On the multilevel internal structure of the asymptotic distribution of resonances

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Abstract

We prove that the asymptotic distribution of resonances has a multilevel internal structure for the following classes of Hamiltonians H: Schrödinger operators with point interactions in $\mathbb{R}^3$, quantum graphs, and 1-D photonic crystals. In the case of $N \geq 2$ point interactions, the set of resonances $\Sigma(H)$ essentially consists of a finite number of sequences with logarithmic asymptotics. We show how the leading parameters μ of these sequences are connected with the geometry of the set $Y = \{y_j\}_{j=1}^N$ of interaction centers $y_j \in \mathbb{R}^3$. The minimal parameter $\mu^{\min}$ corresponds to the sequences with 'more narrow' and so more observable resonances. The asymptotic density of such narrow resonances can be expressed via the multiplicity of $\mu^{\min}$, which occurs to be connected with the symmetries of Y and naturally introduces a finite number of classes of configurations of Y. In the case of quantum graphs and 1-D photonic crystals, the decomposition of $\Sigma(H)$ into a finite number of asymptotic sequences is proved under additional commensurability conditions. To address the case of a general quantum graph, we introduce families of special asymptotic density functions for two classes of strips in $\mathbb{C}$. The obtained results and effects are compared with those of obstacle scattering.

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Contents

1 Introduction

1.1 Main goals and related studies

Let $\Delta = \sum_{j=1}^{N} \partial_{x_j}^2$ be the Laplacian operator in the complex Lebesgue space $L^2_{\mathbb{C}}(\mathbb{R}^m)$ with odd $m \geq 1$. For operators H obtained as various types of perturbations of $(-\Delta)$ on compact subsets of $\mathbb{R}^m$, *resonances* k are defined as poles of the resolvent $(H - z^2)^{-1}$ extended in a generalized sense through $\mathbb{R}$ into the lower complex half-plane $\mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ (this and other types of definitions can be found, e.g., in [4, 17, 55, 56]).

Reviews on resonances for obstacle and geometrical scattering can be found in [17, 55, 56]. Due to various engineering applications, wave equations, resonances, and related optimization problems for noncompact quantum graphs (see [35, 21, 14, 13, 24, 38, 39, 22, 16] and the monographs [10, 43]) and for point interactions (see

[4, 15, 33, 5, 27, 40, 6] and the monographs [2, 7]) attracted a substantial attention during the last decade.

The collection of all resonances $\Sigma(H) \subset \mathbb{C}$ that are associated with an operator H (in short, resonances of H) is a *multiset*, i.e., a set in which an element e can be repeated a finite number $m_e \in \mathbb{N}$ of times (this number m_e is called the multiplicity of e). The *multiplicity of a resonance* k is defined as the multiplicity of the corresponding generalized pole of $(H - z^2)^{-1}$ (e.g., [17, 56]) or as the multiplicity of a certain analytic function built from the resolvent of H and generating resonances as its zeros ([4, 13, 14]).

Presently, there are only few Hamiltonians H for those it is known that $\Sigma(H)$ essentially decomposes into sequences $\{k_n\}_{n\in\mathbb{Z}}$ with prescribed asymptotics ('essentially' here and below means that the decomposition takes place after a possible exclusion of a finite number of resonances). Almost all such H are either one-dimensional [36, 52, 12, 46, 42] or radial symmetric (see [53, 50, 17, 56] and references therein). The exceptions are two examples of 3-dimensional (3-D) Schrödinger Hamiltonians with 2 and 3 point interactions that were considered in [4] and for those $\Sigma(H)$ consists of one and two sequences, resp., both with logarithmic asymptotics of the form $C_1 t - \mathrm{i} C_2 \operatorname{Ln}|t| + C_0 + o(1)$ as integer t goes to $\pm\infty$ (where $C_1, C_2 > 0$ and $C_0 \in \mathbb{C}$ are constants). For the arguments of [4] concerning 3 point interactions the presence of additional symmetries was important, i.e., the example possesses the group of symmetries of a regular triangle embedded into $\mathbb{R}^3$. Here and below under *the group of symmetries of* $Y \subset \mathbb{R}^3$ we understand the group of isometries Is of $\mathbb{R}^3$ such that $\operatorname{Is} Y = Y$.

The first goal of the present paper is to prove that $\Sigma(H)$ essentially decomposes into a finite number of sequences with logarithmic asymptotics for an arbitrary point interaction Hamiltonian $H = H_{a,Y}$ corresponding to the formal differential expression

$$- \Delta u(x) + \text{“}\sum_{j=1}^{N} \mu(a_j)\delta(x - y_j)u(x)\text{“}, \quad x \in \mathbb{R}^3, \ N \in \mathbb{N}, \tag{1.1}$$

with a finite number $N \geq 2$ of *interaction centers* $y_j \in \mathbb{R}^3$ and the tuple $a = (a_j)_{j=1}^N \in \mathbb{C}^N$ of *'strength' parameters* (see [4, 2, 7, 5] and Section 2 for the definition of $H_{a,Y}$). The second goal is to connect the leading parameters of these asymptotic sequences to the geometry of the set $Y = \{y_j\}_{j=1}^N$ (Section 4).

Then we investigate to which extent the result about the structure of $\Sigma(H_{a,Y})$ can be carried over from the case of point interactions to the case of quantum graph Hamiltonians (Section 6).

Our interest in the internal structure of $\Sigma(H)$ is motivated, in particular, by the necessity to consider the notion of high-energy resonance asymptotics from the point of view of Physics , where only the resonances that are closer to $\mathbb{R}$ play role (see [22] and Section 5), and by the recent studies of narrow 'topological' resonances [24, 16]. The rigorous definition of structural parameters of $\Sigma(H)$ can also provide an approach to optimization problems of the type of [12, Section 8], which involve the whole set $\Sigma(H)$ and are much less studied than problems on optimization of an individual resonance [32, 33, 34].

The results on the structure of $\Sigma(H)$ in the cases of point interactions and quantum graphs allow us to consider logarithmic $\{z \in \mathbb{C} \ : \ -C_2 \ln(|\operatorname{Re} z|+1) - C_0 \leq \operatorname{Im} z \leq -C_1 \ln(|\operatorname{Re} z|+1) + C_0\}$ or horizontal $\{-C_2 \leq \operatorname{Im} z \leq -C_1\}$ strips that contain 'more narrow' resonances, to define related asymptotic densities, and, in the case of point interaction Hamiltonians, to connect the density of narrow resonances to the group of symmetries of the set of interaction centers.

Note that various types of strips containing 'more narrow' resonances have been intensively studied in the context of obstacle scattering [47, 48, 49, 55, 28, 56], mainly, with the use of corresponding counting functions.

Recall that the *counting function* $\mathfrak{N}_H(\cdot)$ for resonances is defined by

$$\mathfrak{N}_H(R) := \#\{k \in \Sigma(H) : |k| \leq R\}, \tag{1.2}$$

where $\#E$ is the number of elements of a multiset E. The study of the asymptotics of $\mathfrak{N}(R)$ as $R \to \infty$ for scattering poles associated with compactly supported potentials in $\mathbb{R}^m$ with odd $m \geq 3$ was initiated in [41] (for the relation between the notions of scattering poles and resonances, see [17, 56]). This study was continued [53, 23, 45, 11, 50] and extended to a geometric scattering (see reviews in [55, 17, 56]), to quantum graphs [13, 14, 39, 22], and to point interaction Hamiltonians $H_{a,Y}$ [40, 6].

For any unbounded set $S \subset \mathbb{C}$, one can define a restricted version of the counting function $\mathfrak{N}_H^S(R) := \#\{k \in S \cup \Sigma(H) \ : |k| \leq R\}$. It seems that the study of the asymptotics of $\mathfrak{N}_H^S(R)$ was originally motivated [31] by the fact that in the case of even m, the generalized resolvent has a log-type branching point at $z = 0$. Therefore the counting of resonances was restricted to sectors $\theta \leq \arg z \leq 0$, $\theta \in \mathbb{R}_-$, of the associated Riemannian surface [51] (for the case of odd m, see [47]). On the other side, the existence of infinitely many resonances in a certain horizontal strip S and asymptotic sequences in S were studied for the case of two convex obstacles in connection with trapped broken characteristic rays (see [29, 18, 28] and the reviews in [55, 28]). Trapping effects for obstacle scattering have motivated also the investigation of asymptotics of $\mathfrak{N}_H^S(R)$ for different types of shaped strips S, including semi-logarithmic strips $\{-C_1 \ln(|\operatorname{Re} z|+1) \leq \operatorname{Im} z \leq 0\}$ and cubic strips $\{-C_2|\operatorname{Re} z|^{1/3} - C_0 \leq \operatorname{Im} z \leq -C_1|\operatorname{Re} z|^{1/3} + C_0\}$ (see [47, 48, 49, 28, 30] and references therein; here $C_0, C_1, C_2 > 0$).

In [14], it was shown that the set of resonances $\Sigma(H_\mathcal{G})$ for a noncompact quantum graphs $\mathcal{G}$ with the Kirchhoffs coupling lie in a certain horizontal strip S and the leading term in the asymptotics of $\mathfrak{N}_H(R) = \mathfrak{N}_H^S(R)$ was connected with the structure of the graph. In [13], the study of asymptotics of $\mathfrak{N}_H(R)$ was extended to quantum-graphs with general self-adjoint couplings, and it was shown that the set of resonances is contained in a union of a horizontal strip with a finite number of logarithmic strips.

In Section 7, we discuss some similarities between the structure of $\Sigma(H)$ for point interactions and quantum graphs on one hand, and various cases of obstacle scattering on the other hand.

1.2 Overview of main results and methods of the paper

Our initial step to find asymptotic sequences inside of $\Sigma(H)$ is to use the observation of [13, 14, 40] that the set of resonances is a set of zeroes of a certain exponential polynomial F for the cases of noncompact quantum graphs and of point interaction Hamiltonians (see also [39, 22, 5, 6]). By the Pólya-Dickson theorem, the zeroes of exponential polynomials are concentrated in a finite number of logarithmic strips (see [9, 13, 39]; note that horizontal strips can be considered as a degenerated type of logarithmic ones [8]).

We show that, in the case of a point interaction Hamiltonian $H_{a,Y}$ associated with (1.1), the corresponding F falls into a special class of exponential polynomials whose distribution diagrams generate only retarded logarithmic strips (here and below the terminology of [8] is used). The set of zeroes for such exponential polynomials splits into a finite number of asymptotic sequences. Translating this result to the (multi-)set $\Sigma(H_{a,Y})$ we show that $\Sigma(H_{a,Y})$ essentially consists of sequences that for integer t with large $|t|$ have an asymptotics of the form

$$2\pi\mu_n t - \mathrm{i}\mu_n \operatorname{Ln}|t| + C + o(1)$$

and whose leading parameters μ_n form a finite set $\{\mu_n\}_{n=1}^M \subset \mathbb{R}_+$. It is natural to say that μ_n are the *structural parameters of 1st order* (*the parameters of 2nd order* $\omega_{n,j}$ are hidden inside constants C, see Theorem 3.4 for details). Each of the parameters μ_n participates in a number $r_n \in \mathbb{N}$ of asymptotic sequences. It is natural to call the number r_n the *multiplicity of* μ_n.

To show in Theorems 4.3-4.4 the interplay between $\{\mu_n\}_{n=1}^M$ and the metric geometry of the tuple Y, we introduce a sequence of values $\mathfrak{s}_m$, which are called *m-sizes of* Y and are generalizations of the size of Y introduced in [40]. For integer $m \in [0, N]$, let $S_{N,m}$ be the set of permutations $\sigma \in S_N$ of the symmetric group S_N such that σ has exactly $N - m$ fixed points, i.e., $S_{N,m} := \{\sigma \in S_N \ : \ \#\{i : \sigma(i) \neq i\} = m\}$. For integer $m \in \{0\} \cup [2, N]$, we define *the m-size of* Y by

$$\mathfrak{s}_m = \mathfrak{s}_m(Y) := \max_{\sigma \in S_{N,m}} \sum_{j=1}^{N} |y_j - y_{\sigma(j)}|;$$

$$\mathfrak{s}_1 = \mathfrak{s}_1(Y) := \operatorname{diam} Y, \text{ where } \operatorname{diam}(Y) := \max_{1 \leq j,j' \leq N} |y_j - y_{j'}| \text{ is the diameter of } Y.$$

Then $\mathfrak{s}_0 = 0$, $\mathfrak{s}_2 = 2 \operatorname{diam} Y$, $\mathfrak{s}_3$ is the maximal perimeter of a triangle with vertices in Y; 1-size of Y is defined in a special way since $S_{N,1} = \varnothing$. The logic behind this is that the equality $\mathfrak{s}_1 := (\mathfrak{s}_2 + \mathfrak{s}_0)/2$ simplifies the formulation of Theorem 4.4. The N-size $\mathfrak{s}_N$ was called in [40] the size of Y and used to define Weyl and non-Weyl types of asymptototics for $\mathfrak{N}_{H_{a,Y}}(\cdot)$ (see the beginning of Section 4).

The connection between the parameters μ_n and m-sizes $\mathfrak{s}_m$ is established in Theorems 4.3-4.4. However it is difficult to give an explicit rule of computation of the leading parameters μ_n via m-sizes that works for any Y. The reasons for this are shown by the examples of Section 4.3.

The structural theorem for $\Sigma(H_{a,Y})$ in the case of point interactions is quite special and cannot be directly brought over even to the case of quantum graphs, which is considered in Section 6. However this theorem give a hint how one can define the structural parameters even if the decomposition of $\Sigma(H)$ into asymptotic sequences is not available. With this aim we use counting functions in 'shaped strips' similar to that of [47, 48] and define via them corresponding asymptotic density functions. The main examples of such functions in this paper are the *logarithmic counting functions*

$$\mathfrak{N}^{\log}(\mu, R) := \#\{k \in \Sigma(H) \ : \ -\mu \ln(|\operatorname{Re} k| + 1) \leq \operatorname{Im} k \text{ and } |k| \leq R\}, \qquad (1.3)$$

and the associated *logarithmic density function*

$$\operatorname{Ad}^{\log}(\mu) := \lim_{R\to\infty} \frac{\mathfrak{N}^{\log}(\mu, R)}{R}, \quad \mu \in \mathbb{R}, \text{ and } \operatorname{Ad}^{\log}(+\infty) := \lim_{R\to\infty} \mathfrak{N}_H(R)/R. \qquad (1.4)$$

The above definition of $\operatorname{Ad}^{\log}(+\infty)$ is natural because, for $H_{a,Y}$ and for quantum graphs, this limit exists and has a finite value that was studied in [14, 13, 40, 6] in connection with Weyl-type asymptotics of $\mathfrak{N}_H(\cdot)$.

For $H_{a,Y}$ and for quantum graphs, the function $\operatorname{Ad}^{\log} : (-\infty, +\infty] \to [0, +\infty)$ is bounded, nondecreasing and satisfies $\operatorname{Ad}^{\log}(\mu) = 0$ for $\mu < 0$ and $\operatorname{Ad}^{\log}(\mu) = \operatorname{Ad}^{\log}(+\infty)$ for large enough μ (this follows from the definition, results of [13, 14, 40], and Theorem 3.4). So it defines a bounded measure $d\operatorname{Ad}^{\log}(\cdot)$ on $\mathbb{R}$ with a compact support. The connection of $\operatorname{Ad}^{\log}(\cdot)$ with the structure of $\Sigma(H)$ is obvious in the case of $H_{a,Y}$ from Theorem 3.4 and in the case of quantum graphs from Theorem 6.4 (the latter evolves from [13, Theorem 3.1], but describes $\Sigma(H)$ on a more fine structural level). Namely, the measure $d\operatorname{Ad}^{\log}(\cdot)$ consists of a finite number of point masses, the minimum $\mu^{\min}$ of its support and the height of the corresponding jump $\operatorname{Ad}^{\log}(\mu^{\min}+0) - \operatorname{Ad}^{\log}(\mu^{\min}-0)$ are the parameters describing the high-energy asymptotics of 'most narrow' (and so most physically relevant) resonances (see Section 5 and the discussion in [19]).

For $H_{a,Y}$, the parameter $\mu^{\min}$ is always equal $1/\operatorname{diam} Y$. So the main parameter of the formula (5.1) for *the asymptotic density of narrow resonances* $\operatorname{Ad}^{\log}(\mu^{\min} + 0) - \operatorname{Ad}^{\log}(\mu^{\min} - 0)$ is the multiplicity r^{narrow} of $\mu^{\min}$, which is an integer number between 2 and N and is studied by Theorems 5.2 and 5.4. These theorems naturally lead to a conjecture about the connection of the value of r^{narrow} with the group of symmetries of Y.

The structural description of $\Sigma(H)$ in the case of a quantum graph Hamiltonian H is given in Section 6.1. It is more complex than that of $\Sigma(H_{a,Y})$. In the case where $\mu^{\min} = 0$, a certain neutral strip $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq \widetilde{\gamma}\}$ contains an infinite number of resonances, that are not necessarily decomposable into a finite number of asymptotic sequences (at least we do not know a general enough theorem of such type for exponential polynomials). In the important case of the Kirchhoff coupling, the situation is the worst possible. The measure $d\operatorname{Ad}^{\log}(\cdot)$ consists of one point mass at $\mu^{\min} = 0$, i.e., there exists only one 1st order structural parameter, which does not describe any structure. The description of the structure of $\Sigma(H)$ should be encoded in the measure $d\operatorname{Ad}^{\text{hor}}$ associated with another density function (6.2) built with the use of horizontal strips.

However, the distribution of resonances in the non-decomposable cases is connected with difficult questions arising in the exponential polynomial approach to the study of zeros of Riemann zeta function (see the monograph [9] and the references therein). To obtain a complete decomposition of $\Sigma(H)$ into asymptotic sequences we impose the additional commensurability condition $\ell_{m_1}/\ell_{m_2} \in \mathbb{Q}$ on the lengths ℓ_m of edges of the graph $\mathcal{G}$ (see Theorem 6.1 and the part (iii) of Theorem 6.4, here and below $\mathbb{Q}$ is the set of rational numbers).

In Section 6.2, we brought the above results over the set of resonances of 1-D photonic crystals using the fact that they can be considered as generalized 'weighted' quantum graphs with the Kirchhoff coupling (for another type of coupling for weighted graphs, see [13]).

Notation. We use the convention that if $n_2 < n_1$, then $\{z_n\}_{n=n_1}^{n_2} = \varnothing$. The following standard sets are used: the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ of natural, integer, rational, and real numbers, resp., the lower and upper complex half-planes $\mathbb{C}_\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$, open half-lines $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}$, open discs $\mathbb{D}_\varepsilon(\zeta) := \{z \in \mathbb{C} : |z-\zeta| < \varepsilon\}$, compact $\mathbb{C}$-intervals $[z_1, z_2] := \{sz_1+(1-s)z_2 \ : \ s \in [0,1]\}$ with $z_1, z_2 \in \mathbb{C}$, $\mathbb{C}$-intervals $(z_1, z_2) := [z_1, z_2] \setminus \{z_1, z_2\}$ without endpoints. The above $\mathbb{C}$-intervals are called degenerate if $z_1 = z_2$. In a metric space U with the distance function $\rho_U(\cdot,\cdot)$ (or in a normed space), we use open balls $\mathbb{B}_\varepsilon(u_0) := \{u \in U \ : \ \rho_U(u, u_0) < \varepsilon\}$ always assuming that $\varepsilon > 0$. For a normed space U, $u_0 \in U$, $S \subset U$, and $z \in \mathbb{C}$, we write $zS + u_0 := \{zu + u_0 \ : \ u \in S\}$. For a function g defined on S, $g[S]$ is the image of S. The function $\operatorname{Ln}(\cdot)$ ($\operatorname{Arg}_0(\cdot)$) is the branch of the natural logarithm multi-function $\ln(\cdot)$ (resp., the complex argument $\operatorname{Arg}(\cdot)$) in $\mathbb{C} \setminus (-\infty, 0]$ fixed by $\operatorname{Ln} 1 = \mathrm{i} \operatorname{Arg}_0 1 = 0$. For $z \in \mathbb{R}_-$, we put $\operatorname{Ln} z = \operatorname{Ln}|z| + \mathrm{i} \operatorname{Arg}_0 z = \operatorname{Ln}|z| + \mathrm{i}\pi$. By $\partial_x f$, $\partial_{x_j} f$, etc., we denote (ordinary or partial) derivatives with respect to (w.r.t.) x, x_j, etc.; $\deg p$ stands for the degree of a polynomial p, $\deg X_j$ for the degree of a vertex X_j of a graph; $\mathfrak{e} = [1][2]\dots[N]$ for the identity permutation. Here and below we use the square brackets notation of the textbook [37] for permutation cycles, omitting sometimes, when it is convenient, the degenerate cycles consisting of one element.

2 Resonances of point interaction Hamiltonians

Throughout this paper, the set $Y = \{y_j\}_{j=1}^N$ consist of $N \geq 2$ distinct points y_1, ..., y_N in $\mathbb{R}^3$. Let $a = (a_j)_{j=1}^N \in \mathbb{C}^N$ be the N-tuple of the *'strength' parameters.*

The operator $H_{a,Y}$ associated with (1.1), where $\delta(\cdot - y_j)$ is the Dirac measure placed at the *center* $y_j \in \mathbb{R}^3$ of a *point interaction*, is defined in [1, 2] for the case of real a_j, and in [3, 5] for $a_j \in \mathbb{C}$. It is a closed operator in the complex Hilbert space $L^2_{\mathbb{C}}(\mathbb{R}^3)$ and it has a nonempty resolvent set. The spectrum of $H_{a,Y}$ consists of the essential spectrum $[0, +\infty)$ and an at most finite set of points outside of $[0, +\infty)$ [2, 5] (all of those points are eigenvalues).

The resolvent $(H_{a,Y} - z^2)^{-1}$ of $H_{a,Y}$ is defined in the classical sense on the set of

$z \in \mathbb{C}_+$ such that z^2 is not in the spectrum, and has the integral kernel

$$(H_{a,Y} - z^2)^{-1}(x, x') = G_z(x - x') + \sum_{j,j'=1}^{N} G_z(x - y_j)\, [\Gamma_{a,Y}]^{-1}_{j,j'}\, G_z(x' - y_{j'}), \qquad (2.1)$$

where $x, x' \in \mathbb{R}^3 \backslash Y$ and $x \neq x'$, see e.g. [2, 5]. Here $G_z(x-x') := \frac{e^{\mathrm{i}z|x-x'|}}{4\pi|x-x'|}$ is the integral kernel associated with the resolvent $(-\Delta - z^2)^{-1}$ of the kinetic energy Hamiltonian $-\Delta$; $[\Gamma_{a,Y}]^{-1}_{j,j'}$ denotes the j, j'-element of the inverse to the matrix

$$\Gamma_{a,Y}(z) = \left[\left(a_j - \tfrac{\mathrm{i}z}{4\pi}\right)\delta_{jj'} - \widetilde{G}_z(y_j - y_{j'})\right]_{j,j'=1}^{N}, \text{ where } \widetilde{G}_z(x) := \begin{cases} G_z(x), & x \neq 0 \\ 0, & x = 0 \end{cases}. \qquad (2.2)$$

The Krein-type formula (2.1) for the difference of the perturbed and unperturbed resolvents of operators $H_{a,Y}$ and $-\Delta$ can be used as a definition of $H_{a,Y}$ (see [2]). For other equivalent definitions of H and for the meaning of $\mu(a_j)$ and a_j in (1.1), we refer to [1, 2, 7] in the case $a_j \in \mathbb{R}$, and to [3, 5] in the case $a_j \notin \mathbb{R}$. Note that, in the case $a \in \mathbb{R}^N$, the operator $H_{a,Y}$ is self-adjoint in $L^2_{\mathbb{C}}(\mathbb{R}^3)$; and in the case $a \in (\mathbb{C}_- \cup \mathbb{R})^N$, $H_{a,Y}$ is closed and maximal dissipative (in the sense of [20], or in the sense that $\mathrm{i}H_{a,Y}$ is maximal accretive).

The set of (continuation) resonances $\Sigma(H_{a,Y})$ associated with the operator $H_{a,Y}$ (in short, resonances of $H_{a,Y}$) is by definition the set of zeroes of the determinant $\det \Gamma_{a,Y}(\cdot)$, which we will call the characteristic determinant. This is in agreement with the cut-off resolvent pole definition of [17] and slightly differs from the one used in [4, 2] because isolated eigenvalues are now also included into $\Sigma(H_{a,Y})$. For the origin of this and related approaches to the understanding of resonances, we refer to [4, 17, 44] and the literature therein. The multiplicity of a resonance k will be understood as the multiplicity of a corresponding zero of the determinant $\det \Gamma_{a,Y}$, which is an analytic function in z (see [2]). Equipped with the multiplicity, the set $\Sigma(H_{a,Y})$ becomes a multiset (for the discussions on multiplicities of resonances see e.g. [12, 17, 33, 56]).

The function $\det \Gamma_{a,Y}(\cdot)$ is an exponential polynomial, which after a simple transformation becomes of a special type considered in [8] (for the general theory see [9]). Namely, introducing a new variable $\zeta = -\mathrm{i}z$ and denoting $A_j := 4\pi a_j$, one can see that the modified characteristic determinant $D(\zeta) := (-4\pi)^N \det \Gamma_{a,Y}(\mathrm{i}\zeta)$ can be expanded by the Leibniz formula into the sum of terms

$$e^{\zeta\alpha(\sigma)} p^{[\sigma]}(\zeta) \qquad (2.3)$$

taken over all permutations σ in the symmetric group S_N, Here the constants $\alpha(\sigma) \leq 0$ and the polynomials $p^{[\sigma]}(\cdot)$ have the form

$$\alpha(\sigma) := -\sum_{j:\sigma(j)\neq j} |y_j - y_{\sigma(j)}|, \quad p^{[\sigma]}(\zeta) := \epsilon_\sigma K_1(\sigma) \prod_{j:\sigma(j)=j} (-\zeta - A_j), \qquad (2.4)$$

$$\text{where } K_1(\sigma) := \prod_{j:\sigma(j)\neq j} |y_j - y_{\sigma(j)}|^{-1} > 0 \text{ (in the case } \sigma = \mathfrak{e},\ K_1(\mathfrak{e}) := 1), \qquad (2.5)$$

ϵ_σ is the permutation sign (the Levi-Civita symbol), and $\mathfrak{e}$ is the identity permutation (note that $K_1(\sigma)$ depends also on Y, and $p^{[\sigma]}(\zeta)$ on a and Y).

We will say that an exponential polynomial $g(\cdot)$ is an exp-monomial if it has the form $e^{\beta\zeta}p(\zeta)$, where $\beta \in \mathbb{C}$ and p is a polynomial that is nontrivial in the sense that $p(\cdot) \not\equiv 0$.

3 Logarithmic asymptotic chains of resonances

3.1 The distribution diagram and logarithmic strips

Summing (2.3) and writing the exponential polynomial D in the canonical form [9, 8], one obtains

$$D(\zeta) = \sum_{j=0}^{\nu} P_{\beta_j}(\zeta) e^{\beta_j \zeta}, \tag{3.1}$$

where $\nu \in \mathbb{N} \cup \{0\}$, $\beta_j \leq 0$, and the nontrivial polynomials P_{β_j} (with coefficient depending on a and Y) are such that $\beta_0 < \beta_1 < \cdots < \beta_\nu = 0$. Clearly, $P_0(z) = P_{\beta_\nu} = \prod_{j=1}^{N}(-\zeta - A_j)$. The coefficients β_j in (3.1) (and $\alpha(\sigma)$ in (2.3)) are called frequencies of the corresponding exponential polynomials (resp., exp-monomials). We will use the convention that

$$\text{if } b \notin \{\beta_j\}_{j=0}^{\nu}, \text{ then } P_b(\cdot) \text{ is trivial in the sense that } P_b(\cdot) \equiv 0. \tag{3.2}$$

Since we rely on the terminology and the results of the theory of zeros of exponential polynomials given through [8, Sections 12.4-8], we try to keep our notation as close as possible to that of [8]. It is difficult to achieve this aim completely, in particular, our frequencies β_j are nonpositive, while those of [8] are nonnegative.

It is obvious from the definition of $\mathfrak{s}_m$ (see Section 1) that

$$\begin{gathered}\text{for each } m \in (\mathbb{Z} \cap [0, N]) \setminus \{1\}, \\ \text{there exists } \sigma \in S_N \text{ such that } \alpha(\sigma) = (-\mathfrak{s}_m) \text{ and } \deg p^{[\sigma]} = N - m.\end{gathered} \tag{3.3}$$

In the process of summation of exp-monomials of (2.3) some of the terms may cancel so that, for a certain permutation $\sigma \in S_N$, $\alpha(\sigma)$ is not a frequency of D. If this is the case, we say that there is *frequency cancellation* for the pair $\{a, Y\}$ (for two examples of frequency cancellation see [40] and Example 5.3).

Since $N \geq 2$, there exists a nonzero frequency $\alpha(\sigma)$ that does not cancel for D, i.e., $\nu \geq 1$ and $\alpha(\sigma)$ is a frequency of D for certain $\sigma \neq \mathfrak{e}$. This fact was observed in [5, the proof of Lemma 2.1] and, by a different argument, can also be seen from the next lemma.

Lemma 3.1. *(i) The number $(-\mathfrak{s}_2)$ is a frequency of $D(\cdot)$. Besides,* $\deg P_{-\mathfrak{s}_2} = N - 2$. *(ii) Let $N \geq 3$. Then $(-\mathfrak{s}_3)$ is a frequency of $D(\cdot)$. Moreover,*

$$\mathfrak{s}_3 = \mathfrak{s}_2 \textit{ if all the points of the set } Y \textit{ lie on one line; otherwise, } \mathfrak{s}_3 > \mathfrak{s}_2. \tag{3.4}$$

Proof. *(i)* Consider the class of all transpositions $\sigma = [j_1 j_2]$, $j_1 \neq j_2$, such that

$$|y_{j_1} - y_{j_2}| = \operatorname{diam} Y = \mathfrak{s}_2/2.$$

The corresponding terms of (2.3) are $(-1)e^{-\mathfrak{s}_2\zeta}(\operatorname{diam} Y)^{-2}\prod_{j\neq j_1,j_2}(-\zeta - A_j)$. The highest order coefficients of their polynomial factors $p^{[\sigma]}$ coincide and cannot cancel each other. The exp-monomial produced by summation of these terms has the polynomial part of degree $N-2$ and cannot be canceled by other terms of (2.3) because the other terms either have a lower $\deg p^{[\sigma]}$, or a different frequency $\alpha(\sigma)$. Thus, $(-\mathfrak{s}_2)$ is a frequency of D.

(ii) The statement (3.4) is obvious from the triangle inequality. In the case $\mathfrak{s}_3 = \mathfrak{s}_2$, $(-\mathfrak{s}_2)$ is a frequency of D due to (i). Assume $\mathfrak{s}_3 > \mathfrak{s}_2$. Then it is easy to modify the proof of (i) to show that the frequency $(-\mathfrak{s}_3)$ does not cancel. □

Consider the points $T_{\beta_j} = \beta_j + \mathrm{i}\deg P_{\beta_j} \in \mathbb{C}$, $\quad j = 0, \ldots, \nu$, associated with the canonical form (3.1) of D. Note that $T_{\beta_\nu} = T_0 = \mathrm{i}N$.

The distribution diagram of D (see [8]) is the polygonal line $\mathcal{L}$ in $\mathbb{C}$ determined by the following properties:

(L1) $\mathcal{L}$ joins T_{β_0} with T_{β_ν}.

(L2) $\mathcal{L}$ has vertices only at the points of the set $\{T_{\beta_j}\}_{j=1}^{\nu}$.

(L3) $\mathcal{L}$ is convex upward. (In particular, it is allowed to be of the form of one $\mathbb{C}$-interval $[z_0, z_1]$. This is the case for $N = 2$, but not only, see Section 4.3.)

(L4) There are no points of the set $\{T_{\beta_j}\}_{j=1}^{\nu}$ above $\mathcal{L}$ in the sense that the $\mathbb{C}$-intervals $(T_{\beta_j}, \beta_j - \mathrm{i})$ (with excluded endpoints) do not intersect $\mathcal{L}$.

Let $\mathcal{L}_1$, $\mathcal{L}_2$, ..., $\mathcal{L}_M$ be the successive segments of $\mathcal{L}$ numbered from left to right and such that $\mathcal{L} = \cup_{n=1}^{M}\mathcal{L}_n$ (it is assumed that each segment $\mathcal{L}_n$ is a closed nondegenerate $\mathbb{C}$-interval and the segments are maximal in the sense that two consecutive segments do not belong to the same line). Note that $\mathcal{L}$ has no vertical segments (lying on the lines $\operatorname{Re} z = c$) due to (L4) (see also [8, Section 12.8]).

Let $\{T_{n,h}\}_{h=1}^{\nu_n} := \mathcal{L}_n \cap \{T_{\beta_j}\}_{j=0}^{\nu}$, where $\mathcal{L}_n = [T_{n,1}, T_{n,\nu_n}]$ and the points $T_{n,h}$ on the segment $\mathcal{L}_n$ are numbered from left to right, i.e., $\operatorname{Re} T_{n,1} < \cdots < \operatorname{Re} T_{n,\nu_n}$,

$$T_{1,1} = T_{\beta_0}, \quad T_{M,\nu_M} = T_0 = \mathrm{i}N, \quad \text{and, if } 2 \leq n \leq M,\ T_{n-1,\nu_{n-1}} = T_{n,1}. \tag{3.5}$$

Put $\beta_{n,h} := \operatorname{Re} T_{n,h}$ and $m_{n,h} := \operatorname{Im} T_{n,h} = \deg P_{\beta_{n,h}}$. Then the slopes μ_n of the segments $\mathcal{L}_n$ are defined by

$$\mu_n := \tan(\arg(T_{n,\nu_n} - T_{n,1})) = \frac{m_{n,\nu_n} - m_{n,1}}{\beta_{n,\nu_n} - \beta_{n,1}}, \qquad n = 1, \ldots, M. \tag{3.6}$$

Note that

$$\text{the sequence } \{\mu_j\}_{j=1}^{M} \text{ is strictly decreasing.} \tag{3.7}$$

This follows from the convexity (L3) and the definition of the segments $\mathcal{L}_n$.

The Pólya-Dickson theorem on the zeroes of exponential polynomials [8, 9] states that there exists constants $w_n \geq 0$, $n = 1, \dots, M$, and the logarithmic strips

$$\mathcal{V}(\mu_n, w_n) := \{\zeta \in \mathbb{C} \ : \ |\operatorname{Re}(\zeta + \mu_n \ln \zeta)| \ \leq \ w_n\}, \quad n = 1, \dots, M, \tag{3.8}$$

such that $\mathbb{C} \setminus \bigcup_{j=1}^{M} \mathcal{V}(\mu_n, w_n)$ can be decomposed into a finite number of subsets with the property that in each of them one of the sums $\sum_{T_j \in \mathcal{L}_n} P_{\beta_j}(\zeta) e^{\beta_j \zeta}$ is of predominant order of magnitude over the other terms in (3.1) as $\zeta \to \infty$ [8]. In particular, this implies the following statement.

Lemma 3.2 ([8])**.** *There is only a finite number of zeroes of $D(\cdot)$ in $\mathbb{C} \setminus \bigcup_{j=1}^{n} \mathcal{V}(\mu_n, w_n)$.*

3.2 Bunches of asymptotics chains of resonances

The main point of this subsection is that the modified characteristic determinant D belongs to the more special subclass of exponential polynomials that generate only positive parameters μ_n, i.e., in the terminology of [8], all the logarithmic strips $\mathcal{V}(\mu_n, w_n)$ for D are *retarded*. This allows us to apply the results of [8] about this subclass and in this way to split $\Sigma(H_{a,Y})$ into a finite number of sequences with a prescribed form of asymptotics at ∞.

The fact that $\{\mu_n\}_{n=1}^{M} \subset \mathbb{R}_+$ follows from (3.7) and the following statement.

Proposition 3.3. *$\mu_M = 1/\operatorname{diam} Y = 2/\mathfrak{s}_2$ and $1 \leq M \leq N - 1$.*

Proof. Let us show that $\mu_M = 2/\mathfrak{s}_2$. It follows from Lemma 3.1 that $(-\mathfrak{s}_2)$ is a frequency of D. Further, the arguments of the proof of Lemma 3.1 show that

$$\deg P_{-\mathfrak{s}_2} = N - 2, \quad T_{-\mathfrak{s}_2} = -\mathfrak{s}_2 + \mathrm{i}(N - 2). \tag{3.9}$$

The convexity (L3), (3.5), and the definition of μ_n imply that $\mu_M \leq \frac{\operatorname{Im}(T_{M,\nu_M} - T_{-\mathfrak{s}_2})}{\operatorname{Re}(T_{M,\nu_M} - T_{-\mathfrak{s}_2})} = \frac{2}{\mathfrak{s}_2}$.

Let us prove that $\mu_M \geq 1/\operatorname{diam} Y = 2/\mathfrak{s}_2$ by *reductio ad absurdum*. Assume $\mu_M < 1/\operatorname{diam} Y$. Then there exist $j < \nu$ such that $\frac{N - \deg P_{\beta_j}}{-\beta_j} < \frac{1}{\operatorname{diam} Y}$. So there exists $\sigma \in S_N$ such that $\alpha(\sigma) = \beta_j$ and $(N - \deg p^{[\sigma]}) \operatorname{diam} Y < -\alpha(\sigma)$. However, it follows from (2.4) that $(-1)\alpha(\sigma) \leq \sum_{j : \sigma(j) \neq j} \operatorname{diam} Y = (N - \deg p^{[\sigma]}) \operatorname{diam} Y$, a contradiction.

By Lemma 3.1, $M \geq 1$. Let us show that $M \leq N - 1$. We see that

$$m_{n,h} = \operatorname{Im} T_{n,h} = \deg P_{\beta_{n,h}} \in (\mathbb{Z} \cap [0, N]) \setminus \{N - 1\} \tag{3.10}$$

and, from the definitions of $T_{n,h}$, (3.7), and $\mu_n > 0$, that the sequence $\operatorname{Im} T_{1,1}$, $\operatorname{Im} T_{2,1}$, $\dots$, $\operatorname{Im} T_{M,1}$, $\operatorname{Im} T_{M,\nu_M}$ is strictly increasing. This implies that $\mathcal{L}$ has at most $N - 1$ segments. ∎

The following theorem is the main result of the section.

Theorem 3.4. *There exist numbers $M \in \mathbb{N}$, $M_0 \in \mathbb{N} \cup \{0\}$, finite sequences $\{\mu_n\}_{n=1}^M \subset \mathbb{R}_+$, $\{r_n\}_{n=1}^M \subset \mathbb{N}$, $\{\omega_{n,j}\}_{j=1}^{r_n} \subset \mathbb{C} \setminus \{0\}$ and $\{t_{n,j}\}_{j=1}^{r_n} \subset \mathbb{N}$ for $n = 1, \dots, M$, $\mathcal{K}_0 = \{k_t^{[0]}\}_{t=1}^{M_0} \subset \mathbb{C}$, and infinite sequences $\mathcal{K}_{n,j}^\pm = \{k_{n,j,t}^\pm\}_{t=t_{n,j}}^{+\infty} \subset \mathbb{C}$ such that:*

(i) $\Sigma(H_{a,Y}) = \bigcup_{n=0}^M \mathcal{K}_n$ *(taking into account multiplicities), where*

$$\mathcal{K}_n := \left(\bigcup_{j=1}^{r_n} \mathcal{K}_{n,j}^-\right) \cup \left(\bigcup_{j=1}^{r_n} \mathcal{K}_{n,j}^+\right), \; n = 1, \dots, M.$$

(ii) Each of $\mathcal{K}_{n,j}^\pm$ *has the following asymptotics as* $t \in \mathbb{N}$ *goes to* $+\infty$*:*

$$\frac{k_{n,j,t}^\pm}{\mu_n} = \pm 2\pi t - \mathrm{i}\,\mathrm{Ln}(t) \mp \pi/2 - \mathrm{i}\,\mathrm{Ln}(2\pi\mu_n) + \mathrm{i}\,\mathrm{Ln}(\omega_{n,j}) + o(1). \tag{3.11}$$

(iii) The sequence $\{\mu_n\}_{n=1}^M$ *is strictly decreasing and* $\mu_M = \frac{1}{\operatorname{diam} Y}$.

(iv) $1 \le M \le N - 1$ *and* $2 \le r_M \le \sum_{n=1}^M r_n \le N$.

The proof given below also explains how the parameters of the asymptotics (3.11) can be found from the distribution diagram $\mathcal{L}$.

Proof. We use the notation of Section 3.1 and the parameters defined therein. In particular, $M \ge 1$ is the number of segments of $\mathcal{L}$, $\{\mu_n\}_{n=1}^M$ is the sequence defined by (3.6). Let $c_{n,h}$ be the coefficient corresponding to the monomial of the highest degree for the polynomial $P_{\beta_{n,h}}$, i.e., $c_{n,h} = \lim_{\zeta\to\infty} P_{\beta_{n,h}}(\zeta)/\zeta^{m_{n,h}} \neq 0$, $h = 1, \dots, \nu_n$. Consider the polynomial $q_n(\omega) := \sum_{h=1}^{\nu_n} c_{n,h}\omega^{m_{n,h}-m_{n,1}}$ of the degree

$$r_n := m_{n,\nu_n} - m_{n,1}$$

and the finite sequence of its zeroes $\{\omega_{n,j}\}_{j=1}^{r_n}$ (repeated according to their multiplicities). Note that $c_{n,1} \neq 0$ imply $\omega_{n,j} \neq 0$.

The statement (iii) of the theorem is already proved. Combining (iii) with [8, Theorem 12.8 and 12.10 (d)] (see also [8, (12.8.11-12)], [8, Theorems 12.6-8], and [8, Lemma 12.4] for details), we see that for each $n = 1, \dots, M$, there exists $R_n \ge 0$ and sequences $\{\zeta_{n,j,t}^\pm\}_{t=t_{n,j}^\pm}^{+\infty} \subset \mathbb{C}_\pm$, $j = 1, \dots, r_n$ with the starting numbers $t_{n,j}^\pm \subset \mathbb{N}$ satisfying the following properties:

(a) The sequence $\mathcal{Z}_n^\pm$ of all the zeroes of D in $(\mathcal{V}(\mu_n, w_n) \cap \mathbb{C}_\pm) \setminus \mathbb{D}_{R_n}(0)$ is the union of r_n sequences $\{\zeta_{n,j,t}^\pm\}_{t=t_{n,j}^\pm}^{+\infty}$ (taking into account multiplicities).

(b) Each of the sequences $\{\zeta_{n,j,t}^\pm\}_{t=t_{n,j}^\pm}^{+\infty}$ has the asymptotics (cf. [8, Theorem 12.8])

$$\frac{\zeta_{n,j,t}^\pm}{\mu_n} = \mathrm{Ln}\,\omega_{n,j} - \mathrm{Ln}\,|\pm 2\pi\mu_n t + \mu_n \operatorname{Arg}_0(\omega_{n,j}) \mp \mu_n\pi/2| + \mathrm{i}(\pm 2\pi t \mp \pi/2) + o(1).$$

Due to Lemma 3.2, one sees that only a finite sequence $\mathcal{Z}_0$ of zeroes of D does not get into the multiset $\bigcup_{j=1}^{n} \mathcal{Z}_n^{\pm}$. Passing from ζ to $z = \mathrm{i}\zeta$ and from zeroes of $D(\zeta)$ to that of $\det \Gamma_{a,Y}(z)$, we obtain statements (i)-(ii) of Theorem 3.4. Note that $\mathrm{i}\zeta_{n,j,t}^{\pm} = k_{n,j,t}^{\mp}$.

The part $1 \leq M \leq N-1$ of statement (iv) is proved by Proposition 3.3. To prove $\sum_{n=1}^{M} r_n \leq N$, note that (3.10) and (3.5) imply $\sum_{n=1}^{M} r_n = N - m_{1,1} \leq N$. The fact that $r_M \geq 2$ follows from Lemma 3.1 (i) and Proposition 3.3. This completes the proof of Theorem 3.4. □

4 Geometry of Y and parameters of asymptotics

The parameters μ_n play the role of leading parameters of the asymptotic sequences $\{k_{n,j,t}^{\pm}\}_{t=t_{n,j}}^{+\infty}$ of (3.11). Their role in the distribution of resonances is emphasized by the next statement, which is immediate corollary of Theorem 3.4 (see also [8, formula (12.18.16)] and the correcting remark to it in the Russian edition of [8]).

Corollary 4.1. *For the operator $H_{a,Y}$, the logarithmic asymptotic density function $\mathrm{Ad}^{\log}(\cdot)$ defined by (1.4) is a nondecreasing piecewise constant function with a finite number of jumps. These jumps are exactly at the points of the set $\{\mu_n\}_{n=1}^{M} \subset \mathbb{R}_+$, and the height of each jump is equal to*

$$\mathrm{Ad}^{\log}(\mu_n + 0) - \mathrm{Ad}^{\log}(\mu_n - 0) = \frac{\beta_{n,\nu_n} - \beta_{n,1}}{\pi} = \frac{r_n}{\pi \mu_n}. \tag{4.1}$$

For the following considerations let us recall that the m-sizes $\mathfrak{s}_m$ of Y were defined in Section 1 and that the value $\mathfrak{s}_N$ was introduced in [40] and called the *size of* Y. Let $\mathfrak{N}_{H_{a,Y}}(\cdot)$ be the resonance counting function defined by (1.2) with $H = H_{a,Y}$.

It was shown in [40, Theorem 4.1] (see also comments in [6] about the choice of the 'strength' tuple a) that $\mathfrak{N}_{H_{a,Y}}(R) = \frac{W(a,Y)}{\pi} R + O(1)$, where $0 \leq W(a,Y) \leq \mathfrak{s}_N$, and it was said that $\mathfrak{N}_{H_{a,Y}}(\cdot)$ has the *Weyl-type asymptotics* if $W(a,Y) = \mathfrak{s}_N$ (i.e., roughly speaking, when $W(a,Y)$ attains the maximal possible value). Slightly rephrasing [40], it is natural to say that $W(a,Y)$ is the *effective size of* $H_{a,Y}$.

Taking into account Theorem 3.4, we see that

$$\lim_{\mu \to +\infty} \mathrm{Ad}^{\log}(\mu) = \mathrm{Ad}^{\log}(+\infty) = W(a,Y)/\pi \leq \mathfrak{s}_N/\pi.$$

With the use of Theorem 3.4 (iii)-(iv) one can strengthen [40, Theorem 4.1] by the following inequality

$$W(a,Y)/\pi = \mathrm{Ad}^{\log}(+\infty) \geq \mathrm{Ad}^{\log}(\mu_M + 0) = \frac{r_M}{\pi \mu_M} \geq \frac{2 \operatorname{diam} Y}{\pi}.$$

Thus, $W(a,Y) \geq 2 \operatorname{diam} Y = \mathfrak{s}_2$. The following statement concerns configurations of Y whose effective sizes $W(a,Y)$ take the minimal possible value $\mathfrak{s}_2$.

Corollary 4.2. *If $W(a,Y) = 2 \operatorname{diam} Y$, then all the points of Y lie on one line.*

Proof. If Y does not belong to one line, then Lemma 3.1 (ii) implies that $\mathfrak{s}_3 > \mathfrak{s}_2$ and that β_0 in (3.1) is less or equal than $(-\mathfrak{s}_3)$. Thus, [40, Theorem 2.1] (or, alternatively, Corollary 4.1 and (3.6)) implies $W(a,Y) > \mathfrak{s}_2$. □

The implication converse to Corollary 4.2 does not hold true, as one can see from Section 4.3, where an example of $H_{a,Y}$ with $W(a,Y) = \mathfrak{s}_4 > 2 \operatorname{diam} Y$ can be found for the case $N = 4$.

In this section we show that a stronger connection exists between the family of the parameters μ_n and the metric geometry of the family Y of the interaction centers. This connection is given below by Theorems 4.3-4.4 in terms of the m-sizes $\mathfrak{s}_m$.

4.1 Generic and maximally structured cases

Let $N \geq 2$ be fixed. To parametrize rigorously the family of Hamiltonians $H_{a,Y}$, let us consider in this section Y as a vector in the space $(\mathbb{R}^3)^N$ of ordered N-tuples $y = (y_j)_{j=1}^N$ with the entries $y_j \in \mathbb{R}^3$. We consider $(\mathbb{R}^3)^N$ as a linear normed space with the ℓ^2-norm $|y|_2 = (\sum |y_j|^2)^{1/2}$. Then the ordered collection Y of centers is identified with an element of the family $\mathbb{F} \subset (\mathbb{R}^3)^N$ defined by $\mathbb{F} := \{y \in (\mathbb{R}^3)^N \ : \ y_j \neq y'_j \ \text{ for } \ j \neq j'\}$. We consider $\mathbb{F}$ as a metric space with the distance function induced by the norm $|\cdot|_2$.

We denote by $M(a,Y)$ and $\mu_n(a,Y)$, $n = 1, \dots, M(a,Y)$, the parameter M and μ_n, resp., that are associated with the operator $H_{a,Y}$ in the way described in Section 3 (recall that M is the number of distinct parameters μ_n).

Let us consider the set $\mathbb{F}_{\text{gen}}$ that consists of $Y \subset \mathbb{F}$ that satisfy the following two properties:

(A1) $M(a,Y)$ and $\{\mu_n(a,Y)\}_{n=1}^{M(a,Y)}$ do not depend on the 'strength' tuple $a \in \mathbb{C}^N$,

(A2) $\{\mu_n(a,Y)\}_{n=1}^{M(a,Y)}$ is a subset of

$$\left\{ \frac{m-j}{\mathfrak{s}_m(Y) - \mathfrak{s}_j(Y)} \ : \ 2 \leq m \leq N, \ 0 \leq j \leq N-1, \ j \neq 1, \ \text{and } \mathfrak{s}_j < \mathfrak{s}_m \right\}.$$

The following theorem states that, generically, all μ_n have the form $\frac{m-j}{\mathfrak{s}_m - \mathfrak{s}_j}$.

Theorem 4.3. *There exists a subset of* $\mathbb{F}_{\text{gen}}$ *that is open and dense in* $\mathbb{F}$.

The proof is given in Section 4.2.

The next theorem describe one of the cases where the connection between the sequences of μ_n and $\mathfrak{s}_m$ is especially simple and, simultaneously, the asymptotics of resonances is maximally structured in the sense that the number M of parameters μ_n takes its maximal possible value $N-1$.

Theorem 4.4. *Let* $a \in \mathbb{C}^N$ *and* $Y \in \mathbb{F}$ *be such that the numbers* $\mathfrak{s}_n = \mathfrak{s}_n(Y)$, $n = 1, \dots, N$, *and the polynomials* $P_b(\cdot)$, $b \in \mathbb{R}$, *which are associated with the representation (3.1) for* $D(\zeta) = (-4\pi)^N \det \Gamma_{a,Y}(\mathrm{i}\zeta)$ *and the convention (3.2), have the following properties:*

(A3) For $3 \le n \le N$, $P_{-\mathfrak{s}_n}(\cdot) \not\equiv 0$ *and* $\deg P_{-\mathfrak{s}_n} = N - n$.

(A4) The sequence $\{\mathfrak{s}_m\}_{m=1}^N$ *is increasing and strictly convex upward in the sense that* $\mathfrak{s}_m - \mathfrak{s}_{m-1} > \mathfrak{s}_{m+1} - \mathfrak{s}_m > 0$ *for* $2 \le m \le N-1$ *(if* $N = 2$*, this condition is assumed to be fulfilled automatically; recall also that* $\mathfrak{s}_1 := (\mathfrak{s}_2 + \mathfrak{s}_0)/2 = \operatorname{diam} Y$*).*

Then:

(i) $M(a, Y) = N - 1$ *and* $\mu_{N-m}(a, Y) = \dfrac{1}{\mathfrak{s}_{m+1} - \mathfrak{s}_m}$ *for* $m = 1, \dots, N-1$*;*

(ii) $r_{N-1} = 2$, $r_n = 1$ *for* $1 \le n \le N-2$*, and, after possible exclusion of a finite number of resonances, the multiset* $\Sigma(H_{a,Y})$ *can be decomposed into the* $2N$ *asymptotic sequences (3.11);*

(iii) there exists $R \ge 0$ *such that every resonance* k *with* $|k| \ge R$ *is simple (i.e., of multiplicity 1).*

Examples in Section 4.3 illustrate how large can be the family of sets Y having the property that $H_{a,Y}$ satisfies (A3)-(A4) for every $a \in \mathbb{C}^N$.

4.2 Proofs of Theorems 4.3 and 4.4

Let us introduce the points $\widetilde{T}_m := (-\mathfrak{s}_m) + \mathrm{i}(N - m)$ for $m = 0$ and for $2 \le m \le N$. For $m = 0$ and for $2 \le m \le N$, the point $\widetilde{T}_m$ is not an endpoint of any of the segments $\mathcal{L}_n$, $n = 1, \dots, M$ (and so disappears from the computation of the parameters μ_n) if and only if exactly one of the following conditions hold:

(C1) $\widetilde{T}_m \notin \{T_{\beta_j}\}_{j=0}^{\nu}$.

(C2) $\widetilde{T}_m \in \{T_{\beta_j}\}_{j=0}^{\nu}$, $\widetilde{T}_m \in \mathcal{L}_n$ for a certain n, $1 \le n \le M$, and $\widetilde{T}_m$ is not an endpoint of $\mathcal{L}_n$.

(C3) $\widetilde{T}_m \in \{T_{\beta_j}\}_{j=0}^{\nu}$ and $\widetilde{T}_m$ lies strictly below a certain segment $\mathcal{L}_n$, $1 \le n \le M$, of the distribution diagram (i.e., $(-\mathfrak{s}_m) \in \operatorname{Re}[\mathcal{L}_n]$ and $(\widetilde{T}_m + \mathrm{i}\mathbb{R}_+) \cap \mathcal{L}_n \neq \varnothing$).

The main idea of the proofs of Theorems 4.3 and 4.4 is that some type of control over the cases (C1)-(C3) is needed. While for the proof of Theorem 4.3 we show that generically (C1) does not hold true, the assumptions of Theorem 4.4 exclude the possibility of each of (C1)-(C3).

Let us prove Theorem 4.3. Consider the family $\mathbb{A}_{\mathrm{gen}}$ consisting of all $Y \in \mathbb{F}$ such that, for a certain $a \in \mathbb{C}^N$,

(A5) each of the points $\widetilde{T}_m$ (where $0 \le m \le N$ and $m \neq 1$) belongs to the set $\{T_{\beta_j}\}_{j=0}^{\nu}$ associated with $H_{a,Y}$.

It is easy to note from the form (2.2) of $\det \Gamma_{a,Y}(\cdot)$ that if $Y \in \mathbb{A}_{\mathrm{gen}}$, then (A5) is valid for every $a \in \mathbb{C}^N$. Indeed, (A5) implies that all $\mathfrak{s}_m$, $2 \le m \le N$, are distinct and that there is no cancellation of the leading coefficients of the polynomials $p^{[\sigma]}(\zeta)$ in (2.3) with the degree $N - m$ and σ such that $\alpha(\sigma) = -\mathfrak{s}_m$. On the other hand, these leading coefficients do not depend on a.

Proposition 4.5. *There exists a subset of $\mathbb{A}_{\rm gen}$ that is open and dense in $\mathbb{F}$.*

Proof. It is easy to see that the family $\mathbb{A}_1$ introduced in [6, Section 3.2] is a subset of $\mathbb{A}_{\rm gen}$. So the proposition is an immediate corollary of [6, Lemma 3.4]. □

Lemma 4.6. $\mathbb{A}_{\rm gen} \subset \mathbb{F}_{\rm gen}$

Proof. Let $Y \in \mathbb{A}_{\rm gen}$. Then the validity of conditions (A1)-(A2) of Section 4.1 follows from the definitions of M and μ_n (see Section 3.1) and the following fact: if $Y \in \mathbb{A}_{\rm gen}$, then the set of points $T_{n,h}$, $n = 1, \dots, M$, $h = 1, \dots, \nu_n$, is a subset of the family of points $\widetilde{T}_m$, where $0 \le m \le N$ and $m \ne 1$. Indeed, among all the points T_{β_j} with $\operatorname{Im} T_{\beta_j} = N - m$, the point $\widetilde{T}_m$ has minimal possible real part. Since

$$\widetilde{T}_0 = T_0 = T_{M,\nu_M} \tag{4.2}$$

is the only point of the distribution diagram $\mathcal{L}$ on the line $\operatorname{Im} z = N$, the convexity (L3) of the distribution diagram $\mathcal{L}$ (see Section 3.1) implies the desired statement. □

Combining Proposition 4.5 and Lemma 4.6, *we conclude the proof of Theorem 4.3.*

Let us prove Theorem 4.4. It follows from (A3) that the cancellation (C1) does not happen for $m \ge 3$. For $m = 2$, (C1) does not happen because of Lemma 3.1. For $m = 0$, (C1) does not happen because (4.2) holds for all $H_{a,Y}$. Thus, (A3) implies (A5).

It follows from the proof of Lemma 4.6 that only the points $\widetilde{T}_m$ participate in the computation of M and of the parameters μ_n, $n = 1 \dots, M$. The assumption (A4) ensures that all the parameters $\widetilde{T}_m$, where $0 \le m \le N$ and $m \ne 1$, do participate in this computation. Thus, $M = N - 1$. Now, $\mu_{N-m}(a, Y) = \frac{1}{\mathfrak{s}_{m+1} - \mathfrak{s}_m}$ and the other conclusions of statement (ii) follow directly from (3.6) and Theorem 3.4.

To prove statement (iii) of Theorem 4.4 it is enough to apply Theorem 3.4 and to notice that $\omega_{M,1} \ne \omega_{M,2}$ (this follows from Lemma 3.1 (i) and the procedure for the computation of $\omega_{n,j}$ described in Section 3.2). This completes the proof of Theorem 4.4.

4.3 Examples and the thickness of the case (A3)-(A4).

Let $N = 3$. In this case it is easy to give a complete classification of all possible structures of asymptotics of $\Sigma(H_{a,Y})$ due to the fact that $\mathbb{F}_{\rm gen} = \mathbb{F}$. That is, the distribution diagram $\mathcal{L}$ as well as the parameters μ_n, and $\omega_{n,j}$ of the sequences (3.11) do not depend on the 'strength' tuple a. They also satisfy (A2). Note also that for $N = 3$ the set of tuples Y satisfying (A3)-(A4) is generic in the sense that it is open and dense in $(\mathbb{R}^3)^3$. In more details, there are 3 sub-cases of mutual positions of 3 centers $\{y_j\}_{j=1}^3$ composing Y, which are described by the corresponding lengths $\ell_{j,m} = |y_j - y_m|$:

Case 1. Y consists of the vertices of an equilateral triangle, i.e., $\ell_{1,2} = \ell_{2,3} = \ell_{3,1} = \operatorname{diam} Y$. The asymptotic sequences in this case were found in [4] (formally, under the restriction $a_1 = a_2 = a_3 \in \mathbb{R}$ on the tuple a). Condition (A3) is satisfied, but the strict convexity part of (A4) is not; $M = 1$, $\mu_1 = 1/\operatorname{diam} Y$, and $r_1 = 3$.

Case 2. The points y_1, y_2, and y_3 of Y are on one line. Let us number them such that $\ell_{1,2} + \ell_{2,3} = \ell_{1,3}$. Then $\mathfrak{s}_3 = \mathfrak{s}_2$ and condition (A3) fails; $M = 1$, $\mu_1 = 1/\ell_{1,3}$, and $r_1 = 2$.

Case 3. The generic case, where y_1, y_2, and y_3 do not fit in the Cases 1 and 2. Let us number them such that $\ell_{1,2} + \ell_{2,3} > \ell_{1,3} = \operatorname{diam} Y$. Conditions (A3)-(A4) are satisfied; $M = 2$; $\mu_1 = 1/(\ell_{1,2}+\ell_{2,3}-\ell_{1,3})$ with the multiplicity $r_1 = 1$; $\mu_2 = 1/\ell_{1,3}$ with the multiplicity $r_2 = 2$.

The case of Theorem 4.4 is not generic when $N = 4$. To show this in Proposition 4.8, we start from an example of $H_{a,Y}$ where both of (A3) and (A4) fail.

Example 4.7. Let us fix an arbitrary $a \in \mathbb{C}^4$. Consider $Y = \{y_j\}_{j=1}^4$ with the properties that it is symmetric w.r.t. the origin, lies on one line passing through the origin, and satisfy $|y_1| = |y_4| = c_1$ and $|y_2| = |y_3| = c_2 < c_1$. Then the exp-monomials of (2.3) that have the lowest possible frequency $(-\mathfrak{s}_4) = -4c_1 - 4c_2$ do not cancel each other. Thus, the following statements are valid for this example:

(i) $\Sigma(H_{a,Y})$ has the asymptotics of Weyl type (i.e., $W(a, Y) = \mathfrak{s}_N$);

(ii) each of the statements (A3), (A4), (A5) does not hold (since $\mathfrak{s}_3 = \mathfrak{s}_2$).

Note that, $Y \in \mathbb{F}_{\text{gen}}$.

Proposition 4.8. *Let $a \in \mathbb{C}^4$. Let Y be as in Example 4.7. Then there exists a neighborhood $\mathbb{B}_\varepsilon(Y)$ of Y such that for all $\widetilde{Y} \in \mathbb{B}_\varepsilon(Y)$ at least one of the conditions (A3)-(A4) fails for $H_{a,\widetilde{Y}}$.*

Proof. If the frequency $(-\mathfrak{s}_4)$ cancels, then (A3) fails. Assume now that $\widetilde{Y} \in \mathbb{B}_\varepsilon(Y)$ and (A3) holds for $H_{a,\widetilde{Y}}$. Then it is easy to see that for small enough ε, (C3) is valid and (A4) fails. □

5 High-energy asymptotics of 'physical resonances'

The existence of asymptotic structures means that the main quantities that characterize the high-energy asymptotics of $\Sigma(H_{a,Y})$ from the point of view of Physics are: the minimum $\mu^{\min}$ of the support of the measure $d\mathrm{Ad}^{\log}(\cdot)$ associated with the asymptotic density function $\mathrm{Ad}^{\log}(\cdot)$ and the height of the jump at this point $\mathrm{Ad}^{\log}(\mu^{\min} + 0) - \mathrm{Ad}^{\log}(\mu^{\min} - 0)$. Indeed, in scattering experiments usually only 'sufficiently narrow' and 'threshold' resonances k are detected, and for this reason they are mostly considered to be the 'physically relevant' resonances [19]. Such resonances have small (or, for some models, even zero) value of *resonance width* $|\operatorname{Im} k^2|$ (see [44]).

In the case of the point interaction Hamiltonian $H_{a,Y}$, with the growth of the energy $\operatorname{Re} k^2$, the resonance width growth faster for sequences that have larger leading parameter μ_n. Thus, only the asymptotic sequences corresponding to $\mu^{\min} = \mu_M =$

$1/\operatorname{diam} Y$ have chances to be detected with reasonable confidence. It is natural to call the density

$$\mathrm{Ad}^{\log}(\mu_M + 0) - \mathrm{Ad}^{\log}(\mu_M - 0) = \frac{r_M \operatorname{diam} Y}{\pi} \tag{5.1}$$

of the resonances in the logarithmic strip $\mathrm{i}\mathcal{V}(\mu_M, w_M)$ (see (3.8)) *the asymptotic density of narrow resonances.*

Remark 5.1. The study of the density (5.1) of 'more narrow' resonances has some similarity with one of the questions considered in [25, 48, 49], where in the context of scattering by a strictly convex C^∞-obstacle, a semi-cubic resonance free region $\mathrm{Fr} = \{-C_0 |\operatorname{Re} z|^{1/3} + C_1 \leq \operatorname{Im} z \leq 0\}$ was described [25, 48] and the asymptotics of the counting function in a certain cubic strip $\{-C_2 |\operatorname{Re} z|^{1/3} - C_3 \leq \operatorname{Im} z \leq -C_0 |\operatorname{Re} z|^{1/3} + C_1\}$ (with $C_2 > C_0 > 0$) adjacent to Fr was found [49] under an additional pinching condition on curvatures. Note that in the case of point interactions the information of the structure of $\Sigma(H_{a,Y})$ given in Theorem 3.4 is more detailed than that of [49] in the case of obstacle scattering. In particular, Theorem 3.4 says that the decay rates $(-\operatorname{Im} k_{M,j,t})$ for different indices j of the sequences associated with the smallest leading parameter $\mu_M = \mu^{\min}$ are asymptotically equivalent as $|t| \to \infty$. In other words, the vertical cross sections of $\mathrm{i}\mathcal{V}(\mu_M, w_M)$, which is the closest to $\mathbb{R}$ logarithmic strip containing infinite number of resonances, remain uniformly bounded.

Summarizing, we see that, for $H_{a,Y}$, the diameter $\operatorname{diam} Y$ and the multiplicity parameter $r_M \in \mathbb{N}$, which take values between 2 and N, are the most important characteristics of high-energy asymptotics of 'physical resonances'. The following lemma implies that r_M does not depend on the 'strength' tuple $a \in \mathbb{C}^N$.

Lemma 5.1. *The multiplicity parameter r_M is equal to the maximal integer $m \in [2, N]$ such that $\mathfrak{s}_m = m \operatorname{diam} Y$ and $\widetilde{T}_m \in \mathcal{L}$.*

Proof. The lemma follows immediately from Sections 3.1-3.2 and the definition of the points $\widetilde{T}_m$ in Section 4.2. ☐

Definition 5.1. Let us denote by $r^{\mathrm{narrow}}(Y)$ the value of r_M for $H_{a,Y}$, and by $\mathbb{F}_2$ the set of $Y \in \mathbb{F}$ such that $r^{\mathrm{narrow}}(Y) = 2$ (i.e., such that $r^{\mathrm{narrow}}(Y)$ is minimal possible; we use here the notation of Section 4.1).

The following theorem states that, generically, $r^{\mathrm{narrow}}(Y) = 2$.

Theorem 5.2. *There exists a subset of $\mathbb{F}_2$ that is open and dense in $\mathbb{F}$.*

Proof. Let us denote by $\mathbb{A}_2$ the set of $Y \in \mathbb{F}$ such that $\alpha(\sigma_1) \neq \alpha(\sigma_2)$ holds for any two different transpositions σ_1, $\sigma_2 \in S_{N,2}$ (recall that $\alpha(\sigma)$ was defined by 2.4). Then it follows from [6, Lemma 3.4] that $\mathbb{A}_2$ is open and dense in $\mathbb{F}$. On the other hand, it is obvious from Lemma 5.1 that $\mathbb{A}_2 \subset \mathbb{F}_2$. ☐

So, $r_M > 2$ means that the asymptotic density of narrow resonances is atypically large. Let us consider in more details how this can happen. First, note that for $N \geq 3$,

$$\mathfrak{s}_3 = 3 \operatorname{diam} Y \quad \text{implies} \quad r_M \geq 3. \tag{5.2}$$

Indeed, $\mathfrak{s}_3 = 3 \operatorname{diam} Y$ means exactly that three of the centers y_j form an equilateral triangle with the side of the length $\operatorname{diam} Y$. It follows from Lemma 3.1 (ii), the properties (L3)-(L4) of the distribution diagram $\mathcal{L}$, and Proposition 3.3 that $\widetilde{T}_3 \in \mathcal{L}$, and so there is no need to check this condition in Lemma 5.1 (for $m = 3$).

By the following example we show that for $N \geq 4$ the condition $\mathfrak{s}_4 = 4 \operatorname{diam} Y$ (i.e., Y contains four distinct centers y_{j_n}, $n = 1, \dots, 4$, such that $\operatorname{diam} Y = \ell_{j_1,j_2} = \ell_{j_3,j_4}$) does not imply $r_M \geq 4$, and, moreover, it does not imply $r_M \geq 3$. Here and below, we shall set $\ell_{i,j} = |y_i - y_j|$.

Example 5.3. Let $\widetilde{Y}_4 = \{y_j\}_{j=1}^4$ be the tuple of distinct $\mathbb{R}^3$ points such that

$$\ell_{1,2} = \ell_{2,3} = \ell_{3,4} = \ell_{4,1} = \operatorname{diam} \widetilde{Y}_4 > \ell_{2,4} \geq \ell_{1,3}$$

and let $a \in \mathbb{C}^4$ be arbitrary. Then $\mathfrak{s}_4(\widetilde{Y}_4) = 4 \operatorname{diam} \widetilde{Y}_4$, but the frequency $\mathfrak{s}_4(\widetilde{Y}_4)$ cancels in the process of summation of the exp-monomials (2.3). This can be seen by a simple modification of arguments of [40, Section 5.2] or by direct calculations. Since $\mathfrak{s}_3 < 3 \operatorname{diam} \widetilde{Y}_4$, we see that $M(a, \widetilde{Y}_4) = 2$ and $r^{\text{narrow}}(\widetilde{Y}_4) = 2$.

Let us introduce one more condition on Y with $N \geq 4$:

(A6) there exist four distinct centers $y_{j_n} \in Y$, $n = 1, \dots, 4$, such that $\operatorname{diam} Y = \ell_{j_1,j_2} = \ell_{j_3,j_4}$ and $\{y_{j_i}\}_{i=1}^4$ cannot be reordered to satisfy the condition of Example 5.3.

Theorem 5.4. *Let $N \geq 4$. Assume that condition (A6) holds true or $\mathfrak{s}_3(Y) = 3 \operatorname{diam} Y$. Then $r^{\text{narrow}}(Y) > 2$.*

Proof. The case $\mathfrak{s}_3 = 3 \operatorname{diam} Y$ follows from (5.2). Assume now that $\mathfrak{s}_3 \neq 3 \operatorname{diam} Y$ and that (A6) holds. Note that (A6) implies $\mathfrak{s}_4 = 4 \operatorname{diam} Y$. Therefore, due to Lemma 5.1, it is enough to prove that $(-\mathfrak{s}_4)$ does not cancel in the process of summation of the exp-monomials (2.3) corresponding to permutations $\sigma \in S_{n,4}$.

Consider the family B_1 of unordered sets Y_4 consisting of 4 points of Y such that a certain ordering $\widetilde{Y}_4 := \{y_{m_i}\}_{i=1}^4$ of Y_4 satisfies the conditions of Example 5.3. To such $\widetilde{Y}_4$, we put into correspondence the two families, $E_1(\widetilde{Y}_4)$ and $E_2(\widetilde{Y}_4)$, each of which is an unordered pair of unordered pairs of centers: $E_1(\widetilde{Y}_4) = \{\{y_{m_1}, y_{m_2}\}, \{y_{m_3}, y_{m_4}\}\}$ and $E_2(\widetilde{Y}_4) = \{\{y_{m_2}, y_{m_3}\}, \{y_{m_4}, y_{m_1}\}\}$. The assumption $\mathfrak{s}_3 \neq 3 \operatorname{diam} Y$ implies that for each $Y_4 \in B_1$ there exists a unique unordered pair $E(Y_4)$ of the form $\{E_1(\widetilde{Y}_4), E_2(\widetilde{Y}_4)\}$ i.e., $\mathfrak{s}_3 \neq 3 \operatorname{diam} Y$ implies that various reorderings of $\widetilde{Y}_4$ that preserve the conditions of Example 5.3 produce the same unordered pair $E(Y_4)$ (of pairs of pairs of centers). (For the case where $\mathfrak{s}_3 = 3 \operatorname{diam} Y$ and $\{y_{m_i}\}_{i=1}^4$ are vertices of a regular tetrahedron, the above statement obviously fails.)

The family of all pairs of pairs of centers produced by the maps $E_i(\cdot)$, $i = 1, 2$, shall be denoted by B_2, i.e., $B_2 = E_1[B_1] \cup E_2[B_2]$.

The class $\mathfrak{S}_1$ of all permutations $\sigma \in S_{N,4}$ that have the sign $\epsilon_\sigma = -1$ and produce exp-monomials (2.3) with the frequency $(-\mathfrak{s}_4)$ consists of permutations $[m_1 m_2 m_3 m_4]$ and $[m_4 m_3 m_2 m_1]$ associated with a certain ordered tuple $\widetilde{Y}_4$ having the property that its unordered version Y_4 belongs to B_1. These permutations produce the exp-monomials $\frac{-e^{-\mathfrak{s}_4}}{(\operatorname{diam} Y)^4} \prod_{j=\sigma(j)} (-\zeta - A_j)$. To each pair $[m_1 m_2 m_3 m_4]$ and $[m_4 m_3 m_2 m_1]$ of such permutations we put into correspondence the two permutations $[m_1 m_2][m_3 m_4]$ and $[m_2 m_3][m_4 m_1]$ associated with $E_1(\widetilde{Y}_4)$ and $E_2(\widetilde{Y}_4)$, resp., and we unite all permutations generated in this way by $E_i(\widetilde{Y}_4)$, $i = 1, 2$, into the class $\mathfrak{S}_2$. The permutations $[m_1 m_2][m_3 m_4]$ and $[m_2 m_3][m_4 m_1]$ produce the exp-monomials (2.3) of the form $\frac{e^{-\mathfrak{s}_4}}{(\operatorname{diam} Y)^4} \prod_{j=\sigma(j)} (-\zeta - A_j)$. So after summation of all exp-monomials generated by permutations of the classes $\mathfrak{S}_1$ and $\mathfrak{S}_2$ the coefficients of the highest order cancel and the associated polynomial has a degree $< N - 4$.

Assumption (A6) means that $\{\{y_{j_1}, y_{j_2}\}, \{y_{j_3}, y_{j_4}\}\}$ does not belong to B_2. However, the exp-monomial produced by the associated permutation $\sigma = [j_1 j_2][j_3 j_4]$ has the sign $\epsilon_\sigma = 1$ and the frequency $(-\mathfrak{s}_4)$. Hence the frequency $(-\mathfrak{s}_4)$ does not cancel and, moreover, $\deg P_{-\mathfrak{s}_4} = N - 4$. This proves $\widetilde{T}_4 \in \mathcal{L}$, and due to Lemma 5.1, completes the proof of the theorem. □

Under symmetries of Y we understand isometries Is of $\mathbb{R}^3$ such that $\operatorname{Is}[Y] = Y$. Moreover, two symmetries Is_1 and Is_2 will be considered identical (equivalent) if $\operatorname{Is}_1(y_j) = \operatorname{Is}_2(y_j)$ for all $y_j \in Y$. So the group $\mathfrak{G}$ of symmetries of Y consists of the defined above classes of equivalence of isometries of $\mathbb{R}^3$ that map Y onto Y.

Corollary 5.5. *Let $\mathfrak{G}$ be the group of symmetries of $Y = \{y_j\}_{j=1}^N$. Assume that there exist $\operatorname{Is} \in \mathfrak{G}$ and $y_{j_1}, y_{j_2} \in Y$ with the following properties:*

(i) $|y_{j_1} - y_{j_2}| = \operatorname{diam} Y$,

(ii) the sets $\{y_{j_1}, y_{j_2}\}$ and $\{\operatorname{Is}(y_{j_1}), \operatorname{Is}(y_{j_2})\}$ are disjoint,

(iii) the 4-tuple $\{y_{j_1}, y_{j_2}, \operatorname{Is}(y_{j_1}), \operatorname{Is}(y_{j_2})\}$ cannot be reordered in such a way that it satisfies the conditions of Example 5.3.

Then $r^{\text{narrow}}(Y) > 2$.

Proof. The statement follows easily from Theorem 5.4. □

Theorem 5.4 and Corollary 5.5 suggest that $r^{\text{narrow}}(Y) > 2$ whenever Y possesses a rich enough group of symmetries $\mathfrak{G}$. This claim can be supported by the following examples.

Example 5.6. Let the tuple a be arbitrary. Then $r^{\text{narrow}}(Y) > 2$ in each of the following cases:

(i) Y is the set of all vertices of a Platonic solid;

(ii) Y is the set of all vertices of a right prism.

These statements can be shown by using Theorem 5.4 or by the direct calculation of the exp-monomials with the frequencies $(-m) \operatorname{diam} Y$, $m \in \mathbb{N} \cap [3, N]$.

The $N-1$ possible values $2, 3, \dots, N$ of r^{narrow} naturally split the family $\mathbb{F}$ of all possible tuples Y of interaction centers into $N-1$ classes. In our opinion, the interplay between these classes and the groups of symmetries of Y deserves an additional study.

6 Remarks on resonances of quantum graphs

To study the asymptotic structure of $\Sigma(H)$ for quantum graph Hamiltonians, the above approach requires some adaptation that is done in the next subsection. Subsection 6.2 briefly addresses the resonances of 1-D photonic crystals considering them as a simple case of 'weighted' quantum graphs.

6.1 Structure of resonance asymptotics for quantum graphs

The asymptotics $\mathfrak{N}(R) = \frac{CR}{\pi} + O(1)$ of the resonance counting function $\mathfrak{N}(R)$ for a Hamiltonian H associated with a noncompact quantum graph $\mathcal{G}$ consisting of a finite number of edges have been derived in [13, 14], where, in particular, Weyl and non-Weyl types of asymptotics for $\mathcal{G}$ were introduced depending on the value of the constant $C \geq 0$. The characterizations of these two cases of asymptotics were obtained in terms of types of couplings at the vertices. A number of examples were elaborated (see [39, 22] and the references therein).

We show that, at least under additional conditions, the asymptotic structures which are somewhat similar, but more complex, than those of $\Sigma(H_{a,Y})$, exist in the multiset of resonances $\Sigma(H)$ of a quantum graph. This leads to another versions of the semi-logarithmic strip counting

$$\mathfrak{N}(\mu,\gamma,R) := \#\{k \in \Sigma(H) : -\mu \ln(|\operatorname{Re} k| + 1) - \gamma \leq \operatorname{Im} k,\ |k| \leq R\}, \quad \mu,\gamma \in \mathbb{R},$$
$$\mathfrak{N}(\mu,+\infty,R) := \mathfrak{N}_H(R) \text{ for } \mu \geq 0.$$

and the asymptotic density function

$$\operatorname{Ad}(\mu,\gamma) := \lim_{R\to\infty} \frac{\mathfrak{N}(\mu,\gamma,R)}{R} \tag{6.1}$$

depending now on the two parameters $\mu \geq 0$ and $\gamma \in (-\infty, +\infty]$. (For $\mu < 0$ and $\gamma \neq +\infty$, one can put $\operatorname{Ad}(\mu,\gamma) := 0$ due to the fact that $\#\{k \in \Sigma(H) \cap \mathbb{C}_+\} < \infty$, see Sections 6.1.1-6.1.2).

With such a definition the effective size of H introduced in [13, 14] is equal to $\frac{\pi \operatorname{Ad}(0,+\infty)}{2}$ and [14, Theorem 3.2] and [13, Theorem 3.3] can be rewritten as the inequality

$$\operatorname{Ad}(0,+\infty) = \operatorname{Ad}(\mu,+\infty) \leq \frac{2}{\pi} \sum_{j\in J} \rho_j,$$

where ρ_j are the lengths of the internal (i.e., bounded) edges of the graph, J is the finite set that indexes all such edges, and $\mu \geq 0$ is arbitrary.

6.1.1 The case of Kirchhoff's boundary condition at vertices

In this subsection we follow the settings of [14] and consider on a non-compact quantum graph $\mathcal{G}$ the self-adjoint Hamiltonian $H_{\mathcal{G}}$ associated with the differential expression $-\mathrm{d}^2/\mathrm{d}x^2$ with the continuity condition and the Kirchhoff (boundary) condition at each of the vertices.

The set of resonances $\Sigma(H_{\mathcal{G}})$ is the set of $k \neq 0$ such that there exists a resonant mode $f(\cdot)$ (continuous and $L^2_{\mathbb{C},\mathrm{loc}}$ on $\mathcal{G}$) satisfying $f'' = -k^2 f$ on $\mathcal{G}$, the Kirchhoff conditions at each vertex, and the classical radiation condition on each lead (i.e., semi-infinite edge). The Kirchhoff condition means that the sum of outgoing derivatives of f at a vertex is equal to 0. The radiation condition after the identification of a lead with $[0, +\infty)$ takes the form $f(x) = f(0)\mathrm{e}^{\mathrm{i}kx}$.

To make $\Sigma(H_{\mathcal{G}})$ a multiset, each resonance k is equipped with a multiplicity, which is multiplicity of k as a zero of a specially constructed analytic function $F(z) = \det A(z)$, where $A(z)$ is a matrix produced by a plugging of the exponential fundamental system of solutions $e^{\pm \mathrm{i}kx}$ on each edge into the continuity, Kirchhoff's, and radiation conditions (see [14, Theorem 3.1] for details).

Remark 6.1. In the process of construction in [14] of the matrix-valued function $A(\cdot)$, the Kirchhoff conditions are divided by $\mathrm{i}k$, which is permissible because $k = 0$ is excluded from the consideration. Let us observe that, on one hand, $k = 0$ always satisfies the resonance conditions, and on the other hand after the transition to the settings with the energy-type spectral parameter $\lambda = k^2$, the point $\lambda = 0$ is not a pole, but a branching point of the generalized extension of the resolvent $(H_{\mathcal{G}} - \lambda)^{-1}$. The exclusion of $k = 0$ is usual for some types of 1-D resonance problems (see e.g. [12, 33, 34]) and makes the above definition of resonances slightly different from that of [17, 56], where 0 is permitted to be a resonance of a finite multiplicity. Obviously, this difference does not influence the results of [14, 13] and of the present paper on the asymptotics at ∞; however it influences the formulation of Theorem 6.1 below.

It is shown in [14, Theorem 3.2] that $\Sigma(H_{\mathcal{G}})$ lies in a certain horizontal strip $\{z \in \mathbb{C} : -\gamma \leq \operatorname{Im} z \leq 0\}$ (formally, the theorem states this for a certain strip $\{|\operatorname{Im} z| \leq \gamma\}$, however $H_{\mathcal{G}} = H_{\mathcal{G}}^* \geq 0$, and so $\Sigma(H_{\mathcal{G}}) \cap \mathbb{C}_+ = \varnothing$). Hence, the measure $d\mathrm{Ad}^{\log}$ does not carry information on the internal structure of $\Sigma(H_{\mathcal{G}})$. Indeed, it consists of one mass $\mathrm{Ad}^{\log}(+\infty)$ at the point $\mu = 0$ and so gives only the information about the total asymptotic density $\mathrm{Ad}^{\log}(+\infty)$.

To capture the structure of the asymptotics of $\Sigma(H_{\mathcal{G}})$, let us introduce another asymptotic density function

$$\mathrm{Ad}^{\mathrm{hor}}(\gamma) := \lim_{R\to\infty} \frac{\mathfrak{N}(0, \gamma, R)}{R}, \quad \gamma \in (-\infty, +\infty]. \tag{6.2}$$

To show that $\mathrm{Ad}^{\mathrm{hor}}$ is an adequate tool, we consider the case where the lengths of all internal edges ρ_j are commensurable (in the sense of (6.3)) and make the observation that for this case the measure $d\mathrm{Ad}^{\mathrm{hor}}$ generated by the monotone function $\mathrm{Ad}^{\mathrm{hor}}$ consists of a finite number of point masses.

Theorem 6.1. *Assume that* $\Sigma(H_{\mathcal{G}}) \neq \varnothing$ *and*

$$\text{the lengths } \rho_{j_1} \text{ and } \rho_{j_2} \text{ of two arbitrary internal edges of } \mathcal{G} \text{ satisfy } \rho_{j_1}/\rho_{j_2} \in \mathbb{Q}. \tag{6.3}$$

Then there exist $\widetilde{r} \in \mathbb{N}$, $\beta \in (0,+\infty)$, *and infinite sequences* $\{k_{n,t}\}_{t\in\mathbb{Z}} \subset \mathbb{C}$, $n = 1,\ldots,\widetilde{r}$, *that satisfy*

$$\Sigma(H_{\mathcal{G}}) = \bigcup_{n=1}^{\widetilde{r}} \{k_{n,t}\}_{t\in\mathbb{Z}} \setminus \{0\} \qquad \text{and} \qquad \beta k_{n,t} = 2\pi t - \mathrm{i}\,\mathrm{Ln}\,|\xi_n| + \mathrm{Arg}_0\,\xi_n, \tag{6.4}$$

where ξ_n, $n = 1,\ldots,\widetilde{r}$, *are certain complex algebraic numbers satisfying* $|\xi_n| \geq 1$.

Proof. We can assume that the index set J is of the form $\{1,\ldots,N\}$. The function $F(z) = \det A(z)$ that produces the set of resonances as the set of its zeroes $k \neq 0$ is an exponential polynomial obtained by the summation of exp-monomials of the form $(-1)^{\tau_m} K_m \exp\left(\mathrm{i}z \sum_{j\in J_m} (-1)^{\widetilde{\tau}_{m,j}} \rho_j\right)$ (see [14, Theorem 3.1]), where the index m passes through a certain finite index set $\mathcal{M}$, the sequences $\{\tau_m\}_{m\in\mathcal{M}}$, $\{\widetilde{\tau}_{m,j}\}_{m\in\mathcal{M}}^{j\in J_m}$, $\{K_m\}_{m\in\mathcal{M}}$ are subsets of $\mathbb{Z}$, and, for each $m \in \mathcal{M}$, J_m is a subset of J. Writing F in the canonical form we get

$$F(z) = \sum_{l=0}^{\nu} C_l e^{\mathrm{i}b_l z}, \tag{6.5}$$

where $\{C_l\}_{l=0}^{\nu} \subset \mathbb{Z} \setminus \{0\}$ and all b_l, $l = 0,\ldots,\nu$, have the form $\sum_{j\in\widetilde{J}_l} (-1)^{\tau_{l,j}} \rho_j$ with $\widetilde{J}_l \subset J$ and $\tau_{l,j} \in \mathbb{Z}$. We can also assume that the sequence of b_l is strictly increasing.

Supposing now that (6.3) holds, one can see that the arguments similar to that of [8, Section 12.4] are applicable to the function $\widetilde{F}(z) := e^{-\mathrm{i}b_0 z} F(z)$. Indeed, let $\beta > 0$ be such that $\rho_j = \beta d_j$ with $d_j \in \mathbb{N}$ for all $j \in J$. Then the numbers $\widetilde{b}_l := b_l - b_0$ are also commensurable and $\widetilde{b}_l = \beta \widetilde{d}_l$, $l = 0,\ldots,\nu$, with $\widetilde{d}_l \in \mathbb{N}$. Now one can easily obtain the statement of the theorem writing $F(z) = 0$ as $P(e^{\mathrm{i}\beta z}) = 0$, where $P(\cdot)$ is a polynomial with integer coefficients. □

Thus, in the commensurable case of Theorem 6.1, the asymptotic density $\mathrm{Ad}^{\mathrm{hor}}(\cdot)$ is a piecewise constant function with a finite number of jumps at the points $\beta^{-1}\,\mathrm{Ln}\,|\xi_n|$. Depending on the level of noise, the asymptotic sequences with smaller $\mathrm{Ln}\,|\xi_n|$ have more chances to be detected in scattering experiments because they produce narrower resonances.

It is natural to call the resonances k lying on $\mathbb{R}$ *embedded resonances* because the corresponding points $\lambda = k^2$ in the 'energy' plane are embedded into the continuous spectrum $\sigma_{\mathrm{cont}}(H_{\mathcal{G}}) = [0,+\infty)$ of $H_{\mathcal{G}}$. By [14, Theorem 2.3], every embedded resonance k corresponds to an eigenvalue k^2 of $H_{\mathcal{G}}$ embedded into $\sigma_{\mathrm{cont}}(H_{\mathcal{G}})$. The case $c \in \mathbb{Q} \cup [0,1]$ of the example of [14, Section 6] gives $H_{\mathcal{G}}$ with infinite number of embedded resonances (in more general settings, equispaced sequences of embedded resonances have been considered in [21]).

The following general statement describes in the commensurable case the situation when $\mathrm{Ad}^{\mathrm{hor}}(\cdot)$ has a jump at 0.

Corollary 6.2. *Suppose that (6.3) holds. Then the following statements are equivalent:*
(i) $H_{\mathcal{G}}$ *has an infinite number of embedded eigenvalues (and so, of embedded resonances);*
(ii) $H_{\mathcal{G}}$ *has at least one nonzero eigenvalue or at least one embedded resonance;*
(iii) $\min_{1\le n\le \widetilde{r}} |\xi_n| = 1$ *in the settings of Theorem 6.1.*

Proof. The statement follows immediately from Theorem 6.1. □

In the non-commensurable case, the structure of $\Sigma(H_{\mathcal{G}})$ and the structure of the measure $d\mathrm{Ad}^{\mathrm{hor}}$ generated by the monotone function $\mathrm{Ad}^{\mathrm{hor}}(\cdot)$ deserve an additional study, which is connected with the theory of distribution of zeroes of exponential polynomials (see [9] and references therein). The main parameters of high-energy asymptotics for the general case are: (i) the total asymptotic density $\mathrm{Ad}^{\mathrm{hor}}(\gamma^{\max}+0) = \mathrm{Ad}^{\mathrm{hor}}(+\infty)$, which was considered in [14, Theorem 1.2], (ii) the minimum $\gamma^{\min}$ and the maximum $\gamma^{\max}$ of the (closed) support $\mathrm{supp}(d\mathrm{Ad}^{\mathrm{hor}})$ of the measure $d\mathrm{Ad}^{\mathrm{hor}}$, (iii) $\#\{k \in \Sigma(H_{\mathcal{G}}) : \mathrm{Im}\, k > -\gamma^{\min}\}$, and (iv) $\lim_{\delta\to+0} \frac{\mathrm{Ad}^{\mathrm{hor}}(\gamma^{\min}+\delta)-\mathrm{Ad}^{\mathrm{hor}}(\gamma^{\min})}{\delta}$.

It is possible to strengthen slightly [14, Theorem 1.2] by the following observation.

Corollary 6.3. *Assume that* $\Sigma(H_{\mathcal{G}}) \neq \varnothing$. *Then* $\Sigma(H_{\mathcal{G}})$ *consists of an infinite number of resonances and their asymptotic density* $\mathrm{Ad}^{\mathrm{hor}}(+\infty)(= \lim_{R\to+\infty} \frac{\mathfrak{N}_{H_{\mathcal{G}}}(R)}{R})$ *is positive.*

Proof. It follows from $\Sigma(H_{\mathcal{G}}) \neq \varnothing$ that $\nu \ge 1$ in (6.5). Then the desired statement can be easily obtained from [8, Theorem 12.5] (see also [14, Theorem 3.2] and references in [13, 14]). □

6.1.2 The case of general self-adjoint local coupling

In this subsection the local case of a more general Kostrykin-Schrader-Harmer coupling [35, 26] (in short, local KSH-coupling) is considered with the use of the notation and settings of [13] (see also [39] for a detailed exposition and the literature review).

Let $|\mathcal{G}|$ be the number of vertices of the quantum graph $\mathcal{G}$ and $\{X_n\}_{n=1}^{|\mathcal{G}|}$ be the set of the vertices. Let $\deg X_n$ be the degree of the vertex X_n, i.e, the number of edges connected to X_n.

The quantum graph Hamiltonian $H = H_{\mathcal{G},U}$ and the multiset of its resonances $\Sigma(H_{\mathcal{G},U})$ are defined similarly to Section 6.1.1, but with the Kirchhoff and continuity conditions replaced at each X_i by the local KSH-coupling. The latter means that, at each vertex X_n, the condition $(U_n - I)\Psi_n + \mathrm{i}(U_n + I)\Psi'_n = 0$ is satisfied with a certain unitary $\deg X_n \times \deg X_n$ matrix U_n, where the vector Ψ_n consists of the limits of $f(\cdot)$ at X_n along every edge $\mathcal{E}_j$ connected to X_n, and the vector Ψ'_n consists of the corresponding limits of outwards derivatives. The unitary matrix U in the notation $H_{\mathcal{G},U}$ is composed of the diagonal blocks U_n, see [13, 39] for details.

The next theorem states, roughly speaking, that generally the structure of $\Sigma(H_{\mathcal{G},U})$ is the combination of the two types considered in Section 6.1.1 (with the Kirchhoff and continuity coupling) and Section 3 (for point-interactions).

Let $\widetilde{\mathcal{K}}(\gamma) := \{k \in \Sigma(H_{\mathcal{G},U}) \ : \ \operatorname{Im} k \geq -\gamma\}$ and let $\mu^{\max}$ be the supremum of the support of the measure $d\mathrm{Ad}^{\log}$ for the Hamiltonian $H_{\mathcal{G},U}$ ($\mu^{\max}$ is taken to be equal to $-\infty$ if $\mathrm{Ad}^{\log}(\cdot) \equiv 0$ on $\mathbb{R}$).

Theorem 6.4. *(i) If $\mu^{\max} \leq 0$, then there exists $\widetilde{\gamma} \in \mathbb{R}$ such that $\Sigma(H_{\mathcal{G},U}) = \widetilde{K}(\widetilde{\gamma})$.*
(ii) Assume that $\mu^{\max} > 0$. Then there exists $\widetilde{\gamma} \in \mathbb{R}$, $M \in \mathbb{N}$, a strictly decreasing finite sequence $\{\mu_n\}_{n=1}^M \subset \mathbb{R}_+$, finite sequences $\{r_n\}_{n=1}^M \subset \mathbb{N}$, $\{\omega_{n,j}\}_{j=1}^{r_n} \subset \mathbb{C} \setminus \{0\}$, $\{t_{n,j}^{\pm}\}_{j=1}^{r_n} \subset \mathbb{N}$ for $n = 1,\dots,M$, and infinite sequences $\mathcal{K}_{n,j}^{\pm} = \{k_{n,j,t}^{\pm}\}_{t=t_{n,j}^{\pm}}^{+\infty} \subset \mathbb{C}$, $n = 1$, $\dots$, M, $j = 1$, $\dots$, r_n, with the following properties:

(ii.a) $\Sigma(H_{\mathcal{G},U}) = \widetilde{K}(\widetilde{\gamma}) \cup \bigcup_{n=1}^M \mathcal{K}_n$ (taking into account multiplicities), where $\mathcal{K}_n := (\bigcup_{j=1}^{r_n} \mathcal{K}_{n,j}^-) \cup (\bigcup_{j=1}^{r_n} \mathcal{K}_{n,j}^+)$ for $n = 1$, $\dots$, M.

(ii.b) Each of $\mathcal{K}_{n,j}^{\pm}$ has the asymptotics (3.11) as $t \in \mathbb{N}$ goes to $+\infty$.

(ii.d) If the set $\widetilde{K}(\widetilde{\gamma})$ is infinite, then

$$\lim_{R\to\infty} \frac{\#\{k \in \widetilde{K}(\widetilde{\gamma}) \ : \ |k| \leq R\}}{R} = \mathrm{Ad}^{\mathrm{hor}}(\widetilde{\gamma}+0) = \mathrm{Ad}^{\mathrm{hor}}(+\infty) > 0.$$

(iii) Let additionally the commensurability condition (6.3) hold. Then in each of the cases (i) and (ii) the set $\widetilde{\mathcal{K}}(\widetilde{\gamma})$ is either finite, or satisfies the following property: there exist numbers $\beta > 0$, $\widetilde{r} \in \mathbb{N}$, $M_0 \in \mathbb{N}$, finite sequences $\{\xi_n\}_{n=1}^{\widetilde{r}} \subset \mathbb{C} \setminus \mathbb{D}_1(0)$, $\{\widetilde{t}_n\}_{n=1}^{\widetilde{r}} \subset \mathbb{N}$, $\widetilde{\mathcal{K}}_0 := \{\widetilde{k}_t\}_{t=1}^{M_0}$, and infinite sequences $\{\widetilde{k}_{n,t}^{\pm}\}_{t=\widetilde{t}_n}^{+\infty} \subset \mathbb{C}$ such that

$$\widetilde{\mathcal{K}}(\widetilde{\gamma}) = \widetilde{\mathcal{K}}_0 \cup \bigcup_{n=1}^{\widetilde{r}} \left(\{\widetilde{k}_{n,t}^-\}_{t=\widetilde{t}_{n,j}}^{+\infty} \cup \{\widetilde{k}_{n,t}^+\}_{t=\widetilde{t}_{n,j}}^{+\infty} \right) \quad \textit{and}$$
$$\beta \widetilde{k}_{n,t}^{\pm} = \pm 2\pi t - \mathrm{i}\,\mathrm{Ln}\,|\xi_n| + \mathrm{Arg}_0\,\xi_n + o(1) \qquad \textit{as } t \to +\infty. \tag{6.6}$$

Proof. A function $F(z)$ such that $k \in \Sigma(H_{\mathcal{G},U})$ if and only if $k \neq 0$ and $F(k) = 0$ is constructed in [13]. Moreover, the multiplicities of the resonances are the multiplicities of the corresponding zeros of F. Let us consider a modified version of F given by $\widetilde{F}(\zeta) = F(-\mathrm{i}\zeta)$. Then [13, Theorem 3.1] implies that $\widetilde{F}(\zeta)$ is an exponential polynomial of the form (3.1) with a strictly increasing sequence of frequencies $\widetilde{\beta}_j$ that are not necessarily nonpositive. Now the function $D(\zeta) = e^{-\widetilde{\beta}_\nu} \widetilde{F}(\zeta)$ has the same zeros as $\widetilde{F}(\zeta)$ and has the form (3.1) with a strictly increasing sequence of frequencies β_j so that $\beta_\nu = 0$. The construction of the distribution diagram given in Section 3.1 and in more details in [8] applies to $D(\cdot)$. In particular, in the terminology of [8], the zeros of $D(\cdot)$ lie in a finite number of logarithmic curvilinear strips and possibly one neutral (horizontal) strip $\{|\operatorname{Im} z| \leq \gamma_0\}$ (this is the statement of [13, Theorem 3.1]). The logarithmic strips contain a infinite number of resonances and are necessarily retarded because the self-adjoint and lower semi-bounded from below operator $H_{\mathcal{G},U}$ has at most a finite number of resonances in $\mathbb{C}_+$. Thus, the arguments of Section 3.2 applied to the zeroes of D in logarithmic strips and the arguments of [8, Sections 12.4-6] applied to the zeros of D in the neutral strip easily complete the proof of the theorem. □

We see now that the two level asymptotic structure of $\Sigma(H_{\mathcal{G},U})$ is captured by the asymptotic density function $\mathrm{Ad}(\mu,\gamma)$ of (6.1). The measure $d\mathrm{Ad}^{\log}(\cdot) = d\mathrm{Ad}(\cdot,0)$ has point masses at the numbers μ_n corresponding to the logarithmic asymptotic sequences and, in the case where the strip $\{|\operatorname{Im} z| \le \widetilde{\gamma}\}$ contains an infinite number of resonances, it has also a point mass at 0. If the infimum $\mu^{\min}$ of the support of the measure $d\mathrm{Ad}^{\log}(\cdot)$ is equal to 0, then the measure $d\mathrm{Ad}(0,\cdot) = d\mathrm{Ad}^{\mathrm{hor}}(\cdot)$ is responsible for the internal structure of $\Sigma(H_{\mathcal{G},U})$ in the strip $\{|\operatorname{Im} z| \le \widetilde{\gamma}\}$ and for high-energy asymptotics of narrow resonances.

If $\mu^{\min} > 0$, then $\mu^{\min} = \mu_M$. The high-energy asymptotics of 'physical resonances' is described in this case by the sequences $\mathcal{K}^{\pm}_{M,j}$, $j = 1, \ldots, r_M$, and, on a more rough level, by the asymptotic density $\mathrm{Ad}^{\log}(\mu_M + 0) - \mathrm{Ad}^{\log}(\mu_M - 0)$ in the corresponding logarithmic strip.

6.2 Resonances of 1-D photonic crystals

A typical 1-D photonic crystal (a multi-layer optical cavity) is described by the variable dielectric permittivity $\varepsilon(x) > 0$, $x \in \mathbb{R}$, which is a piecewise constant function on $\mathbb{R}$ with a finite number of steps. That is, there exists a finite partition $-\infty = x_{-1} < x_0 < \cdots < x_N < x_{N+1} = +\infty$ with $N \in \mathbb{N}$ and constants $\epsilon_j \in \mathbb{R}_+$ such that $\varepsilon(x) = \epsilon_j$ for all $x \in (x_{j-1}, x_j)$, $n = 0, \ldots, N+1$. For $1 \le j \le N$, the intervals (x_{j-1}, x_j) represents idealized infinite plane layers of a material with the permittivity ϵ_j. The semi-infinite intervals $(-\infty, x_0)$ and $(x_N, +\infty)$ represents the homogeneous outer medium (it is assumed often that the corresponding permittivities ϵ_0 and ϵ_{N+1} are equal, however this is not important for this section).

For electromagnetic waves that pass normally to the interfaces of the layers, the Maxwell system can be reduced to the wave equation for a nonhomogeneous string $\varepsilon(x)\partial_t^2 v(x,t) = \partial_s^2 v(x,t)$ (see [34] and references therein). The corresponding operator $H_\varepsilon := -\frac{1}{\varepsilon(x)}\partial_x^2$ is self-adjoint and nonnegative in the weighted Hilbert space $L^2_{\mathbb{C}}(\mathbb{R}; \varepsilon(x)dx)$.

The resonances of H_ε are the complex numbers $k \neq 0$ such that there exists a nontrivial solution f to the equation $f'' = -k^2\varepsilon(x)f(x)$ satisfying radiation conditions on the outer intervals $(-\infty, x_0)$ and $(x_N, +\infty)$. The latter means that $f(x) = f(x_0)e^{-\mathrm{i}k(x-x_0)\epsilon_0^{1/2}}$ for $x < x_0$ and $f(x) = f(x_N)e^{\mathrm{i}k(x-x_N)\epsilon_{N+1}^{1/2}}$ for $x > x_N$. The multiplicity of a resonance k can be defined as the multiplicity of the zero at k of the corresponding Keldysh characteristic determinant (see [32, 34]) or, equivalently, via the algebraic multiplicity of the eigenvalue k of a special operator [12]. The multiplicity of each resonance is finite.

The approach to resonances through the corresponding Keldysh characteristic determinant is essentially equivalent to the approach of [14] with the determinant of a matrix-valued function $A(z)$ constructed by the coupling conditions. Indeed, we can consider a 1-D photonic crystal as a *weighted quantum graph* with the simple linear connectivity. It consists of edges $[x_{j-1}, x_j]$, which are internal for $j = 1, \ldots, N$ and external for $j = 0$ and $j = N+1$, equipped with the differential expressions $\frac{1}{\epsilon_j}\partial_x^2$ with the constant coefficient $1/\epsilon_j$. The coupling of the graph is given by the conditions of continuity of f and f', $f(x_j-0)=f(x_j+0)$ and $f'(x_j-0)=f'(x_j+0)$, $j=0,\dots,N$. Note that the latter condition is a simple version of Kirchhoff's condition for the case when only two edges go out of a vertex x_j. (This weighted quantum graph slightly does not fit into the class of the weighted graphs of [13, Section 8] because its coupling conditions for the derivatives are different.)

Theorem 6.5. *Assume that the multiset of resonances $\Sigma(H_\varepsilon)$ of H_ε is nonempty. Then $\#\Sigma(H_\varepsilon)=\infty$ and the following statements hold:*

(i) There exists $\gamma_0>0$ such that $\Sigma(H_\varepsilon)\subset\{z\in\mathbb{C} \ : -\gamma_0\le \operatorname{Im} z<0\}$ and $0<\operatorname{Ad}^{\mathrm{hor}}(\gamma_0)<+\infty$, where $\operatorname{Ad}^{\mathrm{hor}}(\cdot)$ is the asymptotic density function defined by (6.2) for $H=H_\varepsilon$.

(ii) Assume, additionally, that $\frac{(x_j-x_{j-1})\epsilon_j^{1/2}}{(x_{j+1}-x_j)\epsilon_{j+1}^{1/2}}\in\mathbb{Q}$ for $1\le j\le N-1$. Then there exist numbers $\beta>0$, $r\in\mathbb{N}$, and infinite sequences $\{k_{n,t}\}_{t\in\mathbb{Z}}\subset\mathbb{C}$, $n=1,\dots,r$, such that

$$\Sigma(H_\varepsilon)=\bigcup_{n=1}^{r}\{k_{n,t}\}_{t\in\mathbb{Z}} \quad \textit{and} \quad \beta k_{n,t}=2\pi t-\mathrm{i}\operatorname{Ln}|\xi_n|+\operatorname{Arg}_0\xi_n,$$

where ξ_n, $n=1,\dots,r$, are certain complex numbers satisfying $|\xi_n|>1$.

Proof. To obtain the Keldysh characteristic determinant $F(z)$, the construction of the matrix-valued function $A(z)$ from [14] can be applied resulting in $F(z)=\det A(z)$. Since in this process the derivatives $f'(x_j-0)$ are coupled only with the derivatives $f'(x_j+0)$, the multiplicative factors $\mathrm{i}z$ are eliminated from $A(z)$ (similarly to [14] and the definition of resonances in [12, 32, 34]). Hence, $F(z)$ takes the form (6.5) with certain $C_l\in\mathbb{R}\setminus\{0\}$. Thus, the arguments of [14], Theorem 6.1, and Corollary 6.3 can be applied to obtain all the statements of the theorem except the statement that $\Sigma(H_\varepsilon)\cap\mathbb{R}=\varnothing$, which is well-known. □

7 Discussion on other classes of m-D Hamiltonians

We have described two structural levels of the asymptotic distribution of the set of resonances $\Sigma(H)$ for 3-D Schrödinger Hamiltonians $H=H_{a,Y}$ with point interactions and have shown how these results can be adapted to quantum graphs $H_{\mathcal{G},U}$. This allows us to define parameters corresponding to the asymptotics of the 'physically relevant' part of $\Sigma(H)$, which consists of resonances lying in the logarithmic or horizontal strip which situated closest to the real line. The method relies on the fact that, for these two classes, resonances are zeros of exponential polynomials.

One can notice similarities between some of the features of the asymptotic structures described above and the known facts about resonances (or scattering poles) for Dirichlet Laplacians $H_{\mathcal{O}}=-\Delta$ arising in obstacle scattering. The study of resonances of $H_{\mathcal{O}}$ requires more involved analytical tools and the presently available results on the asymptotic structure of the set $\Sigma(H_{\mathcal{O}})$ of resonances of $H_{\mathcal{O}}$ are obtained under various additional assumptions on the types of obstacles (see reviews in [17, 28, 55, 56]). Using the terminology of the present paper, the works [29, 18, 48, 49, 28] give the description of one structural level for the intersection of $\Sigma(H_{\mathcal{O}})$ with a certain horizontal or semi-cubic strip adjacent to the real line.

In more details, the case of two strictly convex obstacles [29, 18, 28] resembles to some extend the case where a quantum graph has $\mu^{\min} = 0$ and commensurable lengths of edges (see Section 6.1.2) because in the both cases there exists a horizontal strip containing an infinite number of resonances composing asymptotically horizontal sequences. One of the differences between these two cases is that, for a quantum graph, there exists a horizontal strip $\{|\operatorname{Im} z| \leq \widetilde{\gamma}\}$ such that the consideration of any wider horizontal strip adds only a finite number of resonances, while in the case of two strictly convex obstacles [18, 28], taking wider and wider horizontal strips one obtains more and more additional asymptotically horizontal sequences.

Reading the works on the case of one strictly convex obstacle $\mathcal{O}$, one could guess in [47, 48, 49] (see also references therein and [30]) a program on the study of the structure of $\Sigma(H_{\mathcal{O}})$ in regions adjacent to $\mathbb{R}$. One of the instruments in this program is the counting function in various shaped strips

$$\widetilde{\mathfrak{N}}_\varphi(R) = \#\{k \in \Sigma(H) \ : \ -\varphi(|\operatorname{Re} k|) \leq \operatorname{Im} k, \ |k| \leq R\}, \tag{7.1}$$

where the function $\varphi : [0, +\infty) \to \mathbb{R}$ describes the shape of the strip $\{-\varphi(|\operatorname{Re} k|) \leq \operatorname{Im} k \leq 0\}$ (see [48]). In particular, [48, 49] work with functions φ of the form $\varphi(\xi) = \varphi_\mu(\xi) = \mu \xi^{1/3}$, $\mu \geq 0$; let us denote the corresponding family of counting functions $\widetilde{\mathfrak{N}}_\varphi(R)$ by $\mathfrak{N}^{\mathrm{cub}}(\mu, \cdot)$. Introducing similarly to (1.4) the asymptotic density functions for cubic semi-strips

$$\operatorname{Ad}^{\mathrm{cub}}_\alpha(\mu) := \limsup_{R \to +\infty} \frac{\mathfrak{N}^{\mathrm{cub}}(\mu, R)}{R^\alpha}, \quad \mu \in \mathbb{R},$$

where an additional parameter $\alpha > 0$ takes into account the possible polynomial growth [41, 53, 47, 49], we can infer from the results of [25, 48, 49] that, in the case of a strictly convex obstacle $\mathcal{O} \subset \mathbb{R}^m$ satisfying additional pinching conditions of [49] on curvatures of the boundary of $\mathcal{O}$, the support of the measure $d\operatorname{Ad}^{\mathrm{cub}}_{m-1}(\mu)$ is separated from 0 and there exists a partition $0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_{2n+1} < \mu_{2n+2} = +\infty$ such that $\operatorname{supp} d\operatorname{Ad}^{\mathrm{cub}}_{m-1}(\mu) \cap [\mu_{2j-1}, \mu_{2j}] \neq \varnothing$ and $\operatorname{supp} d\operatorname{Ad}^{\mathrm{cub}}_{m-1}(\mu) \cap [\mu_{2j}, \mu_{2j+1}] = \varnothing$, $j = 0, \ldots, n$. This suggests that, from the point of view of the asymptotics of 'narrow resonances' (see Section 5), the two following constants have to play a special role: $\mu^{\min} = \inf \operatorname{supp} d\operatorname{Ad}^{\mathrm{cub}}_{m-1}(\mu)$ and $\lim_{\mu \to \mu^{\min}+0} \frac{\operatorname{Ad}^{\mathrm{cub}}_{m-1}(\mu)}{\mu - \mu^{\min}}$.

It seems that very little is known about the internal structure of $\Sigma(H)$ for multi-dimensional Schrödinger operators $H = -\Delta + V$ in $L^2_{\mathbb{C}}(\mathbb{R}^m)$ in the case of real-valued compactly supported potentials $V \in L^\infty_{\mathbb{R},\mathrm{comp}}(\mathbb{R}^m)$ and an odd number $m > 1$. Most of studies of this case were concentrated on the asymptotics of $\mathfrak{N}_H(R)$ for $R \to \infty$. On one hand, it follows from [54] that $\limsup_{R \to \infty} \mathfrak{N}_H(R)/R^m < +\infty$ for every $V \in L^\infty_{\mathbb{R},\mathrm{comp}}(\mathbb{R}^m)$, and from [11] that $\limsup_{R \to \infty} \ln \mathfrak{N}_H(R)/\ln R = m$ for generic V in the same class. However, the best known lower bound for nontrivial smooth $V(\cdot)$ is

$\limsup_{R\to\infty} \frac{\mathfrak{N}_H(R)}{R} > 0$ [45]. A very stimulating and intriguing discussion of the existing gap between known upper and lower bounds on the growth of $\mathfrak{N}_H(R)$ can be found in [56, Section 2.7]. It seems that an example of a nontrivial $V \in L^\infty_{\mathbb{R},\mathrm{comp}}(\mathbb{R}^m)$ with $\limsup_{R\to\infty} \ln \mathfrak{N}_H(R)/\ln R < m$ is not known (see [56, Conjecture 1 in Section 2.7]). Looking from this point of view on the point interaction Hamiltonians $H_{a,Y}$, one sees that, loosely speaking, the lower bound of [45] is achieved on them in the sense that $\lim_{R\to\infty} N_{H_{a,Y}}(R)/R \in \mathbb{R}_+$ [40]. While $H_{a,Y}$ do not belong to the class of operators $H = -\Delta + V$ with $V \in L^\infty_{\mathbb{R},\mathrm{comp}}(\mathbb{R}^3)$, it is reasonable to test on them any prospective method of proving the equality $\limsup_{R\to\infty} \ln \mathfrak{N}_H(R)/\ln R = m$, which was conjectured in [56, Section 2.7].

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References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, The low energy expansion in nonrelativistic scattering theory, Ann. Inst. H. Poincaré Sect. A (N.S.) 37 (1982), no.1, 1–28.

[2] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden, Solvable models in quantum mechanics. 2nd edition, with an appendix by P. Exner. AMS Chelsea Publishing, Providence, RI, 2012.

[3] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, L. Streit, Charged particles with short range interactions, Ann. Inst. H. Poincaré Sect. A (N.S.) 38(1983), no.3, 263–293.

[4] S. Albeverio, R. Høegh-Krohn, Perturbation of resonances in quantum mechanics, J. Math. Anal. Appl. 101 (1984), 491–513.

[5] S. Albeverio, I.M. Karabash, Resonance free regions and non-Hermitian spectral optimization for Schrödinger point interactions, Operators and Matrices 11 (2017), no.4, 1097–1117.

[6] S. Albeverio, I.M. Karabash, Generic asymptotics of resonance counting function for Schrödinger point interactions, to appear in "Analysis as a tool in Mathematical Physics: in Memory of B. Pavlov", ed. Kurasov, P., Laptev, A., Naboko, S., Simon, B., to be published by Birkhäuser, 2020; see also the preprint, arXiv:1803.06039, 2018.

[7] S. Albeverio, P. Kurasov. Singular perturbations of differential operators: solvable Schrödinger-type operators. Cambridge University Press, 2000.

[8] R.E. Bellman, K.L. Cooke, Differential-difference equations. Academic Press. New York, London. 1963.

[9] C.A. Berenstein, R. Gay, Complex analysis and special topics in harmonic analysis. Springer Science & Business Media, 2012.

[10] G. Berkolaiko, P. Kuchment, Introduction to quantum graphs. American Mathematical Soc., 2013.

[11] T.J. Christiansen, P.D. Hislop, The resonance counting function for Schrödinger operators with generic potentials, Math. Res. Lett. 12(6) (2005), 821–826.

[12] S. Cox, E. Zuazua, The rate at which energy decays in a string damped at one end, Indiana Univ. Math. J. 44 (1995), no.2, 545–573.

[13] E.B. Davies, P. Exner, J. Lipovský, Non-Weyl asymptotics for quantum graphs with general coupling conditions, J. Phys. A 43 (2010), 474013, 16 p.

[14] E.B. Davies, A. Pushnitski, Non-Weyl resonance asymptotics for quantum graphs, Analysis & PDE 4 (2011), 729–756.

[15] Y.N. Demkov, P.B. Kurasov, Von Neumann-Wigner theorem: Level repulsion and degenerate eigenvalues, Theoretical and Mathematical Physics 153(1) (2007), 1407-1422.

[16] Y.C. de Verdière, F. Truc. Topological resonances on quantum graphs. Ann. Henri Poincaré 19 (2018), no.5, 1419–1438.

[17] S. Dyatlov, M. Zworski, Mathematical theory of scattering resonances, to be published by American Mathematical Soc..

[18] C. Gérard, Asymptotique des pôles de la matrice de scattering pour deux obstacles strictement convexes. Mém. Soc. Math. France (N.S.), no.31 (1988), 146 pp.

[19] V. Enss, Summary of the conference and some open problems. In: Albeverio S., Ferreira L.S., Streit L. (eds.) "Resonances – Models and Phenomena". Springer, Berlin, Heidelberg, 1984.

[20] P. Exner, Open quantum systems and Feynman integrals. Springer Science & Business Media, Berlin, 2012.

[21] P. Exner, J. Lipovský, Resonances from perturbations of quantum graphs with rationally related edges, J. Phys. A 43(10) (2010), p.105301.

[22] P. Exner, J. Lipovský, Pseudo-orbit approach to trajectories of resonances in quantum graphs with general vertex coupling: Fermi rule and high-energy asymptotics, J. Math. Phys. 58(4) (2017), p.042101.

[23] R. Froese, Upper bounds for the resonance counting function of Schrödinger operators in odd dimensions, Canad. J. Math. 50 (1998), no.3, 538–546.

[24] S. Gnutzmann, H. Schanz, U. Smilansky, Topological resonances in scattering on networks (graphs), Physical Review Letters 110(9) (2013), p.094101.

[25] T. Harge, G. Lebeau, Diffraction par un convexe, Invent. Math. 118(1) (1994), 161–196.

[26] M. Harmer, Hermitian symplectic geometry and extension theory, J. Phys. A 33(50), (2000), p.9193.

[27] M. Holzmann, V. Lotoreichik, Spectral analysis of photonic crystals made of thin rods, Asymptotic Analysis 110 (1-2) (2018), 83–112.

[28] A. Iantchenko, Scattering poles near the real axis for two strictly convex obstacles, Ann. Henri Poincaré 8 (2007), no.3, 513–568.

[29] M. Ikawa, On the poles of the scattering matrix for two strictly convex obstacles, J. Math. Kyoto Univ. 23(1) (1983), 127–194.

[30] Long Jin, Scattering resonances of convex obstacles for general boundary conditions, Comm. Math. Phys. 335(2) (2015), 759–807.

[31] A. Intissar, A polynomial bound on the number of scattering poles for a potential in even dimensional space R^n, Comm. Partial Differential Equations 11 (1986), no.4, 367–396.

[32] I.M. Karabash, Optimization of quasi-normal eigenvalues for 1-D wave equations in inhomogeneous media; description of optimal structures, Asymptotic Analysis 81 (2013) no.3-4, 273-295.

[33] I.M. Karabash, Pareto optimal structures producing resonances of minimal decay under L^1-type constraints, J. Differential Equations 257 (2014), 374–414.

[34] I.M. Karabash, O.M. Logachova, I.V. Verbytskyi, Nonlinear bang-bang eigenproblems and optimization of resonances in layered cavities, Integr. Equ. Oper. Theory 88(1) (2017), 15–44.

[35] V. Kostrykin, R. Schrader, Kirchhoff's rule for quantum wires, J. Phys. A 32(4) (1999), p.595.

[36] A.O. Kravitsky, The two-fold expansion of a certain non-selfadjoint boundary value problem in series of eigenfunctions, Differentsial'nye Uravneniya 4(1) (1968), 165–177 (Russian).

[37] S. Lang, Algebra. Springer-Verlag, New York, 2002.

[38] Minjae Lee, M. Zworski, A Fermi golden rule for quantum graphs, J. Math. Phys. 57(9), (2016), p.092101.

[39] J. Lipovský, Quantum Graphs And Their Resonance Properties, Acta Physica Slovaca 66(4) (2016), 265–363.

[40] J. Lipovský, V. Lotoreichik, Asymptotics of Resonances Induced by Point Interactions, Acta Physica Polonica A 132 (2017), 1677–1682.

[41] R.B. Melrose, Polynomial bound on the number of scattering poles, J. Funct. Anal. 53 (1983), no.3, 287–303.

[42] V.N. Pivovarchik, Inverse problem for a smooth string with damping at one end, J. Operator Theory 38 (1997), no.2, 243–263.

[43] O. Post, Spectral analysis on graph-like spaces. Springer Science & Business Media, 2012.

[44] M. Reed, B. Simon, Analysis of Operators, Vol. IV of Methods of Modern Mathematical Physics. Academic Press, New York, 1978.

[45] A. Sá Barreto, Remarks on the distribution of resonances in odd dimensional Euclidean scattering. Asymptotic Analysis 27(2), (2001), 161–170.

[46] M.A. Shubov, Basis property of eigenfunctions of nonselfadjoint operator pencils generated by the equation of nonhomogeneous damped string, Integr. Equ. Oper. Theory 25(3) (1996), 289–328.

[47] J. Sjöstrand, M. Zworski, Estimates on the number of scattering poles near the real axis for strictly convex obstacles, Ann. Inst. Fourier 43(3), (1993), 769–790.

[48] J. Sjöstrand, M. Zworski, The complex scaling method for scattering by strictly convex obstacles, Arkiv för Matematik 33(1), (1995), 135–172.

[49] J. Sjöstrand, M. Zworski, Asymptotic distribution of resonances for convex obstacles, Acta Mathematica 183(2), (1999), 191–253.

[50] P. Stefanov, Sharp upper bounds on the number of the scattering poles, J. Funct. Anal. 231(1) (2006), 111–142.

[51] G. Vodev, Sharp bounds on the number of scattering poles in even-dimensional spaces, Duke Math. J. 74(1) (1994), 1–17.

[52] M. Zworski, Distribution of poles for scattering on the real line, J. Funct. Anal. 73(2) (1987), 277–296.

[53] M. Zworski, Sharp polynomial bounds on the number of scattering poles of radial potentials, J. Funct. Anal. 82 (1989), no.2, 370-403.

[54] M. Zworski, Sharp polynomial bounds on the number of scattering poles, Duke Math. J. 59(2), (1989), 311–323.

[55] M. Zworski, Resonances in physics and geometry, Notices of the AMS 46(3) (1999), 319–328.

[56] M. Zworski, Mathematical study of scattering resonances, Bull. Math. Sci. 7(1) (2017), 1–85.