

# SYSTEMS OF ERGODIC BSDES ARISING IN REGIME SWITCHING FORWARD PERFORMANCE PROCESSES

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**Abstract.** We introduce and solve a new type of quadratic backward stochastic differential equation systems defined in an infinite time horizon, called *ergodic BSDE systems*. Such systems arise naturally as candidate solutions to characterize forward performance processes and their associated optimal trading strategies in a regime switching market. In addition, we develop a connection between the solution of the ergodic BSDE system and the long-term growth rate of classical utility maximization problems, and use the ergodic BSDE system to study the large time behavior of PDE systems with quadratic growth Hamiltonians.

**Key words.** Infinite horizon BSDE system, ergodic BSDE system, multidimensional comparison theorem, regime switching, forward performance processes, large time behavior of PDE systems.

**AMS subject classifications.** 60H30, 91G10, 93E20

**1. Introduction.** This paper introduces a new class of quadratic backward stochastic differential equation (BSDE for short) systems in an *infinite time horizon*, called *ergodic BSDE systems*. The systems arise in our solution of forward performance processes for portfolio optimization problems in a regime switching market. We show that ergodic BSDE systems are natural candidates for the characterization of forward performance processes and associated optimal strategies in a financial market with multiple regimes.

Let us first recall that an infinite horizon BSDE typically takes the form

$$(1.1) \quad dY_t = -F(t, Y_t, Z_t)dt + (Z_t)^{tr} dW_t, \quad t \geq 0$$

where  $F$  is called the driver of the equation, and  $W$  is a  $d$ -dimensional Brownian motion as the driving noise of the equation. In contrast to the case of a finite time horizon  $[0, T]$ , the infinite horizon BSDE (1.1) is defined over all time horizons and may be ill posed, even if the driver  $F$  is Lipschitz continuous in both  $Y$  and  $Z$ . It has been solved in [8] under a strictly monotone condition on the driver, a typical one of which reads

$$F(t, Y_t, Z_t) = f(t, Z_t) - \rho Y_t,$$

for some constant  $\rho > 0$ . Then, it has been shown in [8] that (1.1) admits a unique bounded solution  $(Y, Z)$  adapted to the Brownian filtration, if  $f$  is Lipschitz continuous in  $Z$ . If  $f$  has a quadratic growth in  $Z$ , it has been further treated in [6].

Note that only bounded solutions are concerned here, for unbounded solutions to BSDE (1.1) are not unique in general. The restriction within bounded solutions

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is also useful in the study of the Markovian BSDE (1.1) and its asymptotic property. Indeed, in a Markovian framework where  $f(t, Z_t) = f(V_t, Z_t)$  with  $V$  being the underlying forward diffusion process, the solution  $(Y, Z)$  admits a Markovian representation  $(Y_t, Z_t) = (y(V_t), z(V_t))$  for some pair of measurable functions  $y(\cdot)$  and  $z(\cdot)$ , and has been shown in [23] and later in [17] that, when  $\rho \rightarrow 0$ , the bounded Markovian solution to BSDE (1.1) converges to the Markovian solution of the *ergodic BSDE*

$$(1.2) \quad dY_t = -(f(V_t, Z_t) - \lambda)dt + (Z_t)^{tr}dW_t, \quad t \geq 0.$$

Here, the constant  $\lambda$  constitutes one part of the solution to (1.2), and has a stochastic control interpretation as the value of an ergodic control problem. The ergodic BSDE (1.2) has been widely used to study the large time behavior of solutions of their finite horizon counterparts (see, for example, [27] and [16]).

Both ergodic BSDE (1.2) and infinite horizon BSDE (1.1) turn out to be natural candidates for the characterization of forward performance processes and their associated optimal portfolio strategies in portfolio optimization problems. Forward performance processes were introduced and developed in [38, 39, 40, 41]. They complement the classical expected utility paradigm in which the utility is a deterministic function chosen at a single terminal time. The value function process is, in turn, constructed backwards in time, as the dynamic programming principle yields. As a result, there is limited flexibility to incorporate updating of risk preferences, rolling horizons, learning, and other realistic “forward in nature” features if one requires that time-consistency is being preserved at all times. Forward performance processes alleviate some of these shortcomings and offer the construction of a genuinely dynamic mechanism for evaluating the performance of investment strategies as the market evolves across (arbitrary) trading horizons. See also [24, 32, 42, 43, 47, 48] for their developments and various applications.

The construction of (*Markovian*) *forward performance processes* is, however, difficult, due to the ill-posed nature and degeneracy of the corresponding (stochastic) partial differential equations (see [21]). This difficulty has been recently overcome in [36], which shows that Markovian forward performance processes in homothetic form can be effectively constructed via the Markovian solutions of the equations like (1.1) and (1.2). It bypasses a number of aforementioned difficulties inherited in the associated SPDE. See also [12] for a further development of this method to study forward entropic risk measures.

Our aim herein is to generalize both (1.1) and (1.2) from scalar-valued to vector-valued equations, i.e. systems of equations. The corresponding BSDE systems are motivated by the construction of *Markovian forward performance processes in a regime switching market*. Due to the interactions of different market regimes through a given Markov chain, the corresponding infinite horizon BSDE system for a Markovian forward performance process is expected to take the form

$$(1.3) \quad dY_t^i = -f^i(V_t, Z_t^i)dt - \sum_{k \in I} q^{ik}(e^{Y_t^k - Y_t^i} - 1)dt + (Z_t^i)^{tr}dW_t,$$

for  $t \geq 0$  and  $i \in I := \{1, 2, \dots, m^0\}$ , where  $q^{ik}$  is the transition rate from market regime  $i$  to  $k$ . The second term on the right hand side of (1.3) couples all the equations together and represents the interaction of different market regimes. A similar feature has also appeared in [3] and [4], where the authors studied classical utility maximization in a regime switching framework and derived a finite horizon BSDE system.

However, different from the finite horizon case, the infinite horizon BSDE system (1.3) is ill posed. Indeed, in a single regime case, (1.3) then reduces to a scalar-valued BSDE, and the strictly monotone condition fails to hold. To overcome this difficulty, we modify (1.3) by adding a discount term  $\rho Y_t^i$  in the driver (see (2.1) in section 2), which serves the role of strict monotonicity. Although this additional discount term makes the modified BSDE system well posed, it however distorts the original problem. As a result, the solution of the modified BSDE system will no longer correspond to a forward performance process.

As a first contribution, we construct Markovian regime switching forward performance processes in homothetic form via the asymptotic limit of the infinite horizon BSDE system (2.1), that is, the ergodic BSDE system (3.7) (see Theorem 4.2). Both BSDE systems (2.1) and (3.7) are new<sup>1</sup>. They are introduced for the first time for the characterization of regime switching forward performance processes. In particular, we show that when there is a single regime, our representation of forward performance processes will recover the ergodic BSDE representation appearing in [36].

Our second contribution is about solvability of the infinite horizon BSDE system (2.1). Since the driver  $f^i$  has quadratic growth in  $Z^i$ , the standard Lipschitz estimates do not apply to our system. Instead, we first apply a truncation technique and derive *a priori* estimates for the solutions, and subsequently show that the truncation constants coincide with the constants appearing in the *a priori* estimates. For this, we make an extensive use of the multidimensional comparison theorem for BSDE systems, which was firstly developed in [29]. An essential idea herein is to use the bounded solution of an auxiliary ODE (not system!) as a universal bound to control all the solution components of the BSDE system.

We then derive the ergodic BSDE system (3.7) as the asymptotic limit of the infinite horizon BSDE system (2.1). This ergodic BSDE system, on one hand, characterizes the regime switching forward performance processes and, on the other hand, is also a natural extension of the ergodic equation introduced in [23]. Herein, a new feature is that all the equation components have a common ergodic constant  $\lambda$  as a part of the solution. Similar to [23], we apply the perturbation technique to construct a sequence of approximate solutions to the ergodic BSDE system. However, the commonly used Girsanov's transformation method does not imply the uniqueness of the solution due to different probability measures induced by each equation component. Instead, we prove the uniqueness of the solution by first converting the ergodic BSDE system (3.7) to a scalar-valued ergodic BSDE driven by the Brownian motion and an exogenously given Markov chain and then using the Girsanov's transformation under the Brownian motion and the Markov chain (see Appendix B).

Our third contribution is about a stochastic control representation for the ergodic constant  $\lambda$  (see Proposition 4.4). We show that it corresponds to the long-term growth rate of a risk-sensitive optimization problem in a regime switching framework. This, in turn, connects with the long-term growth rate of a regime switching utility maximization problem. Thus, our result also unveils an intrinsic connection between forward performance processes and classical expected utilities in a market with multiple regimes.

Our last contribution is using the ergodic BSDE system (3.7) to study the large time behavior of solutions to a class of PDE systems with quadratic growth Hamil-

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<sup>1</sup>Recently, [14] also introduced an ergodic BSDE system motivated from non-zero sum games. However, the structure of their system is different from ours. In particular, there is no comparison theorem for their system.

tonians (see Theorem 5.1). Those PDE systems are often used to characterize the utility indifference prices of financial derivatives in a regime switching market (see [3] and [4]). We show that the solution of the PDE system will converge to the solution of the ergodic BSDE system exponentially fast. To the best of our knowledge, this is the first convergence rate result for the large time behavior of PDE systems.

Turning to literature about the quadratic BSDE (systems), most of the existing results are only for a finite time horizon. The scalar equation with bounded terminal data was first solved in [33] and was applied to solve utility maximization problems in [25]. See also [7, 37, 44] for extensions. The case with unbounded terminal data is more challenging and was solved in [9, 10, 18], with [19] and [20] further showing the uniqueness of the solution. Their applications can be found in [2] and [26]. Recently, there have been a renewed interest in the corresponding quadratic BSDE systems due to their various applications in equilibrium problems, price impact models and non-zero sum games (see, for example, [11, 30, 31, 34, 35, 45] with more references therein). In spite of all the aforementioned results, our paper seems to be the first to introduce and solve quadratic BSDE systems in an infinite time horizon.

The paper is organized as follows. Section 2 introduces an infinite horizon BSDE system with quadratic growth drivers. Section 3 studies its asymptotic limit, which leads to an ergodic BSDE system. Section 4 applies the ergodic BSDE system to construct Markovian forward performance processes in a regime switching market. Section 5 applies the ergodic BSDE system to study the large time behavior of a PDE system. Section 6 then concludes. For the reader's convenience, we also provide a proof of the multidimensional comparison theorem in the appendix.

**2. System of infinite horizon quadratic BSDE.** Let  $W$  be a  $d$ -dimensional Brownian motion on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Denote by  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  the augmented filtration generated by  $W$ . Throughout this paper, we denote by  $A^{tr}$  the transpose of matrix  $A$ . Consider the infinite horizon BSDE system: for  $t \geq 0$  and  $i \in I := \{1, 2, \dots, m^0\}$ ,

$$(2.1) \quad dY_t^i = -f^i(V_t, Z_t^i)dt - \sum_{k \in I} q^{ik} (e^{Y_t^k - Y_t^i} - 1)dt + \rho Y_t^i dt + (Z_t^i)^{tr} dW_t.$$

By a solution to (2.1), we mean a pair of adapted processes  $(Y^i, Z^i)_{i \in I}$  satisfying (2.1) in an arbitrary time horizon.

To solve (2.1), we impose the following assumptions on  $f^i$ .

ASSUMPTION 1. *There exist three constants  $C_v, C_z$  and  $K_f$  such that, for  $i, k \in I$  and  $v, \bar{v}, z, \bar{z} \in \mathbb{R}^d$ ,*

- (i)  $|f^i(v, z) - f^i(\bar{v}, z)| \leq C_v(1 + |z|)|v - \bar{v}|$ ;
- (ii)  $|f^i(v, z) - f^i(v, \bar{z})| \leq C_z(1 + |z| + |\bar{z}|)|z - \bar{z}|$ ;
- (iii)  $|f^i(v, 0)| \leq K_f$ .

Assumption 1(ii) implies that  $f^i(v, z)$  has a quadratic growth in  $z$ . Thus, we are facing a system of quadratic BSDEs defined in an infinite time horizon. The system is coupled through the coefficients  $q^{ik}$ ,  $i, k \in I$ , which satisfy

ASSUMPTION 2. *The square matrix  $\mathcal{Q} := \{q^{ik}\}_{i, k \in I}$  is a transition rate matrix satisfying (i)  $\sum_{k \in I} q^{ik} = 0$ ; (ii)  $q^{ik} \geq 0$  for  $i \neq k$ . Let  $q^{\max}$  be the maximal transition rate, i.e.  $q^{\max} = \max_{i, k} q^{ik}$ .*

The infinite horizon BSDE system (2.1) is coupled with a forward diffusion process  $V$  satisfying

ASSUMPTION 3. *The underlying  $d$ -dimensional forward diffusion process  $V$  is*

given by the solution of the mean-reverting SDE

$$(2.2) \quad dV_t = \eta(V_t)dt + \kappa dW_t,$$

where the drift coefficients  $\eta(\cdot)$  satisfy a dissipative condition, namely, there exists a constant  $C_\eta > C_v$  such that, for  $v, \bar{v} \in \mathbb{R}^d$ ,

$$(\eta(v) - \eta(\bar{v}))^{tr}(v - \bar{v}) \leq -C_\eta |v - \bar{v}|^2.$$

Moreover, the volatility matrix  $\kappa \in \mathbb{R}^{d \times d}$  is positive definite and normalized to  $|\kappa| = 1$ .

The main result of this section is the following existence and uniqueness of the solution to (2.1).

**THEOREM 2.1.** *Let Assumptions 1, 2, and 3 be satisfied. Then, there exists a unique bounded solution  $(Y^i, Z^i)_{i \in I}$  to the infinite horizon BSDE system (2.1) satisfying*

$$(2.3) \quad |Y_t^i| \leq K_y := \frac{K_f}{\rho} \quad \text{and} \quad |Z_t^i| \leq K_z := \frac{C_v}{C_\eta - C_v}.$$

**REMARK 1.** *As explained in the introduction, we restricted our discussion within bounded solutions to BSDE (2.1). This is for the sake of (i) the uniqueness of its adapted solutions; (ii) the discussion of its Markovian solutions; (iii) the discussion of the asymptotic behavior of solutions to (2.1) as the time horizon goes to infinity.*

The rest of this section is devoted to the proof of Theorem 2.1.

**2.1. Sketch of the proof.** To construct a solution of (2.1), we follow a truncation procedure and a stability analysis. To this end, we first define two truncating functions  $p : \mathbb{R} \rightarrow \mathbb{R}$  and  $q : \mathbb{R}^d \rightarrow \mathbb{R}^d$  by

$$(2.4) \quad p(y) := \max\{-K_y, \min\{y, K_y\}\} \quad \text{and} \quad q(z) := \frac{\min\{|z|, K_z\}}{|z|} z \mathbf{1}_{\{z \neq 0\}}.$$

We consider the truncated system of (2.1), namely,

$$(2.5) \quad dY_t^i = -f^i(V_t, q(Z_t^i))dt - \sum_{k \in I} q^{ik}(e^{p(Y_t^k) - p(Y_t^i)} - 1)dt + \rho Y_t^i dt + (Z_t^i)^{tr} dW_t,$$

for  $t \geq 0$  and  $i \in I$ .

Assumption 1 (i) and (ii) imply that the function  $f^i(\cdot, q(\cdot))$  is Lipschitz continuous, i.e.

$$(2.6) \quad |f^i(v, q(z)) - f^i(\bar{v}, q(z))| \leq \frac{C_\eta C_v}{C_\eta - C_v} |v - \bar{v}|,$$

and

$$(2.7) \quad |f^i(v, q(z)) - f^i(v, q(\bar{z}))| \leq C_z \frac{C_\eta + C_v}{C_\eta - C_v} |z - \bar{z}|.$$

It is also immediate to verify that  $\sum_{k \in I} q^{ik}(e^{p(y^k) - p(y^i)} - 1) - \rho y^i$  is continuous and has bounded derivatives except at finite many points. Thus, the driver of the truncated system (2.5) is Lipschitz continuous.

If, moreover, we can show that (2.5) admits a solution, say  $(Y^i, Z^i)_{i \in I}$ , with  $|Y_t^i| \leq K_y$  and  $|Z_t^i| \leq K_z$ , then  $p(Y_t^i) = Y_t^i$  and  $q(Z_t^i) = Z_t^i$ , for  $t \geq 0$  and  $i \in I$ . In turn, the pair of processes  $(Y^i, Z^i)_{i \in I}$  also solve the original infinite horizon BSDE system (2.1).

Next, we construct a solution to (2.5) by an approximation procedure. For  $m \geq 1$  and  $t \in [0, m]$ , we consider the *finite horizon* BSDE system

$$(2.8) \quad Y_t^i(m) = \int_t^m \left[ f^i(V_s, q(Z_s^i(m))) + \sum_{k \in I} q^{ik} (e^{p(Y_s^k(m)) - p(Y_s^i(m))} - 1) - \rho Y_s^i(m) \right] ds - \int_t^m (Z_s^i(m))^{tr} dW_s.$$

For  $t > m$ , we define  $Y_t^i(m) = Z_t^i(m) \equiv 0$ . Note that (2.8) is a standard BSDE system with Lipschitz continuous driver, so it admits a unique solution  $(Y^i(m), Z^i(m))_{i \in I}$ .

We shall first establish uniform bounds (independent of  $m$ ) on  $Y^i(m)$  and  $Z^i(m)$  in Section 2.3. Subsequently, we shall show in Section 2.4 that the pair of processes  $(Y^i(m), Z^i(m))_{m \geq 1}$  is a Cauchy sequence in an appropriate space, whose limit then provides a solution to the infinite horizon BSDE system (2.1). Moreover, the uniqueness of the solution relies on the multidimensional comparison theorem introduced in the next subsection.

**2.2. Multidimensional comparison theorem.** The multidimensional comparison theorem for systems of BSDE, was first established in [29]. A different proof is given in Appendix A for the reader's convenience.

LEMMA 2.2. *For  $T > 0$ , consider a system of BSDEs  $(\xi^i, F^i, G^i)$  with the terminal data  $\xi^i$  and the driver  $(F^i, G^i)$ , namely,*

$$Y_t^i = \xi^i + \int_t^T [F_s^i(Z_s^i) + G_s^i(Y_s^i, Y_s^{-i})] ds - \int_t^T (Z_s^i)^{tr} dW_s, \quad t \in [0, T],$$

where  $Y_s^{-i} := (Y_s^1, \dots, Y_s^{i-1}, Y_s^{i+1}, \dots, Y_s^m)$ . Let  $(\bar{Y}^i, \bar{Z}^i)$  be the solution of the system of BSDEs  $(\bar{\xi}^i, \bar{F}^i, \bar{G}^i)$  with the terminal data  $\bar{\xi}^i$  and the driver  $(\bar{F}^i, \bar{G}^i)$ . Suppose that

(i) both  $\xi^i$  and  $\bar{\xi}^i$  are square integrable and satisfying  $\xi^i \leq \bar{\xi}^i$  for  $i \in I$ ;

(ii) there exist constants  $C_f$  and  $C_g$  such that, for  $i \in I$  and  $z, \bar{z} \in \mathbb{R}^d$ ,  $y = (y^i, y^{-i})$ ,  $\bar{y} = (\bar{y}^i, \bar{y}^{-i}) \in \mathbb{R}^{m^0}$ ,

$$(2.9) \quad |F_s^i(z) - F_s^i(\bar{z})| \leq C_f |z - \bar{z}|,$$

$$(2.10) \quad |G_s^i(y^i, y^{-i}) - G_s^i(\bar{y}^i, \bar{y}^{-i})| \leq C_g |y - \bar{y}|;$$

(iii) the driver  $G_s^i(y^i, y^{-i})$  is nondecreasing in all of its components other than  $y^i$ , i.e. it is nondecreasing in  $y^k$ , for  $k \neq i$ ;

(iv) the following inequalities hold,

$$(2.11) \quad F_s^i(\bar{Z}_s^i) \leq \bar{F}_s^i(\bar{Z}_s^i),$$

$$(2.12) \quad G_s^i(\bar{Y}_s^i, \bar{Y}_s^{-i}) \leq \bar{G}_s^i(\bar{Y}_s^i, \bar{Y}_s^{-i}).$$

Then,  $Y_t^i \leq \bar{Y}_t^i$  for  $t \in [0, T]$  and  $i \in I$ .

REMARK 2. *Lemma 2.2, and its proof, simply correct a minor loss of inefficiency in the arguments developed in [29]. In [29], the Lipschitz conditions (2.9) and*

(2.10) are required to hold also for  $(\bar{F}_i, \bar{G}_i)$ , and both inequalities (2.11) and (2.12) are required to hold for all  $z^i \in \mathbb{R}$  and  $y = (y^i, y^{-i}) \in \mathbb{R}^{m^0}$ . In Lemma 2.2, the Lipschitz conditions on  $(\bar{F}_i, \bar{G}_i)$  are not necessary, and both inequalities (2.11) and (2.12) are required to hold only at the solution  $(\bar{Y}^i, \bar{Z}^i)$ . Such an improvement is crucial and tailor made for our later use.

**2.3. A priori estimates.** We show that the pair of processes  $(Y^i(m), Z^i(m))_{i \in I}$ , as the solution to the finite horizon BSDE system (2.8), have the estimates

$$(2.13) \quad |Y_t^i(m)| \leq K_y \quad \text{and} \quad |Z_t^i(m)| \leq K_z,$$

where the constants  $K_y$  and  $K_z$ , independent of  $m$ , are given in Theorem 2.1.

*The boundedness of  $Y^i(m)$ .* For  $z \in \mathbb{R}^d$  and  $y = (y^i, y^{-i}) \in \mathbb{R}^{m^0}$ , let

$$F_s^i(z) := f^i(V_s, q(z)) \quad \text{and} \quad G_s^i(y^i, y^{-i}) := \sum_{k \in I} q^{ik} (e^{p(y^k) - p(y^i)} - 1) - \rho y^i.$$

Note that both  $F_s^i(z)$  and  $G_s^i(y^i, y^{-i})$  are Lipschitz continuous, and  $G_s^i(y^i, y^{-i})$  is nondecreasing in  $y^k$  for  $k \neq i$ . Moreover, by Assumption 1(iii),  $F_s^i(0) \leq K_f$  and  $G_s^i(\bar{Y}_s, \bar{Y}_s^{-i}) = -\rho \bar{Y}_s$ , where  $\bar{Y}^{-i} := (\bar{Y}, \dots, \bar{Y})$  and  $\bar{Y}$  solves the ODE

$$\bar{Y}_t = \int_t^{m^0-1} (K_f - \rho \bar{Y}_s) ds.$$

Consequently, it follows from Lemma 2.2 that  $Y_t^i(m) \leq \bar{Y}_t \leq \frac{K_f}{\rho}$ , for  $t \in [0, m]$  and  $i \in I$ . Likewise, we also obtain that  $Y_t^i(m) \geq -\frac{K_f}{\rho}$ , so  $|Y_t^i(m)| \leq \frac{K_f}{\rho} = K_y$ . Hence, we have  $p(Y_t^i(m)) \equiv Y_t^i(m)$ , i.e. the truncation function  $p(\cdot)$  does not play a role in BSDE system (2.8).

*The boundedness of  $Z^i(m)$ .* Denote by  $V^{r,v}$  the solution of SDE (2.2) starting from  $v \in \mathbb{R}^d$  at the initial time  $r$ , and by  $(Y_t^{i,r,v}(m), Z_t^{i,r,v}(m)), t \in [r, T]$  the solution of BSDE (2.8) where the process  $V$  is replaced with  $V^{r,v}$ . Identically just as before, we have  $|Y_t^{i,r,v}(m)| \leq K_y$ .

For  $t \in [r, m]$  and  $v, \bar{v} \in \mathbb{R}^d$ , let

$$\delta Y_t^{i,r}(m) := Y_t^{i,r,v}(m) - Y_t^{i,r,\bar{v}}(m) \quad \text{and} \quad \delta Z_t^{i,r}(m) := Z_t^{i,r,v}(m) - Z_t^{i,r,\bar{v}}(m).$$

It then follows from (2.8) that

$$(2.14) \quad \begin{aligned} \delta Y_t^{i,r}(m) &= \int_t^m [f^i(V_s^{r,v}, q(Z_s^{i,r,v}(m))) - f^i(V_s^{r,\bar{v}}, q(Z_s^{i,r,\bar{v}}(m)))] ds \\ &\quad + \int_t^m \sum_{k \in I} \left( q^{ik} (e^{Y_s^{k,r,v}(m) - Y_s^{i,r,v}(m)} - 1) - q^{ik} (e^{Y_s^{k,r,\bar{v}}(m) - Y_s^{i,r,\bar{v}}(m)} - 1) \right) ds \\ &\quad - \int_t^m \rho \delta Y_s^{i,r}(m) ds - \int_t^m (\delta Z_s^{i,r}(m))^{tr} dW_s \\ &= \int_t^m [F_s^{i,r}(\delta Z_s^{i,r}(m)) + G_s^{i,r}(\delta Y_s^{i,r}(m), \delta Y_s^{-i,r}(m))] ds - \int_t^m (\delta Z_s^{i,r}(m))^{tr} dW_s, \end{aligned}$$

where

$$\begin{aligned} F_s^{i,r}(z) &= f^i(V_s^{r,v}, q(Z_s^{i,r,v}(m))) - f^i(V_s^{r,\bar{v}}, q(Z_s^{i,r,v}(m))) \\ &\quad + f^i(V_s^{r,\bar{v}}, q(z + Z_s^{i,r,\bar{v}}(m))) - f^i(V_s^{r,\bar{v}}, q(Z_s^{i,r,\bar{v}}(m))) \end{aligned}$$

and

$$G_s^{i,r}(y^i, y^{-i}) = \sum_{k \in I} q^{ik} \left( e^{y^k - y^i + Y_s^{k,r,\bar{v}}(m) - Y_s^{i,r,\bar{v}}(m)} - e^{Y_s^{k,r,\bar{v}}(m) - Y_s^{i,r,\bar{v}}(m)} \right) - \rho y^i,$$

for  $z \in \mathbb{R}^d$  and  $y = (y^i, y^{-i}) \in \mathbb{R}^{m^0}$ , with  $|y^i| \leq 2K_y$  for  $i \in I$ .

Note that  $F_s^{i,r}(z)$  and  $G_s^{i,r}(y^i, y^{-i})$  are Lipschitz continuous. Moreover,  $G_s^{i,r}(0, 0^{-i}) = 0$  and, by Assumption 1(i) and the Lipschitz estimate (2.6),

$$\begin{aligned} |F_s^{i,r}(0)| &= |f^i(V_s^{r,v}, q(Z_s^{i,r,v}(m))) - f^i(V_s^{r,\bar{v}}, q(Z_s^{i,r,v}(m)))| \\ &\leq \frac{C_v C_\eta}{C_\eta - C_v} |V_s^{r,v} - V_s^{r,\bar{v}}| \leq \frac{C_v C_\eta}{C_\eta - C_v} e^{-C_\eta(s-r)} |v - \bar{v}|, \quad s \in [r, T], \end{aligned}$$

where the last inequality follows from the dissipative condition in Assumption 3 and Gronwall's inequality. Thus,  $(\delta Y_t^{i,r}(m), \delta Z_t^{i,r}(m))_{i \in I}$  is the unique solution to (2.14). Furthermore, note that  $G_s^{i,r}(y^i, y^{-i})$  is nondecreasing in  $y^k$  for  $k \neq i$  and  $G_s^{i,r}(\bar{Y}_s^r, (\bar{Y}_s^r)^{-i}) = -\rho \bar{Y}_s^r$ , where  $\bar{Y}^r$  is the unique solution of the ODE

$$Y_t = \int_t^m \left( \frac{C_v C_\eta}{C_\eta - C_v} e^{-C_\eta(s-r)} |v - \bar{v}| - \rho Y_s \right) ds, \quad t \in [r, T].$$

Consequently, from Lemma 2.2, we have

$$\begin{aligned} \delta Y_t^{i,r}(m) &\leq \bar{Y}_t^r = \frac{C_v C_\eta}{C_\eta - C_v} \frac{e^{\rho t} (e^{-(\rho t + C_\eta(t-r))} - e^{-(\rho m + C_\eta(m-r))})}{\rho + C_\eta} |v - \bar{v}| \\ (2.15) \quad &\leq \frac{C_v}{C_\eta - C_v} |v - \bar{v}| \end{aligned}$$

for  $t \in [r, m]$  and  $i \in I$ . Likewise, we also have

$$(2.16) \quad \delta Y_t^{i,r}(m) \geq \frac{-C_v}{C_\eta - C_v} |v - \bar{v}|.$$

Note that the process  $Y^{i,r,v}(m)$  admits a Markovian representation, i.e. there exists a measurable function  $\mathbf{y}^i(\cdot, \cdot; m)$  such that  $Y_t^{i,r,v}(m) = \mathbf{y}^i(t, V_t^{r,v}; m)$  (see Theorem 4.1 in [22]). If the coefficient  $\eta$  and the driver  $f^i$  are further continuously differentiable functions with bounded derivatives, the function  $v \mapsto \mathbf{y}^i(t, v; m)$  is continuously differentiable such that (see [22, Corollary 4.1])

$$(2.17) \quad \kappa^{tr} \nabla_v \mathbf{y}^i(t, V_t^{r,v}; m) = Z_t^{i,r,v}(m).$$

From (2.15) and (2.16), we have for  $(i, v_1, v_2) \in I \times \mathbb{R}^d \times \mathbb{R}^d$ ,

$$(2.18) \quad |\mathbf{y}^i(t, v_1; m) - \mathbf{y}^i(t, v_2; m)| \leq K_z |v_1 - v_2| \quad \text{with } K_z := \frac{C_v}{C_\eta - C_v}.$$

In view of Assumption 3 on  $\kappa$ , we have  $|Z_t^{i,r,v}(m)| \leq K_z$ , which can be shown (via mollifying the coefficient  $\eta$  and the driver  $f^i$  in a straightforward manner) to hold also for our general  $(\eta, f)$ . Therefore, the *a priori* estimates (2.13) on  $Y^i(m) = Y^{i,0,v}(m)$  and  $Z^i(m) = Z^{i,0,v}(m)$  have been proved.

**2.4. Proof of Theorem 2.1. Existence.** We first prove that  $(Y^i(m))_{m \geq 1}$  is a Cauchy sequence. For  $m \geq n \geq 1$  and  $t \in [0, m]$ , let

$$\delta Y_t^i(m, n) := Y_t^i(m) - Y_t^i(n) \quad \text{and} \quad \delta Z_t^i(m, n) := Z_t^i(m) - Z_t^i(n).$$

Since we have already shown in the last section that  $|Y_t^i(m)| \leq K_y$  and  $|Z_t^i(m)| \leq K_z$ , the truncation functions  $p(\cdot)$  and  $q(\cdot)$  actually do not play any role in (2.8), and we have  $(p(Y_t^i(m)), q(Z_t^i(m))) = (Y_t^i(m), Z_t^i(m))$ . In turn,

$$\begin{aligned} \delta Y_t^i(m, n) &= \int_t^m [f^i(V_s, Z_s^i(m)) - f^i(V_s, Z_s^i(n))] ds + \int_t^m f^i(V_s, 0) \chi_{\{s \geq n\}} ds \\ &\quad + \int_t^m \sum_{k \in I} \left( q^{ik}(e^{Y_s^k(m) - Y_s^i(m)} - 1) - q^{ik}(e^{Y_s^k(n) - Y_s^i(n)} - 1) \right) ds \\ &\quad - \int_t^m \rho \delta Y_s^i(m, n) ds - \int_t^m (\delta Z_s^i(m, n))^{tr} dW_s \\ &= \int_t^m [F_s^i(\delta Z_s^i(m, n)) + G_s^i(\delta Y_s^i(m, n), \delta Y_s^{-i}(m, n))] ds \\ (2.19) \quad &\quad - \int_t^m (\delta Z_s^i(m, n))^{tr} dW_s, \end{aligned}$$

where

$$F_s^i(z) = f^i(V_s, z + Z_s^i(n)) - f^i(V_s, Z_s^i(n)) + f^i(V_s, 0) \chi_{\{s \geq n\}},$$

and

$$G_s^i(y^i, y^{-i}) = \sum_{k \in I} q^{ik} \left( e^{y^k - y^i + Y_s^k(n) - Y_s^i(n)} - e^{Y_s^k(n) - Y_s^i(n)} \right) - \rho y^i,$$

for  $z \in \mathbb{R}^d$  and  $y = (y^i, y^{-i}) \in \mathbb{R}^{m^0}$ , with  $|z| \leq 2K_z$  and  $|y^i| \leq 2K_y$  for  $i \in I$ .

Following along similar arguments as in section 2.3, we deduce that (2.19) is with Lipschitz continuous driver and, therefore,  $(\delta Y^i(m, n), \delta Z^i(m, n))_{i \in I}$  is the unique solution to (2.19). Moreover, by Assumption 1(iii), we have  $F_s^i(0) = f^i(V_s, 0) \chi_{\{s \geq n\}} \leq K_f \chi_{\{s \geq n\}}$  and  $G_s^i(\bar{Y}_s, \bar{Y}_s^{-i}) = -\rho \bar{Y}_s$ , with  $\bar{Y}$  solving the ODE

$$\bar{Y}_t = \int_t^m (K_f \chi_{\{s \geq n\}} - \rho \bar{Y}_s) ds$$

Hence, using Lemma 2.2, we obtain

$$(2.20) \quad \delta Y_t^i(m, n) \leq \bar{Y}_t \leq K_f \int_n^m e^{-\rho(s-t)} ds = \frac{K_f}{\rho} e^{\rho t} (e^{-\rho n} - e^{-\rho m}),$$

for  $t \in [0, m]$  and  $i \in I$ . Likewise, we also have

$$(2.21) \quad \delta Y_t^i(m, n) \geq -\frac{K_f}{\rho} e^{\rho t} (e^{-\rho n} - e^{-\rho m}).$$

Sending  $m, n \rightarrow \infty$ , we obtain that, for any  $T > 0$ ,  $\sup_{t \in [0, T]} |\delta Y_t^i(m, n)| \rightarrow 0$  and, therefore, there exists a limit process  $Y^i$  such that  $Y_t^i(m) \rightarrow Y_t^i$  for almost every  $(t, \omega) \in [0, \infty) \times \Omega$ , with  $|Y_t^i| \leq K_y$ .

To prove that  $Z^i(m)$  is also a Cauchy sequence, we introduce the Banach space

$$\mathcal{L}^{2,\rho} := \left\{ (Z_t)_{t \geq 0} : Z \text{ is progressively measurable and } \mathbb{E} \left[ \int_0^\infty e^{-2\rho s} |Z_s|^2 ds \right] < \infty \right\}.$$

Applying Itô's formula to  $e^{-2\rho t} |\delta Y_t^i(m, n)|^2$  and using (2.19), we get

$$\begin{aligned} & |\delta Y_0^i(m, n)|^2 + \int_0^m e^{-2\rho s} |\delta Z_s^i(m, n)|^2 ds \\ &= \int_0^m \underbrace{2e^{-2\rho s} \delta Y_s^i(m, n) [f^i(V_s, Z_s^i(m)) - f^i(V_s, Z_s^i(n))]}_{(I)} ds \\ & \quad + \int_0^m 2e^{-2\rho s} \delta Y_s^i(m, n) f^i(V_s, 0) \chi_{\{s \geq n\}} ds \\ & \quad + \int_0^m 2e^{-2\rho s} \delta Y_s^i(m, n) \sum_{k \in I} q^{ik} \left( e^{Y_s^k(m) - Y_s^i(m)} - e^{Y_s^k(n) - Y_s^i(n)} \right) ds \\ (2.22) \quad & \quad - \int_0^m 2e^{-2\rho s} \delta Y_s^i(m, n) (\delta Z_s^i(m, n))^{tr} dW_s. \end{aligned}$$

Furthermore, we apply the elementary inequality  $2ab \leq \frac{1}{\epsilon} |a|^2 + \epsilon |b|^2$  to term (I) and obtain

$$\begin{aligned} (I) &\leq \frac{1}{2} e^{-2\rho s} \frac{|f^i(V_s, Z_s^i(m)) - f^i(V_s, Z_s^i(n))|^2}{C_z^2(1 + 2K_z)^2} + 2C_z^2(1 + 2K_z)^2 e^{-2\rho s} |\delta Y_s^i(m, n)|^2 \\ &\leq \frac{1}{2} e^{-2\rho s} |\delta Z_s^i(m, n)|^2 + 2C_z^2(1 + 2K_z)^2 e^{-2\rho s} |\delta Y_s^i(m, n)|^2, \end{aligned}$$

where we also used Assumption 1(ii) and the *a priori* estimate (2.13) on  $Z^i(m)$  in the second equality.

In turn, taking expectation on both sides of (2.22) and using the *a priori* estimate (2.13) on  $Y^i(m)$  yield

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \left[ \int_0^m e^{-2\rho s} |\delta Z_s^i(m, n)|^2 ds \right] \\ &\leq 2C_z^2(1 + 2K_z)^2 \mathbb{E} \left[ \int_0^m e^{-2\rho s} |\delta Y_s^i(m, n)|^2 ds \right] + 2K_f \mathbb{E} \left[ \int_n^m e^{-2\rho s} \delta Y_s^i(m, n) ds \right] \\ & \quad + 4m^0 q^{\max} e^{2K_y} \mathbb{E} \left[ \int_0^m e^{-2\rho s} \delta Y_s^i(m, n) ds \right]. \end{aligned}$$

The dominated convergence theorem then implies  $\delta Z^i(m, n) \rightarrow 0$  in  $\mathcal{L}^{2,\rho}$  and, therefore, there exists a limit process  $Z^i$  such that  $Z^i(m) \rightarrow Z^i$  in  $\mathcal{L}^{2,\rho}$ , with  $|Z_t^i| \leq K_z$ .

It is standard to check that the pair of limit processes  $(Y^i, Z^i)_{i \in I}$  indeed satisfy the infinite horizon BSDE system (2.1). See, for example, section 5 of [8].

*Uniqueness.* Since both  $Y^i$  and  $Z^i$  are bounded, the uniqueness of the bounded solution  $(Y^i, Z^i)_{i \in I}$  to (2.1) follows from the multidimensional comparison theorem in Lemma 2.2. Indeed, suppose  $(Y^i, Z^i)_{i \in I}$  and  $(\bar{Y}^i, \bar{Z}^i)_{i \in I}$  are two bounded solutions to (2.1). For  $t \geq 0$ , let

$$\delta Y_t^i := e^{-\rho t} (Y_t^i - \bar{Y}_t^i) \quad \text{and} \quad \delta Z_t^i := e^{-\rho t} (Z_t^i - \bar{Z}_t^i).$$

For  $T \geq t$ , let  $\varepsilon_T := 2K_y e^{-\rho T}$ . Then, for  $0 \leq t \leq T$ ,

$$\begin{aligned}
\delta Y_t^i &= \delta Y_T^i + \int_t^T e^{-\rho s} [f^i(V_s, Z_s^i) - f^i(V_s, \bar{Z}_s^i)] ds \\
&\quad + \int_t^T e^{-\rho s} \sum_{k \in I} \left( q^{ik} (e^{Y_s^k - Y_s^i} - 1) - q^{ik} (e^{\bar{Y}_s^k - \bar{Y}_s^i} - 1) \right) ds \\
&\quad - \int_t^T (\delta Z_s^i)^{tr} dW_s \\
(2.23) \quad &= \delta Y_T^i + \int_t^T [F_s^i(\delta Z_s^i) + G_s^i(\delta Y_s^i, \delta Y_s^{-i})] ds - \int_t^T (\delta Z_s^i)^{tr} dW_s,
\end{aligned}$$

where

$$F_s^i(z) = e^{-\rho s} [f^i(V_s, e^{\rho s} z + \bar{Z}_s^i) - f^i(V_s, \bar{Z}_s^i)],$$

and

$$G_s^i(y^i, y^{-i}) = e^{-\rho s} \sum_{k \in I} q^{ik} \left( e^{e^{\rho s} (y^k - y^i) + \bar{Y}_s^k - \bar{Y}_s^i} - e^{\bar{Y}_s^k - \bar{Y}_s^i} \right),$$

for  $z \in \mathbb{R}^d$  and  $y = (y^i, y^{-i}) \in \mathbb{R}^{m^0}$ , with  $|z| \leq 2K_z$  and  $|y^i| \leq 2K_y$  for  $i \in I$ .

We apply similar arguments as in section 2.3 to deduce that (2.23) is with Lipschitz continuous driver and, therefore,  $(\delta Y^i, \delta Z^i)_{i \in I}$  is the unique solution to (2.23). Moreover, note that

$$|\delta Y_T^i| \leq 2K_y e^{-\rho T} = \varepsilon_T, \quad F_s^i(0) = 0 \quad \text{and} \quad G_s^i(\varepsilon_T, \varepsilon_T^{-i}) = 0.$$

By Lemma 2.2, we deduce that  $|\delta Y_t^i| \leq \varepsilon_T$  and, therefore,  $\delta Y_t^i = 0$  by sending  $T \rightarrow \infty$ . Consequently,  $\delta Z_t^i = 0$ , which proves the uniqueness of the solution to the infinite horizon BSDE system (2.1).

**3. System of ergodic quadratic BSDEs.** We study the asymptotics of the infinite horizon BSDE system (2.1) when  $\rho \rightarrow 0$ , which leads to a new type of ergodic BSDE systems. The ergodic BSDE system will in turn be used to construct regime switching forward performance processes (section 4) and obtain the large time behavior of PDE systems (section 5). To this end, we require that the transition rate matrix  $\mathcal{Q}$  in Assumption 2 satisfies some sort of irreducible property.

**ASSUMPTION 4.** *The transition rate matrix  $\mathcal{Q}$  satisfies  $q^{ik} > 0$ , for  $i \neq k$ . Let  $q^{\min} > 0$  be the minimal transition rate, i.e.  $q^{\min} = \min_{i \neq k} q^{ik}$ .*

We first show that, under Assumption 4, the difference of any two components, say  $Y^i$  and  $Y^j$ , of the solution to (2.1) is actually bounded uniformly in  $\rho$ .

**LEMMA 3.1.** *Suppose that Assumptions 1-4 are satisfied. For  $i, j \in I$  and  $t \geq 0$ , let  $\Delta Y_t^{ij} = Y_t^i - Y_t^j$ . Then,*

$$(3.1) \quad |\Delta Y_t^{ij}| \leq \frac{1}{q^{\min}} \left( K_f + \frac{C_v C_\eta C_z}{(C_\eta - C_v)^2} \right),$$

with the constants  $K_f, C_v, C_z$  as in Assumption 1, and  $C_\eta$  as in Assumption 3.

*Proof.* It suffices to prove that, for  $m \geq 1$ ,

$$(3.2) \quad |\Delta Y_t^{ij}(m)| := |Y_t^i(m) - Y_t^j(m)| \leq \frac{1}{q^{\min}} \left( K_f + \frac{C_v C_\eta C_z}{(C_\eta - C_v)^2} \right).$$

Then, (3.1) follows by sending  $m \rightarrow \infty$ .

To this end, let  $\Delta Z_t^{ij}(m) = Z_t^i(m) - Z_t^j(m)$ . It is immediate to check that the pair of processes  $(\Delta Y^{ij}(m), \Delta Z^{ij}(m))_{i,j \in I}$  satisfy

$$\begin{aligned}
\Delta Y_t^{ij}(m) &= \int_t^m [f^i(V_s, Z_s^i(m)) - f^j(V_s, Z_s^j(m))] ds \\
&\quad + \int_t^m \sum_{k \in I} \left( q^{ik} (e^{Y_s^k(m) - Y_s^i(m)} - 1) - q^{jk} (e^{Y_s^k(m) - Y_s^j(m)} - 1) \right) ds \\
&\quad - \int_t^m \rho \Delta Y_s^{ij}(m) ds - \int_t^m (\Delta Z_s^{ij})^{tr} dW_s \\
&= \int_t^m [F_s^{ij}(\Delta Z_s^{ij}(m)) + G_s^{ij}(\Delta Y_s^{ij}(m), \Delta Y_s^{-ij}(m))] ds \\
(3.3) \quad &\quad - \int_t^m (\Delta Z_s^{ij}(m))^{tr} dW_s,
\end{aligned}$$

where

$$F_s^{ij}(z) = f^i(V_s, z + Z_s^j(m)) - f^j(V_s, Z_s^j(m)),$$

and

$$G_s^{ij}(y^{ij}, y^{-ij}) = q^{ij} e^{-y^{ij}} - q^{ji} e^{y^{ij}} - \rho y^{ij} + \sum_{k \neq j} q^{ik} e^{y^{ki}} - \sum_{k \neq i} q^{jk} e^{-y^{jk}},$$

for  $z \in \mathbb{R}^d$  and  $y = (y^{ij}, y^{-ij}) \in \mathbb{R}^{m^0}$ , with  $|z| \leq 2K_z$  and  $|y^{ij}| \leq 2K_y$  for  $i, j \in I$ .

Since  $F_s^{ij}(z)$  and  $G_s^{ij}(y^{ij}, y^{-ij})$  are Lipschitz continuous, following along similar arguments as in section 2.3, we deduce that  $(\Delta Y^{ij}(m), \Delta Z^{ij}(m))_{i,j \in I}$  is the unique solution to BSDE system (3.3). Moreover, by Assumption 1(ii)-(iii), we have, for  $v, z \in \mathbb{R}^d$ ,  $|f^i(v, z)| \leq K_f + C_z(|z| + |z|^2)$ , so

$$F_s^{ij}(0) = f^i(V_s, Z_s^j(m)) - f^j(V_s, Z_s^j(m)) \leq 2K_f + 2C_z(K_z + K_z^2).$$

Using  $\sum_{k \neq j} q^{ik} = -q^{ij}$  and  $\sum_{k \neq i} q^{jk} = -q^{ji}$ , we also have

$$G_s^i(\bar{Y}_s, \bar{Y}_s^{-i}) = -(q^{ij} + q^{ji})(e^{\bar{Y}_s} - e^{-\bar{Y}_s}) - \rho \bar{Y}_s,$$

where  $\bar{Y}$  solves the ODE

$$\bar{Y}_t = \int_t^m 2 [K_f + C_z(K_z + K_z^2) - q^{min} \bar{Y}_s] ds.$$

Since  $0 \leq \bar{Y}_t \leq \frac{K_f + C_z(K_z + K_z^2)}{q^{min}}$ , we further have

$$\begin{aligned}
G_s^i(\bar{Y}_s, \bar{Y}_s^{-i}) &\leq -(q^{ij} + q^{ji})(e^{\bar{Y}_s} - e^{-\bar{Y}_s}) \\
&\leq -2q^{min}(\bar{Y}_s + 1 - e^{-\bar{Y}_s}) \leq -2q^{min} \bar{Y}_s,
\end{aligned}$$

and, consequently, using Lemma 2.2 we deduce that

$$\Delta Y_t^{ij}(m) \leq \bar{Y}_t \leq \frac{K_f + C_z(K_z + K_z^2)}{q^{min}}.$$

By the symmetric property, we also have  $\Delta Y_t^{ji}(m) \leq \frac{K_f + C_z(K_z + K_z^2)}{q^{\min}}$ , from which we obtain estimate (3.2).  $\square$

Next, we send  $\rho \rightarrow 0$  in the infinite horizon BSDE system (2.1). To emphasize the dependencies on  $\rho$  and  $V_0 = v$ , we use the notations  $V_t^v, Y_t^{i,\rho,v}$  and  $Z_t^{i,\rho,v}$  in the rest of this section. Sending  $m \rightarrow \infty$  in the estimate (2.18) yields that, for the first component  $Y_t^{i,\rho,v} = \mathbf{y}^{i,\rho}(V_t^v)$  of the solution to (2.1),

$$(3.4) \quad |\mathbf{y}^{i,\rho}(v_1) - \mathbf{y}^{i,\rho}(v_2)| \leq \frac{C_v}{C_\eta - C_v} |v_1 - v_2|, \quad v_1, v_2 \in \mathbb{R}^d.$$

Given a fixed reference point, say  $v_0 \in \mathbb{R}^d$ , we define the processes  $\bar{Y}_t^{i,\rho,v} := Y_t^{i,\rho,v} - Y_0^{m^0,\rho,v_0}$ , for  $t \geq 0$ ,  $i \in I$  and  $v \in \mathbb{R}^d$ , and consider the perturbed version of the infinite horizon BSDE system (2.1)<sup>2</sup>, i.e.

$$(3.5) \quad \begin{aligned} \bar{Y}_t^{i,\rho,v} &= \bar{Y}_T^{i,\rho,v} + \int_t^T \left[ \sum_{k \in I} q^{ik} (e^{\bar{Y}_s^{k,\rho,v} - \bar{Y}_s^{i,\rho,v}} - 1) - \rho \bar{Y}_s^{i,\rho,v} + \rho Y_0^{m^0,\rho,v_0} \right] ds \\ &+ \int_t^T f^i(V_s^v, Z_s^{i,\rho,v}) ds - \int_t^T (Z_s^{i,\rho,v})^{tr} dW_s, \end{aligned}$$

for  $0 \leq t \leq T < \infty$ ,  $i \in I$  and  $v \in \mathbb{R}^d$ . By the Markov property of  $Y^{i,\rho,v}$  (see Theorem 4.1 in [22]), we have  $\bar{Y}_t^{i,\rho,v} = \bar{\mathbf{y}}^{i,\rho}(V_t^v)$  with  $\bar{\mathbf{y}}^{i,\rho}(\cdot) := \mathbf{y}^{i,\rho}(\cdot) - \mathbf{y}^{m^0,\rho}(v_0)$ .

Note that, by estimate (3.4),  $\mathbf{y}^{i,\rho}(\cdot)$  is Lipschitz continuous uniformly in  $\rho$ , and by estimate (3.1),  $\bar{\mathbf{y}}^{i,\rho}(v_0) = \mathbf{y}^{i,\rho}(v_0) - \mathbf{y}^{m^0,\rho}(v_0)$  is bounded uniformly in  $\rho$ . In turn, we deduce that, for  $v \in \mathbb{R}^d$ ,

$$(3.6) \quad \begin{aligned} |\bar{\mathbf{y}}^{i,\rho}(v)| &= |\mathbf{y}^{i,\rho}(v) - \mathbf{y}^{i,\rho}(v_0) + \mathbf{y}^{i,\rho}(v_0) - \mathbf{y}^{m^0,\rho}(v_0)| \\ &\leq \frac{C_v}{C_\eta - C_v} |v - v_0| + \frac{1}{q^{\min}} \left( K_f + \frac{C_v C_\eta C_z}{(C_\eta - C_v)^2} \right). \end{aligned}$$

Moreover, (2.3) implies that  $|\rho \mathbf{y}^{m^0,\rho}(v_0)| \leq \rho K_y = K_f$ . Hence, by a standard diagonal procedure, there exists a sequence, denoted by  $\{\rho_n\}_{n \geq 1}$ , such that, for  $v$  in a dense subset of  $\mathbb{R}^d$ ,

$$\lim_{\rho_n \rightarrow 0} \rho_n \mathbf{y}^{m^0,\rho_n}(v_0) = \lambda, \quad \lim_{\rho_n \rightarrow 0} \bar{\mathbf{y}}^{i,\rho_n}(v) = \mathbf{y}^i(v),$$

for some  $\lambda \in \mathbb{R}$  and the limit function  $\mathbf{y}^i(v)$ .

Since  $\bar{\mathbf{y}}^{i,\rho}(\cdot)$  is Lipschitz continuous uniformly in  $\rho$ , the limit function  $\mathbf{y}^i(\cdot)$  can be further extended to a Lipschitz continuous function defined for all  $v \in \mathbb{R}^d$ , i.e. for  $v \in \mathbb{R}^d$ ,

$$\lim_{\rho_n \rightarrow 0} \bar{\mathbf{y}}^{i,\rho_n}(v) = \mathbf{y}^i(v).$$

Thus, for the infinite horizon BSDE system (3.5), it holds that  $\lim_{\rho_n \rightarrow 0} \bar{Y}_t^{i,\rho_n,v} = \mathbf{y}^i(V_t^v)$  and  $\lim_{\rho_n \rightarrow 0} \rho_n \bar{Y}_t^{i,\rho_n,v} = 0$ .

As a result, by defining the processes  $\mathcal{Y}_t^{i,v} := \mathbf{y}^i(V_t^v)$ , for  $t \geq 0$ ,  $i \in I$  and  $v \in \mathbb{R}^d$ , it is standard to show that (see [17] and [23]) there exist a limit function  $\mathbf{z}^i(\cdot)$  such

<sup>2</sup>There is nothing special about the choice of the reference point  $m^0$ . Any regime  $j \in I$  will also serve the purpose.

that  $Z^{i,\rho_n,v}$  converges to  $Z^{i,v} := \mathbf{z}^i(V_t^v) \in \mathcal{L}^2$  as  $\rho_n \rightarrow 0$ , and  $((\mathcal{Y}^{i,v}, \mathcal{Z}^{i,v})_{i \in I}, \lambda)$  solve the ergodic BSDE system

$$(3.7) \quad d\mathcal{Y}_t^{i,v} = -f^i(V_t^v, \mathcal{Z}_t^{i,v})dt - \sum_{k \in I} q^{ik} (e^{\mathcal{Y}_t^{k,v} - \mathcal{Y}_t^{i,v}} - 1)dt + \lambda dt + (\mathcal{Z}_t^{i,v})^{tr} dW_t,$$

for  $t \geq 0$ ,  $i \in I$  and  $v \in \mathbb{R}^d$ .

The main result of this section is the following existence and uniqueness of the solution to the ergodic BSDE system (3.7). Clearly, ergodic BSDE (3.7) admits multiple (possibly non-Markovian) solutions. It is more reasonable to consider the uniqueness of functions rather than processes (see Remark 4.7 in [23]). This explains why we are only concerned with Markovian solutions of (3.7) in the rest of the paper.

**THEOREM 3.2.** *Suppose that Assumptions 1-4 are satisfied. Then, there exists a unique Markovian solution  $((\mathcal{Y}_t^{i,v}, \mathcal{Z}_t^{i,v})_{i \in I}, \lambda) = ((\mathbf{y}^i(V_t^v), \mathbf{z}^i(V_t^v))_{i \in I}, \lambda)$ ,  $t \geq 0$ , to the ergodic BSDE system (3.7), such that the functions  $(\mathbf{y}^i(\cdot), \mathbf{z}^i(\cdot))$  satisfy*

$$(3.8) \quad |\mathbf{y}^i(v)| \leq C_y(1 + |v|),$$

$$(3.9) \quad |\mathbf{z}^i(v)| \leq K_z = \frac{C_v}{C_\eta - C_v},$$

$$(3.10) \quad |\mathbf{y}^i(v) - \mathbf{y}^j(v)| \leq \frac{1}{q^{\min}} \left( K_f + \frac{C_v C_\eta C_z}{(C_\eta - C_v)^2} \right),$$

for some constant  $C_y > 0$ , where all other constants are given in Lemma 3.1. The function  $\mathbf{y}^i(\cdot)$  is unique up to an additive constant and, without loss of generality, it is set that  $\mathbf{y}^i(0) = 0$ .

*Proof.* We have already shown the existence of a Markovian solution to (3.7). The estimates (3.8), (3.8), and (3.10) follow, respectively, from (3.6), (2.3), and (3.1) by sending  $\rho \rightarrow 0$ . Hence, it remains to show the uniqueness. The idea is to convert the ergodic BSDE system (3.7) to a scalar-valued ergodic BSDE driven by the Brownian motion  $W$  and an exogenously given Markov chain  $\alpha$ . We postpone this part of the proof to Appendix B after we introduce the Markov chain  $\alpha$  in the next section.  $\square$

**REMARK 3.** *The conditions (3.8)-(3.10) are essential for the uniqueness of the Markovian solution to (3.7). We provide examples of Markovian solutions which do not satisfy them. Assume that  $d = m^0 = 1$ ,  $\eta(v) = -\frac{1}{2}v$ , and  $\kappa = 1$ . Then, (3.7) reduces to*

$$d\mathcal{Y}_t^v = -f(V_t^v, \mathcal{Z}_t^v)dt + \lambda dt + \mathcal{Z}^v dW_t,$$

with  $dV_t^v = -\frac{1}{2}V_t^v dt + dW_t$  and  $V_0^v = v$ .

As the first example, we consider  $f(v, z) = \frac{v}{2}e^{-v^2/2}$ . Assumptions 1-4 are then all satisfied. The unique Markovian solution satisfying (3.8)-(3.10) is given by  $(\mathcal{Y}_t^v, \mathcal{Z}_t^v, \lambda) = (\mathbf{y}(V_t^v), \mathbf{z}(V_t^v), 0)$  with

$$(\mathbf{y}(v), \mathbf{z}(v)) = \left( \frac{1}{2} \int_{-\infty}^v e^{-\frac{y^2}{2}} dy, \frac{1}{2} e^{-\frac{v^2}{2}} \right).$$

It is easy to check that both triplets

$$\left( \frac{1}{2} \int_0^v [e^{-\frac{y^2}{2}} - e^{-\frac{v^2}{2}}] dy, \frac{1}{2} [e^{-\frac{v^2}{2}} - e^{-\frac{v^2}{2}}], 0 \right)$$

and

$$\left( \int_0^v e^{\frac{y^2}{2}} \left[ \frac{1}{2} e^{-y^2} + N(y) - 1 \right] dy, e^{\frac{v^2}{2}} \left[ \frac{1}{2} e^{-v^2} + N(v) - 1 \right], \frac{1}{2\sqrt{2\pi}} \right),$$

where  $N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$ , also satisfy (3.7). However, neither of them satisfies the conditions (3.8) and (3.9).

As the second example, we consider  $f(v, z) = \frac{|v|}{2} e^{-v^2/2}$ . The unique Markovian solution satisfying (3.8)-(3.10) is given by the triplet  $(\mathbf{y}(\cdot), \mathbf{z}(\cdot), \frac{1}{2\sqrt{2\pi}})$  with

$$\begin{aligned} \mathbf{y}(v) &= \chi_{\{v \geq 0\}} \int_0^v e^{\frac{y^2}{2}} \left[ \frac{1}{2} e^{-y^2} + N(y) - 1 \right] dy + \chi_{\{v < 0\}} \int_0^v e^{\frac{y^2}{2}} \left[ -\frac{1}{2} e^{-y^2} + N(y) \right] dy; \\ \mathbf{z}(v) &= \chi_{\{v \geq 0\}} e^{\frac{v^2}{2}} \left[ \frac{1}{2} e^{-v^2} + N(v) - 1 \right] + \chi_{\{v < 0\}} e^{\frac{v^2}{2}} \left[ -\frac{1}{2} e^{-v^2} + N(v) \right]. \end{aligned}$$

However, it is easy to check that the triplet  $(\bar{\mathbf{y}}(\cdot), \bar{\mathbf{z}}(\cdot), 0)$  with

$$\begin{aligned} \bar{\mathbf{y}}(v) &= \chi_{\{v \geq 0\}} \int_0^v \left( \frac{1}{2} e^{-\frac{y^2}{2}} - e^{\frac{y^2}{2}} \right) dy - \chi_{\{v < 0\}} \int_0^v \frac{1}{2} e^{-\frac{y^2}{2}} dy; \\ \bar{\mathbf{z}}(v) &= \chi_{\{v \geq 0\}} \left( \frac{1}{2} e^{-\frac{v^2}{2}} - e^{\frac{v^2}{2}} \right) - \chi_{\{v < 0\}} \frac{1}{2} e^{-\frac{v^2}{2}}, \end{aligned}$$

also satisfies (3.7), but fails to satisfy the conditions (3.8) and (3.9).

**4. Application to regime switching forward performance processes.** Let  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  be the filtered probability space introduced in section 2. Assume the probability space also supports a Markov chain  $\alpha$  with its augmented filtration  $\mathbb{H} = \{\mathcal{H}_t\}_{t \geq 0}$  independent of the Brownian filtration  $\mathbb{F}$ . The Markov chain  $\alpha$  has the transition rate matrix  $\mathcal{Q}$  as specified in Assumption 2, and admits the representation

$$d\alpha_t = \sum_{k, k' \in I} (k - k') \chi_{\{\alpha_{t-} = k'\}} dN_t^{k'k},$$

where  $(N^{k'k})_{k', k \in I}$  are independent Poisson processes each with intensity  $q^{k'k}$  (see chapter 9.1.2 in [5]). Let  $T_0 = 0$  and  $T_1, T_2, \dots$  be the jump times of the Markov chain  $\alpha$ , and  $(\alpha^j)_{j \geq 1}$  be a sequence of  $\mathcal{H}_{T_j}$ -measurable random variables representing the position of  $\alpha$  in the time interval  $[T_{j-1}, T_j)$ . Hence,  $\alpha_t = \sum_{j \geq 1} \alpha^{j-1} \chi_{[T_{j-1}, T_j)}(t)$ . Without loss of generality, assume that  $\alpha^0 = i \in I$ . Denote the smallest filtration generated by  $\mathbb{F}$  and  $\mathbb{H}$  as  $\mathbb{G} = \{\mathcal{G}_t\}_{t \geq 0}$ , i.e.  $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ .

We consider a market consisting of a risk-free bond offering zero interest rate and  $n$  risky assets, with  $n \leq d$ . The prices of the  $n$  risky assets are driven by the Markov chain  $\alpha$  and a  $d$ -dimensional stochastic factor process  $V$ , which satisfies Assumption 3.

Each state  $i \in I$  of the Markov chain  $\alpha$  represents a market regime, and in regime  $i$ , the corresponding market price of risk at time  $t$  is  $\theta^i(V_t)$ . The  $n$ -dimensional price process  $S = (S^1, \dots, S^n)^{tr}$  of the risky assets follows

$$(4.1) \quad dS_t = \text{diag}(S_t) \sigma(V_t) (\theta^{\alpha_t} - (V_t) dt + dW_t),$$

where  $\sigma(V_t) \in \mathbb{R}_+^{n \times d}$  is the volatility matrix of the risky assets at time  $t$ , and  $\text{diag}(S_t) = \{\text{diag}(S_t)_{kj}\}_{1 \leq k, j \leq n}$ , with  $\text{diag}(S_t)_{kk} = S_t^k$  and  $\text{diag}(S_t)_{kj} = 0$  for  $k \neq j$ , represents the prices of the risky assets at time  $t$ .

ASSUMPTION 5. *The market coefficients of the  $n$  risky assets satisfy that*

- (i)  $\sigma(v)$  is uniformly bounded in  $v \in \mathbb{R}^d$  and has full rank  $n$ ;
- (ii) for  $i \in I$ ,  $\theta^i(v)$  is uniformly bounded and Lipschitz continuous in  $v \in \mathbb{R}^d$ .

REMARK 4. When  $n < d$ , the financial market is incomplete. A typical example is  $n = 1$  and  $d = 2$  for the following regime switching stochastic volatility model

$$\begin{aligned} dS_t &= S_t \sigma(V_t) (\theta^{\alpha_t} - (V_t) dt + dW_t), \\ dV_t &= \eta(V_t) dt + \kappa dW_t. \end{aligned}$$

Here, the function  $\sigma(\cdot)$  takes values in a two-dimensional row vector space, all the  $m^0 + 1$  functions  $\theta^i(\cdot)$ ,  $i \in I$ , and  $\eta(\cdot)$  take values in a two-dimensional column vector space, and the constant matrix  $\kappa \in \mathbb{R}^{2 \times 2}$  is positive definite and normalized to  $|\kappa| = 1$ .

**4.1. Trading strategies.** In this market environment, an investor trades dynamically among the risk-free bond and the risky assets. Let  $\tilde{\pi} = (\tilde{\pi}^1, \dots, \tilde{\pi}^n)^{tr}$  denote the (discounted by the bond) proportions of her wealth in the risky assets. They are taken to be self-financing and, thus, the (discounted by the bond) wealth process satisfies

$$dX_t(\tilde{\pi}) = X_t(\tilde{\pi}) \tilde{\pi}_t^{tr} \sigma(V_t) (\theta^{\alpha_t} - (V_t) dt + dW_t).$$

As in [36], we work with the trading strategies rescaled by the volatility matrix, namely,  $\pi_t^{tr} := \tilde{\pi}_t^{tr} \sigma(V_t)$ . Then, the wealth process in regime  $i$  satisfies

$$(4.2) \quad dX_t(\pi) = X_t(\pi) \pi_t^{tr} (\theta^{\alpha_t} - (V_t) dt + dW_t).$$

For any  $t \geq 0$ , we denote by  $\mathcal{A}_{[0,t]}^{\mathbb{G}}$  the set of admissible trading strategies in  $[0, t]$ , defined as

$$\begin{aligned} \mathcal{A}_{[0,t]}^{\mathbb{G}} := & \left\{ \pi_s = \pi_0^i \chi_{\{0\}}(s) + \sum_{j \geq 1} \pi_s^{\alpha^{j-1}} \chi_{(T_{j-1}, T_j]}(s), s \in [0, t] : \pi_s^j \in \Pi^j, \right. \\ & \left. \pi^j \text{ is } \mathbb{F}\text{-progressively measurable and } \int_0^t |\pi_s^j|^2 ds < \infty, \mathbb{P}\text{-a.s.} \right\}, \end{aligned}$$

where  $\Pi^j$ ,  $j \in I$ , are closed and convex subsets in  $\mathbb{R}^d$ . So  $\Pi^j$  models the investor's trading constraints, and the investor will adjust her trading constraint sets according to different market regimes.

For  $0 \leq t \leq s$ , the set  $\mathcal{A}_{[t,s]}^{\mathbb{G}}$  is defined in a similar way, and the set of admissible trading strategies for all  $t \geq 0$  is, in turn, defined as  $\mathcal{A}^{\mathbb{G}} = \cup_{t \geq 0} \mathcal{A}_{[0,t]}^{\mathbb{G}}$ .

For the regime switching stochastic volatility model in Remark 4, a typical choice of the trading constraint set  $\Pi^j$  is  $\Pi^j = \mathbb{R} \times \{0\}$  for  $j \in I$ .

**4.2. Regime switching forward performance processes.** The investor uses a forward criterion to measure the performance for her admissible trading strategies. We introduce the definition of regime switching forward performance processes associated with this market.

DEFINITION 4.1. A family of stochastic processes  $(U^i(x, t))_{i \in I}$ , for  $(x, t) \in \mathbb{R}_+^2$ , is a regime switching forward performance process if the following conditions are satisfied:

- (i) For each  $i \in I$  and  $x \in \mathbb{R}_+$ ,  $t \mapsto U^i(x, t)$  is  $\mathbb{F}$ -progressively measurable;
- (ii) For each  $i \in I$  and  $t \geq 0$ , the mapping  $x \mapsto U^i(x, t)$  is strictly increasing and strictly concave;

(iii) Define the process

$$(4.3) \quad U(x, t) := \sum_{j \geq 1} U^{\alpha^{j-1}}(x, t) \chi_{[T_{j-1}, T_j)}(t).$$

Then, for all  $\pi \in \mathcal{A}^{\mathbb{G}}$  and  $0 \leq t \leq s$ ,

$$(4.4) \quad U(X_t(\pi), t) \geq \mathbb{E}[U(X_s(\pi), s) | \mathcal{G}_t],$$

and there exists an optimal  $\pi^* \in \mathcal{A}^{\mathbb{G}}$  such that

$$(4.5) \quad U(X_t(\pi^*), t) = \mathbb{E}[U(X_s(\pi^*), s) | \mathcal{G}_t],$$

with  $X(\pi), X(\pi^*)$  solving (4.2).

The above (super)martingale conditions (4.4) and (4.5) can be restated as follows: For  $j \geq 1$ , on the event  $\{T_{j-1} \leq t < T_j\}$ ,

$$U(x, t) = U^{\alpha^{j-1}}(x, t) = \operatorname{ess\,sup}_{\pi \in \mathcal{A}_{[t, s]}^{\mathbb{G}}} \mathbb{E} \left[ U^{\alpha^{j-1}}(X_s(\pi), s) \chi_{\{s < T_j\}} + U^{\alpha^j}(X_{T_j}(\pi), T_j) \chi_{\{s \geq T_j\}} \middle| \mathcal{F}_t, X_t = x \right],$$

and on  $\{t = T_j\}$ ,  $U(x, t)$  has a jump with size

$$U(x, T_j) - U(x, T_j-) = U^{\alpha^j}(x, T_j) - U^{\alpha^{j-1}}(x, T_j-).$$

Hence, we have the following decomposition formula for  $U(x, t)$  (recall that  $\alpha^0 = i$ ):

$$(4.6) \quad \begin{aligned} U(x, t) &= U(x, 0) + \sum_{j \geq 1} [U(x, t \wedge T_j-) - U(x, t \wedge T_{j-1})] \\ &\quad + \sum_{j \geq 1} [U(x, t \wedge T_j) - U(x, t \wedge T_j-)] \\ &= U^i(x, 0) + \sum_{j \geq 1} [U^{\alpha^{j-1}}(x, t \wedge T_j-) - U^{\alpha^{j-1}}(x, t \wedge T_{j-1})] \\ &\quad + \sum_{j \geq 1} [U^{\alpha^j}(x, T_j) - U^{\alpha^{j-1}}(x, T_j-)] \chi_{\{T_j \leq t\}}. \end{aligned}$$

The first sum on the right hand side of (4.6) is the continuous component of  $U(x, t)$ , while the second sum is the jump component of  $U(x, t)$ .

Here, we focus on *Markovian regime switching forward performance processes in power form*, namely, the processes that are *deterministic* functions of the stochastic factor process  $V$ ,

$$U^i(x, t) = \frac{x^\delta}{\delta} e^{K^i(V_t, t)}$$

for  $\delta \in (0, 1)$  and appropriate function(s)  $K^i : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}$ .

**4.3. Representation via system of ergodic BSDE.** We now characterize Markovian regime switching forward performance processes via the ergodic BSDE system (3.7) introduced in Section 3. For  $i \in I$  and  $(v, z) \in \mathbb{R}^d \times \mathbb{R}^d$ , we consider the driver

$$(4.7) \quad f^i(v, z) = \frac{1}{2}\delta(\delta - 1)\text{dist}^2\left(\Pi, \frac{z + \theta^i(v)}{1 - \delta}\right) + \frac{\delta}{2(1 - \delta)}|z + \theta^i(v)|^2 + \frac{|z|^2}{2}.$$

It is easy to check that  $f^i$  satisfies Assumption 1. Then, from Theorem 3.2, the ergodic BSDE system (3.7) admits a unique Markovian solution  $((\mathcal{Y}^i, \mathcal{Z}^i)_{i \in I}, \lambda)$  satisfying (3.8), (3.9) and (3.10).

**THEOREM 4.2.** *Suppose that Assumptions 1-5 are satisfied. Let  $((\mathcal{Y}_t^i, \mathcal{Z}_t^i)_{i \in I}, \lambda) = ((\mathbf{y}^i(V_t^v), \mathbf{z}^i(V_t^v))_{i \in I}, \lambda)$ ,  $t \geq 0$ , be the unique Markovian solution of the ergodic BSDE system (3.7) with driver  $f^i$  as in (4.7), and satisfy (3.8), (3.9) and (3.10). Then,*

$$(4.8) \quad U^i(x, t) = \frac{x^\delta}{\delta} e^{\mathcal{Y}_t^i - \lambda t} = \frac{x^\delta}{\delta} e^{\mathbf{y}^i(V_t^v) - \lambda t}, \quad i \in I,$$

form a Markovian regime switching forward performance process, and in each regime  $i$ ,

$$(4.9) \quad \pi_t^{i,*} = \text{Proj}_{\Pi^i} \left( \frac{\mathcal{Z}_t^i + \theta^i(V_t)}{1 - \delta} \right)$$

is the associated optimal trading strategy in this regime.

**REMARK 5.** *The boundedness conditions (3.9) and (3.10) are crucial for the verification of the (super)martingale conditions of  $U(x, t)$  (see step 3 in section 4.4), while the linear growth condition (3.8) is used to connect forward performance processes and classical utility maximization (see Proposition 4.4).*

*In particular, if there is only a single regime, i.e.  $m^0 = 1$ , then the ergodic BSDE system (3.7) reduces to*

$$d\mathcal{Y}_t^1 = -f^1(V_t, \mathcal{Z}_t^1)dt + \lambda dt + (\mathcal{Z}_t^1)^{tr} dW_t.$$

*In this case, the Markovian forward performance process has the representation*

$$U^1(x, t) = \frac{x^\delta}{\delta} e^{\mathcal{Y}_t^1 - \lambda t} = \frac{x^\delta}{\delta} e^{\mathbf{y}^1(V_t^v) - \lambda t},$$

*which is precisely the representation formula established in [36, Theorem 3.2].*

To prove Theorem 4.2, we need Itô's formula for the Markov chain  $\alpha$ . We recall it in the following lemma, which will be frequently used in the rest of the paper. Its proof is a straightforward extension of [5] and [46] and is thus omitted here.

**LEMMA 4.3.** *For  $i \in I$ , let  $F_t^i$ ,  $t \geq 0$ , be a family of  $\mathbb{F}$ -progressively measurable and continuous stochastic processes. Then,*

$$\begin{aligned} & \sum_{j \geq 1} \left[ F_{T_j}^{\alpha T_j} - F_{T_j^-}^{\alpha T_j} \right] \chi_{\{T_j \leq t\}} \\ &= \int_0^t \sum_{k \in I} q^{\alpha_s - k} [F_s^k - F_s^{\alpha_s -}] ds + \int_0^t \sum_{k, k' \in I} [F_s^k - F_s^{k'}] \chi_{\{\alpha_s = k'\}} d\tilde{N}_s^{k'k}, \end{aligned}$$

where  $\tilde{N}_t^{k'k} = N_t^{k'k} - q^{k'k}t$ ,  $t \geq 0$ , are the compensated Poisson martingales under the filtration  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ .

**4.4. Proof of Theorem 4.2.** We divide the proof into three steps. The first two steps derive, locally and globally, the stochastic dynamics of the regime switching forward performance process. The last step verifies the super(martingale) conditions in Definition 4.1.

*Step 1.* For  $t \geq 0$  and  $i \in I$ , let  $\bar{\mathcal{Y}}_t^i := \mathcal{Y}_t^i - \lambda t$ . Then, in each time interval  $[T_{j-1}, T_j]$ , we have

$$U(x, t) = U^{\alpha^{j-1}}(x, t) = \frac{x^\delta}{\delta} e^{\bar{\mathcal{Y}}_t^{\alpha^{j-1}}}.$$

On the other hand, for  $t \in (T_{j-1}, T_j]$ , note that any admissible trading strategy  $\pi \in \mathcal{A}^{\mathbb{G}}$  takes the form  $\pi_t = \pi_t^{\alpha^{j-1}}$ , with  $\pi^{\alpha^{j-1}}$  being  $\mathbb{F}$ -progressively measurable. In turn, applying Itô's formula and using the equations (2.1) and (4.2), we obtain

$$\begin{aligned} & \frac{(X_{T_j}(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_{T_j}^{\alpha^{j-1}}} - \frac{(X_{T_{j-1}}(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_{T_{j-1}}^{\alpha^{j-1}}} \\ &= \int_{T_{j-1}}^{T_j} \frac{(X_s(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_s^{\alpha^{j-1}}} \left[ f^{\alpha^{j-1}}(V_s, \mathcal{Z}_s^{\alpha^{j-1}}; \pi_s^{\alpha^{j-1}}) - f^{\alpha^{j-1}}(V_s, \mathcal{Z}_s^{\alpha^{j-1}}) \right] ds \\ & \quad + \int_{T_{j-1}}^{T_j} \frac{(X_s(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_s^{\alpha^{j-1}}} \sum_{k \in I} q^{\alpha^{j-1}k} \left[ 1 - e^{\bar{\mathcal{Y}}_s^k - \bar{\mathcal{Y}}_s^{\alpha^{j-1}}} \right] ds \\ (4.10) \quad & \quad + \int_{T_{j-1}}^{T_j} \frac{(X_s(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_s^{\alpha^{j-1}}} \left( \delta \pi_s^{\alpha^{j-1}} + \mathcal{Z}_s^{\alpha^{j-1}} \right)^{tr} dW_s, \end{aligned}$$

where

$$(4.11) \quad f^i(v, z; \pi) := \frac{1}{2} \delta (\delta - 1) |\pi|^2 + \delta \pi^{tr} \theta^i(v) + \delta \pi^{tr} z + \frac{1}{2} |z|^2,$$

for  $i \in I$  and  $(v, z, \pi) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ .

*Step 2.* Then, for  $t \geq 0$  and  $\pi \in \mathcal{A}^{\mathbb{G}}$ , i.e.  $\pi_t = \pi_0^i + \sum_{j \geq 1} \pi_t^{\alpha^{j-1}} \chi_{(T_{j-1}, T_j]}(t)$ , using the decomposition formula (4.6), we further have

$$\begin{aligned} \frac{(X_t(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_t^{\alpha^i}} - \frac{x^\delta}{\delta} e^{\bar{\mathcal{Y}}_0^i} &= \sum_{j \geq 1} \left[ \frac{(X_{t \wedge T_j}(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_{t \wedge T_j}^{\alpha^{j-1}}} - \frac{(X_{t \wedge T_{j-1}}(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_{t \wedge T_{j-1}}^{\alpha^{j-1}}} \right] \\ & \quad + \sum_{j \geq 1} \left[ \frac{(X_{T_j}(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_{T_j}^{\alpha^j}} - \frac{(X_{T_j}(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_{T_j}^{\alpha^{j-1}}} \right] \chi_{\{T_j \leq t\}} \\ &= (I) + (II). \end{aligned}$$

For the continuous component (I), using (4.10) and the facts that  $\alpha_{s-} = \alpha^{j-1}$ ,  $\pi_s = \pi_s^{\alpha^{j-1}}$ , for  $s \in (t \wedge T_{j-1}, t \wedge T_j]$ , we deduce that

$$\begin{aligned} (I) &= \int_0^t \frac{(X_s(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_s^{\alpha_{s-}}} \left[ f^{\alpha_{s-}}(V_s, \mathcal{Z}_s^{\alpha_{s-}}; \pi_s) - f^{\alpha_{s-}}(V_s, \mathcal{Z}_s^{\alpha_{s-}}) \right] ds \\ & \quad + \int_0^t \frac{(X_s(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_s^{\alpha_{s-}}} \sum_{k \in I} q^{\alpha_{s-}k} \left[ 1 - e^{\bar{\mathcal{Y}}_s^k - \bar{\mathcal{Y}}_s^{\alpha_{s-}}} \right] ds \\ (4.12) \quad & \quad + \int_0^t \frac{(X_s(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_s^{\alpha_{s-}}} \left( \delta \pi_s + \mathcal{Z}_s^{\alpha_{s-}} \right)^{tr} dW_s. \end{aligned}$$

For the jump component (II), using Lemma 4.3, we deduce that

$$(4.13) \quad \begin{aligned} (II) &= \int_0^t \frac{(X_s(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_s^{\alpha_s^-}} \sum_{k, k' \in I} \left[ e^{\bar{\mathcal{Y}}_s^k - \bar{\mathcal{Y}}_s^{k'}} - 1 \right] \chi_{\{\alpha_{s-} = k'\}} d\tilde{N}_s^{k'k} \\ &+ \int_0^t \frac{(X_s(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_s^{\alpha_s^-}} \sum_{k \in I} q^{\alpha_s - k} \left[ e^{\bar{\mathcal{Y}}_s^k - \bar{\mathcal{Y}}_s^{\alpha_s^-}} - 1 \right] ds. \end{aligned}$$

It then follows from (4.12) and (4.13) that

$$\begin{aligned} \frac{(X_t(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_t^{\alpha_t}} - \frac{x^\delta}{\delta} e^{\bar{\mathcal{Y}}_0^i} &= \int_0^t \frac{(X_s(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_s^{\alpha_s^-}} [f^{\alpha_s^-}(V_s, \mathcal{Z}_s^{\alpha_s^-}; \pi_s) - f^{\alpha_s^-}(V_s, \mathcal{Z}_s^{\alpha_s^-})] ds \\ &+ \int_0^t \frac{(X_s(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_s^{\alpha_s^-}} (\delta \pi_s + \mathcal{Z}_s^{\alpha_s^-})^{tr} dW_s \\ &+ \int_0^t \frac{(X_s(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_s^{\alpha_s^-}} \sum_{k, k' \in I} \left[ e^{\bar{\mathcal{Y}}_s^k - \bar{\mathcal{Y}}_s^{k'}} - 1 \right] \chi_{\{\alpha_{s-} = k'\}} d\tilde{N}_s^{k'k}. \end{aligned}$$

In turn,

$$(4.14) \quad \begin{aligned} \frac{(X_t(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_t^{\alpha_t}} &= \frac{x^\delta}{\delta} e^{\mathcal{Y}_0^i} \times e^{\int_0^t f^{\alpha_s^-}(V_s, \mathcal{Z}_s^{\alpha_s^-}; \pi_s) - f^{\alpha_s^-}(V_s, \mathcal{Z}_s^{\alpha_s^-}) ds} \\ &\times \mathcal{E}_t \left( \int_0^\cdot (\delta \pi_s + \mathcal{Z}_s^{\alpha_s^-})^{tr} dW_s \right) \\ &\times \mathcal{E}_t \left( \int_0^\cdot \sum_{k, k' \in I} \left[ e^{\mathcal{Y}_s^k - \mathcal{Y}_s^{k'}} - 1 \right] \chi_{\{\alpha_{s-} = k'\}} d\tilde{N}_s^{k'k} \right), \end{aligned}$$

for any  $\pi \in \mathcal{A}^{\mathbb{G}}$ , where  $\mathcal{E}(\cdot)$  denotes Doléans-Dade stochastic exponential.

*Step 3.* We verify the conditions in Definition 4.1. It is clear that (i) and (ii) hold, so we only verify the super(martingale) conditions in (iii). It follows from (4.7) and (4.11) that

$$f^{\alpha_s^-}(V_s, \mathcal{Z}_s^{\alpha_s^-}; \pi_s) - f^{\alpha_s^-}(V_s, \mathcal{Z}_s^{\alpha_s^-}) \leq 0,$$

for any  $\pi \in \mathcal{A}^{\mathbb{G}}$ . So the process  $\frac{(X_t(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_t^{\alpha_t}}$ ,  $t \geq 0$ , is a local super-martingale (see (4.14)). Next, we verify the second stochastic exponential on the right hand side of (4.14) is a nonnegative bounded  $\mathbb{G}$ -martingale. Indeed, define

$$\eta_s^{k'k} := \left[ e^{\bar{\mathcal{Y}}_s^k - \bar{\mathcal{Y}}_s^{k'}} - 1 \right] \chi_{\{\alpha_{s-} = k'\}} = \left[ e^{\mathcal{Y}_s^k - \mathcal{Y}_s^{k'}} - 1 \right] \chi_{\{\alpha_{s-} = k'\}}, \quad s \geq 0.$$

In turn,

$$\begin{aligned} &\mathcal{E}_t \left( \int_0^\cdot \sum_{k, k' \in I} \left[ e^{\mathcal{Y}_s^k - \mathcal{Y}_s^{k'}} - 1 \right] \chi_{\{\alpha_{s-} = k'\}} d\tilde{N}_s^{k'k} \right) \\ &= \prod_{k, k' \in I} \mathcal{E}_t \left( \int_0^\cdot \eta_s^{k'k} (dN_s^{k'k} - q^{k'k} ds) \right) \\ &= \prod_{k, k' \in I} e^{-\int_0^t \eta_s^{k'k} q^{k'k} ds} \prod_{0 < s \leq t} (1 + \eta_s^{k'k} \Delta N_s^{k'k}). \end{aligned}$$

The estimate (3.10) in Theorem 3.2 implies that the difference of any two components  $\mathcal{Y}^k$  and  $\mathcal{Y}^{k'}$  is bounded:

$$(4.15) \quad |\mathcal{Y}_s^k - \mathcal{Y}_s^{k'}| \leq \frac{1}{q^{\min}} \left( K_f + \frac{C_v C_\eta C_z}{(C_\eta - C_v)^2} \right),$$

so  $\eta^{k'k}$  is bounded. Since  $1 + \eta_s^{k'k} \Delta N_s^{k'k} \geq 0$ , it follows that  $\mathcal{E} \left( \int_0^\cdot \sum_{k,k' \in I} \eta_s^{k'k} d\tilde{N}_s^{k'k} \right)$  is a nonnegative bounded  $\mathbb{G}$ -martingale. In turn,  $\frac{(X_t(\pi))^\delta}{\delta} e^{\bar{\mathcal{Y}}_t^{\alpha t}}$ ,  $t \geq 0$ , is a nonnegative local super-martingale, so it is a super-martingale for any  $\pi \in \mathcal{A}^{\mathbb{G}}$ , and the super-martingale condition (4.4) has been verified.

Finally, note that with  $\pi_s^* = \text{Proj}_{\Pi^{\alpha_s-}} \left( \frac{Z_s^{\alpha_s-} + \theta^{\alpha_s-}(V_s)}{1-\delta} \right)$ , we have

$$f^{\alpha_s-}(V_s, Z_s^{\alpha_s-}; \pi_s^*) - f^{\alpha_s-}(V_s, Z_s^{\alpha_s-}) = 0.$$

The estimate (3.9) in Theorem 3.2 implies that  $Z^i$  is bounded, so the optimal trading strategy  $\pi^*$  is also bounded and therefore  $\pi^* \in \mathcal{A}^{\mathbb{G}}$ . Note that  $\int_0^\cdot (\delta \pi_s^{i,*} + Z_s^i)^{tr} dW_s$  is an  $\mathbb{F}$ -BMO martingale. In turn,  $\mathcal{E} \left( \int_0^\cdot (\delta \pi_s^* + Z_s^{\alpha_s-})^{tr} dW_s \right)$  is a uniformly integrable  $\mathbb{G}$ -martingale. On the other hand, we have shown that  $\mathcal{E} \left( \int_0^\cdot \sum_{k,k' \in I} \eta_s^{k'k} d\tilde{N}_s^{k'k} \right)$  is a nonnegative bounded  $\mathbb{G}$ -martingale. Thus, we easily conclude from (4.14) the martingale condition (4.5) for  $\frac{(X_t(\pi^*))^\delta}{\delta} e^{\bar{\mathcal{Y}}_t^{\alpha t}}$ ,  $t \geq 0$ .

**4.5. Connection with classical utility maximization.** We provide an interpretation of the constant  $\lambda$ , appearing in the representation of the Markovian forward performance process (4.8), as the solution of the risk-sensitive control problem (4.16) below. It turns out that the constant  $\lambda$  is also the optimal long-term growth rate of the utility maximization problem (see (4.17) below). For this, we need to shrink the admissible set  $\mathcal{A}^{\mathbb{G}}$  to  $\bar{\mathcal{A}}^{\mathbb{G}}$  defined as per below:

$$\bar{\mathcal{A}}_{[0,t]}^{\mathbb{G}} = \left\{ \pi \in \mathcal{A}_{[0,t]}^{\mathbb{G}} : \int_0^\cdot (\pi_s^j)^{tr} dW_s \text{ is an } \mathbb{F}\text{-BMO martingale.} \right\}$$

Let  $\bar{\mathcal{A}}^{\mathbb{G}} = \cup_{t \geq 0} \bar{\mathcal{A}}_{[0,t]}^{\mathbb{G}}$ . Note that for  $\pi^*$  given in (4.9), since it is bounded, we also have  $\pi^* \in \bar{\mathcal{A}}^{\mathbb{G}} \subset \mathcal{A}^{\mathbb{G}}$ .

**PROPOSITION 4.4.** *Let  $T > 0$  and  $\pi \in \bar{\mathcal{A}}^{\mathbb{G}}$ . Define the probability measure  $\mathbb{P}^\pi$  as*

$$\frac{d\mathbb{P}^\pi}{d\mathbb{P}} := \mathcal{E}_T \left( \int_0^\cdot \delta \pi_u^{tr} dW_u \right),$$

*and the cost functional*

$$L^i(v; \pi) := \frac{1}{2} \delta (\delta - 1) |\pi|^2 + \delta \pi^{tr} \theta^i(v),$$

*for  $i \in I$  and  $(v, z) \in \mathbb{R}^d \times \mathbb{R}^d$ .*

*Let  $((\mathcal{Y}^i, Z^i)_{i \in I}, \lambda)$  be the unique Markovian solution of the ergodic BSDE system (3.7) with driver  $f^i$  as in (4.7), and satisfy (3.8), (3.9) and (3.10). Then,  $\lambda$  is the long-term growth rate of the risk-sensitive control problem*

$$(4.16) \quad \lambda = \sup_{\pi \in \bar{\mathcal{A}}^{\mathbb{G}}} \limsup_{T \uparrow \infty} \frac{1}{T} \ln \mathbb{E}^{\mathbb{P}^\pi} \left[ e^{\int_0^T L^{\alpha_s-}(V_s, \pi_s) ds} \right],$$

or, alternatively,

$$(4.17) \quad \lambda = \sup_{\pi \in \mathcal{A}^c} \limsup_{T \uparrow \infty} \frac{1}{T} \ln \mathbb{E} \left[ \frac{(X_T(\pi))^\delta}{\delta} \right].$$

For both problems (4.16) and (4.17), the associated optimal control in each regime  $i$  is  $\pi^{i,*}$  as in (4.9).

*Proof.* We first observe that the driver  $f^i$  in (4.7) can be written as

$$f^i(v, z) = \sup_{\pi \in \Pi} \left( L^i(v, \pi) + z^{tr} \delta \pi \right) + \frac{1}{2} |z|^2.$$

Therefore, for arbitrary admissible  $\tilde{\pi}$ , we apply Itô's formula to the ergodic BSDE system (3.7) on  $[T_{j-1}, T_j)$ , and obtain

$$\begin{aligned} & e^{\mathcal{Y}_{T_j^-}^{\alpha^{j-1}}} - e^{\mathcal{Y}_{T_{j-1}}^{\alpha^{j-1}}} \\ &= \int_{T_{j-1}}^{T_j} e^{\mathcal{Y}_s^{\alpha^{j-1}}} \left[ - \sup_{\pi_s^{\alpha^{j-1}} \in \Pi} \left( L^{\alpha^{j-1}}(V_s, \pi_s^{\alpha^{j-1}}) + (\mathcal{Z}_s^{\alpha^{j-1}})^{tr} \delta \pi_s^{\alpha^{j-1}} \right) + (\mathcal{Z}_s^{\alpha^{j-1}})^{tr} \delta \tilde{\pi}_s^{\alpha^{j-1}} \right] ds \\ & \quad + \int_{T_{j-1}}^{T_j} e^{\mathcal{Y}_s^{\alpha^{j-1}}} \left[ \lambda - \sum_{k \in I} q^{\alpha^{j-1}k} (e^{\mathcal{Y}_s^k - \mathcal{Y}_s^{\alpha^{j-1}}} - 1) \right] ds \\ & \quad + \int_{T_{j-1}}^{T_j} e^{\mathcal{Y}_s^{\alpha^{j-1}}} (\mathcal{Z}_s^{\alpha^{j-1}})^{tr} (dW_s - \delta \tilde{\pi}_s^{\alpha^{j-1}} ds). \end{aligned}$$

In general, we decompose  $e^{\mathcal{Y}_T^{\alpha^T}}$  into continuous and jump components as

$$\begin{aligned} e^{\mathcal{Y}_T^{\alpha^T}} - e^{\mathcal{Y}_0^i} &= \sum_{j \geq 1} \left[ e^{\mathcal{Y}_{T \wedge T_j^-}^{\alpha^{j-1}}} - e^{\mathcal{Y}_{T \wedge T_{j-1}}^{\alpha^{j-1}}} \right] + \sum_{j \geq 1} \left[ e^{\mathcal{Y}_{T_j}^{\alpha^j}} - e^{\mathcal{Y}_{T_j^-}^{\alpha^{j-1}}} \right] \chi_{\{T_j \leq T\}} \\ &= (I) + (II). \end{aligned}$$

It follows from the facts that  $\alpha_{s-} = \alpha^{j-1}$ ,  $\pi_s = \pi_s^{\alpha^{j-1}}$  and  $\tilde{\pi}_s = \tilde{\pi}_s^{\alpha^{j-1}}$  for  $s \in (T \wedge T_{j-1}, T \wedge T_j]$  that (I) has the expression

$$\begin{aligned} (I) &= \int_0^T e^{\mathcal{Y}_s^{\alpha_{s-}}} \left[ - \sup_{\pi_s \in \Pi} \left( L^{\alpha_{s-}}(V_s, \pi_s) + (\mathcal{Z}_s^{\alpha_{s-}})^{tr} \delta \pi_s \right) + (\mathcal{Z}_s^{\alpha_{s-}})^{tr} \delta \tilde{\pi}_s + \lambda \right] ds \\ & \quad - \int_0^T e^{\mathcal{Y}_s^{\alpha_{s-}}} \sum_{k \in I} q^{\alpha_{s-}k} (e^{\mathcal{Y}_s^k - \mathcal{Y}_s^{\alpha_{s-}}} - 1) ds \\ (4.18) \quad & + \int_0^T e^{\mathcal{Y}_s^{\alpha_{s-}}} (\mathcal{Z}_s^{\alpha_{s-}})^{tr} (dW_s - \delta \tilde{\pi}_s ds). \end{aligned}$$

Furthermore, it follows from Lemma 4.3 that (II) has the expression

$$\begin{aligned} (II) &= \int_0^T e^{\mathcal{Y}_s^{\alpha_{s-}}} \sum_{k, k' \in I} \left( e^{\mathcal{Y}_s^k - \mathcal{Y}_s^{k'}} - 1 \right) \chi_{\{\alpha_{s-} = k'\}} d\tilde{N}_s^{k'k} \\ (4.19) \quad & + \int_0^T e^{\mathcal{Y}_s^{\alpha_{s-}}} \sum_{k \in I} q^{\alpha_{s-}k} \left( e^{\mathcal{Y}_s^k - \mathcal{Y}_s^{\alpha_{s-}}} - 1 \right) ds. \end{aligned}$$

Consequently, combining (4.18) and (4.19), we obtain

$$\begin{aligned} e^{\mathcal{Y}_T^{\alpha T}} - e^{\mathcal{Y}_0^i} &= \int_0^T e^{\mathcal{Y}_s^{\alpha s-}} \left[ - \sup_{\pi_s \in \Pi} (L^{\alpha s-}(V_s, \pi_s) + (\mathcal{Z}_s^{\alpha s-})^{tr} \delta \pi_s) + (\mathcal{Z}_s^{\alpha s-})^{tr} \delta \tilde{\pi}_s + \lambda \right] ds \\ &\quad + \int_0^T e^{\mathcal{Y}_s^{\alpha s-}} (\mathcal{Z}_s^{\alpha s-})^{tr} dW_s^{\mathbb{P}^{\tilde{\pi}}} \\ &\quad + \int_0^T e^{\mathcal{Y}_s^{\alpha s-}} \sum_{k, k' \in I} \left( e^{\mathcal{Y}_s^k - \mathcal{Y}_s^{k'}} - 1 \right) \chi_{\{\alpha_{s-} = k'\}} d\tilde{N}_s^{k'k}, \end{aligned}$$

where the process  $W_t^{\mathbb{P}^{\tilde{\pi}}} := W_t - \int_0^t \delta \tilde{\pi}_u du$ ,  $t \geq 0$ , is a Brownian motion under  $\mathbb{P}^{\tilde{\pi}}$ . In turn,

$$\begin{aligned} e^{\mathcal{Y}_T^{\alpha T}} &= e^{\mathcal{Y}_0^i + \lambda T} \mathcal{E}_T \left( \int_0^{\cdot} (\mathcal{Z}_s^{\alpha s-})^{tr} dW_s^{\mathbb{P}^{\tilde{\pi}}} \right) \mathcal{E}_T \left( \sum_{k, k' \in I} \left( e^{\mathcal{Y}_s^k - \mathcal{Y}_s^{k'}} - 1 \right) \chi_{\{\alpha_{s-} = k'\}} d\tilde{N}_s^{k'k} \right) \\ &\quad \times e^{-\int_0^T L^{\alpha s-}(V_s, \tilde{\pi}_s) ds} \\ &\quad \times e^{\int_0^T [(L^{\alpha s-}(V_s, \tilde{\pi}_s) + (\mathcal{Z}_s^{\alpha s-})^{tr} \delta \tilde{\pi}_s) - \sup_{\pi_s \in \Pi} (L^{\alpha s-}(V_s, \pi_s) + (\mathcal{Z}_s^{\alpha s-})^{tr} \delta \pi_s)] ds}. \end{aligned}$$

Next, we observe that for any  $\tilde{\pi} \in \bar{\mathcal{A}}^G$ , the last exponential term on the right hand side is bounded above by 1. Taking expectation under  $\mathbb{P}^{\tilde{\pi}}$  then yields

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^{\tilde{\pi}}} \left[ e^{\int_0^T L^{\alpha s-}(V_s, \tilde{\pi}_s) ds} \right] e^{-\mathcal{Y}_0^i - \lambda T} \\ &\leq \mathbb{E}^{\mathbb{P}^{\tilde{\pi}}} \left[ e^{-\mathcal{Y}_T^{\alpha T}} \mathcal{E}_T \left( \int_0^{\cdot} (\mathcal{Z}_s^{\alpha s-})^{tr} dW_s^{\mathbb{P}^{\tilde{\pi}}} \right) \mathcal{E}_T \left( \sum_{k, k' \in I} \left( e^{\mathcal{Y}_s^k - \mathcal{Y}_s^{k'}} - 1 \right) \chi_{\{\alpha_{s-} = k'\}} d\tilde{N}_s^{k'k} \right) \right]. \end{aligned}$$

Define the probability measure  $\mathbb{Q}^{\tilde{\pi}}$  as

$$\frac{d\mathbb{Q}^{\tilde{\pi}}}{d\mathbb{P}^{\tilde{\pi}}} := \mathcal{E}_T \left( \int_0^{\cdot} (\mathcal{Z}_s^{\alpha s-})^{tr} dW_s^{\mathbb{P}^{\tilde{\pi}}} \right) \mathcal{E}_T \left( \int_0^{\cdot} \sum_{k, k' \in I} \left( e^{\mathcal{Y}_s^k - \mathcal{Y}_s^{k'}} - 1 \right) \chi_{\{\alpha_{s-} = k'\}} d\tilde{N}_s^{k'k} \right).$$

Then, it follows from the linear growth condition (3.8) of  $\mathcal{Y}_T^i = \mathbf{y}^i(V_T)$  and Assumption 3 on  $V$  that

$$\frac{1}{C} \leq \mathbb{E}^{\mathbb{Q}^{\tilde{\pi}}} \left( e^{-\mathcal{Y}_T^{\alpha T}} \right) \leq C,$$

for some constant  $C$  independent of  $T$  (see (B.6)). Consequently,

$$\frac{1}{T} \ln \mathbb{E}^{\mathbb{P}^{\tilde{\pi}}} \left[ e^{\int_0^T L^{\alpha s-}(V_s, \tilde{\pi}_s) ds} \right] \leq \lambda + \frac{Y_0^i}{T} + \frac{1}{T} \ln \mathbb{E}^{\mathbb{Q}^{\tilde{\pi}}} \left( e^{-\mathcal{Y}_T^{\alpha T}} \right).$$

Sending  $T \rightarrow \infty$ , we obtain, for any  $\tilde{\pi} \in \bar{\mathcal{A}}^G$ ,

$$\lambda \geq \limsup_{T \uparrow \infty} \frac{1}{T} \ln \mathbb{E}^{\mathbb{P}^{\tilde{\pi}}} \left[ e^{\int_0^T L^{\alpha s-}(V_s, \tilde{\pi}_s) ds} \right],$$

with equality choosing  $\tilde{\pi}_s = \pi_s^*$ , with  $\pi_s^*$  as in (4.9).

To show that  $\lambda$  also solves (4.17), we observe that for  $\pi \in \bar{\mathcal{A}}^G$ , we have

$$\begin{aligned} \mathbb{E} \left[ \frac{(X_T^\pi)^\delta}{\delta} \right] &= \frac{X_0^\delta}{\delta} \mathbb{E} \left[ e^{\int_0^T L^{\alpha_s - (V_s, \pi_s)} ds} \mathcal{E}_T \left( \int_0^\cdot \delta \pi_s^{tr} dW_s \right)_T \right] \\ &= \frac{x^\delta}{\delta} \mathbb{E}^{\mathbb{P}^\pi} \left[ e^{\int_0^T L^{\alpha_s - (V_s, \pi_s)} ds} \right], \end{aligned}$$

and the rest of the arguments follow.  $\square$

**5. Application to the large time behavior of PDE systems with quadratic growth Hamiltonians.** As the second application, we use the ergodic BSDE system (3.7) to study the large time behavior of the PDE system with quadratic growth Hamiltonians, namely

$$(5.1) \quad \begin{aligned} -\partial_t \mathbf{y}^i(t, v) + \frac{1}{2} \text{Trace}(\kappa^{tr} \kappa \nabla_v^2 \mathbf{y}^i(t, v)) + \eta(v)^{tr} \nabla_v \mathbf{y}^i(t, v) \\ + f^i(v, \kappa^{tr} \nabla_v \mathbf{y}^i(t, v)) + \sum_{k \in I} q^{ik} \left( e^{(\mathbf{y}^k - \mathbf{y}^i)(t, v)} - 1 \right) = 0, \end{aligned}$$

with initial condition  $\mathbf{y}^i(0, v) = h^i(v)$ , for  $(t, v) \in \mathbb{R}_+ \times \mathbb{R}^d$  and  $i \in I$ . The data  $\kappa, \eta(\cdot), f^i(\cdot, \cdot)$  and  $q^{ik}$  of the PDE system are assumed to satisfy Assumptions 1-4 and, moreover, the initial condition  $h^i(\cdot)$  is bounded and Lipschitz continuous. Due to Assumption 1(ii), the Hamiltonians  $f^i(\cdot, \cdot)$  has quadratic growth in the gradients  $\nabla_v \mathbf{y}^i(t, v)$ . For this reason, (5.1) is dubbed as *a PDE system with quadratic growth Hamiltonians*. A special case of the above PDE system (5.1) has been considered in [3] and [4] to study the utility indifference prices of financial derivatives in a regime switching market.

The scalar case of (5.1) and its large time behavior has been studied in [27] using the ergodic BSDE approach. We extend their result from the scalar case to the system of equations. First, we provide a probabilistic representation for the PDE system (5.1). For  $T > 0$ , let  $(\mathcal{Y}^{i,v}(T), \mathcal{Z}^{i,v}(T))_{i \in I}$  be a solution to the finite horizon BSDE system

$$(5.2) \quad \begin{aligned} \mathcal{Y}_t^{i,v}(T) &= h^i(V_T^v) + \int_t^T \left[ f^i(V_s^v, \mathcal{Z}_s^{i,v}(T)) + \sum_{k \in I} q^{ik} (e^{\mathcal{Y}_s^{k,v}(T) - \mathcal{Y}_s^{i,v}(T)} - 1) \right] ds \\ &\quad - \int_t^T (\mathcal{Z}_s^{i,v}(T))^{tr} dW_s. \end{aligned}$$

Following along the similar arguments used to solve the finite horizon BSDE system (2.8) (see section 2.3 with  $\rho = 0$ ), we deduce that  $(\mathcal{Y}^{i,v}(T), \mathcal{Z}^{i,v}(T))_{i \in I}$  is actually the unique bounded solution of (5.2) with

$$(5.3) \quad |\mathcal{Z}_t^{i,v}(T)| \leq \frac{C_v}{C_\eta - C_v} + C_h.$$

Note that the bound of  $\mathcal{Y}^{i,v}(T)$  may depend on  $T$ . Furthermore, following from [1, Theorems 3.4 and 3.5], we deduce that  $\mathbf{y}^i(\cdot, \cdot)$ , defined as  $\mathbf{y}^i(T - t, V_t^v) := \mathcal{Y}_t^{i,v}(T)$ , is the unique viscosity solution to the PDE system (5.1). Since the monotone condition for  $\mathbf{y}^k$  in the last nonlinear term of (5.1) holds, a comparison result similar to Lemma 2.2 also holds for (5.1) (see Remark 3.9 in [1]).

**THEOREM 5.1.** *Suppose that Assumptions 1-4 hold, and  $h^i(\cdot)$ ,  $i \in I$ , is bounded by a constant  $K_h$  and Lipschitz continuous with its Lipschitz constant  $C_h$ .*

Let  $((\mathcal{Y}^{i,v}, \mathcal{Z}^{i,v})_{i \in I}, \lambda)$  be the unique Markovian solution of the ergodic BSDE system (3.7) with  $\mathcal{Y}_t^{i,v} = \mathbf{y}^i(V_t^v)$  and  $\mathcal{Z}_t^{i,v} = \mathbf{z}^i(V_t^v)$  satisfying (3.8), (3.9) and (3.10). Let  $\mathbf{y}^i(\cdot, \cdot)$  be the unique viscosity solution to the PDE system (5.1). Then, there exists a constant  $L$ , independent of  $v \in \mathbb{R}^d$  and  $i \in I$ , such that

$$(5.4) \quad \lim_{T \rightarrow \infty} (\mathbf{y}^i(T, v) - \lambda T - \mathbf{y}^i(v)) = L,$$

and moreover, there exist constants  $C$  and  $K_v$ , independent of  $T$ , such that

$$(5.5) \quad |\mathbf{y}^i(T, v) - \lambda T - \mathbf{y}^i(v) - L| \leq C(1 + |v|^2)e^{-K_v T}.$$

*Proof.* The proof is adapted from the arguments in [27, Section 4.2] (see also [28]). In the following, we only highlight the key difference from their proof.

We first convert the BSDE system (5.2) to a scalar-valued BSDE driven by the Brownian motion  $W$  and the Markov chain  $\alpha$ . To this end, similar to Appendix B, for  $t \in [0, T]$  and  $v \in \mathbb{R}^d$ , we introduce

$$\mathcal{Y}_t^v(T) := \mathcal{Y}_t^{\alpha_t, v}(T) = \mathbf{y}^{\alpha_t}(T - t, V_t^v),$$

$$\mathcal{Z}_t^v(T) := \mathcal{Z}_t^{\alpha_t, v}(T) = \mathbf{z}^{\alpha_t}(T - t, V_t^v),$$

and for  $k', k \in I$ ,

$$\mathcal{U}_t^v(k', k; T) := \mathcal{Y}_t^{k, v}(T) - \mathcal{Y}_t^{k', v}(T) = (\mathbf{y}^k - \mathbf{y}^{k'})(T - t, V_t^v).$$

Then, following along the similar arguments in the proof of Lemma 3.1, we deduce that

$$(5.6) \quad |\mathcal{U}_t^v(k', k; T)| \leq \frac{1}{q^{\min}} \left( K_f + \frac{C_v C_\eta C_z}{(C_\eta - C_v)^2} \right) + K_h.$$

In turn, using Lemma 4.3, we deduce that  $(\mathcal{Y}^v(T), \mathcal{Z}^v(T), (\mathcal{U}^v(k', k; T))_{k', k \in I})$  satisfies the scalar-valued BSDE driven by  $W$  and  $\alpha$ , i.e. for  $t \in [0, T]$ ,

$$(5.7) \quad \begin{aligned} \mathcal{Y}_t^v(T) &= h^{\alpha_t}(V_t^v) + \int_t^T f^{\alpha_s}(V_s^v, \mathcal{Z}_s^v(T)) dt - \int_t^T (\mathcal{Z}_s^v(T))^{tr} dW_s \\ &\quad + \int_t^T \sum_{k \in I} q^{\alpha_s - k} \left[ e^{\mathcal{U}_s^v(\alpha_s, k; T)} - 1 - \mathcal{U}_s^v(\alpha_s, k; T) \right] ds \\ &\quad - \int_t^T \sum_{k, k' \in I} \mathcal{U}_s^v(k', k; T) \chi_{\{\alpha_s = k'\}} d\tilde{N}_s^{k'k}. \end{aligned}$$

Next, we define  $\delta \mathcal{Y}_t^v(T) := \mathbf{y}^{\alpha_t}(T - t, V_t^v) - \mathbf{y}^{\alpha_t}(V_t^v) - \lambda(T - t)$  for  $t \in [0, T]$ . Then, we have the following key estimates.

LEMMA 5.2. *The function  $\delta \mathcal{Y}_0^v(T) = \mathbf{y}^i(T, v) - \mathbf{y}^i(v) - \lambda T$ ,  $v \in \mathbb{R}^d$ , admits the following properties: There exist constants  $C$  and  $K_v$ , independent of  $T$ , such that for arbitrary  $v_1, v_2 \in \mathbb{R}^d$ ,*

- (i)  $|\delta \mathcal{Y}_0^{v_1}(T)| \leq C(1 + |v_1|)$ ;
- (ii)  $|\delta \mathcal{Y}_0^{v_1}(T) - \delta \mathcal{Y}_0^{v_2}(T)| \leq C|v_1 - v_2|$ ;
- (iii)  $|\delta \mathcal{Y}_0^{v_1}(T) - \delta \mathcal{Y}_0^{v_2}(T)| \leq C(1 + |v_1|^2 + |v_2|^2)e^{-K_v T}$ .

*Proof.* First, we prove Assertion (ii). When  $\eta$  and  $f$  are continuously differentiable functions with bounded derivatives, noting

$$\kappa^{tr} \nabla_v \mathbf{y}^i(T-t, V_t^v) = \mathcal{Z}_t^{i,v}(T) \quad \text{and} \quad \kappa^{tr} \nabla_v \mathbf{y}^i(V_t^v) = \mathcal{Z}_t^{i,v},$$

the desired assertion follows from the boundedness of both  $\mathcal{Z}_t^{i,v}(T)$  and  $\mathcal{Z}_t^{i,v}$  (cf. (5.3) and (3.9)) and Assumption 3 on  $\kappa$ . For our general  $\eta$  and  $f$ , Assertion (ii) can be proved by a standard mollification argument.

Next, we prove the assertions (i) and (iii). To this end, for  $t \in [0, T]$ , define

$$\delta \mathcal{Z}_t^v(T) := \mathcal{Z}_t^v(T) - \mathcal{Z}_t^v, \quad \text{and} \quad \delta \mathcal{U}_t^v(k', k; T) := \mathcal{U}_t^v(k', k; T) - \mathcal{U}_t^v(k', k)$$

with  $k', k \in I$ . Then, we deduce from (5.7) and (B.1) that  $(\delta \mathcal{Y}^v(T), \delta \mathcal{Z}^v(T), (\delta \mathcal{U}^v(k', k; T))_{k', k \in I})$  satisfies

$$\begin{aligned} (5.8) \quad \delta \mathcal{Y}_0^v(T) &= h^{\alpha_T}(V_T^v) - \mathbf{y}^{\alpha_T}(V_T^v) \\ &+ \int_0^T [f^{\alpha_{s-}}(V_s^v, \mathcal{Z}_s^v(T)) - f^{\alpha_{s-}}(V_s^v, \mathcal{Z}_s^v)] ds - \int_0^T (\delta \mathcal{Z}_s^v(T))^{tr} dW_s \\ &+ \int_0^T \sum_{k \in I} q^{\alpha_{s-} - k} [g(\mathcal{U}_s^v(\alpha_{s-}, k; T)) - g(\mathcal{U}_s^v(\alpha_{s-}, k))] ds \\ &- \int_0^T \sum_{k, k'} \delta \mathcal{U}_s^v(k', k; T) \chi_{\{\alpha_{s-} = k'\}} d\tilde{N}_s^{k'k}, \end{aligned}$$

where  $g(\cdot)$  is given in (B.3). Since  $\mathcal{Z}^v(T)$ ,  $\mathcal{Z}^v$ ,  $\mathcal{U}^v(k', k; T)$  and  $\mathcal{U}^v(k', k)$  are all uniformly bounded (cf. (5.3), (3.9), (5.6) and (3.10)), analogous to Appendix B, we may introduce an equivalent probability measure  $\mathbb{Q}$ , under which we have

$$(5.9) \quad \delta \mathcal{Y}_0^v(T) = \mathbb{E}^{\mathbb{Q}} [h^{\alpha_T}(V_T^v) - \mathbf{y}^{\alpha_T}(V_T^v)].$$

Since both  $h^i(\cdot)$  and  $\mathbf{y}^i(\cdot)$  with  $i \in I$ , have at most a linear growth, we deduce assertion (i) from the estimate in (B.6).

To prove assertion (iii), from (5.9), we have, for  $v, \bar{v} \in \mathbb{R}^d$ ,

$$\delta \mathcal{Y}_0^v(T) - \delta \mathcal{Y}_0^{\bar{v}}(T) = \mathbb{E}^{\mathbb{Q}} [(h^{\alpha_T}(V_T^v) - \mathbf{y}^{\alpha_T}(V_T^v)) - (h^{\alpha_T}(V_T^{\bar{v}}) - \mathbf{y}^{\alpha_T}(V_T^{\bar{v}}))].$$

The conclusion then follows from the linear growth of both  $h^i(\cdot)$  and  $\mathbf{y}^i(\cdot)$  for  $i \in I$ , and the estimate in (B.7).  $\square$

Let us return to the proof of Theorem 5.1. Using the first estimate (i) in Lemma 5.2, by a standard diagonal procedure, we may construct a sequence  $\{T_k\}$  such that

$$\lim_{T_k \rightarrow \infty} (\mathbf{y}^i(T_k, v) - \mathbf{y}^i(v) - \lambda T_k) = L(v)$$

for some limit function  $L(v)$ . Moreover, the second estimate (ii) in Lemma 5.2 implies that the limit function  $L(v)$  can be extended to a Lipschitz continuous function, and the third estimate (iii) in Lemma 5.2 further implies that the limit actually satisfies  $L(v) = L$  with  $L$  being a constant. This establishes the limit (5.4).

To show the convergence rate (5.5), we deduce from (5.4) and (5.9) that, for  $T' > T$ ,

$$\begin{aligned} |\delta \mathcal{Y}_0^v(T) - L| &= \lim_{T' \rightarrow \infty} |\delta \mathcal{Y}_0^v(T) - \delta \mathcal{Y}_0^v(T')| \\ &= \lim_{T' \rightarrow \infty} \left| \delta \mathcal{Y}_0^v(T) - \mathbb{E}^{\mathbb{Q}} \left[ h^{\alpha_{T'}^m}(V_{T'}^v) - \mathbf{y}^{\alpha_{T'}^m}(V_{T'}^v) \right] \right|, \end{aligned}$$

where  $m(T') := 2i - \alpha_{T'-T}^i$ . Here we use  $\alpha^i$  to emphasize the initial data of the Markov chain  $\alpha_0 = i$ . It then follows from the tower property of conditional expectations that,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \left[ h_{T'}^{\alpha_{T'}^{m(T')}}(V_{T'}^v) - \mathbf{y}_{T'}^{\alpha_{T'}^{m(T')}}(V_{T'}^v) \right] &= \mathbb{E}^{\mathbb{Q}} \left[ \mathbb{E}^{\mathbb{Q}} \left[ h_{T'}^{\alpha_{T'}^{m(T')}}(V_{T'}^v) - \mathbf{y}_{T'}^{\alpha_{T'}^{m(T')}}(V_{T'}^v) \middle| \mathcal{G}_{T'-T} \right] \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \mathbf{y}_{T'-T}^{\alpha_{T'-T}^{m(T')}}(T, V_{T'-T}^v) - \mathbf{y}_{T'-T}^{\alpha_{T'-T}^{m(T')}}(V_{T'-T}^v) - \lambda T \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \mathbf{y}^i(T, V_{T'-T}^v) - \mathbf{y}^i(V_{T'-T}^v) - \lambda T \right] \end{aligned}$$

where we also used the relationship  $\alpha_{T'-T}^{m(T')} = \alpha_{T'-T}^{i - (\alpha_{T'-T}^i - i)} = i$  in the last equality. In turn, using the definition  $\delta \mathcal{Y}_0^v(T) = \mathbf{y}^i(T, v) - \mathbf{y}^i(v) - \lambda T$ , we obtain

$$\begin{aligned} |\delta \mathcal{Y}_0^v(T) - L| &= \lim_{T' \rightarrow \infty} \left| \delta \mathcal{Y}_0^v(T) - \mathbb{E}^{\mathbb{Q}} \left[ h_{T'}^{\alpha_{T'}^{m(T')}}(V_{T'}^v) - \mathbf{y}_{T'}^{\alpha_{T'}^{m(T')}}(V_{T'}^v) \right] \right| \\ &= \lim_{T' \rightarrow \infty} \mathbb{E}^{\mathbb{Q}} \left[ \mathbf{y}^i(T, v) - \mathbf{y}^i(v) - (\mathbf{y}^i(T, V_{T'-T}^v) - \mathbf{y}^i(V_{T'-T}^v)) \right], \\ &\leq \lim_{T' \rightarrow \infty} C (1 + |v|^2 + \mathbb{E}^{\mathbb{Q}} [ |V_{T'-T}^v|^2 ]) e^{-K_v T}, \end{aligned}$$

where the assertion (iii) in Lemma 5.2 is used in the last inequality. The convergence rate then follows from the moment estimate (B.6). The proof of Theorem 5.1 is complete.  $\square$

**6. Conclusions.** In this paper, we introduced and solved a new type of quadratic BSDE systems in an infinite time horizon and, subsequently, derived their asymptotic limit as ergodic BSDE systems. The ergodic BSDE system is used to characterize Markovian regime switching forward performance processes and their associated optimal portfolio strategies. We have also shown a connection between Markovian regime switching forward performance processes and their classical expected utility counterparts via the constant  $\lambda$  in the corresponding ergodic BSDE system. Finally, we use the ergodic BSDE system to study the large time behavior for a class of PDE systems with quadratic growth Hamiltonians.

**Appendix A. Proof of Lemma 2.2.** The idea of the proof is adapted from the arguments used in [29]. For  $t \in [0, T]$ , let

$$\delta Y_t^i := Y_t^i - \bar{Y}_t^i, \quad \delta Z_t^i := Z_t^i - \bar{Z}_t^i \quad \text{and} \quad \delta \xi^i := \xi^i - \bar{\xi}^i.$$

Applying Itô's formula to  $(\delta Y_t^{i+})^2$  yields

$$\begin{aligned} (\delta Y_t^{i+})^2 &= (\delta \xi^{i+})^2 + \int_t^T 2\delta Y_s^{i+} [F_s^i(Z_s^i) - \bar{F}_s^i(\bar{Z}_s^i)] ds \\ &\quad + \int_t^T 2\delta Y_s^{i+} [G_s^i(Y_s^i, Y_s^{-i}) - \bar{G}_s^i(\bar{Y}_s^i, \bar{Y}_s^{-i})] ds \\ &\quad - \int_t^T \chi_{\{\delta Y_s^i > 0\}} |\delta Z_s^i|^2 ds - \int_t^T 2\delta Y_s^{i+} (\delta Z_s^i)^{tr} dW_s. \end{aligned}$$

Using (2.9) and (2.11), we obtain

$$F_s^i(Z_s^i) - \bar{F}_s^i(\bar{Z}_s^i) = F_s^i(Z_s^i) - F_s^i(\bar{Z}_s^i) + F_s^i(\bar{Z}_s^i) - \bar{F}_s^i(\bar{Z}_s^i) \leq C_f |\delta Z_s^i|.$$

Using (2.10) and (2.12), together with the monotone condition of  $G_s^i$ , we further obtain

$$\begin{aligned} & G_s^i(Y_s^i, Y_s^{-i}) - \bar{G}_s^i(\bar{Y}_s^i, \bar{Y}_s^{-i}) \\ &= G_s^i(Y_s^i, Y_s^{-i}) - G_s^i(\bar{Y}_s^i, \bar{Y}_s^{-i}) + G_s^i(\bar{Y}_s^i, \bar{Y}_s^{-i}) - \bar{G}_s^i(\bar{Y}_s^i, \bar{Y}_s^{-i}) \\ &\leq C_g \left( |\delta Y_s^i| + \sum_{k \neq i} \delta Y_s^{k+} \right). \end{aligned}$$

In turn, since  $\delta \xi^{i+} = 0$ , we have

$$\begin{aligned} & \mathbb{E}[(\delta Y_t^{i+})^2] \\ &\leq \mathbb{E} \left[ \int_t^T \left( 2C_f \delta Y_s^{i+} |\delta Z_s^i| + 2C_g \delta Y_s^{i+} (|\delta Y_s^i| + \sum_{k \neq i} \delta Y_s^{k+}) - \chi_{\{\delta Y_s^i > 0\}} |\delta Z_s^i|^2 \right) ds \right] \\ &\leq \mathbb{E} \left[ \int_t^T \chi_{\{\delta Y_s^i > 0\}} \left( -|\delta Z_s^i|^2 + 2C_f \delta Y_s^i |\delta Z_s^i| - C_f^2 (\delta Y_s^i)^2 \right) ds \right] \\ &\quad + \mathbb{E} \left[ \int_t^T \left( (2C_g + C_f^2) (\delta Y_s^{i+})^2 + C_g^2 (\delta Y_s^{i+})^2 + \sum_{k \neq i} (\delta Y_s^{k+})^2 \right) ds \right]. \end{aligned}$$

Thus, there exists a constant  $C$  such that

$$\sum_{i \in I} \mathbb{E}[(\delta Y_t^{i+})^2] \leq C \int_t^T \sum_{i \in I} \mathbb{E}[(\delta Y_s^{i+})^2] ds.$$

It then follows from Gronwall's inequality that  $\mathbb{E}[(\delta Y_t^i)^2] = 0$ , for  $t \in [0, T]$  and  $i \in I$ , so  $Y_t^i \leq \bar{Y}_t^i$  and we conclude.

**Appendix B. Proof of Theorem 3.2.** Let  $\alpha$  be the Markov chain introduced in section 4 satisfying Assumptions 2 and 4. Let  $((\mathcal{Y}^{i,v}, \mathcal{Z}^{i,v})_{i \in I}, \lambda)$  and  $((\bar{\mathcal{Y}}^{i,v}, \bar{\mathcal{Z}}^{i,v})_{i \in I}, \bar{\lambda})$  be two Markovian solutions to the ergodic BSDE system (3.7) both satisfying (3.8), (3.9) and (3.10).

For  $t \geq 0$  and  $v \in \mathbb{R}^d$ , define

$$\mathcal{Y}_t^v := \mathcal{Y}_t^{\alpha_t, v} = \mathbf{y}^{\alpha_t}(V_t^v),$$

$$\mathcal{Z}_t^v := \mathcal{Z}_t^{\alpha_t-, v} = \mathbf{z}^{\alpha_t-}(V_t^v),$$

and for  $k', k \in I$ ,

$$\mathcal{U}_t^v(k', k) := \mathcal{Y}_t^{k, v} - \mathcal{Y}_t^{k', v} = (\mathbf{y}^k - \mathbf{y}^{k'})(V_t^v).$$

We may also define  $(\bar{\mathcal{Y}}^v, \bar{\mathcal{Z}}^v, (\bar{\mathcal{U}}^v(k', k))_{k', k \in I})$  in an analogous way. Furthermore, let  $\delta \mathcal{Y}_t^v := \mathcal{Y}_t^v - \bar{\mathcal{Y}}_t^v$ ,  $\delta \mathcal{Z}_t^v := \mathcal{Z}_t^v - \bar{\mathcal{Z}}_t^v$ ,  $\delta \mathcal{U}_t^v(k', k) := \mathcal{U}_t^v(k', k) - \bar{\mathcal{U}}_t^v(k', k)$  and  $\delta \lambda := \lambda - \bar{\lambda}$ .

First, using Lemma 4.3, we deduce that  $(\mathcal{Y}^v, \mathcal{Z}^v, (\mathcal{U}^v(k', k))_{k', k \in I}, \lambda)$  satisfies the scalar-valued ergodic BSDE driven by the Brownian motion  $W$  and the Markov chain

$\alpha$ , i.e. for  $t \geq 0$ ,

$$(B.1) \quad \begin{aligned} d\mathcal{Y}_t^v &= -f^{\alpha t-}(V_t^v, \mathcal{Z}_t^v)dt - \sum_{k \in I} q^{\alpha t-k} \left[ e^{\mathcal{U}_t^v(\alpha_{t-}, k)} - 1 - \mathcal{U}_t^v(\alpha_{t-}, k) \right] dt + \lambda dt \\ &+ (\mathcal{Z}_t^v)^{tr} dW_t + \sum_{k, k' \in I} \mathcal{U}_t^v(k', k) \chi_{\{\alpha_{t-}=k'\}} d\tilde{N}_t^{k'k}. \end{aligned}$$

In turn,  $(\delta\mathcal{Y}^v, \delta\mathcal{Z}^v, (\delta\mathcal{U}^v(k', k))_{k', k \in I}, \delta\lambda)$  satisfies

$$(B.2) \quad \begin{aligned} d(\delta\mathcal{Y}_t^v) &= -[f^{\alpha t-}(V_t^v, \mathcal{Z}_t^v) - f^{\alpha t-}(V_t^v, \bar{\mathcal{Z}}_t^v)] dt + (\delta\mathcal{Z}_t^v)^{tr} dW_t \\ &- \sum_{k \in I} q^{\alpha t-k} [g(\mathcal{U}_t^v(\alpha_{t-}, k)) - g(\bar{\mathcal{U}}_t^v(\alpha_{t-}, k))] dt \\ &+ \sum_{k, k'} \delta\mathcal{U}_t^v(k', k) \chi_{\{\alpha_{t-}=k'\}} d\tilde{N}_t^{k'k} + \delta\lambda dt, \end{aligned}$$

where

$$(B.3) \quad g(x) := e^x - 1 - x, \text{ with } |x| \leq \frac{1}{q^{\min}} \left( K_f + \frac{C_v C_\eta C_z}{(C_\eta - C_v)^2} \right).$$

Next, we introduce

$$\delta f^{\alpha t-}(V_t^v) := \frac{f^{\alpha t-}(V_t^v, \mathcal{Z}_t^v) - f^{\alpha t-}(V_t^v, \bar{\mathcal{Z}}_t^v)}{|\delta\mathcal{Z}_t^v|^2} \delta\mathcal{Z}_t^v \chi_{\{\delta\mathcal{Z}_t^v \neq 0\}},$$

and, for  $k \in I$ ,

$$\delta g^{\alpha t-k}(V_t^v) := \frac{g(\mathcal{U}_t^v(\alpha_{t-}, k)) - g(\bar{\mathcal{U}}_t^v(\alpha_{t-}, k))}{\delta\mathcal{U}_t^v(\alpha_{t-}, k)} \chi_{\{\delta\mathcal{U}_t^v(\alpha_{t-}, k) \neq 0\}}.$$

Note that Assumption 1(ii) and (3.9) imply that  $\delta f^{\alpha t-}(V_t^v)$ ,  $t \geq 0$ , is uniformly bounded. Moreover, the mean value theorem (applied to the function  $g(\cdot)$ ) and (3.10) imply that  $\delta g^{\alpha t-k}(V_t^v)$ ,  $t \geq 0$ , is also uniformly bounded. Thus, for any  $T > 0$ , define an equivalent probability measure  $\mathbb{Q}$  as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \mathcal{E}_T \left( \int_0^\cdot (\delta f^{\alpha s-}(V_s^v))^{tr} dW_s \right) \mathcal{E}_T \left( \int_0^\cdot \sum_{k, k' \in I} \delta g^{k'k}(V_s^v) \chi_{\{\alpha_{s-}=k'\}} d\tilde{N}_s^{k'k} \right),$$

so that under  $\mathbb{Q}$ , we have

$$(B.4) \quad \delta\lambda = \frac{\mathbb{E}^{\mathbb{Q}}[\delta\mathcal{Y}_T^v - \delta\mathcal{Y}_0^v]}{T} = \frac{\mathbb{E}^{\mathbb{Q}}[\mathbf{y}^{\alpha T}(V_T^v) - \bar{\mathbf{y}}^{\alpha T}(V_T^v)] - [\mathbf{y}^i(v) - \bar{\mathbf{y}}^i(v)]}{T}.$$

Since both  $\mathbf{y}^k(\cdot)$  and  $\bar{\mathbf{y}}^k(\cdot)$ ,  $k \in I$ , have at most linear growth (cf. (3.8)), it follows from (B.6) that  $\delta\lambda = 0$  by sending  $T \rightarrow \infty$  in (B.4).

We are left to show that  $\mathbf{y}^i(\cdot) = \bar{\mathbf{y}}^i(\cdot)$  and  $\mathbf{z}^i(\cdot) = \bar{\mathbf{z}}^i(\cdot)$  for  $i \in I$ . To this end, it suffices to show that

$$(B.5) \quad \delta\mathcal{Y}_0^v = \mathcal{Y}_0^v - \bar{\mathcal{Y}}_0^v = (\mathbf{y}^i - \bar{\mathbf{y}}^i)(v) = 0.$$

The rest of the proof then follows from Theorem 3.11 in [17]. To prove (B.5), we have, from (B.2), that

$$\delta\mathcal{Y}_0^v = \mathbb{E}^{\mathbb{Q}}[\delta\mathcal{Y}_T^v] = \mathbb{E}^{\mathbb{Q}}[\mathbf{y}^{\alpha T}(V_T^v) - \bar{\mathbf{y}}^{\alpha T}(V_T^v)].$$

Using (B.7) and the fact that  $\mathbf{y}^i(0) = \bar{\mathbf{y}}^i(0) = 0$ , we obtain

$$\mathbb{E}^{\mathbb{Q}}[\mathbf{y}^{\alpha T}(V_T^v) - \bar{\mathbf{y}}^{\alpha T}(V_T^v)] \leq C(1 + |v|^2)e^{-K_v T}.$$

Hence, (B.5) follows by sending  $T \rightarrow \infty$  in the above inequality.

To conclude the paper, we recall the following moment estimate and coupling estimate, which can be proved in a similar way to [17] (Proposition 2.3 and Theorem 2.4 for the Brownian motion case), [15] (section 3 for the Markov chain case) and [13] (section 3.2 for the Lévy process case).

**PROPOSITION B.1.** *Let  $T > 0$  be fixed. Let  $H^i : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $G^{ik} : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $i, k \in I$ , be measurable bounded functions. Under Assumption 3, suppose that the processes  $(V^v, \alpha)$  follow*

$$dV_t^v = [\eta(V_t^v) + H^{\alpha_{t-}}(V_t^v)]dt + \kappa dW_t^{\mathbb{Q}},$$

and

$$d\alpha_t = \sum_{k \in I} q^{\alpha_{t-} - k} (k - \alpha_{t-}) (1 + G^{\alpha_{t-} - k}(V_t^v)) dt + \sum_{k, k' \in I} (k - k') \chi_{\{\alpha_{t-} = k'\}} d\tilde{N}_t^{\mathbb{Q}, k'k},$$

where  $\mathbb{Q}$  is an equivalent probability measure defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \mathcal{E}_T \left( \int_0^\cdot (H^{\alpha_s -} (V_s^v))^{\text{tr}} dW_s \right) \mathcal{E}_T \left( \int_0^\cdot \sum_{k, k' \in I} G^{k'k}(V_s^v) \chi_{\{\alpha_s = k'\}} d\tilde{N}_s^{k'k} \right),$$

with  $W^{\mathbb{Q}} := W - \int_0^\cdot H^{\alpha_{t-}}(V_t^v) dt$  and  $\tilde{N}^{\mathbb{Q}, k'k} := \tilde{N}^{k'k} - \int_0^\cdot q^{k'k} G^{k'k}(V_t^v) dt$ ,  $k', k \in I$ , being the corresponding Brownian motion and compensated Poisson martingales under  $\mathbb{Q}$ , respectively. Then, there exists a constant  $C > 0$  such that for any measurable functions  $\phi^i : \mathbb{R}^d \rightarrow \mathbb{R}$  ( $i \in I$ ) with a polynomial growth rate  $\mu > 0$ ,

$$(B.6) \quad \mathbb{E}^{\mathbb{Q}}[\phi^{\alpha T}(V_T^v)] \leq C(1 + |v|^\mu), \quad v \in \mathbb{R}^d.$$

Furthermore, there exists a constant  $K_v > 0$  such that for  $v_1, v_2 \in \mathbb{R}^d$ ,

$$(B.7) \quad \mathbb{E}^{\mathbb{Q}}[\phi^{\alpha T}(V_T^{v_1}) - \phi^{\alpha T}(V_T^{v_2})] \leq C(1 + |v_1|^{1+\mu} + |v_2|^{1+\mu})e^{-K_v T}.$$

The constants  $C$  and  $K_v$  depend on the functions  $H^i(\cdot)$  and  $G^{ik}(\cdot)$ ,  $i, k \in I$ , through their supremum norms.

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#### REFERENCES

- [1] Barles, G., Buckdahn, R., and Pardoux, E. (1997). Backward stochastic differential equations and integral-partial differential equations. *Stochastics* **60**(1-2), 57–83.
- [2] Barrieu, P. and El Karoui, N. (2013). Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. *Ann. Probab.* **41**(3) 1831–1863.
- [3] Becherer, D. (2004). Utility indifference hedging and valuation via reaction diffusion systems. *Proceedings of the Royal Society, Series A* **460** 27–51.

- [4] Becherer, D. and Schweizer, M. (2005). Classical solutions to reaction diffusion systems for hedging problems with interacting Ito and point processes. *Ann. Appl. Probab.* **15(2)** 1111–1144.
- [5] Brémaud, P. (2013). *Markov Chains: Gibbs Fields, Monte Carlo Simulation, and Queues*. Springer-Verlag, New York.
- [6] Briand, P. and Confortola, F. (2008). Quadratic BSDEs with random terminal time and elliptic PDEs in infinite dimension. *Electronic Journal of Probability* **13(54)** 1529–1561.
- [7] Briand, P. and Elie, R. (2013). A simple constructive approach to quadratic BSDEs with or without delay. *Stochastic Process. Appl.* **123(8)** 2921–2939.
- [8] Briand, P. and Hu, Y. (1998). Stability of BSDEs with random terminal time and homogenization of semilinear elliptic PDEs. *J. Funct. Anal.* **155(2)** 455–494.
- [9] Briand, P. and Hu, Y. (2006). BSDE with quadratic growth and unbounded terminal value. *Probab. Theory Related Fields* **136(4)** 604–618.
- [10] Briand, P. and Hu, Y. (2008). Quadratic BSDEs with convex generators and unbounded terminal conditions. *Probab. Theory Related Fields* **141(3-4)** 543–567.
- [11] Cheridito, P. and Nam, K. (2015). Multidimensional quadratic and subquadratic BSDEs with special structure. *Stochastics* **87(5)** 871–884.
- [12] Chong, W.F., Hu, Y., Liang, G. and Zariphopoulou, T. (2019) An ergodic BSDE approach to forward entropic risk measures: representation and large-maturity behavior. *Finance and Stochastics* **23(1)** 239–273.
- [13] Cohen, S. N., and Fedyashov, V. (2014). Ergodic BSDEs with jumps and time dependence. *arXiv preprint* arXiv:1406.4329.
- [14] Cohen, S. N. and Fedyashov, V. (2017). Nash equilibria for nonzero-sum ergodic stochastic differential games. *Journal of Applied Probability* **54(4)** 977–994.
- [15] Cohen, S. N. and Hu, Y. (2013). Ergodic BSDEs driven by Markov chains. *SIAM J. Control Optim.* **51(5)** 4138–4168.
- [16] Cosso, A., Fuhrman, M. and Pham, H. (2016). Long time asymptotics for fully nonlinear Bellman equation: a backward SDE approach. *Stochastic Process. Appl.* **126(7)** 1932–1973.
- [17] Debussche, A., Hu, Y., and Tessitore, G. (2011). Ergodic BSDEs under weak dissipative assumptions. *Stochastic Process. Appl.* **121(3)** 407–426.
- [18] Delbaen, F., Hu, Y. and Bao, X. (2011). Backward SDEs with superquadratic growth. *Probab. Theory Related Fields* **150(1-2)** 145–192.
- [19] Delbaen, F., Hu, Y. and Richou, A. (2011). On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions. *Annales de l'institut Henri Poincaré(B)* **47(2)** 559–574.
- [20] Delbaen, F., Hu, Y. and Richou, A. (2015). On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions: The critical case. *Discrete and Continuous Dynamical Systems-Series A* **35(11)** 5273–5283.
- [21] El Karoui, N. and Mrad, M. (2014). An exact connection between two solvable SDEs and a non linear utility stochastic PDE. *SIAM J. Financ Math.* **4(1)** 697–736.
- [22] El Karoui, N., Peng, S., and Quenez, M. (1997). Backward SDEs in finance, *Mathematical Finance*, **7(1)** 1–71.
- [23] Fuhrman, M., Hu, Y. and Tessitore, G. (2009). Ergodic BSDEs and optimal ergodic control in Banach spaces. *SIAM J. Control Optim.* **48(3)** 1542–1566.
- [24] Henderson, V. and Hobson, D. (2007). Horizon-unbiased utility functions. *Stochastic Process. Appl.* **117(11)** 1621–1641.
- [25] Hu, Y., Imkeller P. and Muller, M. (2005). Utility maximization in incomplete markets. *Ann. Appl. Probab.* **15(3)** 1691–1712.
- [26] Hu, Y., Liang, G. and Tang, S. (2017). Exponential utility maximization and indifference valuation with unbounded payoffs. *arXiv preprint* arXiv:1707.00199.
- [27] Hu, Y., Madec, P. and Richou, A. (2015). A probabilistic approach to large time behaviour of mild solutions of HJB equations in infinite dimension. *SIAM J. Control Optim.* **53(1)** 378–398.
- [28] Hu, Y. and Madec, P. (2016). A probabilistic approach to large time behaviour of viscosity solutions of parabolic equations with Neumann boundary conditions. *Appl. Math. Optim.* **74(2)** 345–374.
- [29] Hu, Y., and Peng, S. (2006). On the comparison theorem for multidimensional BSDEs. *Comptes Rendus Mathématique* **343(2)** 135–140.
- [30] Hu, Y. and Tang, S. (2016). Multi-dimensional backward stochastic differential equations of diagonally quadratic generators. *Stochastic Process. Appl.* **126(4)** 1066–1086.
- [31] Jamneshan, A., Kupper, M., and Luo, P. (2017). Multidimensional quadratic BSDEs with

- separated generators. *Electronic Communications in Probability* **22(58)** 1–10.
- [32] Kallblad, S., Oblój, J. and Zariphopoulou, T. (2018). Dynamically consistent investment under model uncertainty: the robust forward criteria. *Finance and Stochastics* **22(4)** 879–918.
- [33] Kobylanski, M. (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.* **28(2)** 558–602.
- [34] Kramkov, D. and Pulido, S. (2016). A system of quadratic BSDEs arising in a price impact model. *Ann. Appl. Probab.* **26(2)** 794–817.
- [35] Kramkov, D. and Pulido, S. (2016). Stability and analytic expansions of local solutions of systems of quadratic BSDEs with applications to a price impact model. *SIAM J. Financ Math.* **7(1)** 567–587.
- [36] Liang, G. and Zariphopoulou, T. (2017). Representation of homothetic forward performance processes in stochastic factor models via ergodic and infinite horizon BSDE. *SIAM J. Financ Math.* **8(1)** 344–372.
- [37] Morlais, M. A. (2009). Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem. *Finance and Stochastics* **13(1)** 121–150.
- [38] Musiela, M. and Zariphopoulou, T. (2007). Investment and valuation under backward and forward dynamic exponential utilities in a stochastic factor model. *Advances in Mathematical Finance* 303–334.
- [39] Musiela, M. and Zariphopoulou, T. (2008). Optimal asset allocation under forward exponential performance criteria, *Markov Processes and Related Topics: A Festschrift for T. G. Kurtz, Lecture Notes-Monograph Series, Institute for Mathematical Statistics* **4** 285–300.
- [40] Musiela, M. and Zariphopoulou, T. (2009). Portfolio choice under dynamic investment performance criteria. *Quantitative Finance* **9(2)** 161–170.
- [41] Musiela, M. and Zariphopoulou, T. (2010). Portfolio choice under space-time monotone performance criteria. *SIAM J. Financ Math.* **1(1)** 326–365.
- [42] Nadtochiy, S. and Tehranchi, M. (2017). Optimal investment for all time horizons and Martin boundary of space-time diffusions. *Mathematical Finance* **27(2)** 438–470.
- [43] Shkolnikov, M., Sircar, R., and Zariphopoulou, T. (2016). Asymptotic analysis of forward performance processes in incomplete markets and their ill-posed HJB equations, *SIAM J. Financ Math.* **7(1)**, 588–618.
- [44] Tevzadze, R. (2008). Solvability of backward stochastic differential equations with quadratic growth. *Stochastic Process. Appl.* **118(3)** 503–515.
- [45] Xing, H. and Žitković, G. (2018). A class of globally solvable Markovian quadratic BSDE systems and applications. *Ann. Probab.* **46(1)** 491–550.
- [46] Yin, G. and Zhang, Q. (2012). *Continuous-Time Markov Chains and Applications: A Two-Time-Scale Approach*. Springer-Verlag, New York.
- [47] Zariphopoulou, T. and Žitković, G. (2010). Maturity-independent risk measures. *SIAM J. Financ Math.* **1(1)** 266–288.
- [48] Žitković, G. (2009). A dual characterization of self-generation and exponential forward performances. *Ann. Appl. Probab.* **19(6)** 2176–2210.