}.$$

Since $C^\alpha \subseteq A^\alpha$ (Notations, (iv)), we deduce a morphism:

$$\mathcal{F}^\bullet[-n+2q] \to R(\imath\circ\rho)_*\mathbb{Q}_{\widetilde{\Delta}},$$

and by (5) we get a sequence:

$$\mathcal{F}^\bullet[-n+2q] \to R\pi_*\mathbb{Q}_{\widetilde{X}}[2q] \to R(\imath\circ\rho)_*\mathbb{Q}_{\widetilde{\Delta}}[2q] \cong \left(\sum_{\alpha=0}^{2p} A^\alpha \otimes R\imath_*\mathbb{Q}_\Delta[-\alpha]\right)[2q].$$

By the self-intersection formula, and the assumption $(a3)$ (compare also with Notations, (iv)), the composite of these morphisms sends $\mathcal{F}^\bullet[-n+2q]$ isomorphically onto a subcomplex $\mathcal{G}^\bullet$ of

$$\left(\sum_{\alpha=0}^{2p} A^\alpha \otimes R\imath_*\mathbb{Q}_\Delta[-\alpha]\right)[2q] = \sum_{\alpha=-2q}^{2p-2q} A^{\alpha+2q} \otimes R\imath_*\mathbb{Q}_\Delta[-\alpha],$$

which, up to change the cohomology extension θ, identifies with:

$$\mathcal{G}^\bullet \cong \left(\sum_{\alpha=0}^{p-q} B^{\alpha+2q} \otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[-\alpha]\right) \oplus \left(\sum_{\alpha=p-q+1}^{2p-2q} A^{\alpha+2q} \otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_\Delta[-\alpha]\right).$$

It follows that the morphism $\mathcal{F}^\bullet[-n+2q] \to R\pi_*\mathbb{Q}_{\widetilde{X}}[2q]$ has a section. Therefore $\mathcal{F}^\bullet[-n+2q]$ is a direct summand of $R\pi_*\mathbb{Q}_{\widetilde{X}}[2q]$, i.e. $\mathcal{F}^\bullet$ is a direct summand of $R\pi_*\mathbb{Q}_{\widetilde{X}}[n]$.

(c) By [11, Theorem 2.3.26, (i), p. 41], it follows that

$$\mathcal{H}^\alpha\left(R\pi_*\mathbb{Q}_{\widetilde{X}}[n]\right)_x \cong A^{\alpha+n}$$

for every $x \in \Delta$ and every $\alpha \in \mathbb{Z}$. On the other hand, for every $\alpha, \beta \in \mathbb{Z}$, one has:

$$\mathcal{H}^{\alpha}\left(R\imath_*\mathbb{Q}_{\Delta}[n-2q-\beta]\right)_x \cong \begin{cases} \mathbb{Q} & \text{if } \alpha+n-2q-\beta=0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that, for every $x \in \Delta$ and every $\alpha \geq -m$, one has:

$$\mathcal{H}^{\alpha}\left(\mathcal{F}^{\bullet}\right)_x \cong \begin{cases} A^{p-q} & \text{if } \alpha=-m \\ C^{\alpha+n-2q} & \text{if } \alpha>-m. \end{cases}$$

We are done because, by (Notations, (iv)) and (2), in the case $\alpha=-m$, we have: $A^{p-q} \cong C^{p-q} \cong A^{p+q} = A^{-m+n} = A^{\alpha+n}$, and, in the case $\alpha > -m$, we have $C^{\alpha+n-2q} \cong A^{\alpha+n}$.

(d) First we analyze the summand on the left of $\mathcal{F}^{\bullet}[-n]$. Set

$$h^{\alpha}(\Delta)=\dim_{\mathbb{Q}} H^{\alpha}(\Delta).$$

We have:

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{Z}} \mathbf{h}^{\alpha}\left(\sum_{\beta=0}^{p-q} A^{\beta}\otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_{\Delta}[-2q-\beta]\right)t^{\alpha} \\
&=\sum_{\alpha\in\mathbb{Z}}\left(\sum_{\beta=0}^{p-q} a^{\beta}\cdot h^{\alpha-\beta-2q}(\Delta)\right)t^{\alpha} \\
&=\sum_{\alpha\in\mathbb{Z}}\left(\sum_{\beta=0}^{p-q}\left(a^{\beta}t^{\beta+2q}\right)\cdot\left(h^{\alpha-\beta-2q}(\Delta)t^{\alpha-\beta-2q}\right)\right) \\
&=\sum_{\beta=0}^{p-q}\left(\sum_{\alpha\in\mathbb{Z}}\left(a^{\beta}t^{\beta+2q}\right)\cdot\left(h^{\alpha-\beta-2q}(\Delta)t^{\alpha-\beta-2q}\right)\right) \\
&=\left(\sum_{\beta=0}^{p-q} a^{\beta}t^{\beta+2q}\right)H_{\Delta}(t).
\end{aligned}$$

As for the summand on the right of $\mathcal{F}^{\bullet}[-n]$, we have:

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{Z}} \mathbf{h}^{\alpha}\left(\sum_{\beta=p-q+1}^{2p-2q} A^{\beta+2q}\otimes_{\mathbb{Q}} R\imath_*\mathbb{Q}_{\Delta}[-2q-\beta]\right)t^{\alpha} \\
&=\sum_{\alpha\in\mathbb{Z}}\left(\sum_{\beta=p-q+1}^{2p-2q} a^{\beta+2q}\cdot h^{\alpha-\beta-2q}(\Delta)\right)t^{\alpha} \\
&=\sum_{\alpha\in\mathbb{Z}}\left(\sum_{\beta=p-q+1}^{2p-2q}\left(a^{\beta+2q}t^{\beta+2q}\right)\cdot\left(h^{\alpha-\beta-2q}(\Delta)t^{\alpha-\beta-2q}\right)\right) \\
&=\sum_{\beta=p-q+1}^{2p-2q}\left(\sum_{\alpha\in\mathbb{Z}}\left(a^{\beta+2q}t^{\beta+2q}\right)\cdot\left(h^{\alpha-\beta-2q}(\Delta)t^{\alpha-\beta-2q}\right)\right) \\
&=\left(\sum_{\beta=p-q+1}^{2p-2q} a^{\beta+2q}t^{\beta+2q}\right)H_{\Delta}(t).
\end{aligned}$$

Putting together we get:

$$f(t)=\left(\sum_{\beta=0}^{p-q}a^{\beta}t^{\beta+2q}+\sum_{\beta=p-q+1}^{2p-2q}a^{\beta+2q}t^{\beta+2q}\right)H_{\Delta}(t).$$

Now we notice that:

$$\sum_{\beta=0}^{p-q}a^{\beta}t^{\beta+2q}=t^{2q}\left(\sum_{\beta=0}^{p-q}a^{\beta}t^{\beta}\right).$$

On the other hand, when $p-q+1\leq\beta\leq 2p-2q$, by the Poincaré Duality Theorem, we have:

$$a^{\beta+2q}t^{\beta+2q}=\frac{a^{\beta+2q}}{t^{2p-2q-\beta}}t^{2p}=\frac{a^{2p-2q-\beta}}{t^{2p-2q-\beta}}t^{2p},$$

with $0\leq 2p-2q-\beta\leq p-q-1$. Therefore, we have:

$$\sum_{\beta=p-q+1}^{2p-2q}a^{\beta+2q}t^{\beta+2q}=t^{2p}\left(\sum_{\beta=0}^{p-q-1}\frac{a^{\beta}}{t^{\beta}}\right).$$

It follows that

$$f(t)=\left[t^{2q}\left(\sum_{\beta=0}^{p-q}a^{\beta}t^{\beta}\right)+t^{2p}\left(\sum_{\beta=0}^{p-q-1}\frac{a^{\beta}}{t^{\beta}}\right)\right]H_{\Delta}(t)=g(t)H_{\Delta}(t).$$

□

We are in position to prove Theorem 3.1 and Corollary 3.2.

Proof of Theorem 3.1. By Lemma 3.3, (b), there exists a complex $\mathcal{K}^{\bullet}$ such that

(6) $$R\pi_{*}\mathbb{Q}_{\widetilde{X}}[n]\cong\mathcal{K}^{\bullet}\oplus\mathcal{F}^{\bullet}.$$

Therefore, we only have to prove that:

$$\mathcal{K}^{\bullet}\cong IC^{\bullet}_{X}.$$

Observe that $\mathcal{K}^{\bullet}$ is self-dual, because, by [11, p. 69, Proposition 3.3.7] and Lemma 3.3, (a), so are both $R\pi_{*}\mathbb{Q}^{\bullet}_{\widetilde{X}}[n]$ and $\mathcal{F}^{\bullet}$. Now set $U:=X\backslash\Delta$, and denote by $\jmath_U:U\hookrightarrow X$ the inclusion. Since the complex $\mathcal{F}^{\bullet}$ is supported on Δ, by (Notations, (iii), $(a1)$), it follows that the restriction $(\jmath_U)^{-1}\mathcal{K}^{\bullet}$ of $\mathcal{K}^{\bullet}$ to U is $\mathbb{Q}_U[n]$. Moreover, by (6), we have $\mathcal{K}^{\bullet}\in D^b_c(X)$ [11]. Therefore, $\mathcal{K}^{\bullet}$ is an extension of $\mathbb{Q}_U[n]$. Hence, to prove that $\mathcal{K}^{\bullet}\cong IC^{\bullet}_X$, it suffices to prove that $\mathcal{K}^{\bullet}\cong(\jmath_U)_{!*}\mathbb{Q}_U[n]$, i.e. that $\mathcal{K}^{\bullet}$ is the intermediary extension of $\mathbb{Q}_U[n]$ [11, p.156 and p.135]. Taking into account that $\mathcal{K}^{\bullet}$ is self-dual, this in turn reduces to prove that, for every $x\in\Delta$ and every $\alpha\geq -m$, one has $\mathcal{H}^{\alpha}(\mathcal{K}^{\bullet})_x=0$ [11, Proposition 5.2.8., p.135, and Remark 5.4.2., p. 156]. This follows from (6), and Lemma 3.3, (c). □

Proof of Corollary 3.2. It follows from Theorem 3.1 and Lemma 3.3, (d), taking into account that $IH^{\alpha}(X)=\mathbb{H}^{\alpha}(IC^{\bullet}_X[-n])$. □

4. Example: single condition Schubert varieties with two strata.

Fix integers i, j, k, l such that:

$$0 \leq i \leq j \leq l, \qquad 0 \leq i \leq k \leq l, \qquad \min\{j,k\} = i+1.$$

Let $F^j \subseteq \mathbb{C}^l$ denote a fixed j-dimensional subspace, and let $\mathbb{G}_k(\mathbb{C}^l)$ denote the Grassmann variety of k planes in $\mathbb{C}^l$. Define

$$\mathcal{S} := \left\{ V^k \in \mathbb{G}_k(\mathbb{C}^l) : \dim V^k \cap F^j \geq i \right\}.$$

$\mathcal{S}$ is called a *single condition Schubert variety* [4, p. 328], and we say *with two strata* because $\min\{j,k\} = i+1$ (see (10) below).

Our aim is to compute the Poincaré polynomial $IH_{\mathcal{S}}(t)$ of the intersection cohomology of $\mathcal{S}$, using Corollary 3.2 with $X = \mathcal{S}$.

To this purpose, consider the map [4, p. 328]:

$$\pi : \widetilde{\mathcal{S}} \to \mathcal{S},$$

where

$$\widetilde{\mathcal{S}} := \left\{ (W^i, V^k) \in \mathbb{G}_i(F^j) \times \mathbb{G}_k(\mathbb{C}^l) : W^i \subseteq V^k \right\}, \quad \text{and} \quad \pi(W^i, V^k) = V^k.$$

The map π is a resolution of singularities of $\mathcal{S}$. We have:

$$\mathrm{Sing}(\mathcal{S}) = \left\{ V^k \in \mathbb{G}_k(\mathbb{C}^l) : \dim V^k \cap F^j > i \right\}.$$

Since $\min\{j,k\} = i+1$, it follows that:

$$\mathrm{Sing}(\mathcal{S}) \cong \begin{cases} \mathbb{G}_{k-j}(\mathbb{C}^{l-j}) & \text{if } i+1 = j \\ \mathbb{G}_k(\mathbb{C}^j) & \text{if } i+1 = k. \end{cases} \tag{7}$$

Therefore, $\mathrm{Sing}(\mathcal{S})$ is nonsingular. Moreover, π induces an isomorphism

$$\pi^{-1}(\mathcal{S} \backslash \mathrm{Sing}(\mathcal{S})) \cong \mathcal{S} \backslash \mathrm{Sing}(\mathcal{S}), \tag{8}$$

and

$$\pi^{-1}\left(\mathrm{Sing}(\mathcal{S})\right) \to \mathrm{Sing}(\mathcal{S})$$

is a smooth fibration, with

$$\pi^{-1}(x) \cong \mathbb{P}^i \tag{9}$$

for every $x \in \mathrm{Sing}(\mathcal{S})$. So, the flag

$$\mathcal{S} \supseteq \mathrm{Sing}(\mathcal{S}) \tag{10}$$

is a stratification of $\mathcal{S}$ adapted to π [4], [27]. Observe that the natural projection:

$$(W^i, V^k) \in \widetilde{\mathcal{S}} \to W^i \in \mathbb{G}_i(F^j)$$

is a smooth fibration, with base space $\mathbb{G}_i(F^j)$ and fiber $\mathbb{G}_{k-i}(\mathbb{C}^{l-i})$. Therefore, the Poincaré polynomial $H_{\widetilde{\mathcal{S}}}(t)$ of the cohomology of $\widetilde{\mathcal{S}}$ is (compare with (4)):

$$H_{\widetilde{\mathcal{S}}}(t) = Q_i^j Q_{k-i}^{l-i} = \frac{P_j}{P_i P_{j-i}} \cdot \frac{P_{l-i}}{P_{k-i} P_{l-k}}. \tag{11}$$

The map π is said a *small resolution* of $\mathcal{S}$ if and only if, for every $x \in \mathrm{Sing}(\mathcal{S})$, one has

$$\dim \pi^{-1}(x) < \frac{1}{2}\left(\dim \mathcal{S} - \dim \mathrm{Sing}(\mathcal{S})\right).$$

Since

$$\dim \mathcal{S} = i(j-i) + (k-i)(l-k),$$

and

$$\dim \mathcal{S} - \dim \mathrm{Sing}(\mathcal{S}) = 2i + 1 + l - j - k,$$

it follows that

$$\pi \textit{ is a small resolution of } \mathcal{S} \textit{ if and only if } \; l - j - k \geq 0.$$

In this case, one knows that $IH_{\mathcal{S}}(t)$ is equal to the Poincaré polynomial $H_{\widetilde{\mathcal{S}}}(t)$ of the cohomology of $\widetilde{\mathcal{S}}$ [4], [15]:

$$\pi \quad \text{small} \implies IH_{\mathcal{S}}(t) = H_{\widetilde{\mathcal{S}}}(t).$$

Hence, if π is small, i.e. if $l - j - k \geq 0$, by (11) we get:

$$IH_{\mathcal{S}}(t) = Q_i^j Q_{k-i}^{l-i} = \frac{P_j}{P_i P_{j-i}} \cdot \frac{P_{l-i}}{P_{k-i} P_{l-k}}. \tag{12}$$

This argument appears in [4, p. 329] (see also [18, p. 110-113]). It applies only if π is a small resolution, bypassing the Decomposition Theorem.

When π is non-small, we may apply our Corollary 3.2. In fact, if we set $\Delta = \mathrm{Sing}(\mathcal{S})$, then the map $\pi : \widetilde{\mathcal{S}} \to \mathcal{S}$ verifies all the assumptions $(a1)$, $(a2)$, $(a3)$ stated in Notations, (iii) (see Lemma 4.1 below), and therefore, by Corollary 3.2, we get:

$$IH_{\mathcal{S}}(t) = H_{\widetilde{\mathcal{S}}}(t) - H_{\Delta}(t) g(t). \tag{13}$$

In order to explicit this formula, we distinguish the cases $i + 1 = j$ and $i + 1 = k$.

• In the case $i + 1 = j$, comparing with the invariants defined in Notations, we have:

1) $\Delta \cong \mathbb{G}_{k-j}(\mathbb{C}^{l-j})$ and, by (9), $G \cong \mathbb{P}^{j-1}$;
2) $n = j - 1 + (k - j + 1)(l - k)$;
3) $m = (l - k)(k - j)$;
4) $p = j - 1$;
5) $q = l - k$;
6) $p - q = j + k - l - 1$ and therefore

$$\pi \quad \text{is small} \quad \iff \quad l - j - k \geq 0 \quad \iff \quad p - q < 0;$$

7) $a^{\alpha} = \dim H^{\alpha}(\mathbb{P}^{j-1})$;
8) $g(t) = t^{2(l-k)} + t^{2(l-k+1)} + \cdots + t^{2(j-1)}$.

By (4), (11), and (13), we deduce (recall that $P_1 = 1$):

$$IH_{\mathcal{S}}(t) = \frac{P_j}{P_{j-1}} \cdot \frac{P_{l-j+1}}{P_{k-j+1} P_{l-k}} - \left(t^{2(l-k)} + t^{2(l-k+1)} + \cdots + t^{2(j-1)}\right) \cdot \frac{P_{l-j}}{P_{k-j} P_{l-k}}, \tag{14}$$

where $t^{2(l-k)}+t^{2(l-k+1)}+\cdots+t^{2(j-1)}$ denotes the zero polynomial when $j+k\leq l$ (compare with Notations, (vii)). Hence, previous formula reduces to (12) in the small case.

- In the case $i+1=k$, the invariants are:

1) $\Delta\cong\mathbb{G}_k(\mathbb{C}^j)$ and, by (9), $G\cong\mathbb{P}^{k-1}$;
2) $n=(k-1)(j-k+1)+l-k$;
3) $m=k(j-k)$;
4) $p=k-1$;
5) $q=l-j$;
6) $p-q=j+k-l-1$ and therefore

$$\pi \quad \text{is small} \quad \iff \quad l-j-k\geq 0 \quad \iff \quad p-q<0;$$

7) $a^{\alpha}=\dim H^{\alpha}(\mathbb{P}^{k-1})$;
8) $g(t)=t^{2(l-j)}+t^{2(l-j+1)}+\cdots+t^{2(k-1)}$.

By (4), (11), and (13), we deduce:

(15)

$$IH_{\mathcal{S}}(t)=\frac{P_j}{P_{k-1}P_{j-k+1}}\cdot\frac{P_{l-k+1}}{P_{l-k}}-\left(t^{2(l-j)}+t^{2(l-j+1)}+\cdots+t^{2(k-1)}\right)\cdot\frac{P_j}{P_kP_{j-k}},$$

where $t^{2(l-j)}+t^{2(l-j+1)}+\cdots+t^{2(k-1)}$ denotes the zero polynomial when $j+k\leq l$, and previous formula reduces to (12) in the small case.

Lemma 4.1. *The resolution of singularities $\pi:\widetilde{\mathcal{S}}\to\mathcal{S}$, with $\Delta=Sing(\mathcal{S})$, verifies all the assumptions $(a1)$, $(a2)$, $(a3)$ stated in Notations, (iii).*

Proof. In view of the description of the map $\pi:\widetilde{\mathcal{S}}\to\mathcal{S}$ given in (7), (8), and (9), we only have to verify the assumption $(a3)$.

First we examine the case $i+1=j$.

In this case, we have

$$\widetilde{\mathcal{S}}=\left\{(W^{j-1},V^k)\in\mathbb{G}_{j-1}(F^j)\times\mathbb{G}_k(\mathbb{C}^l): W^{j-1}\subseteq V^k\right\},$$

and

$$\widetilde{\Delta}=\left\{(W^{j-1},V^k)\in\mathbb{G}_{j-1}(F^j)\times\mathbb{G}_k(\mathbb{C}^l): F^j\subseteq V^k\right\}.$$

Let S_{j-1} denote the tautological bundle on $\mathbb{G}_{j-1}(F^j)\cong\mathbb{P}^{j-1}$, and S_k the tautological bundle on $\mathbb{G}_k(\mathbb{C}^l)$. Let S'_{j-1} and S'_k denote the pull-back of S_{j-1} and S_k via the natural projections $\widetilde{\Delta}\to\mathbb{G}_{j-1}(F^j)$ and $\widetilde{\Delta}\to\mathbb{G}_k(\mathbb{C}^l)$. We have identifications with Grassmann bundles [13, p. 434, B.5.7]:

$$\widetilde{\mathcal{S}}\cong\mathbb{G}_{k-j+1}(\mathbb{C}^l/S_{j-1}) \quad \text{and} \quad \widetilde{\Delta}\cong\mathbb{G}_{k-j}(\mathbb{C}^l/F^j),$$

where $\mathbb{C}^l$ and F^j denote the trivial vector bundles (in this case, on $\mathbb{G}_{j-1}(F^j)$). The relative tangent bundles are (compare with [13, p. 435, B.5.8]):

$$j^*T_{\widetilde{\mathcal{S}}/\mathbb{G}_{j-1}(F^j)}\cong\mathrm{Hom}(S'_k/S'_{j-1},\mathbb{C}^l/S'_k),$$

and

$$T_{\widetilde{\Delta}/\mathbb{G}_{j-1}(F^j)}\cong\mathrm{Hom}(S'_k/F^j,\mathbb{C}^l/S'_k),$$

where $\jmath$ denotes the inclusion $\widetilde{\Delta} \hookrightarrow \widetilde{\mathcal{S}}$, and $\mathbb{C}^l$ and F^j denote the trivial vector bundles on $\widetilde{\Delta}$. Therefore, applying $\mathrm{Hom}(\cdot\,, \mathbb{C}^l/S'_k)$ to the exact sequence

$$0 \to F^j/S'_{j-1} \to S'_k/S'_{j-1} \to S'_k/F^j \to 0,$$

we get the exact sequence:

$$0 \to T_{\widetilde{\Delta}/\mathbb{G}_{k-1}(F^j)} \to \jmath^* T_{\widetilde{\mathcal{S}}/\mathbb{G}_{k-1}(F^j)} \to \mathrm{Hom}(F^j/S'_{j-1}, \mathbb{C}^l/S'_k) \to 0.$$

It enables us to identify the normal bundle N of $\widetilde{\Delta}$ in $\widetilde{\mathcal{S}}$ [13, p. 438, B.7.2]:

$$N \cong \mathrm{Hom}(F^j/S'_{j-1}, \mathbb{C}^l/S'_k).$$

It follows that the restriction $N_{|G}$ of N to the fiber $G \cong \mathbb{P}^{j-1}$ of $\rho : \widetilde{\Delta} \hookrightarrow \Delta$ is:

$$N_{|G} \cong \mathcal{O}_G(-1) \otimes \mathbb{C}^q.$$

Hence:

$$0 \neq c = c_q(N_{|G}) = (-h)^q \in H^{2q}(\mathbb{P}^{j-1}) \cong H^{2q}(G),$$

where $h \in H^{2q}(\mathbb{P}^{j-1})$ denotes the hyperplane class. This is enough to prove $(a3)$ because $G \cong \mathbb{P}^{j-1}$ is a projective space.

Now we turn to the case $i + 1 = k$.

In this case, we have

$$\widetilde{\mathcal{S}} = \left\{(W^{k-1}, V^k) \in \mathbb{G}_{k-1}(F^j) \times \mathbb{G}_k(\mathbb{C}^l) : W^{k-1} \subseteq V^k\right\},$$

and

$$\widetilde{\Delta} = \left\{(W^{k-1}, V^k) \in \mathbb{G}_{k-1}(F^j) \times \mathbb{G}_k(F^j) : W^{k-1} \subseteq V^k\right\}.$$

Let S_{k-1} denote the tautological bundle on $\mathbb{G}_{k-1}(F^j)$, and S_k the tautological bundle on $\mathbb{G}_k(F^j)$. Let S'_{k-1} and S'_k denote the pull-back of S_{k-1} and S_k via the natural projection $\widetilde{\Delta} \to \mathbb{G}_{k-1}(F^j)$. We have identifications with projective bundles:

$$\widetilde{\mathcal{S}} \cong \mathbb{P}(\mathbb{C}^l/S_{k-1}), \quad \text{and} \quad \widetilde{\Delta} \cong \mathbb{P}(F^j/S_{k-1}).$$

The relative tangent bundles are [13, p. 435, B.5.8]:

$$\jmath^* T_{\widetilde{\mathcal{S}}/\mathbb{G}_{k-1}(F^j)} \cong \mathrm{Hom}(S'_k/S'_{k-1}, \mathbb{C}^l/S'_k),$$

and

$$T_{\widetilde{\Delta}/\mathbb{G}_{k-1}(F^j)} \cong \mathrm{Hom}(S'_k/S'_{k-1}, F^j/S'_k),$$

where $\jmath$ denotes the inclusion $\widetilde{\Delta} \hookrightarrow \widetilde{\mathcal{S}}$. Therefore, applying $\mathrm{Hom}(S'_k/S'_{k-1}, \cdot)$ to the exact sequence

$$0 \to F^j/S'_k \to \mathbb{C}^l/S'_k \to \mathbb{C}^q \to 0,$$

we get the exact sequence:

$$0 \to T_{\widetilde{\Delta}/\mathbb{G}_{k-1}(F^j)} \to \jmath^* T_{\widetilde{\mathcal{S}}/\mathbb{G}_{k-1}(F^j)} \to \mathrm{Hom}(S'_k/S'_{k-1}, \mathbb{C}^q) \to 0.$$

Hence, the normal bundle N of $\widetilde{\Delta}$ in $\widetilde{\mathcal{S}}$ [13, p. 438, B.7.2] is:

$$N \cong \mathrm{Hom}(S'_k/S'_{k-1}, \mathbb{C}^q).$$

It follows that the restriction $N_{|G}$ of N to the fiber $G \cong \mathbb{P}^{k-1}$ of $\rho : \widetilde{\Delta} \hookrightarrow \Delta$ is:

$$N_{|G} \cong \mathcal{O}_G(-1) \otimes \mathbb{C}^q.$$

Hence:

$$0 \neq c = c_q(N_{|G}) = (-h)^q \in H^{2q}(\mathbb{P}^{k-1}) \cong H^{2q}(G),$$

where $h \in H^{2q}(\mathbb{P}^{k-1})$ denotes the hyperplane class. □

Remark 4.2. (i) Another resolution of $\mathcal{S}$ is given by

$$\pi_1 : (V^k, U^{k+j-i}) \in \widetilde{\mathcal{S}_1} \to V^k \in \mathcal{S},$$

where

$$\widetilde{\mathcal{S}_1} := \left\{ (V^k, U^{k+j-i}) \in \mathbb{G}_k(\mathbb{C}^l) \times \mathbb{G}_{k+j-i}(\mathbb{C}^l) : V^k + F^j \subseteq U^{k+j-i} \right\}.$$

A similar argument as before shows that

π_1 *is a small resolution of* $\mathcal{S}$ *if and only if* $l - j - k \leq 0$,

and, in this case, we have:

$$IH_{\mathcal{S}}(t) = H_{\widetilde{\mathcal{S}_1}}(t) = Q_{k-i}^{l-j} Q_k^{k+j-i} = \frac{P_{l-j}}{P_{k-i}P_{l-j-k+i}} \cdot \frac{P_{k+j-i}}{P_k P_{j-i}}. \tag{16}$$

This is another way to compute $IH_{\mathcal{S}}(t)$ when π is non-small, relying on the same argument as in [4].

(ii) Comparing (14), (15) and (16), in the case $l \leq j + k$ we obtain the following polynomial identities, that one may easily verify with a direct computation:

if $i + 1 = j$ then:

$$\frac{P_{l-j}}{P_{k-j+1}P_{l-k-1}} \cdot \frac{P_{k+1}}{P_k}$$

$$= \frac{P_j}{P_{j-1}} \cdot \frac{P_{l-j+1}}{P_{k-j+1}P_{l-k}} - \left(t^{2(l-k)} + t^{2(l-k+1)} + \cdots + t^{2(j-1)} \right) \cdot \frac{P_{l-j}}{P_{k-j}P_{l-k}};$$

if $i + 1 = k$ then:

$$\frac{P_{l-j}}{P_1 P_{l-j-1}} \cdot \frac{P_{j+1}}{P_k P_{j-k+1}}$$

$$= \frac{P_j}{P_{k-1}P_{j-k+1}} \cdot \frac{P_{l-k+1}}{P_{l-k}} - \left(t^{2(l-j)} + t^{2(l-j+1)} + \cdots + t^{2(k-1)} \right) \cdot \frac{P_j}{P_k P_{j-k}}.$$

5. Example: hypersurfaces of $\mathbb{P}^5$ with one-dimensional singular locus

Fix a smooth threefold $T \subset \mathbb{P}^5$, complete intersection, with equations $t_1 = t_2 = 0$. Let $X \subset \mathbb{P}^5$ be a general hypersurface containing T, with equation $t_1t_3 - t_2t_4 = 0$. By Bertini's theorem, the singular locus of X is contained in T. Actually, since T is smooth, $\mathrm{Sing}(X)$ is equal to the smooth complete intersection curve Δ, defined by $t_1 = t_2 = t_3 = t_4 = 0$. Set:

$$d_i := \deg t_i, \quad x := \deg X = d_1 + d_3 = d_2 + d_4, \quad \delta := \deg \Delta = d_1d_2d_3d_4.$$

Observe that:

$$\mathcal{O}_\Delta(K_\Delta) \cong \mathcal{O}_\Delta(2x-6), \quad 2g - 2 = (2x-6)\delta,$$

where K_Δ denotes the canonical divisor of Δ, and g the genus.

Let $\sigma : \mathbb{P} \to \mathbb{P}^5$ be the blowing-up of $\mathbb{P}^5$ along Δ. Let $E \cong \Delta \times \mathbb{P}^3 \subset \mathbb{P}$ be the exceptional divisor. Let $\widetilde{X} \subset \mathbb{P}$ be the strict transform of X, which is the blowing-up of X along Δ. The restriction of σ to $\widetilde{X}$:

$$\pi : \widetilde{X} \to X$$

is a resolution of singularities of X. The exceptional divisor $\widetilde{\Delta}$ of $\widetilde{X}$ is:

$$\widetilde{\Delta} \cong \Delta \times G,$$

where G is the smooth quadric surface in $\mathbb{P}^3$. The resolution π verifies all the assumptions $(a1)$, $(a2)$, $(a3)$, and therefore, by Corollary 3.2, we get:

$$IH_X(t) = H_{\widetilde{X}}(t) - H_\Delta(t)g(t).$$

In order to explicit this formula, first we notice that, in this example, the invariants are: $n = 4$, $m = 1$, $p = 2$, $q = 1$. So, we have:

$$H_\Delta(t) = 1 + 2gt + t^2, \quad g(t) = t^2 + t^4.$$

It remains to compute $H_{\widetilde{X}}(t)$, i.e. the Betti numbers $b_i(\widetilde{X})$ of $\widetilde{X}$.

To this purpose, we recall some properties of $\mathbb{P}$, which we will use in the sequel. We refer to [16, p. 605], [2, p. 592, Lemma 1.4 and Proof of Theorem 1.2], and [13, p. 67, Example 3.3.4] for more details. Set $\mathcal{O}_\mathbb{P}(H) = \sigma^*\mathcal{O}_{\mathbb{P}^5}(1)$. We have:

$$\mathcal{O}_\mathbb{P}(K_\mathbb{P}) \cong \mathcal{O}_\mathbb{P}(-6H + 3E), \quad \mathcal{O}_\mathbb{P}(\widetilde{X}) \cong \mathcal{O}_\mathbb{P}(xH - 2E), \tag{17}$$

and so

$$\mathcal{O}_\mathbb{P}(K_\mathbb{P} + \widetilde{X}) \cong \mathcal{O}_\mathbb{P}((x-6)H + E), \quad \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) \cong \mathcal{O}_\mathbb{P}((x-6)H + E) \otimes \mathcal{O}_{\widetilde{X}}. \tag{18}$$

We also have:

$$H^5 = 1, \quad H^4E = H^3E^2 = H^2E^3 = 0, \quad HE^4 = -\delta, \tag{19}$$

$$E^5 = -c_1(N_{\Delta,\mathbb{P}^5}) = 2 - 2g - 6\delta$$

($N_{\Delta,\mathbb{P}^5}$ denotes the normal bundle of Δ in $\mathbb{P}^5$).

Moreover:

$$H^\alpha(\mathbb{P}, \sigma^*M \otimes \mathcal{O}_\mathbb{P}(iE)) \cong H^\alpha(\mathbb{P}^5, M) \tag{20}$$

for every vector bundle M on $\mathbb{P}^5$, every α, and every $0 \leq i \leq 3$, and

$$H^\alpha(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(iH - E)) \cong H^\alpha(\mathbb{P}^5, \mathcal{I}_{\Delta,\mathbb{P}^5}(i)), \tag{21}$$

for every α and every i ($\mathcal{I}_{\Delta,\mathbb{P}^5}$ denotes the ideal sheaf of Δ in $\mathbb{P}^5$).

We are in position to compute the Betti numbers of $\widetilde{X}$.

Lemma 5.1.

$$b_1(\widetilde{X}) = 0, \quad b_2(\widetilde{X}) = 3, \quad b_3(\widetilde{X}) = 4g,$$
$$b_4(\widetilde{X}) = (x-2)(x^2-3x+3)(x^2-x+1) - (g-1) + 3(2-\delta).$$

Corollary 5.2.

$$IH_X(t) = 1 + 2t^2 + 2gt^3$$
$$+[(x-2)(x^2-3x+3)(x^2-x+1) - (g-1) + (4-3\delta)]t^4 + 2gt^5 + 2t^6 + t^8.$$

Proof of the Lemma 5.1. For every $\alpha \in \mathbb{Z}$, consider the following natural commutative diagram:

$$\begin{array}{ccccccc}
H_{\alpha+1}(\widetilde{X}, \widetilde{\Delta}) & \longrightarrow & H_\alpha(\widetilde{\Delta}) & \longrightarrow & H_\alpha(\widetilde{X}) & \longrightarrow & H_\alpha(\widetilde{X}, \widetilde{\Delta}) \\
\| & & \downarrow & & \downarrow & & \| \\
H_{\alpha+1}(X, \Delta) & \longrightarrow & H_\alpha(\Delta) & \longrightarrow & H_\alpha(X) & \longrightarrow & H_\alpha(X, \Delta),
\end{array}$$

where the horizontal rows are the homology exact sequences of the couple, and the vertical maps are induced by π. As for the isomorphism $H_*(\widetilde{X}, \widetilde{\Delta}) \cong H_*(X, \Delta)$, see [19, p. 23]. By the Lefschetz Hyperplane Theorem, we know that

$$b_1(X) = b_3(X) = 0, \quad b_2(X) = 1.$$

Combining with the Künneth formula for $\widetilde{\Delta} \cong \Delta \times G$, by a simple diagram chase we deduce:

- $b_1(\widetilde{X}) = 0$;
- the push-forward $H_2(\widetilde{\Delta}) \to H_2(\widetilde{X})$ is an isomorphism, therefore $b_2(\widetilde{X}) = b_2(\widetilde{\Delta}) = 3$;
- the push-forward $H_3(\widetilde{\Delta}) \to H_3(\widetilde{X})$ is onto.

In particular, the pull-back $H^3(\widetilde{X}) \to H^3(\widetilde{\Delta})$ is injective. Since

$$H^3(\widetilde{\Delta}) \cong H^1(\Delta) \otimes H^2(G) \cong H^{1,2}(\widetilde{\Delta}) \oplus H^{2,1}(\widetilde{\Delta}) \cong \mathbb{C}^{4g},$$

it follows that $H^3(\widetilde{X}) = H^{1,2}(\widetilde{X}) \oplus H^{2,1}(\widetilde{X})$. Hence, in order to prove that $b_3(\widetilde{X}) = 4g$, it suffices to prove that

$$h^{1,2}(\widetilde{X}) \geq 2g. \tag{22}$$

To this aim, let

$$N_{\widetilde{X},\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(\widetilde{X}) \otimes \mathcal{O}_{\widetilde{X}}$$

be the normal bundle of $\widetilde{X}$ in $\mathbb{P}$. From the natural exact sequence

$$0 \to N^\vee_{\widetilde{X},\mathbb{P}} \to \Omega_{\mathbb{P}} \otimes \mathcal{O}_{\widetilde{X}} \to \Omega_{\widetilde{X}} \to 0,$$

we get the following exact sequence:

$$H^1(\widetilde{X}, \Omega_{\widetilde{X}}) \to H^2(\widetilde{X}, N^\vee_{\widetilde{X},\mathbb{P}}) \to H^2(\widetilde{X}, \Omega_{\mathbb{P}} \otimes \mathcal{O}_{\widetilde{X}}) \to H^2(\widetilde{X}, \Omega_{\widetilde{X}}). \tag{23}$$

In order to identify the first map $H^1(\widetilde{X},\Omega_{\widetilde{X}})\to H^2(\widetilde{X},N^{\vee}_{\widetilde{X},\mathbb{P}})$, first notice that

$$H^1(\widetilde{X},\Omega_{\widetilde{X}})\cong H^2(\widetilde{\Delta}) \tag{24}$$

because

$$H^2(\widetilde{X})\cong H^2(\widetilde{\Delta})\cong (H^0(\Delta)\otimes H^2(G))\oplus(H^2(\Delta)\otimes H^0(G))\cong H^{1,1}(\widetilde{\Delta}).$$

On the other hand, by the Serre Duality Theorem and (18), we have:

$$H^2(\widetilde{X},N^{\vee}_{\widetilde{X},\mathbb{P}})\cong H^2(\widetilde{X},\mathcal{O}_{\mathbb{P}}((2x-6)H-E)\otimes\mathcal{O}_{\widetilde{X}})^{\vee}.$$

Tensoring the exact sequence

$$0\to\mathcal{O}_{\mathbb{P}}(-\widetilde{X})\to\mathcal{O}_{\mathbb{P}}\to\mathcal{O}_{\widetilde{X}}\to 0$$

with $\mathcal{O}_{\mathbb{P}}((2x-6)H-E)$, we get the exact sequence

$$H^2(\mathbb{P},\mathcal{O}_{\mathbb{P}}((x-6)H+E))\to H^2(\mathbb{P},\mathcal{O}_{\mathbb{P}}((2x-6)H-E))\to$$
$$\to H^2(\widetilde{X},\mathcal{O}_{\mathbb{P}}((2x-6)H-E)\otimes\mathcal{O}_{\widetilde{X}})\to H^3(\mathbb{P},\mathcal{O}_{\mathbb{P}}((x-6)H+E)).$$

Now by (20) and (21) we have:

$$H^2(\mathbb{P},\mathcal{O}_{\mathbb{P}}((x-6)H+E))\cong H^2(\mathbb{P}^5,\mathcal{O}_{\mathbb{P}^5}(x-6))=0,$$
$$H^3(\mathbb{P},\mathcal{O}_{\mathbb{P}}((x-6)H+E))\cong H^3(\mathbb{P}^5,\mathcal{O}_{\mathbb{P}^5}(x-6))=0,$$

and

$$H^2(\mathbb{P},\mathcal{O}_{\mathbb{P}}((2x-6)H-E))\cong H^2(\mathbb{P}^5,\mathcal{I}_{\Delta,\mathbb{P}^5}(2x-6))\cong$$
$$\cong H^1(\Delta,\mathcal{O}_{\Delta}(2x-6))\cong H^0(\Delta,\mathcal{O}_{\Delta})^{\vee},$$

because $\mathcal{O}_{\Delta}(2x-6)\cong\mathcal{O}_{\Delta}(K_{\Delta})$. Summing up, we get

$$H^2(\widetilde{X},N^{\vee}_{\widetilde{X},\mathbb{P}})\cong H^0(\Delta,\mathcal{O}_{\Delta})\cong H^0(\Delta)\cong\mathbb{C}. \tag{25}$$

By (24) and (25), it follows that the map $H^1(\widetilde{X},\Omega_{\widetilde{X}})\to H^2(\widetilde{X},N^{\vee}_{\widetilde{X},\mathbb{P}})$ identifies with the surjective projection $H^2(\widetilde{\Delta})\to H^0(\Delta)$ given by the Künneth formula. By (23), it follows an injective map

$$0\to H^2(\widetilde{X},\Omega_{\mathbb{P}}\otimes\mathcal{O}_{\widetilde{X}})\to H^2(\widetilde{X},\Omega_{\widetilde{X}}).$$

Hence, by (22), to prove that $b_3(\widetilde{X})=4g$, it suffices to prove that

$$\dim H^2(\widetilde{X},\Omega_{\mathbb{P}}\otimes\mathcal{O}_{\widetilde{X}})\geq 2g. \tag{26}$$

To this aim, consider again the exact sequence

$$0\to\mathcal{O}_{\mathbb{P}}(-\widetilde{X})\to\mathcal{O}_{\mathbb{P}}\to\mathcal{O}_{\widetilde{X}}\to 0.$$

Tensoring with $\Omega_{\mathbb{P}}$, and taking the cohomology, we get the exact sequence:

$$H^2(\mathbb{P},\Omega_{\mathbb{P}}\otimes\mathcal{O}_{\mathbb{P}}(-\widetilde{X}))\to H^2(\mathbb{P},\Omega_{\mathbb{P}})\to H^2(\widetilde{X},\Omega_{\mathbb{P}}\otimes\mathcal{O}_{\widetilde{X}}).$$

Since $\dim H^2(\mathbb{P},\Omega_{\mathbb{P}})=2g$ [26, p. 180, Theorem 7.31], to prove (26) (hence (22)), it is enough to prove that

$$H^2(\mathbb{P},\Omega_{\mathbb{P}}\otimes\mathcal{O}_{\mathbb{P}}(-\widetilde{X}))=0,$$

i.e., by the Serre Duality Theorem, that

$$H^3(\mathbb{P},\mathcal{T}_{\mathbb{P}}\otimes\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}}+\widetilde{X}))=0. \tag{27}$$

Consider the exact sequence [13, p. 299]

$$0 \to \mathcal{T}_{\mathbb{P}} \to \sigma^*\mathcal{T}_{\mathbb{P}^5} \to j_*(F) \to 0,$$

where $j : E \to \mathbb{P}$ denotes the inclusion, and F the universal quotient bundle on E. Tensoring with $\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}}+\widetilde{X})$, and taking the cohomology, we get the exact sequence:

$$H^2(\mathbb{P}, \sigma^*\mathcal{T}_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X})) \to H^2(\mathbb{P}, j_*(F) \otimes \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X})) \to$$

$$\to H^3(\mathbb{P}, \mathcal{T}_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X})) \to H^3(\mathbb{P}, \sigma^*\mathcal{T}_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X})).$$

By (18) and (20) we have:

$$H^2(\mathbb{P}, \sigma^*\mathcal{T}_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X})) \cong H^2(\mathbb{P}^5, \mathcal{T}_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}^5}(x-6)) = 0,$$

$$H^3(\mathbb{P}, \sigma^*\mathcal{T}_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X})) \cong H^3(\mathbb{P}^5, \mathcal{T}_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}^5}(x-6)) = 0,$$

and, by the projection formula, we have

$$H^2(\mathbb{P}, j_*(F) \otimes \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X})) \cong H^2(E, F \otimes j^*(\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X}))).$$

It follows that

$$H^3(\mathbb{P}, \mathcal{T}_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X})) \cong H^2(E, F \otimes j^*(\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X}))).$$

So, in order to prove (27), it suffices to prove that

$$H^2(E, F \otimes j^*(\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X}))) = 0. \tag{28}$$

Consider the exact sequence [13, loc. cit.]

$$0 \to N_{E,\mathbb{P}} \to \tau^* N_{\Delta,\mathbb{P}^5} \to F \to 0,$$

where $N_{E,\mathbb{P}} \cong \mathcal{O}_E \otimes \mathcal{O}_{\mathbb{P}}(E)$ is the normal bundle of E in $\mathbb{P}$, $\tau : E \to \Delta$ is the natural projection, and $N_{\Delta,\mathbb{P}^5} \cong \oplus_{i=1}^4 \mathcal{O}_\Delta(d_i)$ is the normal bundle of Δ in $\mathbb{P}^5$. Tensoring with $j^*(\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + \widetilde{X}))$, and taking into account (18), we deduce that the proof of the vanishing (28) (hence the proof of (22)) amounts to show that:

$$H^2(E, j^*(\mathcal{O}_{\mathbb{P}}((d_i + x - 6)H + E))) = H^3(E, j^*(\mathcal{O}_{\mathbb{P}}((x-6)H + 2E))) = 0, \tag{29}$$

for every $i = 1, 2, 3, 4$. Consider the exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}}(-E) \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_E \to 0.$$

Tensoring with $\mathcal{O}_{\mathbb{P}}((d_i + x - 6)H + E)$, we have the exact sequence:

$$H^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}((d_i + x - 6)H + E)) \to H^2(E, j^*(\mathcal{O}_{\mathbb{P}}((d_i + x - 6)H + E))) \to$$

$$\to H^3(\mathbb{P}, \mathcal{O}_{\mathbb{P}}((d_i + x - 6)H)),$$

and tensoring with $\mathcal{O}_{\mathbb{P}}((x-6)H + 2E)$, we have the exact sequence:

$$H^3(\mathbb{P}, \mathcal{O}_{\mathbb{P}}((x-6)H + 2E)) \to H^3(E, j^*(\mathcal{O}_{\mathbb{P}}((x-6)H + 2E))) \to$$

$$\to H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}((x-6)H + E)).$$

By (20) we have:

$$H^2(\mathbb{P}, \mathcal{O}_{\mathbb{P}}((d_i + x - 6)H + E)) \cong H^2(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(d_i + x - 6)) = 0,$$

$$H^3(\mathbb{P}, \mathcal{O}_{\mathbb{P}}((d_i + x - 6)H)) \cong H^3(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(d_i + x - 6)) = 0,$$

$$H^3(\mathbb{P}, \mathcal{O}_{\mathbb{P}}((x-6)H + 2E)) \cong H^3(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(x-6)) = 0,$$

$$H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}((x-6)H+E)) \cong H^4(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(x-6)) = 0.$$

This proves the vanishing (29), and concludes the proof of the equality $b_3(\widetilde{X}) = 4g$.

Now we turn to $b_4(\widetilde{X})$.

By the Gauss-Bonnet Formula [16, p. 416], we know that

$$c_4(\mathcal{T}_{\widetilde{X}}) = \chi_{\rm top}(\widetilde{X}).$$

Therefore, by the previous computations of $b_i(\widetilde{X})$, $i = 1, 2, 3$, we have:

$$b_4(\widetilde{X}) = c_4(\mathcal{T}_{\widetilde{X}}) + 4(2g-2).$$

Hence, the computation of $b_4(\widetilde{X})$ amounts to that of $c_4(\mathcal{T}_{\widetilde{X}})$. By the exact sequence:

$$0 \to \mathcal{T}_{\widetilde{X}} \to \mathcal{T}_{\mathbb{P}} \otimes \mathcal{O}_{\widetilde{X}} \to N_{\widetilde{X},\mathbb{P}} \to 0,$$

we get

$$c_4(\mathcal{T}_{\widetilde{X}}) = \widetilde{X} \cdot c_4(\mathcal{T}_{\mathbb{P}}) - \widetilde{X}^2 \cdot c_3(\mathcal{T}_{\mathbb{P}}) + \widetilde{X}^3 \cdot c_2(\mathcal{T}_{\mathbb{P}}) - \widetilde{X}^4 \cdot c_1(\mathcal{T}_{\mathbb{P}}) + \widetilde{X}^5. \tag{30}$$

On the other hand, using [13, p. 300, Example 15.4.2], we find:

$$c_1(\mathcal{T}_{\mathbb{P}}) = 6H - 3E, \quad c_2(\mathcal{T}_{\mathbb{P}}) = 15H^2 + 2(x-9)HE + 2E^2,$$

$$c_3(\mathcal{T}_{\mathbb{P}}) = 20H^3 + 8x(x-3)H^2E + 4(3-x)HE^2 + 2E^3,$$

$$c_4(\mathcal{T}_{\mathbb{P}}) = 15H^4 + 12HE^3 - 3E^4.$$

Inserting previous data into (30), and taking into account (17) and (19), we get:

$$c_4(\mathcal{T}_{\widetilde{X}}) = (x-2)(x^2-3x+3)(x^2-x+1) - 9(g-1) + 3(2-\delta).$$

□

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