

Lefschetz exceptional collections in S_k -equivariant categories of $(\mathbb{P}^n)^k$

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Abstract

We consider the bounded derived category of S_k -equivariant coherent sheaves on $(\mathbb{P}^n)^k$. The goal of this paper is to construct in this category a rectangular Lefschetz exceptional collection when this is possible, or a minimal Lefschetz exceptional collection when a rectangular one does not exist. The main results of the paper include the construction of a rectangular Lefschetz exceptional collection in the case $k = 3$ and in the case $n = 1$ when $\gcd(n + 1, k) = 1$. We also construct minimal Lefschetz exceptional collection for $n = 1$ and even k , and for $n = 2$ and $k = 3$.

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1 Introduction

The bounded derived category of coherent sheaves is the main homological invariant of an algebraic variety which captures the most essential geometric information. It stands in the focus of many recent research papers. One of the ways to describe it is via an exceptional collection.

Recall that an object E in a \mathbb{C} -linear triangulated category \mathcal{T} is *exceptional* if $\text{Ext}^0(E, E) = \mathbb{C}$ and $\text{Ext}^i(E, E) = 0$ for $i \neq 0$. Furthermore, a collection E_1, \dots, E_r of objects in \mathcal{T} is an *exceptional collection* if each E_i is an exceptional object and $\text{Ext}^\bullet(E_i, E_j) = 0$ for $i > j$. An exceptional collection is *full* if the smallest full triangulated subcategory of \mathcal{T} containing all E_i coincides with \mathcal{T} .

Recently a special class of exceptional collections attracted much attention. Recall that an exceptional collection E_1, \dots, E_r in the bounded derived category of coherent sheaves $\mathcal{D}(X)$ of a smooth projective variety X is *Lefschetz* with respect to a line bundle \mathcal{L} if there is a partition $r = r_0 + r_1 + \dots + r_d$ with $r_0 \geq r_1 \geq \dots \geq r_d$ such that

$$E_{r_0+r_1+\dots+r_{i-1}+t} \cong E_t \otimes \mathcal{L}^i \quad \text{for all } 1 \leq t \leq r_i \quad \text{and } 1 \leq i \leq d.$$

In other words, if the objects of the collection are obtained by \mathcal{L} -twists from the subcollection of the first r_0 objects according to the pattern provided by the partition.

As it is clear from the definition, a Lefschetz collection is determined by its *starting block* E_1, \dots, E_{r_0} and the partition (r_0, r_1, \dots, r_d) . It is less evident, but is still true, that if a Lefschetz collection is full, then the partition is itself determined by the starting block of the collection [6, Lemma 4.5]. Thus, extendability to a Lefschetz collection is just a property of an exceptional collection E_1, \dots, E_{r_0} .

It follows that there is a natural partial order on the set of all Lefschetz collections in $\mathcal{D}(X)$ — a Lefschetz collection with a starting block E_1, \dots, E_{r_0} is *smaller* than a Lefschetz collection with a starting block E'_1, \dots, E'_{s_0} if E_1, \dots, E_{r_0} is a subcollection in E'_1, \dots, E'_{s_0} , see [9, Definition 1.4].

A Lefschetz collection E_1, \dots, E_r with partition r_0, r_1, \dots, r_d is called *rectangular of length $d+1$* , if $r_0 = r_1 = \dots = r_d$ (equivalently, if the Young diagram representing the partition is a rectangle of length $d+1$). Of course, a necessary condition for the existence of a rectangular Lefschetz collection in $\mathcal{D}(X)$ is a factorization

$$\text{rk}(K_0(\mathcal{D}(X))) = r_0(d+1) \tag{1.1}$$

for the rank of the Grothendieck group of X . On the other hand, if a rectangular Lefschetz decomposition in $\mathcal{D}(X)$ exists, and if its length $d+1$ has the property that $\mathcal{L}^{d+1} \cong \omega_X^{-1}$ where ω_X is the canonical bundle of X , that is $d+1$ equals the

index of X with respect to \mathcal{L} , then this collection is automatically minimal (this follows easily from Serre duality, see [9, Subsection 2.1]).

Lefschetz collections have many nice properties and are very important for homological projective duality and categorical resolutions of singularities [7]. Especially nice and important are rectangular (resp. minimal) Lefschetz collections. So, the following problem is very interesting.

Problem 1.1. *Given a smooth projective variety X and a line bundle \mathcal{L} , construct a full rectangular Lefschetz collection in $\mathcal{D}(X)$ with respect to \mathcal{L} of length equal to the index of X , or, if the above is impossible, a minimal Lefschetz collection.*

There are many varieties X for which the above problem was solved. Among these are projective spaces, most of the Grassmannians, and some other homogeneous spaces [2]. In this paper we discuss Problem 1.1 for a very simple variety

$$X = X_k^n := \underbrace{\mathbb{P}^n \times \mathbb{P}^n \times \cdots \times \mathbb{P}^n}_{k \text{ copies}},$$

but replace the category $\mathcal{D}(X_k^n)$ with the equivariant derived category $\mathcal{D}_{S_k}(X_k^n)$ with respect to the natural action of the symmetric group S_k (by permutation of factors). Note that this category can be considered as the derived category of the *quotient stack* $[X_k^n/S_k]$. The line bundle \mathcal{L} here is, of course, the ample generator $\mathcal{O}(1, 1, \dots, 1)$ of the invariant Picard group $\text{Pic}(X_k^n)^{S_k}$. Note that the index of X_k^n with respect to \mathcal{L} is equal to $n + 1$, so the goal of the paper can be formulated as follows.

Problem 1.2. *Find a full rectangular Lefschetz collection of length $n + 1$ in $\mathcal{D}_{S_k}(X_k^n)$ with respect to the line bundle $\mathcal{O}(1, 1, \dots, 1)$ or a minimal Lefschetz collection if the above is impossible.*

Note that without passing to the equivariant category the problem becomes trivial. To construct a rectangular Lefschetz collection in $\mathcal{D}(X_k^n)$ one can just choose any full exceptional collection in $\mathcal{D}(X_{k-1}^n)$ and consider its pullback to X_k^n as the starting block. It is elementary to check that it extends to a rectangular Lefschetz collection of length $n + 1$. However, the S_k -symmetry in this construction is broken, and it cannot be performed in the equivariant category.

For $k = 1$ the Problem 1.2 is trivial (the desired collection is just the Beilinson exceptional collection $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)$ of line bundles on \mathbb{P}^n). Furthermore, for $k = 2$ the Problem 1.2 was essentially solved in [10].

The main result of our paper is a partial solution to the Problem 1.2.

First, we construct in Theorem 3.2 a rectangular S_k -invariant Lefschetz exceptional collection in $\mathcal{D}(X_k^n)$ whose cardinality in case of coprime k and $n + 1$ equals the rank of the Grothendieck group of X_k^n (by Elagin's Theorem, see Theorem 2.4, this gives an exceptional collection in the equivariant category, whose length equals the rank of its Grothendieck group). So, it is natural to expect that this collection is full and (in the coprime case) gives a solution to Problem 1.2. However, in general we could not prove its fullness.

Our second main result is a proof of fullness of the above collection for $k = 3$ and $n = 3p$ or $n = 3p + 1$ (this ensures that k and $n + 1$ are coprime).

We also perform a first step in the direction of non-coprime k and $n + 1$ by constructing a minimal S_3 -invariant Lefschetz exceptional collection in $\mathcal{D}(X_3^2)$ (including a proof of its fullness).

Besides that we also solve the Problem 1.2 for $n = 1$, that is, construct a rectangular S_k -invariant Lefschetz collection of length 2 in $\mathcal{D}(X_k^1)$ when k is odd, and a minimal Lefschetz collection when k is even. However, this case is much more simple than the case $k = 3$ discussed above.

An interesting feature of the Lefschetz collections that we construct in Theorem 3.2 is that they resemble very much the minimal Lefschetz collections in the derived categories of the Grassmannians $\text{Gr}(k, n+1+k)$ constructed by Anton Fonarev, see [2]. It would be very interesting to understand the relations between these, since on one hand, this suggests a possible solution to the Problem 1.2 for other values of k (by considering analogues of Fonarev's collections), and on the other hand, a solution to the Problem 1.2 can help in dealing with the Grassmannians $\text{Gr}(k, n)$ when k and n are not coprime (in this case there is no rectangular collection on the Grassmannian, and a minimal collection is not quite known).

This paper is organized as follows. In Section 2 we recall the definitions of full exceptional collections, Lefschetz and rectangular decompositions, and Elagin's Theorem. In Section 3 we construct an S_k -invariant exceptional collection in $\mathcal{D}(X_k^n)$ and discuss numerical restrictions for the existence of a rectangular Lefschetz collection and some numerical bounds for a minimal Lefschetz collection. Finally, in Section 4 we prove fullness of the constructed collections for X_k^1 , X_3^{3p} , X_3^{3p+1} and X_3^2 respectively.

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2 Preliminaries

Given an algebraic variety X we denote the bounded derived category $\mathcal{D}^b(\text{coh}(X))$ of coherent sheaves on X by $\mathcal{D}(X)$. In this paper we concentrate on the case when X

is a power of a projective space

$$X = X_k^n = (\mathbb{P}^n)^k,$$

In some cases, we will omit the indices k and n and write $\mathcal{D}(X)$ instead $\mathcal{D}(X_k^n)$.

2.1 Exceptional collections in $\mathcal{D}(X_k^n)$

Clearly, X_k^n is a smooth projective variety with $\dim(X) = kn$. Its Picard group is isomorphic to $\text{Pic}(X_k^n) \cong \mathbb{Z}^k$ and has a basis consisting of the pullbacks of hyperplane classes of the factors. For $a = (a_1, \dots, a_k) \in \mathbb{Z}^k$ we write

$$\mathcal{O}(a) = \mathcal{O}(a_1, \dots, a_k) = \mathcal{O}(a_1) \boxtimes \dots \boxtimes \mathcal{O}(a_k)$$

for the corresponding line bundle on X_k^n . We note that by the Künneth formula

$$\text{Ext}^\bullet(\mathcal{O}(a), \mathcal{O}(b)) \cong \bigotimes_{i=1}^k \text{Ext}^\bullet(\mathcal{O}(a_i), \mathcal{O}(b_i)). \quad (2.1)$$

In particular, any line bundle on X_k^n is exceptional, and the line bundles $\mathcal{O}(a)$ and $\mathcal{O}(b)$ are semiorthogonal, i.e., $\text{Ext}^\bullet(\mathcal{O}(a), \mathcal{O}(b))$ is equal to 0, if and only if the pair $(\mathcal{O}(a_i), \mathcal{O}(b_i))$ on \mathbb{P}^n is semiorthogonal for at least one i . In view of Bott's formula for the cohomology of line bundles on a projective space, we can rewrite the semiorthogonality condition as

$$\text{Ext}^\bullet(\mathcal{O}(a), \mathcal{O}(b)) = 0 \text{ if and only if } 0 < a_i - b_i \leq n \text{ for some } 1 \leq i \leq k. \quad (2.2)$$

This property allows to verify easily semiorthogonality of collections of line bundles. For fullness, the following observations are useful.

For a subset $I \subset \{1, \dots, k\}$ of indices define the set $[0, n]^I \subset \text{Pic}(X_k^n)$ as

$$[0, n]^I = \{a \in \mathbb{Z}^k \mid a_i \in [0, n] \text{ if } i \in I \text{ and } a_i = 0 \text{ if } i \notin I\}.$$

If $I = \{1, \dots, k\}$, then denote $[0, n]^I$ by $[0, n]^k$.

Theorem 2.1. *The collection $\{\mathcal{O}(a)\}_{a \in [0, n]^k}$ (lexicographically ordered) is a full exceptional collection in $\mathcal{D}(X_k^n)$.*

Proof. Semiorthogonality of the collection follows easily from (2.2). For fullness we refer to [11]. \square

We will also need the following simple consequence of the fullness of the above collection.

Corollary 2.2. *Let \mathcal{T} be a triangulated subcategory of $\mathcal{D}(X_k^n)$. Assume that for some subset $I \subset \{1, \dots, k\}$ and some $a \in \text{Pic}(X_k^n)$ one has $\mathcal{O}(a+b) \in \mathcal{T}$ for any $b \in [0, n]^I$. Then the same holds true for any $b \in \mathbb{Z}^I$.*

Proof. First assume $a = 0$. Then the collection $\{\mathcal{O}(b)\}_{b \in [0, n]^I}$ is just the pullback of the full exceptional collection in

$$X_I^n = \prod_{i \in I} \mathbb{P}^n$$

with respect to the natural projection $X_k^n \rightarrow X_I^n$. Consequently, by Theorem 2.1 the category \mathcal{T} contains the pullback of any line bundle on X_I^n , and this is just the claim of the lemma in this case.

For arbitrary a just note that $\{\mathcal{O}(a+b)\}_{b \in [0, n]^I}$ is the twist of $\{\mathcal{O}(b)\}_{b \in [0, n]^I}$ by $\mathcal{O}(a)$. Since a line bundle twist is an autoequivalence of $\mathcal{D}(X_k^n)$, the general claim follows. \square

2.2 Semiorthogonal and Lefschetz decompositions

In some cases it is slightly more convenient to work with semiorthogonal decompositions than with exceptional collections. Here, we remind the corresponding definitions.

Definition 2.3. Suppose $\mathcal{B}_0, \dots, \mathcal{B}_d$ are full triangulated subcategories of \mathcal{T} such that $\text{Hom}(\mathcal{B}_i, \mathcal{B}_j) = 0$ for all $i > j$. We say that $\mathcal{B}_0, \dots, \mathcal{B}_d$ form a *semiorthogonal decomposition* of \mathcal{T} if the smallest full triangulated subcategory of \mathcal{T} containing \mathcal{B}_i for all i coincides with \mathcal{T} .

We will denote a semiorthogonal decomposition by

$$\langle \mathcal{B}_0, \dots, \mathcal{B}_d \rangle = \mathcal{T}.$$

Assume that $\mathcal{T} = \mathcal{D}(X)$ and a line bundle \mathcal{L} on X is given. For an object F in $\mathcal{D}(X)$ we denote

$$F(i) := F \otimes \mathcal{L}^i$$

the image of F under the autoequivalence of \mathcal{T} given by the \mathcal{L}^i -twist, and for a subcategory $\mathcal{A} \subset \mathcal{T}$ we denote

$$\mathcal{A}(i) := \{F(i) \mid F \in \mathcal{A}\} \subset \mathcal{T}$$

the image of \mathcal{A} under this autoequivalence.

A semiorthogonal decomposition

$$\mathcal{D}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_d(d) \rangle \quad (2.3)$$

is called *Lefschetz decomposition* if $\mathcal{A}_{i+1} \subset \mathcal{A}_i$ for all $0 \leq i < d$.

We say that a Lefschetz decomposition (2.3) is *rectangular* if $\mathcal{A}_0 = \dots = \mathcal{A}_d$. A rectangular decomposition can be simply written as

$$\mathcal{D}(X) = \langle \mathcal{A}, \mathcal{A}(1), \dots, \mathcal{A}(d) \rangle, \quad (2.4)$$

where $\mathcal{A} = \mathcal{A}_0$.

2.3 Exceptional collections in equivariant derived categories

Assume a finite group G acts on a smooth projective variety X . The following result of Alexei Elagin gives a way to construct an exceptional collection in the equivariant derived category $\mathcal{D}_G(X)$.

Theorem 2.4 ([1, Theorem 2.3]). *Assume that E_1, \dots, E_r is a full G -invariant exceptional collection in $\mathcal{D}(X)$, that is, the G -action induces a permutation of objects of the collection. Assume s is the number of G -orbits on $\{E_1, \dots, E_r\}$ and let E_{i_1}, \dots, E_{i_s} , $i_1 < \dots < i_s$ be their representatives. For each $1 \leq t \leq r$ let H_t be the stabilizer of E_{i_t} and assume that for each t the object E_{i_t} admits an H_t -equivariant structure. Then there exists a full exceptional collection of the equivariant category*

$$\mathcal{D}_G(X) = \langle \bar{E}_{i_1}^{(1)}, \dots, \bar{E}_{i_1}^{(m_1)}, \dots, \bar{E}_{i_s}^{(1)}, \dots, \bar{E}_{i_s}^{(m_s)} \rangle.$$

Here $\bar{E}_{i_t}^{(j)} = E_{i_t} \otimes V_t^{(j)}$, where $V_t^{(1)}, \dots, V_t^{(m_t)}$ are all irreducible representations of H_t up to isomorphism, and we consider the natural G -equivariant structure on $\bar{E}_{i_t}^{(j)}$.

We note that any line bundle on $X = X_k^n$ has a natural equivariant structure with respect to the subgroup of S_k that stabilizes it. Indeed, for this it is enough to note that the line bundle $\mathcal{O}(i, i, \dots, i)$ is S_k -equivariant for each i . Thus, the above theorem applies to any exceptional collection formed by line bundles on X_k^n as soon as it is S_k -invariant. To ensure that the resulting collection in the equivariant category is Lefschetz we will use the following evident observation.

Corollary 2.5. *Assume that L is a G -equivariant line bundle on X and E_1, \dots, E_r is a Lefschetz exceptional collection with respect to L which satisfies the assumptions of Theorem 2.4. Then the corresponding exceptional collection in the equivariant category is also Lefschetz. Moreover, if the original collection is rectangular then so is the equivariant one with the same number of blocks.*

Proof. Let E_1, \dots, E_{r_0} be the starting block of the original Lefschetz collection and s_0 be the number of G -orbits in the block E_1, \dots, E_{r_0} . Then it is straightforward to check that $\bar{E}_{i_1}^{(1)}, \dots, \bar{E}_{i_1}^{(m_1)}, \dots, \bar{E}_{i_{s_0}}^{(1)}, \dots, \bar{E}_{i_{s_0}}^{(m_{s_0})}$ can serve as the starting block of a Lefschetz collection in $\mathcal{D}_G(X)$. From the equivariance of L it is also clear that the property of being rectangular is preserved by this construction. \square

Thus, to construct a (rectangular) Lefschetz collection in $\mathcal{D}_{S_k}(X_k^n)$ it is enough to construct a (rectangular) S_k -invariant Lefschetz collection in $\mathcal{D}(X)$ consisting of line bundles. This is what we do in the next sections.

3 A Lefschetz collection and numerical minimality

In this section we construct a Lefschetz S_k -invariant exceptional collection on X_k^n and find some numerical conditions for minimality of a Lefschetz exceptional collection. In what follows we always denote

$$h := n + 1.$$

3.1 A Lefschetz collection

We consider the following two S_k -invariant subsets of the lattice $\text{Pic}(X_k^n) = \mathbb{Z}^k$:

$$\widehat{\mathbb{E}}_k^n = \{S_k \cdot (c_1, \dots, c_k) \mid c_1 \geq \dots \geq c_k = 0 \text{ and } kc_i \leq h(k - i)\}, \quad (3.1)$$

and

$$\mathbb{E}_k^n = \{S_k \cdot (c_1, \dots, c_k) \mid c_1 \geq \dots \geq c_k = 0 \text{ and } kc_i < h(k - i) \text{ for } i \neq k\}. \quad (3.2)$$

Note that the only difference in the definitions of \mathbb{E}_k^n and $\widehat{\mathbb{E}}_k^n$ is that a non-strict inequality in (3.1) is replaced by a strict one in (3.2). In particular,

$$\mathbb{E}_k^n \subset \widehat{\mathbb{E}}_k^n,$$

and if all the fractions $h(k - i)/k$ for $1 \leq i \leq k - 1$ are non-integer, i.e., when h and k are coprime, we have an equality $\mathbb{E}_k^n = \widehat{\mathbb{E}}_k^n$.

We consider the above two sets with the lexicographical order restricted from \mathbb{Z}^k .

Lemma 3.1. *The set of line bundles $\mathcal{O}(c)$ for $c \in \widehat{\mathbb{E}}_k^n$ is an exceptional S_k -invariant collection with respect to the lexicographical order on $\widehat{\mathbb{E}}_k^n$.*

Proof. Follows from the evident inclusion $\widehat{\mathbb{E}}_k^n \subset [0, n]^k$ and Theorem 2.1. \square

Since the set \mathbb{E}_k^n is an S_k -invariant subset in $\widehat{\mathbb{E}}_k^n$, the collection of line bundles $\mathcal{O}(c)$ for $c \in \mathbb{E}_k^n$ is also an exceptional S_k -invariant collection with respect to the lexicographical order on \mathbb{E}_k^n .

We denote by

$$\mathcal{A} = \langle \mathcal{O}(c) \rangle_{c \in \mathbb{E}_k^n} \quad \text{and} \quad \widehat{\mathcal{A}} = \langle \mathcal{O}(c) \rangle_{c \in \widehat{\mathbb{E}}_k^n} \quad (3.3)$$

the subcategories in $\mathcal{D}(X_k^n)$ generated by the above exceptional collections. Furthermore, for each $c = (c_1, c_2, \dots, c_k) \in \mathbb{Z}^k$ we denote

$$c(i) = (c_1 + i, c_2 + i, \dots, c_k + i),$$

so that $\mathcal{O}(c(i)) \cong \mathcal{O}(c) \otimes \mathcal{O}(i, i, \dots, i)$.

Theorem 3.2. *For any $h > i > j \geq 0$ we have $\text{Hom}(\mathcal{A}(i), \widehat{\mathcal{A}}(j)) = 0$.*

In particular, the category

$$\mathcal{T} := \langle \widehat{\mathcal{A}}, \mathcal{A}(1) \dots \mathcal{A}(n) \rangle \subset \mathcal{D}(X_k^n) \quad (3.4)$$

is generated by an S_k -invariant Lefschetz collection.

Proof. Obviously, it is enough to prove the theorem for $j = 0, i > 0$. In other words, it is enough to prove that for any $a \in \mathbb{E}_k^n, b \in \widehat{\mathbb{E}}_k^n$ we have $\text{Hom}(\mathcal{O}(a(i)), \mathcal{O}(b)) = 0$. Furthermore, by S_k -invariance of the set \mathbb{E}_k^n , we can assume that $a_1 \geq \dots \geq a_k = 0$.

First, assume that $a_1 + i < h$. Then $a_t + i < h$ for all t . On the other hand, by definition of $\widehat{\mathbb{E}}_k^n$ we have $b_t = 0$ for some t . Then $0 < (a_t + i) - b_t \leq n$, hence we have $\text{Hom}(\mathcal{O}(a(i)), \mathcal{O}(b)) = 0$ by (2.2).

So, from now on we can assume that $a_1 + i \geq h$. At the same time $a_k + i = i < h$. Let r be the maximal index such that

$$a_r + i \geq h \quad \text{and} \quad a_{r+1} + i < h.$$

The first of these inequalities implies

$$i \geq h - a_r > h - h(k - r)/k = hr/k.$$

On the other hand, consider all t such that

$$b_t \leq hr/k.$$

Note that by definition of $\widehat{\mathbb{E}}_k^n$ there are at least $r + 1$ such t (corresponding to the smallest $r + 1$ values of b_t), hence for some of these we have $t \geq r + 1$. For such t we have

$$hr/k < i \leq a_t + i \leq a_{r+1} + i < h \quad \text{and} \quad 0 \leq b_t \leq hr/k.$$

In particular, $0 < a_t + i - b_t < h$. Hence $\text{Hom}(\mathcal{O}(a(i)), \mathcal{O}(b)) = 0$ by (2.2). \square

Below we will prove that the category \mathcal{T} defined by (3.4) is equal to $\mathcal{D}(X_k^n)$ in case $n \not\equiv 2 \pmod{3}$, $k = 3$ (Subection 4.2) and $n = 1$ and any k (Subection 4.1). In particular, for $\text{gcd}(h, k) = 1$ the right side of (3.4) gives a rectangular Lefschetz decomposition of $\mathcal{D}(X_k^n)$.

However, in general the sum of the ranks of the Grothendieck groups of the components of (3.4) is less than the rank of the Grothendieck group of $\mathcal{D}(X_k^n)$, so it requires a modification. In Subsection 4.3 we show how such a modification can be performed for $n = 2$ and $k = 3$.

3.2 Numerical restrictions

We keep the notation $h = n + 1$ and let V be a vector space of dimension h , so that $\mathbb{P}(V) = \mathbb{P}^n$. Denote by

$$K_{\mathbb{C}} := K_0(\mathbb{P}(V)) \otimes \mathbb{C},$$

the complexified Grothendieck group of coherent sheaves on $\mathbb{P}(V)$. It is also a vector space of dimension h . Moreover, we have

$$K_0(X_k^n) \otimes \mathbb{C} = K_0(\mathbb{P}(V)^k) \otimes \mathbb{C} \cong K_{\mathbb{C}}^{\otimes k}.$$

The group $\text{GL}(K_{\mathbb{C}})$ acts naturally on the vector space $K_{\mathbb{C}}^{\otimes k}$, and the group S_k acts on $K_{\mathbb{C}}^{\otimes k}$ by permutation of factors (this action is induced by the action of S_k on X_k^n). These two actions commute, therefore $K_{\mathbb{C}}^{\otimes k}$ is a $(\text{GL}(K_{\mathbb{C}}), S_k)$ -bimodule. In the next lemma we describe a decomposition of $K_{\mathbb{C}}^{\otimes k}$ into a direct sum of irreducible representations, provided by the Schur–Weyl duality.

We denote by $\rho(h, k)$ the set of all Young diagrams of k boxes with at most h rows, by $\Sigma^\lambda K_{\mathbb{C}}$ the irreducible representation of $\text{GL}(K_{\mathbb{C}})$ corresponding to the Young diagram λ , (it is also known as the Schur functor associated with λ), and by R_{λ^T} the irreducible representation of S_k corresponding to the transposed Young diagram λ^T .

Lemma 3.3 (Schur–Weyl duality, [4]). *There exists an isomorphism of $\text{GL}(K_{\mathbb{C}}) \times S_k$ representations:*

$$K_{\mathbb{C}}^{\otimes k} = \bigoplus_{\lambda \in \rho(h, k)} \Sigma^\lambda K_{\mathbb{C}} \otimes R_{\lambda^T}.$$

In other words, the decomposition of $K_{\mathbb{C}}^{\otimes k}$ into a direct sum of irreducible S_k -representations contains $\dim(\Sigma^\lambda K_{\mathbb{C}})$ copies of the irreducible representation R_{λ^T} .

The above decomposition allows to give a simple necessary condition for the existence of a rectangular S_k -invariant Lefschetz collection in $\mathcal{D}(X_k^n)$. In what follows we call it the *divisibility criterion*.

Corollary 3.4. *If a rectangular S_k -invariant Lefschetz decomposition of length h of $\mathcal{D}(X_k^n)$ exists, then h divides $\dim \Sigma^\lambda K_{\mathbb{C}}$ for all $\lambda \in \rho(h, k)$.*

Proof. Assume $\mathcal{D}(X_k^n) = \langle \mathcal{A}_0, \mathcal{A}_0(1), \dots, \mathcal{A}_0(n) \rangle$ is a rectangular S_k -invariant Lefschetz decomposition. Then we have

$$K_{\mathbb{C}}^{\otimes k} = K_0(X_k^n) \otimes \mathbb{C} = (K_0(\mathcal{A}_0) \otimes \mathbb{C})^{\oplus h}.$$

Since \mathcal{A}_0 is S_k -invariant, $K_0(\mathcal{A}_0) \otimes \mathbb{C} \subset K_0(X_k^n) \otimes \mathbb{C}$ is an S_k -subrepresentation, so the above equality shows that the multiplicity of each irreducible summand of $K_{\mathbb{C}}^{\otimes k}$ is divisible by h . \square

The same argument as above gives the following bound for the ranks of the Grothendieck groups of components of an arbitrary S_k -invariant Lefschetz decomposition of $\mathcal{D}(X_k^n)$. Denote by $\lfloor t \rfloor$ and $\lceil t \rceil$ the lower and upper integral parts of t .

Corollary 3.5. *Suppose $\mathcal{D}(X_k^n) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_n(n) \rangle$ is a Lefschetz S_k -invariant decomposition. Let r_i be the rank of $K_0(\mathcal{A}_i)$. Then*

$$r_0 \geq \sum_{\lambda \in \rho(h, k)} \left\lfloor \frac{\dim \Sigma^\lambda K_{\mathbb{C}}}{h} \right\rfloor \dim R_{\lambda^T} \quad \text{and} \quad r_n \leq \sum_{\lambda \in \rho(h, k)} \left\lceil \frac{\dim \Sigma^\lambda K_{\mathbb{C}}}{h} \right\rceil \dim R_{\lambda^T}.$$

Proof. Suppose that

$$K_0(\mathcal{A}_i) \otimes \mathbb{C} = \sum_{\lambda \in \rho(h, k)} R_{\lambda^T}^{\oplus a_i^\lambda}.$$

From 3.3 we get that $\sum_{0 \leq i \leq n} a_i^\lambda = \dim \Sigma^\lambda K_{\mathbb{C}}$ for any λ . Since $\mathcal{A}_j \subset \mathcal{A}_i$ for any $i < j$, we have $a_j^\lambda \leq a_i^\lambda$ for any λ and $i < j$. Thus

$$a_0^\lambda \geq \left\lfloor \frac{\dim \Sigma^\lambda K_{\mathbb{C}}}{h} \right\rfloor \quad \text{and} \quad a_n^\lambda \leq \left\lceil \frac{\dim \Sigma^\lambda K_{\mathbb{C}}}{h} \right\rceil.$$

This completes the proof. \square

As an example we consider the case $n = 1$ and $k = 2m$.

Corollary 3.6. *Suppose $\mathcal{D}(X_{2m}^1) = \langle \mathcal{A}_0, \mathcal{A}_1(1) \rangle$ is a Lefschetz S_{2m} -invariant decomposition. Let r_i be the rank of $K_0(\mathcal{A}_i)$. Then $r_0 - r_1 \geq \binom{2m}{m}$.*

In Subsection 4.1 we will show that the above inequality is sharp.

Proof. Any diagram in $\rho(2, 2m)$ is of the shape

$$\lambda(l) := (2m - l, l).$$

for some $0 \leq l \leq m$. By Weyl dimension formula we have

$$\dim(\Sigma^{\lambda(l)} K_{\mathbb{C}}) = 2m - 2l + 1,$$

and by the hook-length formula

$$\dim R_{\lambda(l)^T} = \frac{2m! (2m - 2l + 1)}{(2m - l + 1)! l!} = \frac{2m - 2l + 1}{2m + 1} \binom{2m + 1}{l}.$$

For each l we have $\lceil \frac{2m-2l+1}{2} \rceil - \lfloor \frac{2m-2l+1}{2} \rfloor = 1$, hence by Corollary 3.5, we have

$$\begin{aligned} r_0 - r_1 &\geq \sum_{l=0}^m \dim R_{\lambda(l)^T} = \sum_{l=0}^m \frac{2m - 2l + 1}{2m + 1} \binom{2m + 1}{l} \\ &= \sum_{l=0}^m \binom{2m + 1}{l} - 2 \sum_{l=0}^m \binom{2m}{l-1}. \end{aligned}$$

The first sum is equal to 2^{2m} , and the second is equal to $2^{2m} - \binom{2m}{m}$, so we conclude that $r_0 - r_1 \geq \binom{2m}{m}$. \square

If we restrict to the case of S_k -invariant Lefschetz collections, the inequalities of Corollary 3.5 can be, in general, improved, because in this case each $K_0(\mathcal{A}_i)$ is a *permutation representation* of S_k .

As an example, we consider the case $k = 3$, $n = 2$ (so that $h = 3$). In this case the set $\rho(3, 3)$ consists of three Young diagrams: (3) , $(2, 1)$, and $(1, 1, 1)$, and

$$\dim \Sigma^{(3)} K_{\mathbb{C}} = 10, \quad \dim \Sigma^{(2,1)} K_{\mathbb{C}} = 8, \quad \dim \Sigma^{(1,1,1)} K_{\mathbb{C}} = 1,$$

while

$$\dim R_{(3)^T} = 1, \quad \dim R_{(2,1)^T} = 2, \quad \dim R_{(1,1,1)^T} = 1.$$

Consequently, if $\mathcal{D}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2) \rangle$ is an S_k -invariant Lefschetz decomposition and r_i is the rank of $K_0(\mathcal{A}_i)$, then by Corollary 3.5 we have

$$r_0 \geq \left\lfloor \frac{10}{3} \right\rfloor \cdot 1 + \left\lfloor \frac{8}{3} \right\rfloor \cdot 2 + \left\lfloor \frac{1}{3} \right\rfloor \cdot 1 = 4 + 6 + 1 = 11$$

and

$$r_2 \leq \left\lfloor \frac{10}{3} \right\rfloor \cdot 1 + \left\lfloor \frac{8}{3} \right\rfloor \cdot 2 + \left\lfloor \frac{1}{1} \right\rfloor \cdot 1 = 3 + 4 + 0 = 7.$$

On the other hand, we can prove the following result.

Proposition 3.7. *Assume $\langle \mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2) \rangle = \mathcal{D}(X_3^2)$ is a Lefschetz decomposition, such that each component \mathcal{A}_i is generated by an S_3 -invariant exceptional collection $\{E_{i,j}\}_{j=1}^{r_i}$. Then $r_0 \geq 13$ and $r_2 \leq 7$.*

Proof. The classes of exceptional objects $E_{i,j}$ form a basis of the Grothendieck group $K_0(\mathcal{A}_i)$. Since the collection is S_3 -invariant, this basis is permuted by the group action, i.e., $K_0(\mathcal{A}_i) \otimes \mathbb{C}$ is a sum of permutation representations. There are three such representations:

$$\mathbb{C}[S_3] \cong R_{(3)^T} \oplus R_{(2,1)^T}^{\oplus 2} \oplus R_{(1,1,1)^T}, \quad \mathbb{C}[S_3/S_2] \cong R_{(3)^T} \oplus R_{(2,1)^T}, \quad \mathbb{C}[S_3/S_3] \cong R_{(3)^T}.$$

Note that $R_{(1,1,1)^T}$ only appears as a summand of $\mathbb{C}[S_3]$.

On the other hand, by Lemma 3.3 we have

$$K_0(\mathcal{A}_0) \oplus K_0(\mathcal{A}_1) \oplus K_0(\mathcal{A}_2) \cong K_0(\mathcal{D}(X_3^2)) \cong R_{(3)^T}^{\oplus 10} \oplus R_{(2,1)^T}^{\oplus 8} \oplus R_{(1,1,1)^T}.$$

Finally, by the Lefschetz property, we have $K_0(\mathcal{A}_2) \subset K_0(\mathcal{A}_1) \subset K_0(\mathcal{A}_0)$. This means that $R_{(1,1,1)^T}$ has to be a direct summand of $K_0(\mathcal{A}_0)$, hence $K_0(\mathcal{A}_0)$ contains the entire regular representation $\mathbb{C}[S_3]$, and implies $r_0 \geq r_1 + 6 \geq r_2 + 6$. Therefore

$$3r_0 \geq r_0 + (r_1 + 6) + (r_2 + 6) = 27 + 6 + 6 = 39,$$

and hence $r_0 \geq 13$.

Since

$$2r_2 \leq r_1 + r_2 = 27 - r_0 \leq 27 - 13 = 14,$$

we have $r_2 \leq 7$. □

In Section 4.3 we will construct a full S_3 -invariant Lefschetz exceptional collection in $\mathcal{D}(X_3^2)$ with $(r_0, r_1, r_2) = (13, 7, 7)$.

3.3 Verifications of divisibility

To check divisibility of the dimensions of $\Sigma^\lambda K_{\mathbb{C}}$ the following corollary of Littlewood–Richardson rule is useful.

Lemma 3.8 ([4]). *Let $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ be a Young diagram. Then*

$$\Lambda^{\mu_1} K_{\mathbb{C}} \otimes \Lambda^{\mu_2} K_{\mathbb{C}} \otimes \dots \otimes \Lambda^{\mu_m} K_{\mathbb{C}} \cong \Sigma^{\mu^T} K_{\mathbb{C}} \oplus \left(\bigoplus_{\lambda < \mu^T} (\Sigma^\lambda K_{\mathbb{C}})^{\oplus c(\lambda, \mu)} \right),$$

where $<$ stands for the dominance order [3, Section 2.2], and $c(\lambda, \mu)$ are nonnegative integers.

The next proposition gives some necessary and sufficient conditions for divisibility.

Proposition 3.9. (1) *If h divides k , then $\dim \Sigma^\lambda K_{\mathbb{C}}$ is not divisible by h for some Young diagram $\lambda \in \rho(h, k)$.*

(2) *If k is not divisible by h and for any integer r such that $1 \leq r \leq \min(k, h-1)$ the binomial coefficient $\binom{h}{r}$ is divisible by h , then $\dim(\Sigma^\lambda K_{\mathbb{C}})$ is divisible by h for any Young diagram $\lambda \in \rho(h, k)$.*

Proof. (1) Suppose $k = ht$. Consider the Young diagram ξ with t columns of height h . Then $\Sigma^\xi K_{\mathbb{C}} \cong (\det K_{\mathbb{C}})^{\otimes t}$, hence $\dim(\Sigma^\xi K_{\mathbb{C}}) = 1$. Since $h = n + 1 \geq 2$, we see that $\dim(\Sigma^\xi K_{\mathbb{C}})$ is not divisible by h .

(2) We use ascending induction on Young diagrams in $\rho(h, k)$ with respect to the dominance order.

Base. Suppose $k = ht + r$, where $t \in \mathbb{Z}_{\geq 0}$ and $1 \leq r \leq h-1$. It is clear that the smallest diagram $\omega \in \rho(h, k)$ is the diagram with t columns of height h and one column of height r . Then $\Sigma^\omega K_{\mathbb{C}} \cong (\det K_{\mathbb{C}})^{\otimes t} \otimes \Lambda^r K_{\mathbb{C}}$, hence $\dim(\Sigma^\omega K_{\mathbb{C}}) = \binom{h}{r}$, which is divisible by h by the assumption of the proposition.

Induction step. Consider a diagram μ such that $\mu^T \in \rho(h, k)$. Suppose that for any $\lambda < \mu^T$, $\lambda \in \rho(h, k)$, the dimension $\dim(\Sigma^\lambda K_{\mathbb{C}})$ is divisible by h . Let us prove that $\dim(\Sigma^{\mu^T} K_{\mathbb{C}})$ is also divisible by h . Using Lemma 3.8, we get

$$\begin{aligned} \dim(\Sigma^{\mu^T} K_{\mathbb{C}}) &= \prod_{i=1}^m \dim(\Lambda^{\mu_i} K_{\mathbb{C}}) - \dim \left(\bigoplus_{\lambda < \mu^T} (\Sigma^\lambda K_{\mathbb{C}})^{\oplus c(\lambda)} \right) = \\ &= \prod_{i=1}^m \binom{h}{\mu_i} - \sum_{\lambda < \mu^T} c(\lambda, \mu) \cdot \dim(\Sigma^\lambda K_{\mathbb{C}}). \end{aligned}$$

By induction hypothesis, $\sum_{\lambda < \mu^T} c(\lambda, \mu) \cdot \dim(\Sigma^\lambda K_{\mathbb{C}})$ is divisible by h . Since k is not divisible by h we see that there exist i such that $1 \leq \mu_i \leq h - 1$. Clearly, $\mu_i \leq k$. Therefore, $1 \leq \mu_i \leq \min(k, h - 1)$. Thus $\binom{h}{\mu_i}$ is divisible by h by the assumption of the theorem. Hence $\prod_{i=1}^m \binom{h}{\mu_i}$ and consequently $\dim(\Sigma^{\mu^T} K_{\mathbb{C}})$ is divisible by h . \square

Note that to prove the inductive step we need only one $\binom{h}{\mu_i}$ to be divisible by h for each $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ with $\mu^T \in \rho(h, k)$. This suggests that the assumption of Theorem 3.9(2) can be weakened.

Next, we discuss consequences of the above in the case $k = 3$.

Proposition 3.10. *If $n = 3p + 2$, then the category $\mathcal{D}(X_3^n)$ does not have a rectangular S_3 -invariant Lefschetz decomposition of length $n + 1$.*

Proof. If $n = 3p + 2$, then $n > 1$ and $h = n + 1 \geq 3$. Thus for $k = 3$ all Young diagrams of three boxes are in $\rho(h, k)$. These diagrams are $(1, 1, 1)$, $(2, 1)$, and (3) . The dimensions of the corresponding Schur functors are given by

$$\begin{aligned} \dim \Sigma^{(3)} K_{\mathbb{C}} &= \frac{(h+2)(h+1)h}{6}, \\ \dim \Sigma^{(2,1)} K_{\mathbb{C}} &= \frac{(h+1)h(h-1)}{3}, \\ \dim \Sigma^{(1,1,1)} K_{\mathbb{C}} &= \frac{h(h-1)(h-2)}{6}. \end{aligned}$$

(see for instance the dimension formula from [4, Exercise 6.4]).

Thus, a necessary condition for the existence of a rectangular S_3 -invariant Lefschetz decomposition of length h is that the three numbers above are divisible by h . This is equivalent to the integrality of the fractions

$$\frac{(h-1)(h-2)}{6}, \quad \frac{(h+1)(h-1)}{3}, \quad \text{and} \quad \frac{(h+2)(h+1)}{6}.$$

It is easy to see that this condition holds if and only if h is not divisible by 3. Since $h = n + 1$ we obtain that this condition holds if and only if $n \neq 3p + 2$. \square

In other words, we can expect the existence of the desired rectangular decomposition only if $n = 3p$ or $n = 3p + 1$. In the Subsection 4.2 we prove that the desired rectangular decomposition exists in these cases.

4 Fullness

In this section we prove that the S_k -invariant Lefschetz collection (3.4) generates the category $\mathcal{D}(X_k^n)$ when $n = 1$ and any k (Subsection 4.1) or $k = 3$ and $n \not\equiv 2 \pmod{3}$ (Subsection 4.2) and moreover provides a minimal S_k -invariant Lefschetz collection in it. We also discuss the case $k = 3$, $n = 2$ (Subsection 4.3) that shows that in general collection (3.4) needs a modification.

4.1 Minimal Lefschetz decomposition for $\mathcal{D}(X_k^1)$

First, we consider the case $n = 1$. Recall the definition (3.3) of S_k -invariant subcategories $\mathcal{A} \subset \widehat{\mathcal{A}} \subset \mathcal{D}(X_k^1)$. In the case $n = 1$ it can be rewritten as

$$\mathcal{A} = \{a \in [0, 1]^k \mid \text{Card} \{i \mid a_i = 0\} > k/2\}, \quad (4.1)$$

$$\widehat{\mathcal{A}} = \{a \in [0, 1]^k \mid \text{Card} \{i \mid a_i = 0\} \geq k/2\}, \quad (4.2)$$

where Card stands for the cardinality of a set. If k is odd, $\mathcal{A} = \widehat{\mathcal{A}}$.

Theorem 4.1. *We have S_k -invariant Lefschetz decompositions*

$$\mathcal{D}(X_k^1) = \begin{cases} \langle \mathcal{A}, \mathcal{A}(1) \rangle, & \text{if } k = 2m + 1, \\ \langle \widehat{\mathcal{A}}, \mathcal{A}(1) \rangle, & \text{if } k = 2m. \end{cases}$$

Moreover, these are minimal Lefschetz collections.

Proof. By definition both subcategories \mathcal{A} and $\widehat{\mathcal{A}}$ are generated by S_k -invariant exceptional collections. Moreover, by Theorem 3.2 they are semiorthogonal. Thus for the first part of the theorem it is enough to show that $\widehat{\mathcal{A}}$ and $\mathcal{A}(1)$ generate $\mathcal{D}(X_k^1)$. For this we show that

$$\mathcal{O}(b) \in \mathcal{A}(1) \quad \text{if } b \in [0, 2]^k \text{ and } \text{Card} \{i \mid b_i = 1\} \geq m + 1. \quad (4.3)$$

Indeed, by definition of $\mathcal{A}(1)$ we have

$$\mathcal{O}(b) \in \mathcal{A}(1) \quad \text{if } b \in [1, 2]^k \text{ and } \text{Card} \{i \mid b_i = 1\} \geq m + 1. \quad (4.4)$$

Note that $\mathcal{O}(1, \dots, 1) \in \mathcal{A}(1)$. We apply Corollary 2.2 to $a = (1, \dots, 1)$ and any I of cardinality m . It proves that for any $b \in [0, 2]^k$ such that $b_i = 1$ for $i \notin I$ we have $\mathcal{O}(b) \in \mathcal{A}(1)$. This proves (4.3).

Combining (4.3) with the definition of $\widehat{\mathcal{A}}$, we deduce that all line bundles $\mathcal{O}(a)$ with $a \in [0, 1]^k$ are contained in the subcategory of $\mathcal{D}(X_k^1)$ generated by $\widehat{\mathcal{A}}$ and $\mathcal{A}(1)$. By Theorem 2.1 this proves the first part of Theorem 4.1.

It remains to show the minimality of the constructed Lefschetz collection. For odd k the collection is rectangular of length $d = 2$, hence minimal (see [9, Subsection 2.1]), so there is nothing to prove. For even k we note that the ranks of the Grothendieck groups of $\widehat{\mathcal{A}}$ and \mathcal{A} are given by

$$r_0 = 2^{2m} + \frac{1}{2} \binom{2m}{m} \quad \text{and} \quad r_1 = 2^{2m} - \frac{1}{2} \binom{2m}{m}$$

respectively. In particular, $r_0 - r_1 = \binom{2m}{m}$, hence the collection is minimal by Corollary 3.6. \square

4.2 Lefschetz decompositions for $\mathcal{D}(X_3^{3p})$ and $\mathcal{D}(X_3^{3p+1})$

In this subsection we prove the following

Theorem 4.2. *Let $n = 3p$ or $n = 3p+1$. The categories \mathcal{A} defined by (3.3) and (3.2) generate an S_3 -invariant rectangular Lefschetz collection*

$$\mathcal{D}(X_3^n) = \langle \mathcal{A}, \mathcal{A}(1), \dots, \mathcal{A}(n) \rangle.$$

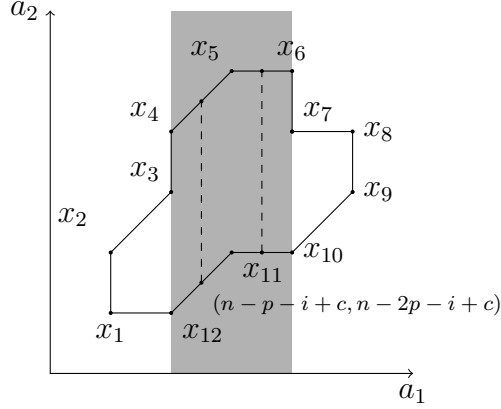
The proof takes the rest of the section. As in the case of Theorem 4.6 we denote by \mathcal{T} the triangulated subcategory of $\mathcal{D}(X)$ generated by the above Lefschetz collection. Note that \mathcal{T} is S_3 -invariant. By subsequent applications of Corollary 2.2 we will show that many other line bundles are contained in \mathcal{T} , until in the end we have $\mathcal{O}(a) \in \mathcal{T}$ for all $a \in [0, n]^3$ and conclude by Theorem 2.1.

We will prove the statement of Theorem 4.2 for $n = 3p$ and $n = 3p+1$ in parallel. Denote by T the set of all $a \in \mathbb{Z}^3$ such that $\mathcal{O}(a) \in \mathcal{T}$. Note that T is S_3 -invariant.

Proposition 4.3. *For each $i \in [n-p, n]$ and $a \in \mathbb{Z}^3$ with $a_3 = i$, we have $a \in T$.*

Proof. Let us fix $i \in [n-p, n]$. Consider a plane and mark on it all integral points (a_1, a_2) such that $(a_1, a_2, i) \in T$. By definition (3.2) all integral points of the polygon in Figure 1 are marked. The coordinates of its vertices x_1, \dots, x_{12} are listed in the table below.

Figure 1: Illustration for Step 1 of Proposition 4.3.



	$n = 3p$	$n = 3p + 1$
x_1	$(i - 2p, i - 2p)$	$(i - 2p - 1, i - 2p - 1)$
x_2	$(i - 2p, i - p)$	$(i - 2p - 1, i - p - 1)$
x_3	$(i - p, i)$	$(i - p, i)$
x_4	$(i - p, i + p)$	$(i - p, i + p + 1)$
x_5	$(i, i + 2p)$	$(i, i + 2p + 1)$
x_6	$(i + p, i + 2p)$	$(i + p, i + 2p + 1)$
x_7	$(i + p, i + p)$	$(i + p, i + p)$
x_8	$(i + 2p, i + p)$	$(i + 2p + 1, i + p)$
x_9	$(i + 2p, i)$	$(i + 2p + 1, i)$
x_{10}	$(i + p, i - p)$	$(i + p + 1, i - p)$
x_{11}	$(i, i - p)$	$(i, i - p)$
x_{12}	$(i - p, i - 2p)$	$(i - p - 1, i - 2p - 1)$

Our goal is to show that all integral points of the plane are in T . We do this in several steps.

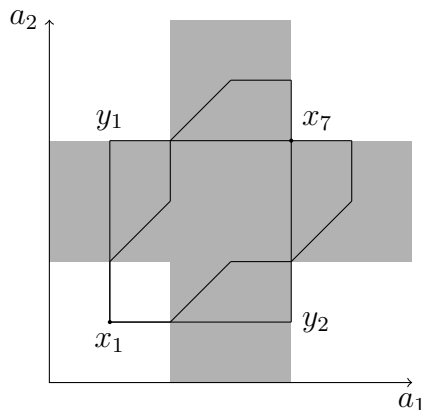
Step 1. For each $c \in [0, p]$ we apply Corollary 2.2 with a any integral point on the union of the edges $[x_{10}, x_{11}]$ and $[x_{11}, x_{12}]$ of the polygon in Figure 1, i.e., with $a = (i + c, i - p, i)$, $I = \{2\}$ or $a = (i + p + c, i - 2p + c, i)$, $I = \{2\}$. Each dashed segment in Figure 1 contains n integral points corresponding to line bundles contained in \mathcal{T} .

By Corollary 2.2 we conclude that all points $(i + c, t, i)$, $(i + p + c, i - 2p + c, i)$ are in T for any $t \in \mathbb{Z}$. In other words, all points in the grey vertical stripe in Figure 1 are in T .

Step 2. Using S_3 -symmetry of T we conclude that all points in the horizontal grey stripe on Figure 2 are in T .

Step 3. Combining the results of Step 1 and Step 2 above, we see that $a \in T$ for any a such that $(a_1, a_2) \in [i + p - n, i + p]^2$, $a_3 = i$. In other words, all points in the square with vertices x_1, y_1, x_7, y_2 in Figure 2 are in T . Therefore we can apply Corollary 2.2 with $a = (i + p - n, i + p - n, i)$ and $I = \{1, 2\}$. We conclude that if $a_3 = i$, then $a \in T$.

Figure 2: Illustration for Steps 2–3 of Proposition 4.3.

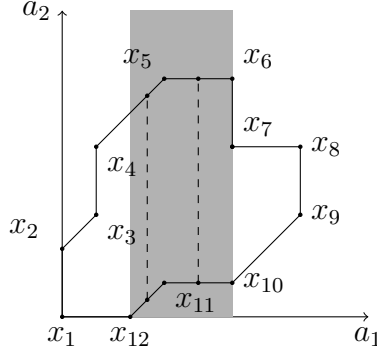


This completes the proof of Proposition 4.3. □

Proposition 4.4. For any $i \in [p, n - p - 1]$, $a \in \mathbb{Z}^3$ such that $a_3 = i$, we have $a \in T$.

Proof. Let us fix $i \in [p, n - p - 1]$. Consider a plane and mark on it all integral points (a_1, a_2) such that $\mathcal{O}(a_1, a_2, i) \in \mathcal{T}$. By definition of (3.2) all integral points of the polygon in Figure 3 are marked. The coordinates of its vertices x_1, \dots, x_{12} are listed in the table below.

Figure 3: Illustration for Step 1 of Proposition 4.4.



	$n = 3p$	$n = 3p + 1$
x_1	$(0, 0)$	$(0, 0)$
x_2	$(0, p)$	$(0, p)$
x_3	$(i - p, i)$	$(i - p, i)$
x_4	$(i - p, i + p)$	$(i - p, i + p + 1)$
x_5	$(i, i + 2p)$	$(i, i + p + 1)$
x_6	$(i + p, i + 2p)$	$(i + p, i + 2p + 1)$
x_7	$(i + p, i + p)$	$(i + p, i + p)$
x_8	$(i + 2p, i + p)$	$(i + 2p + 1, i + p)$
x_9	$(i + 2p, i)$	$(i + 2p + 1, i)$
x_{10}	$(i + p, i - p)$	$(i + p + 1, i - p)$
x_{11}	$(i, i - p)$	$(i, i - p)$
x_{12}	$(p, 0)$	$(p, 0)$

Our goal is to show that all integral points of the plane are in T . We do this in several steps.

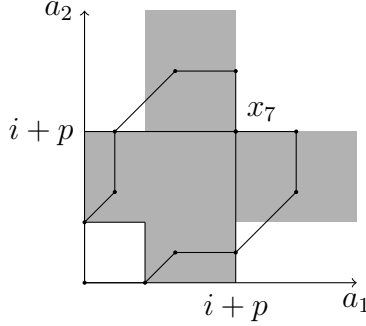
Step 1. For any $c \in [0, i - p]$ we apply Corollary 2.2 with a any integral point on the union of the edges $[x_{10}, x_{11}]$ and $[x_{11}, x_{12}]$ of the polygon in Figure 3, i.e., with $a = (i + c, i - p, i)$, $I = \{2\}$ or $a = (p + c, c, n - i)$, $I = \{2\}$. Each dashed segment in Figure 3 contains n integral points corresponding to line bundles contained in \mathcal{T} .

By Corollary 2.2 we conclude that all points $(t, p + c, i)$, $(p + c, t, i)$ are in T for any $t \in \mathbb{Z}$. In other words, all points in the grey vertical stripe in Figure 3 are in T .

Step 2. Using S_3 -symmetry of T we conclude that all points in the horizontal grey stripe on Figure 4 are in T .

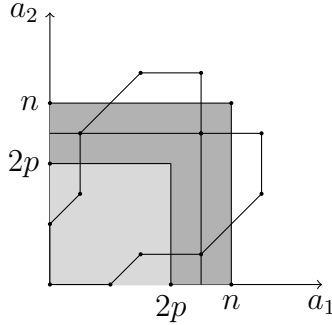
Combining the results of Step 1 and Step 2 above, we see that $a \in T$ for any a such that $(a_1, a_2) \in [0, i + p]^2$, $a_3 = i$. Since $i \in [p, n - p - 1]$, we have $i + p \geq 2p$.

Figure 4: Illustration for Step 2 of Proposition 4.4.



Step 3. Note that by Proposition 4.3 and S_3 -symmetry of T we have $a \in T$ if $a_1 \in [n-p, n]$ or $a_2 \in [n-p, n]$. Using Step 2 and the inequality $n-p \leq 2p+1$ we get that $a \in T$ for any a such that $(a_1, a_2) \in [0, n]^2, a_3 = i$. Therefore we can apply Corollary 2.2 with $a = (0, 0, i)$ and $I = \{1, 2\}$. We conclude that if $a_3 = i$, then $a \in T$.

Figure 5: Illustration for Step 3 of Proposition 4.4.

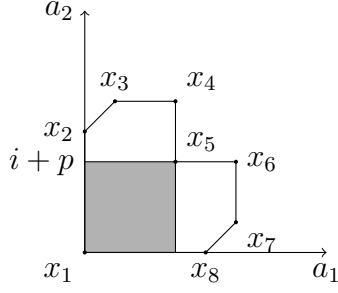


This completes the proof of Proposition 4.4. □

Proposition 4.5. For any $i \in [0, p-1]$ and $a \in \mathbb{Z}^3$ with $a_3 = i$, we have $a \in T$.

Proof. Let us fix $i \in [0, p-1]$. Consider a plane and mark on it all integral points (a_1, a_2) such that $\mathcal{O}(a_1, a_2, i) \in \mathcal{T}$. By definition of (3.2) all integral points of the polygon in Figure 6 are marked. The coordinates of its vertices x_1, \dots, x_8 are listed in the table below.

Figure 6: Illustration for Proposition 4.5.



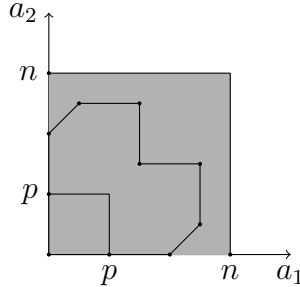
	$n = 3p$	$n = 3p + 1$
x_1	$(0, 0)$	$(0, 0)$
x_2	$(0, 2p)$	$(0, 2p + 1)$
x_3	$(i, i + 2p)$	$(i, i + 2p + 1)$
x_4	$(i + p, i + 2p)$	$(i + p, i + 2p + 1)$
x_5	$(i + p, i + p)$	$(i + p, i + p)$
x_6	$(i + 2p, i + p)$	$(i + 2p + 1, i + p)$
x_7	$(i + 2p, i)$	$(i + 2p + 1, i)$
x_8	$(2p, 0)$	$(2p + 1, 0)$

Our goal is to show that all integral points of the plane are in T .

We see that $a \in T$ for any a such that $(a_1, a_2) \in [0, i + p]^2, a_3 = i$. Since i is in $[0, p - 1]$, we have $i + p \geq p$.

Note that by Propositions 4.3 and 4.4 and S_3 -symmetry of T we have $a \in T$ if $a_1 \in [p, n]$ or $a_2 \in [p, n]$. Thus we get that $a \in T$ for any a such that (a_1, a_2) is in $[0, n]^2, a_3 = i$. In other words, all points in the grey square in Figure 7 are in T . Therefore we can apply Corollary 2.2 with $a = (0, 0, i)$ and $I = \{1, 2\}$. We conclude that if $a_3 = i$, then $a \in T$.

Figure 7: Illustration for Proposition 4.5.



This completes the proof of Proposition 4.5. \square

Proof of Theorem 4.2. We combine Propositions 4.3–4.5 to conclude that if a_3 belongs to $[0, n]$, then $a \in T$. Therefore we can apply Corollary 2.2 with $a = (0, 0, 0)$ and $I = \{1, 2, 3\}$. This concludes the proof of Theorem 4.2. \square

4.3 Minimal Lefschetz decomposition for $\mathcal{D}(X_3^2)$

Consider the case $n = 2$, $k = 3$. We have $h = n + 1 = 3$, $\dim K_{\mathbb{C}} = 3$. By Proposition 3.10, there is no rectangular S_3 -invariant Lefschetz decomposition of $\mathcal{D}(X_3^2)$. In this section we construct a minimal (non-rectangular) S_3 -invariant Lefschetz decomposition of $\mathcal{D}(X_3^2)$. In particular, we prove its fullness. The same method was used for proving fullness for any $n \not\equiv 2 \pmod{3}$.

As we proved in Proposition 3.7, an S_3 -invariant exceptional collection in $\mathcal{D}(X_3^2)$ cannot have less than 13 exceptional objects in the starting block.

We consider the category \mathcal{B} , generated by S_3 -orbits of the following line bundles: $\mathcal{O}(0, 0, 0)$, $\mathcal{O}(1, 0, 0)$, $\mathcal{O}(1, 1, 0)$. We consider the category $\widehat{\mathcal{B}}$, generated \mathcal{B} and S_3 -orbit of the line bundle $\mathcal{O}(2, 1, 0)$.

Take the collection with the following components:

$\widehat{\mathcal{B}}$	$\mathcal{B}(1)$	$\mathcal{B}(2)$
$\mathcal{O}(0, 0, 0)$	$\mathcal{O}(1, 1, 1)$	$\mathcal{O}(2, 2, 2)$
$\mathcal{O}(1, 0, 0)$	$\mathcal{O}(2, 1, 1)$	$\mathcal{O}(3, 2, 2)$
$\mathcal{O}(1, 1, 0)$	$\mathcal{O}(2, 2, 1)$	$\mathcal{O}(3, 3, 2)$
$\mathcal{O}(2, 1, 0)$		

Note that $\widehat{\mathcal{B}} \subset \widehat{\mathcal{A}}$ (they differ by S_3 -orbit of $\mathcal{O}(2, 0, 0)$) and $\mathcal{A} \subset \mathcal{B}$ (they differ by S_3 -orbit of $\mathcal{O}(1, 1, 0)$). In other words, we remove one orbit from $\widehat{\mathcal{A}}$ and add one orbit to \mathcal{A} .

The starting component $\widehat{\mathcal{B}}$ is generated by $1+3+3+6 = 13$ line bundles, while the other two components are generated by $1+3+3 = 7$ line bundles. Evidently, $\mathcal{B} \subset \widehat{\mathcal{B}}$.

Theorem 4.6. *The categories $\widehat{\mathcal{B}}$, $\mathcal{B}(1)$ and $\mathcal{B}(2)$ described above generate a minimal S_3 -invariant Lefschetz collection in $\mathcal{D}(X_3^2)$. In particular,*

$$\mathcal{D}(X_3^2) = \langle \widehat{\mathcal{B}}, \mathcal{B}(1), \mathcal{B}(2) \rangle. \quad (4.5)$$

Proof. Obviously, the categories $\widehat{\mathcal{B}}$, $\mathcal{B}(1)$ and $\mathcal{B}(2)$ are S_3 -invariant.

Let us prove that $(\widehat{\mathcal{B}}, \mathcal{B}(1), \mathcal{B}(2))$ is semiorthogonal. Since $\widehat{\mathcal{B}} \subset \widehat{\mathcal{A}}$ and $\mathcal{B} = \langle \mathcal{A}, \mathcal{C} \rangle$, where \mathcal{A} and $\widehat{\mathcal{A}}$ are the components of (3.4) and \mathcal{C} is the category generated by the S_3 -orbit of $\mathcal{O}(1, 1, 0)$, it is enough to check that

$$\begin{aligned}\mathrm{Ext}^\bullet(\mathcal{C}(1), \widehat{\mathcal{B}}) &= 0, \\ \mathrm{Ext}^\bullet(\mathcal{C}(2), \widehat{\mathcal{B}}) &= 0, \\ \mathrm{Ext}^\bullet(\mathcal{C}(2), \mathcal{C}(1)) &= 0.\end{aligned}$$

These equalities can be easily checked by inspection using (2.2).

We conclude that $(\widehat{\mathcal{B}}, \mathcal{B}(1), \mathcal{B}(2))$ is S_3 -invariant and semiorthogonal. Let us show that it generates $\mathcal{D}(X_3^2)$.

For this we denote by \mathcal{T} the triangulated subcategory of $\mathcal{D}(X_3^2)$ generated by the categories $\widehat{\mathcal{B}}, \mathcal{B}(1), \mathcal{B}(2)$. Applying Corollary 2.2 several times we will show that more line bundles are contained in \mathcal{T} . We note \mathcal{T} is S_3 -invariant, so as soon as a line bundle is proved to be contained in \mathcal{T} , its entire S_3 -orbit is also contained in \mathcal{T} .

Step 1. We note that $\mathcal{O}(2, 2, 1)$, $\mathcal{O}(2, 2, 2)$, and $\mathcal{O}(2, 2, 3)$ are all in \mathcal{T} (the first is in $\mathcal{B}(1)$, while the other two are in $\mathcal{B}(2)$). Applying Corollary 2.2 with $a = (2, 2, 1)$ and $I = \{3\}$ we conclude that all line bundles $\mathcal{O}(2, 2, t)$ are in \mathcal{T} . In particular,

$$\mathcal{O}(2, 2, 0) \in \mathcal{T}.$$

Step 2. We note that $\mathcal{O}(1, 2, 0)$, $\mathcal{O}(1, 2, 1)$, and $\mathcal{O}(1, 2, 2)$ are in \mathcal{T} (the first is in $\widehat{\mathcal{B}}$, while the other two are in $\mathcal{B}(1)$). Applying Corollary 2.2 with $a = (1, 2, 0)$ and $I = \{3\}$ we conclude that all line bundles $\mathcal{O}(1, 2, t)$ are in \mathcal{T} . In particular,

$$\mathcal{O}(1, 2, 3) \in \mathcal{T}.$$

Step 3. We note that $\mathcal{O}(3, 2, 1)$, $\mathcal{O}(3, 2, 2)$, and $\mathcal{O}(3, 2, 3)$ are in \mathcal{T} (for the first of them we use the result of Step 2). Applying Corollary 2.2 with $a = (3, 2, 1)$ and $I = \{3\}$ we conclude that all line bundles $\mathcal{O}(3, 2, t)$ are in \mathcal{T} . In particular,

$$\mathcal{O}(3, 2, 0) \in \mathcal{T}.$$

Step 4. We note that $\mathcal{O}(2, 0, 1)$, $\mathcal{O}(2, 0, 2)$, and $\mathcal{O}(2, 0, 3)$ are in \mathcal{T} (for the last two of them we use the results of Step 1 and Step 3 and S_3 -invariance of \mathcal{T}). Applying Corollary 2.2 with $a = (2, 0, 1)$ and $I = \{3\}$ we conclude that all line bundles $\mathcal{O}(2, 0, t)$ are in \mathcal{T} . In particular,

$$\mathcal{O}(2, 0, 0) \in \mathcal{T}.$$

Combining the original collection with the results of Steps 1–4 above and S_3 -invariance, we see that all line bundles $\mathcal{O}(a)$ with $a \in [0, 2]^3$ are contained in \mathcal{T} . Therefore, by Theorem 2.1 we have $\mathcal{T} = \mathcal{D}(X_3^2)$.

Finally, the minimality of the constructed Lefschetz collection follows from Proposition 3.7. \square

Definition 4.7 ([9, Definition 1.3]). The *rectangular part* of Lefschetz decomposition $\langle \mathcal{B}_0, \mathcal{B}_1(1), \dots, \mathcal{B}_d(d) \rangle = \mathcal{D}(X)$ is $(\mathcal{B}_d, \mathcal{B}_d(1), \dots, \mathcal{B}_d(d))$. The subcategory of $\mathcal{D}(X)$ orthogonal to the rectangular part of a given Lefschetz decomposition is called its *residual category*:

$$\mathcal{R}_{\mathcal{B}^\bullet} = \langle \mathcal{B}_d, \mathcal{B}_d(1), \dots, \mathcal{B}_d(d) \rangle^\perp.$$

Theorem 4.8. *The residual category of the Lefschetz decomposition (4.5) is generated by S_3 -orbit of the line bundle $\mathcal{O}(1, -1, 0)$.*

Proof. Denote by \mathcal{R} the category generated by S_3 -orbit of the line bundle $\mathcal{O}(1, -1, 0)$. Firstly, we need to check that

$$\begin{aligned} \text{Ext}^\bullet(\mathcal{B}, \mathcal{R}) &= 0, \\ \text{Ext}^\bullet(\mathcal{B}(1), \mathcal{R}) &= 0, \\ \text{Ext}^\bullet(\mathcal{B}(2), \mathcal{R}) &= 0. \end{aligned}$$

These equalities can be easily checked by inspection using (2.2).

Secondly, we prove that $\widehat{\mathcal{B}} \subset \langle \mathcal{R}, \mathcal{B} \rangle$. Clearly, for that it is enough to prove that the line bundle $\mathcal{O}(2, 1, 0)$ is in $\langle \mathcal{R}, \mathcal{B} \rangle$. Indeed, we note that $\mathcal{O}(-1, 1, 0)$, $\mathcal{O}(0, 1, 0)$, and $\mathcal{O}(1, 1, 0)$ are in $\langle \mathcal{R}, \mathcal{B} \rangle$. Applying Corollary 2.2 with $a = (-1, 1, 0)$ and $I = \{1\}$ we conclude that the line bundle $\mathcal{O}(2, 1, 0)$ is in $\langle \mathcal{R}, \mathcal{B} \rangle$.

Thus

$$\langle \mathcal{R}, \mathcal{B}, \mathcal{B}(1), \mathcal{B}(2) \rangle \supset \langle \widehat{\mathcal{B}}, \mathcal{B}(1), \mathcal{B}(2) \rangle = \mathcal{D}(X_3^2).$$

Therefore \mathcal{R} is the residual category of the Lefschetz decomposition (4.5). \square

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