QUASI-COMPACT GROUP SCHEMES, HOPF SHEAVES, AND THEIR REPRESENTATIONS

ALVARO RITTATORE, PEDRO LUIS DEL ÁNGEL, AND WALTER FERRER SANTOS

Abstract. We explore the notion of *representation of an affine extension of an abelian variety* — such an extension is a faithfully flat affine morphism of $\Bbbk$–group schemes $q : G \to A$, where A is an abelian variety. We characterize the categories that arise as the category of representations of an affine extension $q : G \to A$, generalizing the classical results of Tannaka Duality established for affine $\Bbbk$–group schemes (that is, when $A = \operatorname{Spec}(\Bbbk)$). We also prove the existence of a contravariant equivalence between the category of affine extensions of a given A and the category of *faithful commutative Hopf sheaves* on A, generalizing in this manner the well-known op–equivalence between affine group schemes and commutative Hopf algebras. If $\mathcal{H}_q$ is the Hopf sheaf on A associated to q, the category of representations of q is equivalent to the category of $\mathcal{H}_q$–comodules.

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Contents

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1. Introduction

Roughly speaking, given a certain family of objects $\mathcal{R}$ (the "representable objects") and a fixed basic monoidal category $\mathcal{C}$, a "representation theory" consists in the association to an element $r \in \mathcal{R}$, of a pair $\big(\operatorname{Rep}(r),\, U : \operatorname{Rep}(r) \to \mathcal{C}\big)$, where $\operatorname{Rep}(r)$ is a monoidal category — the *category of representations* of r —, and U a monoidal functor (the *forgetful functor*) — eventually with certain additional properties depending on the situation under consideration. One aspires to "reconstruct" each $r \in \mathcal{R}$ in terms of the corresponding pair $\big(\operatorname{Rep}(r), U\big)$, and also to describe intrinsically all the pairs $(\mathcal{D}, U : \mathcal{D} \to \mathcal{C})$ that are equivalent to pairs of the above form for some r. For example, such a platform has been developed within the following frameworks: categories of groups (abstract, topological, Lie, affine algebraic, differential); of general algebras; of Lie algebras, Hopf algebras. In this context the notion of *Tannaka Duality* is generally presented as an answer to the following questions (see [38], [53] or [58] for a precise formulation):

The Reconstruction Problem: can a representable object be described in terms of its category of representations?

The Recognition Problem: can a category of representations be described intrinsically?

It is worth mentioning that the theory of Tannaka Duality was generalized to a categorical context: the relevant concept of "tannakian adjunction" was developed and some of the classical results were generalized and clarified (see [58] and [38]).

In the case of affine algebraic group schemes over a field (even over a commutative ring) the theory of *rational representations* has achieved a considerable degree or maturity and many of its main problems have been solved and important advances have been done in the theory of its actions on general schemes. Examples of significant accomplishments in the area are: the completion of the structure theory of reductive affine group schemes (see [20]), the development of the geometric methods in invariant theory (see [44]), or more generally the theory of transformation groups (see [39]). In particular both the reconstruction and recognition questions were answered positively: Saavedra first presented a proof in [50] which was later observed to have some mistakes. A correct proof of the result was produced afterwards by Deligne and Milne in [21]; see also [22].

Temporarily, call $\operatorname{Rep}(G)$, the category of finite dimensional rational representations of an affine group scheme; it is well known that $\operatorname{Rep}(G)$ is monoidal, rigid, abelian and $\Bbbk$-linear. Denote as $\omega : \operatorname{Rep}(G) \to \operatorname{Vect}_{\Bbbk}$ the corresponding monoidal forgetful functor. In this context Saavedra-Deligne-Milne's result can be stated as follows (see also [38], [53] or [58] for other perspectives of the same problem):

Theorem (Tannaka Duality for affine group schemes, [21, Prop. 2.8, Thm. 2.11])**.**

(1) Reconstruction Theorem. *Let G be an affine group scheme. Consider the pair* $(\operatorname{Rep}(G), \omega)$ *and the group scheme of tensor automorphisms of ω, denoted as* $\operatorname{Aut}^{\otimes}(\omega)$ *— see [21, page 20] for a definition of this group and compare with Definition 3.66 below. Then* $\operatorname{Aut}^{\otimes}(\omega)$ *is an affine group scheme isomorphic to G.*

In particular, if G and G' are affine group schemes such that $(\operatorname{Rep}(G), \omega)$ *and* $(\operatorname{Rep}(G'), \omega')$ *are equivalent as monoidal $\Bbbk$–linear categories with a forgetful functor, then G and G' are isomorphic group schemes.*

(2) Recognition Theorem. *Let $\mathcal{C}$ be a monoidal, abelian, rigid $\Bbbk$–linear category such that $\Bbbk = \operatorname{End}(\mathbb{I})$, together with an exact, faithful, $\Bbbk$–linear monoidal functor $\omega : \mathcal{C} \to$* $\operatorname{Vect}_{f,\Bbbk}$*. Then $(\mathcal{C}, \omega)$ is equivalent (as a monoidal category with forgetful functor) to the category of rational representations of the affine group scheme* $\operatorname{Aut}^{\otimes}(\omega)$*.*

Let G be a group scheme of finite type (see Definition 2.1); it is easy to see that the naive attempt to define the category of representations of G as a direct generalization of the affine situation, yields a category which does not fulfill our needs, as it is too small to determine G — for example, if G is an anti-affine group (see Definition 2.36), then the only morphism of group schemes $G \to \operatorname{GL}(V)$ is the trivial morphism.

Motivated by previous work of Brion, Rittatore and others on the structure of group and monoid schemes (see for example [10], [11], [12], [14]) and on their actions ([8], [15]), and taking into account the mentioned obstruction to the naive approach, we propose a representation theory *not* for isolated group schemes, but for what we call *affine extensions* of abelian varieties. Roughly speaking, an affine extension is a generalization of the so-called Chevalley decomposition for algebraic groups (see Theorem 2.26): an affine extension $\mathcal{S}$ is an exact sequence of group schemes $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$, where A is an abelian variety and H is an affine group scheme — equivalently, q is an affine, faithfully flat morphism of quasi-compact group schemes and $H = \operatorname{Ker}(q)$ (see definitions 2.12 and 2.15). It follows that $q : G \to A$ is an H–torsor (see definitions 2.11–2.15). A *representation* for $\mathcal{S}$ is built on an homogeneous vector bundle over A — by homogeneous we mean that vector bundle $E \to A$ is such that $E_{\overline{\Bbbk}} \cong t_a^* E_{\overline{\Bbbk}}$ for any translation $t_a : A \to A$

by a geometric point $a \in A(\overline{\Bbbk})$ (see [18] and Definition 3.9 for a more conceptual and precise definition).

The basic nomenclature of the paper as well as the necessary properties of the category of affine extensions of a fixed abelian variety A are presented in Chapter 2. Therein, we also present minor indispensable complements to some of the classical results on the theory of group schemes.

In order to capture (i.e. *to reconstruct*) the complete structure of the affine extension $\mathcal{S}$ with a representation theory supported on a category with *homogeneous vector bundles as objects*, we need to consider "more morphisms" than the usual ones between vector bundles. This new category $\mathrm{HVB}_{\mathrm{gr}}(A)$ is an *enriched category* over the monoidal category $\big(\mathrm{Sch}\,|\Bbbk, \times, \{*\} = \mathrm{Spec}(\Bbbk)\big)$. Moreover, the (scheme of) morphisms between two objects, denoted as $\mathrm{Hom}_{\mathrm{gr}}(E, E')$, is also a homogeneous vector bundle over A; we call its structure morphism $d : \mathrm{Hom}_{\mathrm{gr}}(E, E') \to A$ the *degree map.*

The *automorphism group* $\mathrm{Aut}_{\mathrm{gr}}(E)$ of a general vector bundle (see Definition 3.18) is a smooth group scheme of finite type, and the *degree map* $d : \mathrm{Aut}_{\mathrm{gr}}(E) \to A$ is an affine morphism of group schemes, with kernel $\mathrm{Aut}_0(E)$ — the group of "classical" automorphisms of E. This result is well known for algebraically closed fields; more recently it was generalized for arbitrary fields (see [18, Lemma 2.8]). We say that the bundle E is homogeneous when the sequence

$$Aut_{\mathrm{gr}}(E) : \qquad 1 \longrightarrow \mathrm{Aut}_0(E) \longrightarrow \mathrm{Aut}_{\mathrm{gr}}(E) \xrightarrow{\ d\ } A \longrightarrow 0,$$

is exact — and hence $Aut_{\mathrm{gr}}(E)$ is an affine extension, see 3.9 for a precise definition.

A *representation* of an affine extension $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{\ q\ } A \longrightarrow 0$ or an *$\mathcal{S}$–module*, is a morphism of group schemes $\rho : G \to \mathrm{Aut}_{\mathrm{gr}}(E)$, where E is a homogeneous vector bundle over A, such that ρ induces the identity on A (see Definition 3.33). The *category* $\mathrm{Rep}(\mathcal{S})$ *of $\mathcal{S}$–modules* is the category enriched over $\mathrm{Sch}\,|\Bbbk$ that has as objects the representation of $\mathcal{S}$ and as hom-objects the scheme (in fact a homogeneous vector bundle) of G–equivariant graded morphisms of homogeneous vector bundles (see Definition 3.41 and Lemma 3.37).

If G is an affine group scheme G (i.e. when $A = \mathrm{Spec}(\Bbbk)$) then the above definition corresponds to the category of finite dimensional rational G–modules and the G–equivariant morphisms.

The category of representations of an affine extension is monoidal "in degree zero", see Remark 3.27 and Lemma 3.28; and this *weaker* monoidality condition becomes a decisive ingredient in the proof of the Tannaka Duality Theorem in our context. We establish a) a version of Tannaka's reconstruction Theorem 4.6) proving that from the category of representations of an extension $\mathcal{S}$ we can define $\mathcal{S}$ itself; b) we prove that the mentioned reconstruction is accomplished in the best possible way when using the category of representations (Recognition Theorem 5.1).

Moreover, as expected in view of the results and methods of the affine case, in order to establish a version of the Tannaka Duality in our context one should deal with affine extensions as limits of affine extensions of finite type (that is, such that the corresponding groups H and G are group schemes of finite type); in Theorem 2.55 we prove that *any* affine extension is such a limit. On the other hand, by a result of D. Perrin, if G is a connected group scheme then G is quasi-compact and it is a limit of a family $\{G_\alpha\}_{\alpha\in I}$ of group schemes of finite type (see [45, théorèmes II.2.4 and IV.3.2]). Moreover, G fits into an affine extension of an abelian variety — this result, stated without a complete proof in [45, Corollary V.4.3.1], is proved in Corollary 2.27 below.

The op-equivalence between the category of affine group schemes and commutative Hopf algebras — that to a given affine group scheme G associates the algebra of global section $\mathcal{O}_G(G)$ with a structure of Hopf algebra induced by the multiplication and inversion morphisms in G — is not only an important viewpoint but also a powerful tool in the study of the representation theory of affine group schemes. Thus, once we have constructed an adequate representation theory for affine extensions — in the sense that the representation theory satisfies a full Tannaka Duality Theorem —, we undertake the generalization of this well-known equivalence to the context of affine extensions, by developing the notion of *Hopf sheaf over an abelian variety*.

As we are working with quasi-compact morphisms of group schemes $q : G \to A$, we consider the categories $\mathrm{Sch}|_{\mathrm{qc}}A$ (of quasi-compact schemes over A), and QA-alg (of quasi-coherent sheaves of $\mathcal{O}_A$–algebras) and the well known covariant equivalence associated to the functors $\mathcal{P} : \mathrm{Sch}|_{\mathrm{qc}}A \to QA\text{-alg}^{\mathrm{op}}$, $\mathcal{P}(x : X \to A) = x_*(\mathcal{O}_X)$ and $\mathrm{Spec} : QA\text{-alg}^{\mathrm{op}} \to \mathrm{Sch}|_{\mathrm{qc}}A$, where $\mathrm{Spec}(\mathcal{F})$ is the affine scheme over A associated to the sheaf of $\mathcal{O}_A$–algebras $\mathcal{F}$. In general these functors are adjoint (see Remark 6.31 and [31, §1.2,§1.3]), but they establish a covariant equivalence when restricted to the situation that the objects $x : X \to A$ are affine morphisms. Moreover, if $A = \mathrm{Spec}(\Bbbk)$, then we obtain the usual equivalence mentioned above.

In order to obtain the necessary generalization of the notion of Hopf algebra to this context, one needs to observe that $\mathrm{Sch}|_{\mathrm{qc}}A$ and QA-alg are in fact duoidal categories (see Definition 6.7) and that the adjoint functors $\mathcal{P}$ and Spec are (strong, colax) monoidal for the given monoidal structures (see Theorem 6.46 for a precise statement). The construction of such duoidal structures is known in the setting of slice categories, but we take an explicit approach in order to identify the affine extensions as a certain type of bimonoids in the category (see Lemma 6.10, Proposition 6.12 and Theorem 6.18). It its worth noticing that the construction of the duoidal structure on $\mathrm{Sch}|_{\mathrm{qc}}A$ relies heavily in the fact that A supports a commutative sum (see Definition 6.4).

In the general setting of arbitrary duoidal categories there is no canonical way to define an antipode, or more generally the notion of group (Hopf) object. However, in our particular category $\mathrm{Sch}|_{\mathrm{qc}}A$ we have an obvious candidate for a group type object, namely the quasi-compact morphisms of group schemes $q : G \to A$. Thus, the affine morphisms of group schemes $q : G \to A$ are the affine group type objects for the duoidal structure on $\mathrm{Sch}|_{\mathrm{qc}}A$; under the functor $\mathcal{P}$, these group objects are in bijection with the (faithful, commutative) group type objects for the duoidal structure on QA-alg, that we call *faithful commutative Hopf sheaves* (see Definition 6.58 and Theorem 6.62).

A drawback of the proposed definition of $\mathcal{S}$–module is that it only contemplates the finite dimensional objects — for affine group schemes, the notion of *rational G–module* allows to take into account the infinite dimensional case (see for example [26, Definition 5.3.7]). Indeed, whereas an infinite dimensional $\Bbbk$–space is a colimit of finite dimensional sub-spaces — a directed union of finite dimensional sub-spaces —, we need an adequate notion of "rational infinite dimensional vector bundle", convenient for our purposes. The op-equivalence of the category of affine extensions and the category of flat commutative Hopf sheaves allows the lifting of this obstruction, by considering the $\mathcal{S}$–modules as sheaves.

More precisely, we proceed as follows: given an affine scheme $X = \operatorname{Spec}(B)$, J.-P. Serre proposed in [56] the category of projective B-modules as a generalization of the notion of vector bundle. In [25], V. Drinfeld generalizes in turn Serre's proposal, by considering quasi-coherent, flat sheaves on an scheme X — recall that if X is a nœtherian scheme, then the category of coherent flat $\mathcal{O}_A$–modules, being the category locally free of finite rank $\mathcal{O}_A$–modules, is equivalent to the category of vector bundles (see [56, Proposition 2], Remark 7.1 and Proposition7.5). Hence, we establish the notion of comodule of a Hopf sheaf taking as support the quasi-coherent, flat sheaves on A. However, in order to exploit the well known equivalence between the category of vector bundles and the category of coherent flat $\mathcal{O}_A$–modules and to establish a notion of *$\mathcal{S}$–linearized sheaves* for an affine extension $\mathcal{S}$, we need to develop the notions of *graded morphisms of sheaves* and *homogeneous sheaf* (see definitions 7.6 and 7.17); we define in this way the *category of homogeneous sheaves on A with graded morphisms* as a $\operatorname{Sch}|\Bbbk$–category — in Lemma 7.22, we prove that the category of homogeneous vector bundles with graded morphism is equivalent to the category of homogeneous quasi-coherent, flat sheaves with graded morphisms.

Once the categorical framework above is established, we can consider the categories of *$\mathcal{S}$–linearized sheaves with graded morphisms* and of *$\mathcal{H}_{\mathcal{S}}$–comodules with graded morphisms* (here $\mathcal{H}_{\mathcal{S}}$ denotes the Hopf sheaf associated to $\mathcal{S}$ — this is done in sections 7.3 and 7.1 respectively —, and prove the equivalence between the category $\operatorname{Rep}(\mathcal{S})$ and the categories of coherent flat $\mathcal{S}$–linearized sheaves and $\mathcal{H}_{\mathcal{S}}$–comodules with graded morphisms (see Theorem 7.35 and Proposition 7.38). Finally, we also propose a notion of *rational sheaf* that could be useful in the study of these categories (see definitions 7.34 and 7.37).

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2. Extensions of abelian varieties by affine group schemes

2.1. Group schemes and their actions.

In this section we present some basic definitions and known results on quasi-compact group schemes.

Definition 2.1. (1) A $\Bbbk$*–monoid scheme* M — or *monoid scheme over* $\Bbbk$ — is a $\Bbbk$–scheme together with two $\Bbbk$–morphisms $m_M : M \times M \to M$ and $e_M : \operatorname{Spec}(\Bbbk) = \{*\} \to M$ (called the *multiplication*, and the *unit* respectively), satisfying the usual commutative diagrams (of associativity of m_M and unitality of e_M).

(2) A $\Bbbk$–*group scheme* G — or *group scheme over* $\Bbbk$ — is a $\Bbbk$–monoid scheme together with an *inversion morphism* $\iota_G : G \to G$ (defined over $\Bbbk$ and satisfying the corresponding commutative diagrams).

(3) A *morphism of monoid schemes* between M and M' is a morphism of $\Bbbk$–schemes $f : M \to M'$ satisfying the usual commutative diagrams:

$$\begin{tikzcd} M \times M \arrow[r, "m_M"] \arrow[d, "f\times f"'] & M \arrow[d, "f"] \\ M' \times M' \arrow[r, "m_{M'}"'] & M' \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}(\Bbbk) \arrow[r, "e_M"] \arrow[dr, "e_{M'}"'] & M \arrow[d, "f"] \\ & M' \end{tikzcd}$$

If both M and M' are group schemes, we say that f is a *morphism of group schemes* (in this case, $f \circ i_G = i_{G'} \circ f$).

(4) A monoid scheme M is affine (resp. of finite type, resp. smooth) if M is so as scheme.

(5) An *abelian variety* is a smooth, connected, proper $\Bbbk$–group scheme of finite type — an abelian variety is necessarily a commutative group. An *isogeny* of abelian varieties is a group homomorphism that is surjective and has finite kernel.

Most of the time we abbreviate and omit the mention to the base field e.g. a $\Bbbk$–scheme is called simply a scheme, and a $\Bbbk$–group scheme of finite type is referred as a *group scheme of finite type*.

Whenever it is convenient or necessary, we will interpret a group scheme G as a representable functor $G : (\mathrm{Sch}\,|\Bbbk)^{\mathrm{op}} \to \mathrm{Groups}$ — where $\mathrm{Sch}\,|\Bbbk$ is the category of $\Bbbk$–schemes and Groups the category of abstract groups. If T is a $\Bbbk$–scheme, then $G(T)$ together with $m(T), i_G(T), e_G(T)$ is called the group of the T–points of the scheme G.

Remark 2.2. Traditionally, group schemes of finite type were called "algebraic groups" (cf. [23, 60]), but currently this nomenclature does not seem to have a unique connotation (e.g. in [44] an algebraic group is a *smooth* group scheme of finite type). In order to avoid confusion we prefer to use a more explicit, unambiguous, name.

Definition 2.3. (1) An *action of a* $\Bbbk$*–group scheme* G *on a* $\Bbbk$*-scheme* X is a morphism of schemes $a : G \times X \to X$, satisfying the usual commutative diagrams. In this situation the scheme X is said to be a G*–scheme*.

(2) Given two G–schemes X and Y, a morphism $f : X \to Y$ is G*–equivariant* (or a *morphism of* G*–schemes*) if the following diagram is commutative, where the horizontal arrows are the corresponding G–actions:

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "\mathrm{id}\times f"'] & X \arrow[d, "f"] \\ G \times Y \arrow[r] & Y \end{tikzcd}$$

Remark 2.4. (1) It is well known (see for example [41]) that to give an action of G on X is equivalent to give a morphism of functors (that is, a natural transformation) $\phi : G \to \mathrm{Aut}_X$, where $\mathrm{Aut}_X : (\mathrm{Sch}\,|\Bbbk)^{\mathrm{op}} \to \mathrm{Groups}$ is the so called *automorphism group functor*. Recall that given a scheme T, the group $\mathrm{Aut}_X(T) \subseteq \mathrm{Aut}_{\mathrm{Sch}\,|T}(X \times$

T), where $\mathrm{Sch}\,|T$ denotes the category of schemes over T and we consider $X \times T$ as an T–scheme given by the projection on the second coordinate, is defined as follows:

$$\mathrm{Aut}_X(T) = \big\{ f : X \times T \to X \times T \text{ isomorphism } : f(x,t) = \big(\widetilde{f}(x,t), t\big)\ ,\ \widetilde{f} : X \times T \to X \big\}.$$

Equivalently, $f \in \mathrm{Aut}_X(T)$ if $f : X \times T \to X \times T$ is an isomorphism and the following diagram commutes

$$\begin{tikzcd} X \times T \arrow[r, "f", "\cong"'] \arrow[d, "p_2"'] & X \times T \arrow[d, "p_2"] \\ T \arrow[r, "\mathrm{id}_T"] & T \end{tikzcd}$$

in which case $\widetilde{f} = p_1 \circ f$ (see for example [41] or [14]).

(2) In particular, when Aut_X is a group scheme, we have a canonical action $a : \mathrm{Aut}_X \times X \to X$, induced by $\mathrm{id}_{\mathrm{Aut}_X} : \mathrm{Aut}_X \to \mathrm{Aut}_X$: if $f = (\widetilde{f}, p_2) \in \mathrm{Aut}_X(T)$ and $x \in X(T)$, then $a(f,x) = \widetilde{f} \circ (x, \mathrm{id}_T)$.

Remark 2.5. (1) If G is a connected group scheme, then G is quasi-compact, as follows from [45, Théorème IV.3.2]).

(2) Let $f : G \to G'$ be a morphism of group schemes. Then the *scheme theoretic image* of f, denoted as $f(G)$, is the smallest closed subgroup scheme of G' containing the image of f.

(3) Any group scheme is separated (because $e_G : \mathrm{Spec}(\Bbbk) \to G$ is a closed immersion, see for example [57, Tag 045W]).

(4) Since any morphism of separated schemes is separated, any morphism of group schemes $f : G \to G'$ is separated.

We finish this section by recalling some fundamental results on the structure of quasi-compact group schemes due to D. Perrin (see [45, 46]).

Theorem 2.6 (Perrin, [45, Théorème II.2.4 and Théorème V.1.1])**.** *Let G be a quasi-compact group scheme. Then*

(1) There is a unique irreducible component of G passing trough e_G — this component is called the neutral component of G *and denoted as G^0 —; moreover G^0 is geometrically irreducible;*

(2) the inclusion $i : G^0 \to G$ is a flat closed immersion;

(3) G^0 is a normal (quasi-compact) subgroup of G;

(4) the quotient G/G^0 exists and is an affine group scheme, with fields as local rings. Moreover, G/G^0 is compact, totally discontinuous, and limit of étale finite groups (see Section 2.5). □

Theorem 2.7 (Perrin, [45, Corollaire V.3.2])**.** *Let G be a quasi-compact $\Bbbk$–group scheme, and $K \subset G$ a closed subgroup scheme. Then, in the following two situations the quotient G/K exists in the category of $\Bbbk$–schemes:*

(1) K is defined by a sheaf of finitely generated ideals, in which case G/K is of finite type;

(2) K is a normal subgroup of G, in which case G/K is a group scheme. □

Proposition 2.8. *Let M be a monoid scheme, G a reduced group scheme and $f : M \to G$ a quasi-compact dominant morphism of monoid schemes. Then f is flat.*

Proof. If M is a group scheme, this result is proved in [45, Proposition II.1.3]. An inspection of the proof presented therein, shows that it is still valid for M a monoid scheme. $\square$

Theorem 2.9 (Perrin, [45, Proposition II.1.5, Lemme V.3.3.1 and Corollaire V.3.3])**.**

Let $f : G \to K$ be a quasi-compact morphism of group schemes; let $f(G)$ be the scheme theoretic image of f. Then $f(G) \cong G/\operatorname{Ker} f$ and the induced morphism $\widetilde{f} : G \to f(G)$ is faithfully flat. In particular, the induced morphism $G/\operatorname{Ker} f \to K$ is a closed immersion. $\square$

Corollary 2.10. *Let $f : G \to K$ be a quasi-compact morphism of group schemes, with K reduced. Then the following three assertions are equivalent: (a) f is faithfully flat; (b) the map associated to f at the level of sets is surjective; (c) the map $f(\overline{\Bbbk}) : G(\overline{\Bbbk}) \to K(\overline{\Bbbk})$ is surjective.*

Moreover if f is as above and K is connected, then the restriction $f|_{G^0} : G^0 \to K$ is faithfully flat.

Proof. Indeed, under the hypothesis of this corollary, $K(\overline{\Bbbk})$ is dense in the base space of K.

If K is connected, since f is faithfully flat, it follows that G^0 dominates K (see for example [32, Proposition IV.2.3.4]). Since $f|_{G^0}$ factors through a closed immersion, the result follows. $\square$

2.2. **Extensions of abelian varieties by affine group schemes.**

Definition 2.11. Let H be a $\Bbbk$–group scheme, X an H–scheme with action a and $f : X \to Y$ an H–invariant morphism of schemes, f is an (H, a)*–torsor* if:

(1) f is quasi-compact and faithfully flat;

(2) The morphism $H \times X \to X \times_Y X$ induced by a and the projection over the second coordinate, is an isomorphism; in other words, the commutative diagram below is cartesian:

$$\begin{tikzcd} H \times X \arrow[r, "a"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

When no confusion arises, we will say that f is an H*–torsor* of a *torsor under* H.

Definition 2.12. Let $j : N \to G$ and $q : G \to Q$ be two morphisms of group schemes. The sequence

$$\mathcal{S} : \qquad 1 \longrightarrow N \xrightarrow{j} G \xrightarrow{q} Q \longrightarrow 1$$

is a *short exact sequence of group schemes* if and only if the following two conditions are satisfied:

(1) The sequence $\mathcal{S}$ is left exact; that is, the sequence $1 \to N(T) \to G(T) \to Q(T)$ is exact for every $\Bbbk$-scheme T — equivalently, $\operatorname{Ker} j$ is trivial and j induces an isomorphism $\operatorname{Ker} q \cong N$.

(2) If T is a scheme and $y \in Q(T)$, then there exists a faithfully flat, quasi-compact morphism $f : T' \to T$ and $x \in G(T')$ such that $q_{T'}(x) = Q(f)(y) \in Q(T')$.

Remark 2.13. (1) Notice that condition (2) of Definition 2.12 holds whenever $q : G \to Q$ is an fpqc (i.e. faithfully flat quasi-compact) morphism.

(2) Moreover, if $q : G \to Q$ is an fpqc morphism, then clearly q is an N–torsor — the second condition of Definition 2.11 is easily proved due to the fact that all the schemes involved are group schemes. In particular, since q is an N–torsor, q is a categorical quotient (see for example [14, §2.6]).

Example 2.14. Let G be a connected group scheme and $H \subset G$ a normal closed subgroup scheme. Then it follows from [45, Corollaire IV.3.3] that G/H is a group scheme and the quotient map $q : G \to G/H$ is a faithfully flat quasi-compact morphism. In particular, the sequence $1 \longrightarrow H \longrightarrow G \xrightarrow{q} G/H \longrightarrow 0$ is exact.

Definition 2.15. Let A be an abelian variety. A *group extension of* A is a short exact sequence $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$. If moreover $q : G \to A$ is a faithfully flat quasi-compact morphism we say that $\mathcal{S}$ is a *quasi-compact group extension of* A; if q is a faithfully flat affine morphism, we say that $\mathcal{S}$ is an *an affine group extension of* A.

Remark 2.16. Let A be an abelian variety. If $q : G \to A$ is a surjective quasi-compact morphism of group schemes, then G is a quasi-compact group scheme and q is a faithfully flat morphism by [45, Proposition II.1.3], since A is a reduced group scheme (see Corollary 2.10). It follows that

$$\mathcal{S}_q :\ 1 \longrightarrow \operatorname{Ker}(q) \longrightarrow G \xrightarrow{q} A \longrightarrow 0$$

is a quasi-compact extension of A.

On the other hand, if G is a quasi-compact group scheme and $H \subset G$ is a normal subgroup scheme such that $A = G/H$ is an abelian variety, then the canonical projection $q : G \to A$ is an H–torsor, and the corresponding exact sequence is a quasi-compact extension (see [45, Corollaire IV.3.3 and Proposition II.1.3]).

Remark 2.17. (1) Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be a quasi-compact extension. Then $\mathcal{S}$ is an affine extension if and only if H is an affine group scheme. See [23, III, § 3,2.5/6], or [37, § I.5.7] for a similar result for H–torsors.

(2) By definition, if a group scheme G fits into an affine extension then G is quasi-compact; see Corollary 2.27 below for a partial converse due to D. Perrin ([45]).

(3) It is well known that if $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ is an affine extension then G is of finite type if and only if H is of finite type, see for example [14, Proposition 2.6.5]; if this is the case, we say that the extension is of *finite type*.

(4) Let A be an abelian variety and $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ a short exact sequence. It follows from Remark 2.16 that a $\mathcal{S}$ is a quasi-compact (resp. affine) extension if and only if $\mathcal{S}$ is left exact and q a surjective quasi-compact (resp. affine) morphism.

We complete the definition of the category of quasi-compact (resp. affine) extensions of an affine variety A by defining its morphisms (see also Section 6.1).

Definition 2.18. Let A be an abelian variety.

(1) The *category* $\mathrm{GE}|_{\mathrm{qc}} A$ *of quasi-compact group extensions of* A has as objects the quasi-compact extensions of A and as *morphisms* $\phi : \mathcal{S} \to \mathcal{S}'$ between two quasi-compact extensions of A, the commutative diagrams of the form:

$$\begin{tikzcd} \mathcal{S}: \arrow[d, "\phi"] & 1 \arrow[r] & N \arrow[r, "j"] \arrow[d, "f_N"'] & G \arrow[r, "q"] \arrow[d, "f"'] & A \arrow[r] \arrow[d, equal] & 0 \\ \mathcal{S}': & 1 \arrow[r] & N' \arrow[r, "j'"] & G' \arrow[r, "q'"] & A \arrow[r] & 0 \end{tikzcd} \tag{2.1}$$

where f_N and f are morphisms of group schemes.

(2) The *category* $\mathrm{GE}|_{\mathrm{aff}} A$ *of affine extensions of* A is defined as the full subcategory of $\mathrm{GE}|_{\mathrm{qc}} A$ with objects the affine extensions.

(3) If $\mathbf{P}$ is a class of morphisms of schemes (affine, quasi-compact, finitely presented, etc.) we say that the morphism $\phi : \mathcal{S} \to \mathcal{S}'$ is of class $\mathbf{P}$ if and only if f is of class $\mathbf{P}$.

Notation 2.19. In the situation of a diagram such as (2.1), the morphism $f : G \to G'$ will be called, the *mid morphism* of ϕ.

Remark 2.20. It is evident that it is equivalent to give a diagram as (2.1) or a commutative triangle of morphisms of group schemes as below, with q and q' affine morphisms.

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "q"'] & & G' \arrow[dl, "q'"] \\ & A & \end{tikzcd}$$

Indeed, if $f : G \to G'$ is as above, then $\mathrm{Ker}(f) \subset \mathrm{Ker}(q)$ and the restriction $f|_{\mathrm{Ker}(q)} : \mathrm{Ker}(q) \to \mathrm{Ker}(q')$ makes sense as $f\big(\mathrm{Ker}(q)\big) \subset \mathrm{Ker}(q')$.

In particular, notice that if $\phi : \mathcal{S} \to \mathcal{S}'$ is a morphism of affine extensions then the mid morphism f is an affine morphism, since it is a morphism in the category $\mathrm{Sch}|_{\mathrm{aff}} A$.

Remark 2.21. (1) The composition of morphisms in $\mathrm{GE}|_{\mathrm{qc}} A$ and the identity morphism are defined in the obvious manner.

(2) Clearly $\mathcal{S}$ and $\mathcal{S}'$ are *isomorphic* if and only if the maps f_N and f are isomorphisms — this last assertion is equivalent the assertion that f is an isomorphism (compare with Theorem 2.9, Remark 2.16 and § 2.3).

Definition 2.22. If in Definition 2.15 the group scheme H is smooth, then the canonical projection $q : G \to A$ is a smooth morphism; in this situation we say that the extension is *smooth*.

Examples 2.23. (1) If G is an affine group scheme, then G can be viewed in a canonical way as an affine extension of the trivial abelian variety $\operatorname{Spec} \Bbbk = \{*\}$ by means of the short exact sequence $1 \longrightarrow G \xrightarrow{\mathrm{id}} G \longrightarrow \operatorname{Spec}(\Bbbk) \longrightarrow 0$.

(2) An abelian variety A can be thought as an affine extension in a natural way as: $0 \longrightarrow 0 \longrightarrow A \xrightarrow{\mathrm{id}} A \longrightarrow 0$.

(3) If $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ is an affine extension and $f : A \to A$ is an isogeny (i.e. a surjective morphism of abelian varieties with finite kernel), then $1 \longrightarrow \mathrm{Ker}(f \circ q) = q^{-1}\big(\mathrm{Ker}(f)\big) \longrightarrow G \xrightarrow{f \circ q} A \longrightarrow 0$ is an affine extension of A (see Remark 2.17).

In particular if $f : A \to A$ is an isogeny, then $0 \longrightarrow \mathrm{Ker}(f) \longrightarrow A \xrightarrow{f} A \longrightarrow 0$ is an affine extension.

Remark 2.24. Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension and $\ell : H \to H'$ be a morphism of affine group schemes. Assume moreover that $H \subset G$ is central in G and that $\ell(H) \subset H'$ is central in H'. Then $\Gamma_\ell(H)$, the scheme theoretic image of the "graph" morphism $\Gamma_\ell = (\mathrm{inc}, \ell \circ \iota_H) = (\mathrm{inc} \times \ell) \circ \Delta_H : H \to G \times H'$, is a central subgroup scheme of $G \times H'$ — here $\Delta_H : H \to H \times H$ denotes the diagonal embedding, $\Delta_H(h) = (h, h^{-1})$. Therefore, the quotient $G \times^H H' = (G \times H')/\Gamma_\ell(H)$ is a quasi-compact group scheme and fits into an affine extension, that we denote $\ell_* \mathcal{S}$ — it is also possible to deduce the existence of $\ell_* \mathcal{S}$ from the properties of the *induced space* (see Theorem 2.61). Moreover, ℓ yields a morphism $\lambda : \mathcal{S} \to \ell_* \mathcal{S}$ of affine extensions:

$$\begin{tikzcd}
\mathcal{S}: \arrow[d, "\lambda"] & 1 \arrow[r] & H \arrow[r] \arrow[d, "\ell"'] & G \arrow[r, "q"] \arrow[d, "j"] & A \arrow[r] \arrow[d, equal] & 0 \\
\ell_* \mathcal{S}: & 1 \arrow[r] & H' \arrow[r] & G \times^H H' \arrow[r, "\pi_{H'}"'] & A \arrow[r] & 0
\end{tikzcd}$$

where $j : G \to G \times^H H'$ is given by $j(g) = [g, 1] := \pi_{G \times H'}(g, 1)$, with $\pi_{G \times H'} : G \times H' \to G \times^H H'$ the canonical projection, and $\pi_{H'} : G \times^H H' \to A$ is given by $\pi_{H'}\big([g, h']\big) = q(g)$. Indeed, note that $\pi_{H'}$ is well defined and that $\pi_{H'}\big([g, h']\big) = 0$ if and only if $g \in H$, therefore $\mathrm{Ker}(\pi_{H'}) = \big\{[1, h'] : h' \in H'\big\} = H'$.

Note that if H' is smooth, then $\ell_* \mathcal{S}$ is a smooth extension.

Definition 2.25. A *closed immersion* of the affine extension $\mathcal{T}$ into the affine extension $\mathcal{S}$ (both extensions of A) is a morphism $\phi : \mathcal{T} \to \mathcal{S}$ of affine extensions

$$\begin{tikzcd}
\mathcal{T}: \arrow[d, "\phi"'] & 1 \arrow[r] & H' \arrow[r] \arrow[d, hook, "f|_{H'}"'] & G' \arrow[r, "q'"] \arrow[d, hook, "f"] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{S}: & 1 \arrow[r] & H \arrow[r] & G \arrow[r, "q"'] & A \arrow[r] & 0
\end{tikzcd}$$

such that the vertical morphism $f : G' \to G$ (and therefore $f|_{H'} : H' \to H$) is a closed immersion.

In particular, if $G' \subset G$ is a closed subgroup scheme such that $q(G') = A$ and $H' = \mathrm{Ker}(q|_{G'})$, then $\mathcal{T}$: $1 \longrightarrow H' \longrightarrow G' \xrightarrow{q|_{G'}} A \longrightarrow 0$ is an affine extension and the inclusion $\mathcal{T} \hookrightarrow \mathcal{S}$ is a closed immersion, we say that $\mathcal{T}$ is a *(closed, affine) sub-extension* of $\mathcal{S}$.

The following theorem was first announced by C. Chevalley in the 1950s and published in 1960 in [19]. We present here a slightly more general version, due to M. Brion (see [14, Theorem 2] and Corollary 2.27 below).

Theorem 2.26 (Chevalley, Raynaud, Brion)**.** *Every $\Bbbk$–group scheme of finite type G has a smallest normal subgroup scheme G_{aff} such that the quotient G/G_{aff} is proper. Moreover, G_{aff} is affine and connected, and the associated short sequence of group schemes over $\Bbbk$ is exact (see Definition 2.12)*

$$1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \xrightarrow{q} G/G_{\mathrm{aff}} \longrightarrow 0. \tag{2.2}$$

If $\Bbbk$ is perfect and G is smooth, then G_{aff} is smooth as well, and its formation commutes with field extensions — that is, if $\Bbbk \subseteq \mathbb{K}$, then $G(\mathbb{K})_{\mathrm{aff}} = G_{\mathrm{aff}}(\mathbb{K})$. In particular, if G is a connected group scheme of finite type over a perfect field, then G fits in a (smooth) affine extension of the abelian variety $A = G/G_{\mathrm{aff}}$. □

Since every group scheme of finite type is an extension of a smooth group scheme by an infinitesimal group scheme (see [14, Proposition 2.9.2]), Theorem 2.26 implies that any connected group scheme of finite type fits in an affine extension.

Corollary 2.27. *Let G be a connected group scheme of finite type. Then there exists an affine extension of an abelian variety $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$, with H a connected affine group scheme.*

Proof. See [14, Corollary 4.3.7]. □

Remark 2.28. It follows from Perrin's Approximation Theorem ([45, Théorème V.3.1], see Theorem 2.54 below) that Theorem 2.26 and its Corollary 2.27 imply that *any connected* group scheme fits into an affine extension (see [45, Corollary V.4.3.1]).

Notation 2.29. If G is a smooth group scheme of finite type, then the sequence (2.2) is known as the *Chevalley decomposition* of G.

Remark 2.30. If G is a smooth group scheme of finite type over a perfect field $\Bbbk$, then G_{aff} is the largest normal, affine, connected, smooth, subgroup scheme of G (see for example [14]).

The following uniqueness result (for $\Bbbk$ a perfect field) follows easily. Assume that a given smooth group scheme G fits in an exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0,$$

with H affine connected and G/H an abelian variety. Then there are isomorphisms $f_1 : H \cong G_{\mathrm{aff}}$ and $f_2 : G/H \cong A$, such that the diagram of short exact sequences is commutative:

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] \arrow[d, "f_1"'] & G \arrow[r] \arrow[d, "\mathrm{id}"'] & G/H \arrow[r] \arrow[d, "f_2"'] & 0 \\
1 \arrow[r] & G_{\mathrm{aff}} \arrow[r] & G \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

It follows that G_{aff} is the *unique* normal, affine, connected, smooth, subgroup scheme H such that G/H is proper. Indeed, if H is such a group, then $G_{\mathrm{aff}} \subset H$ by the Chevalley decomposition theorem 2.26, and $H \subset G_{\mathrm{aff}}$ by the preceding remark.

Lemma 2.31. *Let G be a group scheme of finite type over a perfect field $\Bbbk$ and assume that G fits in an exact sequence of group schemes*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{q_G} G/H \longrightarrow 0$$

with H an affine connected normal subgroup scheme and G/H proper. Then

$$1 \longrightarrow H_{\mathrm{red}} \longrightarrow G_{\mathrm{red}} \longrightarrow (G/H)_{\mathrm{red}} \longrightarrow 0$$

is the Chevalley decomposition of G_{red}.

Proof. By construction $G_{\mathrm{aff}} \subset H$ and it follows that $(G_{\mathrm{aff}})_{\mathrm{red}} \subset H_{\mathrm{red}}$. Now, since H_{red} is affine and connected, then its Albanese variety is $\mathcal{A}lb(H_{red}) = \mathrm{Spec}(\Bbbk)$ and so $q_G(H_{red}) = \{0\} \subset A$; therefore, $H_{\mathrm{red}} \subset G_{\mathrm{aff}}$. $\square$

Remark 2.32. In Lemma 2.31 the condition that $\Bbbk$ is a perfect field cannot be omitted. Indeed, if $\Bbbk$ is not perfect, then G_{red} is not necessarily a group scheme, as it is shown in [14, Example 2.5.5].

The Chevalley decomposition of smooth group schemes has the following functorial property.

Lemma 2.33. *If $f : G \to G'$ is a morphism of smooth group schemes of finite type, then their Chevalley decompositions fit in the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{G}: \arrow[d, "\phi"] & 1 \arrow[r] & G_{\mathrm{aff}} \arrow[r] \arrow[d, dotted, "f|_{G_{\mathrm{aff}}}"'] & G \arrow[r, "q"] \arrow[d, "f"] & Q \arrow[r] \arrow[d, dotted, "\widetilde{f}"] & 0 \\
\mathcal{G}': & 1 \arrow[r] & G'_{\mathrm{aff}} \arrow[r] & G' \arrow[r, "q'"'] & Q' \arrow[r] & 0
\end{tikzcd} \tag{2.3}$$

If f is a faithfully flat morphism, then the vertical arrows of the diagram above are faithfully flat morphisms. Moreover, if f is affine and faithfully flat, then $\widetilde{f}$ is an isogeny.

Proof. Since G is a group scheme of finite type the image of $q' \circ f$ is closed in Q'; thus $q' \circ f(G) \subset Q'$ is a proper group and it follows that $G_{\mathrm{aff}} \subset \mathrm{Ker}(q' \circ f)$. By the universal property of the quotient, it follows that $q' \circ f$ induces a morphism $\widetilde{f} : Q \to Q'$ that fits in Diagram (2.3).

Assume now that f is faithfully flat. Then $\widetilde{f} \circ q = q' \circ f$ is faithfully flat, q being a faithfully flat morphism it follows that $\widetilde{f}$ is faithfully flat. Since $f(G_{\mathrm{aff}}) \subset G'_{\mathrm{aff}}$ is a closed (therefore affine) subscheme and f is faithfully flat, it follows that $f(G_{\mathrm{aff}})$ is an affine normal subgroup scheme of G' — recall that if $g' \in G'(T)$ then there exists a faithfully flat quasi-compact morphism $f : T' \to T$ and a point $g \in G(T')$ such that $f(T')(g) = g'$. The faithfully flat morphism $G \to G'/f(G_{\mathrm{aff}})$ factors through Q and so $G'/f(G_{\mathrm{aff}})$ is a proper group scheme. The minimality of G'_{aff} then implies that $f(G_{\mathrm{aff}}) = G'_{\mathrm{aff}}$; that is, $f|_{G_{\mathrm{aff}}}$ is faithfully flat.

If f is an affine morphism, then $\mathrm{Ker}(q' \circ f)$ is an affine closed subgroup scheme of G. It follows that $\mathrm{Ker}(\widetilde{f}) = q\big(\mathrm{Ker}(q' \circ f)\big) \cong \mathrm{Ker}(q' \circ f)/G_{\mathrm{aff}}$ is a closed affine subgroup scheme of an abelian variety, and therefore is an affine (and hence a finite) subgroup scheme of Q. $\square$

2.3. **Quasi-compact extensions as schemes over an abelian variety.**

As follows from Remark 2.16, to give a quasi-compact (resp. affine) extension over an abelian variety A is equivalent to give a surjective, quasi-compact (resp. affine), morphism of group schemes $q : G \to A$.

On the other hand and concerning the arrows, given two surjective quasi-compact (resp. affine) morphisms of group schemes $q : G \to A$ and $q' : G' \to A$, and a morphism of group schemes $f : G \to G'$ such that $q' \circ f = q$. It easily follows that $f\big(\operatorname{Ker}(q)\big) \subset \operatorname{Ker}(q')$. Hence, f induces a morphism of extensions

$$\begin{tikzcd}
\mathcal{S}: \arrow[d, "\widetilde{f}"] & 1 \arrow[r] & \operatorname{Ker}(q) \arrow[r] \arrow[d, "f|_{\operatorname{Ker}(q)}"'] & G \arrow[r, "q"] \arrow[d, "f"'] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{S}': & 1 \arrow[r] & \operatorname{Ker}(q') \arrow[r] & G' \arrow[r, "q'"'] & A \arrow[r] & 0
\end{tikzcd}$$

Therefore, the category $\operatorname{GE}|_{\mathrm{qc}}A$ of quasi-compact (resp. $\operatorname{GE}|_{\mathrm{aff}}A$ of affine) extensions is equivalent to a subcategory of $\operatorname{Sch}|_{\mathrm{qc}}A$ (resp. $\operatorname{Sch}|_{\mathrm{aff}}A$), that has as objects the separated (see Remark 2.5) quasi-compact (resp. affine) surjective (and therefore faithfully flat) morphisms of group schemes $q : G \to A$ and as morphisms $f : (q : G \to A) \to (q' : G \to A)$, the morphisms in $\operatorname{Sch}|_{\mathrm{qc}}A$ (resp. $\operatorname{Sch}|_{\mathrm{aff}}A$) that are also morphisms of group schemes $f : G \to G'$.

Remark 2.34. The reader should be aware that under the above equivalence, affine extensions *do not* correspond to affine *group schemes over the scheme* A — recall that the product of a group scheme over A is a morphism $m : G \times_A G \to G$.

In Section 6.1 we will introduce a structure of duoidal category on $\operatorname{Sch}|_{\mathrm{qc}}A$ (see Definition 6.7 and Lemma 6.10), such that the quasi-compact (resp. affine) extensions correspond to the group type objects for this category (resp. the group type objects that are affine over A). See Proposition 6.12 and Theorem 6.18.

Notation 2.35. In view of the above equivalence, we will abuse of notation and say that a surjective, quasi-compact (resp. affine) morphism of group schemes $q : G \to A$ is a *quasi-compact* (resp. *affine*) *extension* of A.

In what follows, we will freely use both points of view (affine extensions as short exact sequences or as surjective affine morphism of group schemes) depending on which one is better adapted to the particular result or definition.

2.4. **Rosenlicht decomposition for affine extensions.**

Definition 2.36. (1) A group scheme G defined over a field $\Bbbk$ is called *anti-affine* if $\mathcal{O}_G(G) = \Bbbk$.

(2) An affine extension $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$, with G anti-affine, is said to be of *anti-affine type*. Equivalently, an affine extension of anti-affine type is a surjective affine morphism $q : G \to A$, with G an anti-affine group scheme.

Whereas the notion of anti-affine group scheme already appeared (implicitly) in the work of Rosenlicht ([48]) and Serre ([54]) in the late 50s, it was not regularly studied until about 10 years ago. In [11], Brion began a thorough study of anti-affine group schemes, generalizing earlier results by Rosenlicht on the decomposition of a group scheme of finite type (see [48] and [49] and Theorem 2.42 below) — the classification of anti-affine groups was obtained simultaneously by Brion (*op. cit.*) and C. Sancho de Salas and F. Sancho de Salas ([52], see also [51]).

In this section (see Theorem 2.42), we present a generalization of the Rosenlicht decomposition to the setting of affine extensions of an abelian variety as well as some related properties. We begin by recalling the results on anti-affine group schemes that will be used in what follows; for other properties, in particular for a complete classification theorem, see [11], [12] and [14].

Remark 2.37. (1) It is well known (see for example [12, Chapters 2 and 5]), that an anti-affine group scheme is connected and commutative.

(2) If G is an anti-affine group scheme of finite type, then G is smooth (see for example [14, Lemma 3.3.2]).

(3) In particular, if G is an anti-affine smooth group scheme of finite type, using the Chevalley decomposition (Theorem 2.26) we deduce that G is the extension of a proper group scheme A by a commutative affine group scheme of finite type. This result was much improved by Brion in [14, Section 5.5] (see also [11]): the affine subgroup G_{aff} and the group scheme A appearing therein are smooth (i.e. A is an abelian variety).

Definition 2.38. The *affinization functor* $\mathrm{Aff} : \mathrm{Sch}|_{\mathrm{qc}}\Bbbk \to \mathrm{Sch}|_{\mathrm{aff}}\Bbbk$ (the codomain is the category of affine $\Bbbk$–schemes) defined at the level of objects as $\mathrm{Aff}(X) = \mathrm{Spec}\big(\mathcal{O}_X(X)\big)$, and $\mathrm{Aff}(f : X \to Y) : \mathrm{Aff}(X) \to \mathrm{Aff}(Y)$ is defined as the morphism $\mathrm{Spec}(f_Y^{\#}) : \mathrm{Spec}\big(f_*(\mathcal{O}_X)(Y)\big) = \mathrm{Spec}\big(\mathcal{O}_X(X)\big) \to \mathrm{Spec}\big(\mathcal{O}_Y(Y)\big)$. In this situation $\mathrm{Aff}(X)$ is called the affinization of X.

Remark 2.39. We list some of the properties of this functor for immediate use, see [23, III.3.8], [45, § V.4.2], and [14, § 3.2]) for general references. Later in 6.34 we deal with the properties of this functor in the more general context of schemes over a fixed scheme S.

(1) There is an adjunction $\mathrm{Aff} \dashv \mathrm{inc}$ as: $\begin{tikzcd} \mathrm{Sch}|_{\mathrm{qc}}\Bbbk \arrow[r, shift left, "\mathrm{Aff}"] \arrow[r, phantom, "\perp"] & \mathrm{Sch}|_{\mathrm{aff}}\Bbbk \arrow[l, shift left, "\mathrm{inc}"] \end{tikzcd}$.

(2) The counit of this adjunction is an isomorphism and the unit is given by a family of morphisms of schemes $\eta_X : X \to \mathrm{Aff}(X) : \mathrm{Sch}|\Bbbk \to \mathrm{Sch}|_{\mathrm{aff}}\Bbbk$ that satisfies the following universal property.

For any morphism $f : X \to Y$ with $X \in \mathrm{Sch}, Y \in \mathrm{Sch}|_{\mathrm{aff}}\Bbbk$ there is a unique morphism $\widehat{f}$ that makes commutative the diagram below:

$$\begin{tikzcd} X \arrow[rr, "\eta_X"] \arrow[dr, "f"'] & & \mathrm{Aff}(X) \arrow[dl, dotted, "\widehat{f}"] \\ & Y & \end{tikzcd}$$

The morphism $\eta_X : X \to \mathrm{Aff}(X)$ is called the *affinization morphism* of X; if $U = \mathrm{Spec}\big(\mathcal{O}_X(U)\big) \subset X$ is an affine open subset, then $\eta_X|_U$ is the morphism induced by the restriction $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$.

(3) In the case that G is a quasi-compact group scheme so is $\mathrm{Aff}(G)$, and the adjunction restricts to the category of group schemes. In particular the unit η_G is a morphism of group schemes and its kernel is a closed subgroup scheme of G, that we call G_{ant}.

In [45], Perrin considered the affinization morphism in the situation of quasi-compact group schemes:

Proposition 2.40. *Let G be a quasi-compact group scheme. Then*

(1) The affinization morphism is a faithfully flat morphism (of group schemes).

(2) $\operatorname{Ker}(\eta_G)$ is a geometrically reduced, connected anti-affine group scheme, contained in the center of G^0.

(3) $\operatorname{Ker}(\eta_G)$ is the smallest normal subgroup scheme K such that the quotient G/K is affine.

(4) $\operatorname{Ker}(\eta_G)$ is the largest anti-affine subgroup scheme of G.

Proof. Assertions (1) and (2) are the content of [45, Théorème V.4.2]. The remaining assertions can be proved easily using the universal property of the affinization morphism η. Indeed, let $K \subset G$ be a normal subgroup scheme such that G/K is an affine group; let $q_K : G \to G/K$ be the canonical projection. Then there exists a morphism of group schemes $f : \operatorname{Aff}(G) \to G/K$ such that $f \circ \eta_G = q_K$. It follows that $G_{\mathrm{ant}} = \operatorname{Ker}(\eta_G) \subset \operatorname{Ker}(q_K) = K$.

On the other hand, it is clear that if we consider a subgroup $L \subseteq G$ that is anti-affine, then the inclusion $L \subset G$ induces a closed immersion $\mathrm{inc} : \operatorname{Aff}(L) \hookrightarrow \operatorname{Aff}(G)$, such that $\eta_G|_L = \mathrm{inc} \circ \eta_L$. Since L is anti-affine, it follows that η_L is the trivial morphism; thus, $L \subseteq \operatorname{Ker}(\eta_G)$. □

As noted in the introduction of this section, Rosenlicht decomposition was generalized by Brion (for smooth group schemes of finite type, see [14, theorems 1 and 5.1.1, Proposition 3.3.5]). In Theorem 2.42 we produce a Rosenlicht decomposition for affine extensions — that will be improved afterwards in Theorem 2.60. Before proving the existence of such decomposition, we mention an easy technical result on faithfully flat morphisms for which we write the proof for the lack of an adequate reference.

Lemma 2.41. *Let $i : X \to Y$ be a closed immersion of $\Bbbk$–schemes and $f : T' \to T$ a faithfully flat morphism. If $y : T \to Y$ is a T–point of Y, and $x : T' \to X$ a T' point of X such that $i \circ x = y \circ f$, then there exists a T–point $\widetilde{x} : T \to X$ such that the following diagram is commutative*

$$\begin{tikzcd} T' \arrow[r, "f"] \arrow[d, "x"'] & T \arrow[d, "y"] \arrow[dl, "\widetilde{x}"'] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

Proof. By the universal property of the fibred product we obtain a commutative diagram:

$$\begin{tikzcd} T' \arrow[dr, "\ell"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "x"'] & & \\ & X \times_Y T \arrow[r, hook, "p_2"] \arrow[d, "p_1"'] & T \arrow[d, "y"] \\ & X \arrow[r, hook, "i"'] & Y. \end{tikzcd}$$

where p_2 is again a closed immersion. Since f is faithfully flat, then p_2 is an isomorphism (since p_2 is also surjective with $p_2^\#$ injective) and the result follows. □

Theorem 2.42 (Rosenlicht decomposition of affine extensions)**.**
Let $\mathcal{S}\colon\ 1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ *be an affine extension and* $G_{\rm ant} = \operatorname{Ker}(\eta_G)$. *Then:*

(1) The restriction $m|_{G_{\rm ant}\times H} : G_{\rm ant} \times H \to G$ *is a faithfully flat morphism of group schemes, with kernel* $\Delta(G_{\rm ant} \cap H)$*, the image of the diagonal embedding* $\Delta : G_{\rm ant} \cap H \to G_{\rm ant} \times H$, $\Delta(z) = (z, z^{-1})$*:*

$$1 \longrightarrow G_{\rm ant} \cap H \longrightarrow G_{\rm ant} \times H \xrightarrow{m|_{G_{\rm ant}\times H}} G \longrightarrow 1.$$

In particular, $G = G_{\rm ant}H = HG_{\rm ant} \cong (G_{\rm ant} \times H)/(G_{\rm ant} \cap H)$.

(2) The restriction $q|_{G_{\rm ant}} : G_{\rm ant} \to A$ *is faithfully flat and induces a closed sub-extension of anti-affine type of* $\mathcal{S}$*:*

$$\begin{tikzcd}
\mathcal{S}_{\rm ant}: \arrow[d] & 1 \arrow[r] & G_{\rm ant}\cap H \arrow[r] \arrow[d, hook] & G_{\rm ant} \arrow[r, "q|_{G_{\rm ant}}"] \arrow[d, hook] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{S}: & 1 \arrow[r] & H \arrow[r] & G \arrow[r, "q"] & A \arrow[r] & 0
\end{tikzcd}$$

(3) If moreover G is connected of finite type, then:

(a) $(G_{\rm ant})_{\rm aff} \subset G_{\rm ant} \cap H$*, and* $(G_{\rm ant})_{\rm aff}$ *is a normal subgroup of finite index.*

(b) If $\Delta = (\mathrm{id}_{(G_{\rm ant})_{\rm aff}}, i|_{(G_{\rm ant})_{\rm aff}}) : (G_{\rm ant})_{\rm aff} \to G_{\rm ant} \times H$, $\Delta(z) = (z, z^{-1})$*, is the diagonal embedding of* $(G_{\rm ant})_{\rm aff}$ *in* $G_{\rm ant} \times H$*, then the quotient* $G' = (G_{\rm ant} \times H)/\Delta\big((G_{\rm ant})_{\rm aff}\big)$ *exists and it is a group scheme of finite type. Moreover, the canonical morphism* $f : G' \to G$ *is an isogeny — in the terminology of Section 2.6,* $G' = G_{\rm ant} \times^{(G_{\rm ant})_{\rm aff}} H$*, the* induced space *for the canonical action of* $(G_{\rm ant})_{\rm aff}$ *on* H*.*

Proof. We first prove Assertion (1) in the case that G is connected, situation in which one can follow closely the proof of [14, Theorem 5.1.1], where it is proved for the case that G is a smooth scheme of finite type — the fact that the quotient $G/H = A$ is an abelian variety allows this transcription.

Since $G_{\rm ant}$ is central in G, it is clear that $m|_{G_{\rm ant}\times H} : G_{\rm ant} \times H \to G$ is a quasi-compact morphism of group schemes, with $\operatorname{Ker}(m|_{G_{\rm ant}\times H}) = \Delta_{G_{\rm ant}\cap H}(G_{\rm ant} \cap H) = \{(h, h^{-1}) : h \in G_{\rm ant} \cap H\}$; therefore its schematic image is the image of the closed immersion $(G_{\rm ant} \times H)/\Delta_{G_{\rm ant}\cap H}(G_{\rm ant} \cap H)$ by [45, Corollaire V.3.3]. Hence, $G_{\rm ant}H$ is a normal closed subgroup of G and the quotient $G \to G/(G_{\rm ant}H)$ factors through morphisms $G/H \to G/(G_{\rm ant}H)$ and $G/G_{\rm ant} \to G/(G_{\rm ant}H)$

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & A = G/H \arrow[d] \\
\operatorname{Aff}(G) = G/G_{\rm ant} \arrow[r] & G/(G_{\rm ant}H)
\end{tikzcd}$$

It follows that $G/(G_{\rm ant}H)$ is on the one hand a quotient of the (connected) abelian variety A — and hence it is a connected abelian variety — and on the other hand $G/(G_{\mathrm{ant}}H)$ is a quotient of an affine group scheme by a normal closed subgroup — and hence affine by [24, VIb 11.17]. We deduce that $G/(G_{\mathrm{ant}}H) = \operatorname{Spec}(\Bbbk)$ and therefore $G = G_{\mathrm{ant}}H$.

In order to prove Assertion (2) we can assume that G is connected, since $G_{\mathrm{ant}} \subset G^0$, and $1 \longrightarrow H \cap G^0 \longrightarrow G^0 \xrightarrow{q|_{G^0}} A \longrightarrow 0$ is an affine extension (see Corollary 2.10). Since $A = q(G) = q(G_{\mathrm{ant}} \cdot H) = q(G_{\mathrm{ant}})$, it follows that $A \cong G_{\mathrm{ant}}/(G_{\mathrm{ant}} \cap H)$, and hence Assertion (2) is proved.

In order to prove (1) in the non–connected situation, assume that G is not necessarily connected and let $g : T \to G \in G(T)$ be a T–point. Since $\mathcal{S}_{\mathrm{ant}}$ is an affine extension, there exists a faithfully flat morphism $h : T' \to T$ and $b : T' \to G_{\mathrm{ant}} \in G_{\mathrm{ant}}(T')$ such that $q \circ b = q \circ g \circ h : T' \to A \in A(T')$. It follows that $(g \circ h)b^{-1} \in H(T')$; therefore, $g \circ h \in G_{\mathrm{ant}}(T')H(T') \subset G_{\mathrm{ant}}H(T')$. But the morphism $m|_{G_{\mathrm{ant}} \times H} : G_{\mathrm{ant}} \times H \to G$ is a morphism of quasi-compact group schemes and therefore has closed image $G_{\mathrm{ant}}H$. Applying Lemma 2.41, we deduce that $g \in G_{\mathrm{ant}}H(T)$. Hence $G_{\mathrm{ant}}H = G$. $\square$

The proof of part (a) of assertion (3) appears in [14, Theorem 5.1.1]. In order to prove part (b) consider $\mathcal{G}_{\mathrm{ant}}$, the Chevalley decomposition of G_{ant}:

$$\mathcal{G}_{\mathrm{ant}} : \qquad 0 \longrightarrow (G_{\mathrm{ant}})_{\mathrm{aff}} \longrightarrow G_{\mathrm{ant}} \xrightarrow{\widetilde{q}} A \longrightarrow 0$$

Notice first that in general $\widetilde{q} \neq q$, since $(G_{\mathrm{ant}})_{\mathrm{aff}} \neq G_{\mathrm{ant}} \cap H$. As observed in Remark 2.24, since $(G_{\mathrm{ant}})_{\mathrm{aff}}$ is central in G as well as in H, $\widetilde{q}$ induces an affine extension of finite type $\mathcal{S}'$: $1 \longrightarrow H \longrightarrow G' \xrightarrow{q'} A \longrightarrow 0$. It is clear that the multiplication morphism $G_{\mathrm{ant}} \times H \to G$ induces a morphism of group schemes $f : G' = (G_{\mathrm{ant}} \times H)/\Delta\big((G_{\mathrm{ant}})_{\mathrm{aff}}\big) \to G$, with finite Kernel $\operatorname{Ker}(f) \cong (G_{\mathrm{ant}} \cap H)/(G_{\mathrm{ant}})_{\mathrm{aff}}$. $\square$

Notation 2.43. In view of Proposition 2.40, Remark 2.39 and Theorem 2.42, from now on if G is a quasi-compact group scheme, we denote $G_{\mathrm{ant}} = \operatorname{Ker}(\eta_G)$.

Remark 2.44. Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension. Then $G^0 = (G^0 \cap H)G_{\mathrm{ant}}$ and $G = HG^0 = HG_{\mathrm{ant}}$.

Indeed, $1 \longrightarrow H \cap G^0 \longrightarrow G^0 \xrightarrow{q|_{G^0}} A \longrightarrow 0$ is a closed sub-extension of $\mathcal{S}$ and the result follows from Theorem 2.42.

Remark 2.45. Let G be a connected group scheme of finite type. Consider the Chevalley decomposition of G: $1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \xrightarrow{q} A \longrightarrow 0$.

In this situation we have that: the Rosenlicht decomposition of G is $G = G_{\mathrm{aff}}G_{\mathrm{ant}}$; $(G_{\mathrm{ant}})_{\mathrm{aff}} \subseteq G_{\mathrm{aff}} \cap G_{\mathrm{ant}}$ is of finite index; and the product morphism $m : G_{\mathrm{ant}} \times G_{\mathrm{aff}} \to G$ induces an isogeny $(G_{\mathrm{ant}} \times G_{\mathrm{aff}})/\Delta\big((G_{\mathrm{ant}})_{\mathrm{aff}}\big) \to G$.

Hence, Theorem 2.42 can be viewed as a generalization to affine extensions of the above well known decomposition (see [14][Thm. 5.1.1]).

The following lemma is an easy consequence of Lemma 2.33.

Lemma 2.46. *If $f : G \to G'$ is a morphism of quasi-compact group schemes, then $f(G_{\mathrm{ant}}) \subset G'_{\mathrm{ant}}$. If G, G' are of finite type, then the morphism f induces the* *following commutative diagram of Chevalley decompositions:*

$$\begin{tikzcd}
\mathcal{G}_{\mathrm{ant}}: \arrow[d, "\phi"] & 0 \arrow[r] & (G_{\mathrm{ant}})_{\mathrm{aff}} \arrow[r] \arrow[d, "f|_{(G_{\mathrm{ant}})_{\mathrm{aff}}}"'] & G_{\mathrm{ant}} \arrow[r, "\widetilde{q}"] \arrow[d, "f|_{G_{\mathrm{ant}}}"] & A \arrow[r] \arrow[d, "\widetilde{f}"] & 0 \\
\mathcal{G}'_{\mathrm{ant}}: & 0 \arrow[r] & (G'_{\mathrm{ant}})_{\mathrm{aff}} \arrow[r] & G'_{\mathrm{ant}} \arrow[r, "\widetilde{q}'"'] & A' \arrow[r] & 0
\end{tikzcd}$$

Moreover, if f is faithfully flat, then the vertical arrows of the diagram above are also faithfully flat. In particular, if f is an affine faithfully flat morphism, then $\widetilde{f}$ is an isogeny.

PROOF. The morphism $\eta_{G'} \circ f : G \to \mathrm{Aff}(G')$ factors through η_G. It follows that $f(G_{\mathrm{ant}}) \subset G'_{\mathrm{ant}}$. Since in the finite type case G_{ant} is a smooth group scheme, all assertions follow from Lemma 2.33. $\square$

2.5. **Filtered systems of affine extensions.**

We begin with some considerations about limits of filtered systems of schemes over $\Bbbk$. Recall that a *filtered system of schemes* consists of a family $\big\{(X_\alpha, f_{\alpha,\beta}) : \alpha, \beta \in I\,,\ \alpha \geq \beta\,,\ f_{\alpha,\beta} : X_\alpha \to X_\beta\big\}$, where I is a finitely upper bounded — or filtered — poset, $\{X_\alpha : \alpha \in I\}$ is a family of schemes and $f_{\alpha,\beta}$ are morphisms of schemes — called the *connecting (or transition) morphisms* — such that if $\alpha \geq \beta \geq \gamma$, then $f_{\alpha,\gamma} = f_{\beta,\gamma} \circ f_{\alpha,\beta}$ and $f_{\alpha,\alpha} = \mathrm{id}_{X_\alpha}$. If the limit of such a system of schemes exists, we use the following notation: $(X, f_\alpha : \alpha \in I) = \lim_{\alpha \in I}\{X_\alpha\,,\ f_{\alpha,\beta} : \alpha, \beta \in I\}$ and when the rest of the ingredients are clear we write $X = \lim X_\alpha$. A family of morphisms $g_\alpha : Z \to X_\alpha\,,\ \alpha \in I$ is said to be *compatible* with the filtered system $\big\{(X_\alpha, f_{\alpha,\beta})\big\}$ if for all $\alpha \geq \beta$, $f_{\alpha,\beta} g_\alpha = g_\beta$.

Next, we recall some known properties of the limit in the above situation; the basic references are [33, §8] and [57, Tag 01YT].

Remark 2.47. Let $\big\{(X_\alpha, f_{\alpha,\beta})\big\}_{\alpha,\beta \in I}$ be a filtered system of schemes, and assume that the transition morphisms $f_{\alpha,\beta} : X_\alpha \to X_\beta$ are affine. Then:

(1) $X = \lim X_\alpha$ exists in the category of schemes and the induced morphisms f_α are affine. Moreover, if all the X_α are affine, so is the limit X (see [57, Tag 01YX]).

(2) If the transition morphisms $f_{\alpha,\beta}$ are surjective for all $\alpha, \beta \in I$ then the morphisms $f_\alpha : \lim X_\alpha \to X_\alpha$ are surjective for all $\alpha \in I$ — this is an easy exercise on limits on the category of topological spaces.

Moreover, if Z is a quasi-compact $\Bbbk$–scheme and $g_\alpha : Z \to X_\alpha$, $\alpha \in I$, is a family of compatible morphisms that are also surjective, then the induced morphism $g : Z \to X$ is surjective.

(3) If the transition morphisms $f_{\alpha,\beta}$ are faithfully flat for all $\alpha, \beta \in I$, then the morphisms f_α are faithfully flat for all α.

Moreover, if Z is a quasi-compact scheme and $g_\alpha : Z \to X_\alpha$, $\alpha \in I$, is a family of faithfully flat compatible morphisms, then the induced morphism $g : Z \to X$ is faithfully flat.

(4) For any $\alpha \in I$ and $U_\alpha \subseteq X_\alpha$ open subscheme, we have that $f_\alpha^{-1}(U_\alpha) = \lim_{\beta \geq \alpha} f_{\beta,\alpha}^{-1}(U_\alpha)$ as schemes.

Notation 2.48. In what follows we work with filtered systems in the categories of group schemes over $\Bbbk$ and $\mathrm{GE}\,|_{\mathrm{aff}}A$, with the additional condition that the transition morphisms are affine faithfully flat. We introduce for clarity the following notations.

(1) A filtered system of group schemes $\{G_\alpha, f_{\alpha,\beta}\}$ is *affine* (resp. *faithfully flat*) if the transition morphisms are affine (resp. faithfully flat) morphisms of group schemes.

(2) A *filtered system of affine extensions* is a family $\big\{(\mathcal{S}_\alpha, \phi_{\alpha,\beta}) : \alpha, \beta \in I\big\}$, where I is a filtered poset, such that for all $\alpha \leq \beta$ the transition morphisms $\phi_{\alpha,\beta}$

$$\begin{tikzcd}
\mathcal{S}_\alpha : \arrow[d, "\phi_{\alpha,\beta}"'] & 1 \arrow[r] & H_\alpha \arrow[r] \arrow[d] & G_\alpha \arrow[r, "q_\alpha"] \arrow[d, "f_{\alpha,\beta}"] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{S}_\beta : & 1 \arrow[r] & H_\beta \arrow[r] & G_\beta \arrow[r, "q_\beta"'] & A \arrow[r] & 0
\end{tikzcd}$$

are morphisms of affine extensions, and the family $\{G_\alpha, f_{\alpha,\beta}\}$ is a filtered system of group schemes — notice that in particular this implies that $\phi_{\alpha,\gamma} = \phi_{\beta,\gamma}\circ\phi_{\alpha,\beta}$ and $\phi_{\alpha,\alpha} = \mathrm{id}_{\mathcal{S}_\alpha}$ for all $\alpha \geq \beta \geq \gamma$.

A filtered system of affine extensions is *affine* (resp. *faithfully flat*) if the filtered system of group schemes $\{G_\alpha, f_{\alpha,\beta}\}$ is so — that is, in accordance with Definition 2.18, the transition morphisms $\phi_{\alpha,\beta}$ are affine, faithfully flat.

Remark 2.49. It follows from Remark 2.20 that a filtered system of affine extensions is always affine.

Proposition 2.50. *Let $\big\{(\mathcal{S}_\alpha, \phi_{\alpha,\beta} : \mathcal{S}_\alpha \to \mathcal{S}_\beta) : \alpha \geq \beta \in I\big\}$ be an (affine) faithfully flat filtered system in $\mathrm{GE}\,|_{\mathrm{aff}}A$ (see Notation 2.48). Then:*

(1) The limits $G := \lim G_\alpha$ and $H := \lim H_\alpha$ exist in the category of group schemes and H is affine. Moreover, the morphisms $f_\alpha : G \to G_\alpha$ are affine and faithfully flat.

(2) If we call $q : G \to A$ the morphism induced by the compatible morphisms $q_\alpha : G_\alpha \to A$, then $\mathrm{Ker}(q) = H$ and the sequence $\mathcal{S} :\ 1 \longrightarrow H \longrightarrow G \xrightarrow{\ q\ } A \longrightarrow 0$ coincides with $\lim \mathcal{S}_\alpha$ in $\mathrm{GE}\,|_{\mathrm{aff}}A$.

Proof. (1) The existence of the limits in the category of schemes and the affineness of H as well as the assertions about the maps f_α follow from Remark 2.47, taking also into account [45, Proposition II.1.3] (see Remark 2.16).

(2) First observe that $q : G \to A$ is affine and faithfully flat, since $q = q_\alpha f_\alpha$ with q_α and f_α affine and faithfully flat.

Next, we prove that the sequence $\mathcal{S}$ is left exact, that is $\mathrm{Ker}(q) = H$. The commutative diagrams

$$\begin{tikzcd}
H \arrow[r] \arrow[dr] & H_\alpha \arrow[r] \arrow[d] & G_\alpha \arrow[d] \\
& H_\beta \arrow[r] & G_\beta
\end{tikzcd}$$

induce an injective morphism of group schemes $H \to G$. As $H_\alpha = \mathrm{Ker}(q_\alpha)$ for all α, then $H \subset \mathrm{Ker}(q)$. Moreover, if $h : K \to G$ is a morphism of group schemes such that $q\circ h = 0$, then $q_\alpha\circ f_\alpha\circ h = 0$ for all α. Therefore, $\mathrm{Im}(f_\alpha\circ h) \subset H_\alpha$ for all α

and $f_{\alpha,\beta}\circ f_\alpha\circ h = f_\beta\circ h$, so $h : K \to G$ factors through H and the proof of the left exactness is finished.

Thus, $\mathcal{S} \in \mathrm{GE}\,|_{\mathrm{aff}} A$ and for all $\alpha \in I$ we have morphisms of affine extensions $\phi_\alpha : \mathcal{S} \to \mathcal{S}_\alpha$, compatible with the transition morphisms $\phi_{\alpha,\beta}$:

$$\begin{tikzcd}
\mathcal{S}: \arrow[d, "\phi_\alpha"] & 1 \arrow[r] & H \arrow[r] \arrow[d, "f_\alpha|_H"] & G \arrow[r, "q"] \arrow[d, "f_\alpha"] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{S}_\alpha: & 1 \arrow[r] & H_\alpha \arrow[r] & G_\alpha \arrow[r, "q_\alpha"'] & A \arrow[r] & 0
\end{tikzcd}$$

where f_α, $\alpha \in I$, are the morphisms associated to $G = \lim_\alpha G_\alpha$.

In order to prove that $(\mathcal{S}, \phi_\alpha) = \lim \mathcal{S}_\alpha$, let $\mathcal{S}'$: $1 \longrightarrow H' \longrightarrow G' \xrightarrow{q'} A \longrightarrow 0$ be an affine extension and $\phi'_\alpha : \mathcal{S}' \to \mathcal{S}_\alpha$ be a compatible family of morphisms of affine extensions. Then we have compatible families of morphisms $f'_\alpha : G' \to G_\alpha$ and $f'_\alpha|_{H'} : H' \to H_\alpha$, for the filtered systems $\{G_\alpha, f_{\alpha,\beta}\}$ and $\{H_\alpha, f_{\alpha,\beta}|_{H_\alpha}\}$ respectively, that factors through morphisms $f' : G' \to G$ and $f'' : H' \to H$. It is easy to see that $f'' = f'_{H'}$, and therefore f' induces a morphism of extensions $\phi' : \mathcal{S}' \to \mathcal{S}$, that factors the morphisms ϕ'_α. $\square$

Definition 2.51. An affine extension $\mathcal{S} \in \mathrm{GE}\,|_{\mathrm{aff}} A$ is called *pro-algebraic* if there exists an (affine) faithfully flat filtered system of affine extensions of finite type $\{\mathcal{S}_\alpha, \phi_{\alpha,\beta} : \alpha, \beta \in I\}$ such that such that $\mathcal{S} = \lim \mathcal{S}_\alpha$.

Remark 2.52. The term *pro-algebraic* has its roots in the fact that usually group schemes of finite type are called algebraic groups, see Remark 2.2.

Examples 2.53. (1) Any affine group scheme G is the limit of an affine faithfully flat filtered system of affine group schemes of finite type (see for example [60, Page 24]). In terms of affine extensions, this well known result reads as follows: let $G = \lim G_\alpha$ and consider the affine extensions $\mathcal{G}_{\mathrm{aff}}$: $1 \longrightarrow G \xrightarrow{\mathrm{id}} G \longrightarrow 0 \longrightarrow 0$, and $\mathcal{G}_{\alpha,\mathrm{aff}}$: $1 \longrightarrow G_\alpha \xrightarrow{\mathrm{id}} G_\alpha \longrightarrow 0 \longrightarrow 0$. Then $\mathcal{G}_{\mathrm{aff}} = \lim \mathcal{G}_{\alpha,\mathrm{aff}}$.

(2) Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension, with H central. Then $H = \lim H_\alpha$, where $\{H_\alpha\}_{\alpha\in I}$ is an (affine) faithfully flat filtered system of affine group schemes of finite type; denote the transition morphisms by $f_{\alpha,\beta}$: $H_\alpha \to H_\beta$ and the canonical projections by $f_\alpha : H \to H_\alpha$. Since H is central and the canonical maps are faithfully flat, we can apply Remark 2.24 and construct the push-forward by f_α, obtaining an affine faithfully flat filtered system of affine extensions as follows:

$$\begin{tikzcd}
(f_\alpha)_*(\mathcal{S}): \arrow[d, "\phi_{\alpha,\beta}"] & 1 \arrow[r] & H_\alpha \arrow[r] \arrow[d, "f_{\alpha,\beta}"'] & G \times^H H_\alpha \arrow[r, "\pi_{H_\alpha}"] \arrow[d, "\overline{f_{\alpha,\beta}}"] & A \arrow[r] \arrow[d, equal] & 0 \\
(f_\beta)_*(\mathcal{S}): & 1 \arrow[r] & H_\beta \arrow[r] & G \times^H H_\alpha \arrow[r, "\pi_{H_\alpha}"] & A \arrow[r] & 0
\end{tikzcd}$$

where $G \times^H H_\alpha = (G \times H_\alpha)/\Delta(H)$ and $\overline{f_{\alpha,\beta}} : G \times^H H_\alpha \to G \times^H H_\beta$ is the morphism induced by $G \times H_\alpha \to G \times^H H_\beta$, $(g, h_\alpha) \mapsto \big[g, f_{\alpha,\beta}(h_\alpha)\big]$ — notice that if $h \in H$, then $\big[gh^{-1}, f_{\alpha,\beta}(f_\alpha(h)h_\alpha)\big] = \big[gh^{-1}, f_\beta(h)f_{\alpha,\beta}(h_\alpha)\big] = \big[g, f_{\alpha,\beta}(h_\alpha)\big] \in G \times^H H_\beta$.

Thus, if H is central, the affine extension $\mathcal{S}$ is pro-algebraic. In particular, commutative affine extensions are pro-algebraic.

Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension; then G is a quasi-compact group scheme (see Remark 2.17). Conversely, Perrin proved in [45, Corollary V.4.3.1] that if G is a *connected* quasi-compact group scheme, then G fits into an affine extension. This result is a consequence of the Chevalley decomposition of group schemes of finite type (Theorem 2.26), together with Perrin's Approximation Theorem below, that in particular shows that any quasi-compact connected group scheme G is pro-algebraic, that is G is the limit of a directed system of group schemes of finite type.

Theorem 2.54 (Perrin's Approximation Theorem, [45, Théorème V.3.1])**.** *Let G be a quasi-compact group scheme. Then there exists an affine faithfully flat filtered system of group schemes of finite type $\{G_\alpha, f_{\alpha,\beta} : alpha, \beta \in I\}$, such that $G \cong \lim_\alpha G_\alpha$. In particular, the canonical morphisms $f_\alpha : G \to G_\alpha$ are faithfully flat, with* $\operatorname{Ker}(f_\alpha)$ *an affine closed subgroup.* □

Combining Theorem 2.42 with the well known approximation theorem for affine group schemes, we can refine Perrin's approximation Theorem and show that any affine extension is pro-algebraic.

Theorem 2.55. *Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension. Then $\mathcal{S}$ is pro-algebraic.*

Proof. Let $G = G_{\mathrm{ant}}H = G_{\mathrm{ant}} \times^{G_{\mathrm{ant}}\cap H} H$ be the Rosenlicht decomposition of $\mathcal{S}$ (Theorem 2.42). Since H is an affine scheme, there exists an affine faithfully flat filtered system $\{H_\alpha, p_{\alpha,\beta} : \alpha, \beta \in I\}$, such that $H = \lim H_\alpha$; let $p_\alpha : H \to H_\alpha$ be the canonical morphisms (of group schemes, faithfully flat). Since G_{ant} is central, $G_{\mathrm{ant}} \cap H$ is central in H; it follows that $p_\alpha(G_{\mathrm{ant}} \cap H)$ is central in H_α. By Remark 2.24, we have morphisms of affine extensions

$$\begin{array}{ccccccccccc}
\mathcal{S}_{\mathrm{ant}} & & 1 & \longrightarrow & G_{\mathrm{ant}} \cap H & \longrightarrow & G_{\mathrm{ant}} & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow{\scriptstyle \phi_\alpha} & & & & {\scriptstyle p_\alpha}\downarrow & & {\scriptstyle f_\alpha}\downarrow & & \| & & \\
\mathcal{S}_\alpha & & 1 & \longrightarrow & H_\alpha & \longrightarrow & G_{\mathrm{ant}} \times^{G_{\mathrm{ant}}\cap H} H_\alpha & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

where $f_\alpha(z) = [z, 1]$ for all $z \in G_{\mathrm{ant}}$ (see Example 2.53). These morphisms clearly extend to morphisms

$$\begin{array}{ccccccccccc}
\mathcal{S} & & 1 & \longrightarrow & H & \longrightarrow & G = G_{\mathrm{ant}}H & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow{\scriptstyle \lambda_\alpha} & & & & {\scriptstyle p_\alpha}\downarrow & & {\scriptstyle \ell_\alpha}\downarrow & & \| & & \\
\mathcal{S}_\alpha & & 1 & \longrightarrow & H_\alpha & \longrightarrow & G_{\mathrm{ant}} \times^{G_{\mathrm{ant}}\cap H} H_\alpha & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

where $\ell_\alpha : G \to G_{\mathrm{ant}} \times^{G_{\mathrm{ant}}\cap H} H_\alpha$ is the morphism of group schemes induced by $\pi_{G_{\mathrm{ant}}\times H_\alpha} \circ (\mathrm{id}_{G_{\mathrm{ant}}} \times p_\alpha) : G_{\mathrm{ant}} \times H \to G_{\mathrm{ant}} \times H_\alpha \to G_{\mathrm{ant}} \times^{G_{\mathrm{ant}}\cap H} H_\alpha$.

Moreover, the morphisms $\mathrm{id}_{G_{\mathrm{ant}}} \times p_{\alpha\beta}$ induce affine, faithfully flat morphisms of group schemes $\ell_{\alpha,\beta} : G_\alpha = G_{\mathrm{ant}} \times^{G_{\mathrm{ant}}\cap H} H_\alpha \to G_\beta = G_{\mathrm{ant}} \times^{G_{\mathrm{ant}}\cap H} H_\beta$. Therefore, the family $\{\mathcal{S}_\alpha, \ell_{\alpha,\beta}\}$ conforms an affine faithfully flat filtered system of affine extensions of finite type; let $\mathcal{G}$ be its limit.

We prove now that $\mathcal{G} \cong \mathcal{S}$. Indeed, the morphisms λ_α induce a faithfully flat morphism

$$\begin{tikzcd} \mathcal{S} \arrow[d, "\lambda"] & 1 \arrow[r] & H \arrow[r] \arrow[d, "\ell|_H"'] & G = G_{\mathrm{ant}}H \arrow[r] \arrow[d, "\ell"'] & A \arrow[r] \arrow[d, equal] & 0 \\ \mathcal{G} & 1 \arrow[r] & \widetilde{H} \arrow[r] & \widetilde{G} \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

But by construction $\widetilde{H} = H = \lim H_\alpha$ and $\ell|_H = \mathrm{id}_H$. Moreover, by the commutativity of the diagram above, $\mathrm{Ker}(\ell) \subset H$. Hence, ℓ is injective and $\mathcal{S} \cong \mathcal{G}$. $\square$

Once we have established that any affine extension is pro-algebraic, we can estate Rosenlicht decomposition (Theorem 2.42) in terms of affine sub-extensions.

Lemma 2.56. *Let $\{\mathcal{S}_\alpha, \phi_{\alpha,\beta} : \mathcal{S}_\alpha \to \mathcal{S}_\beta ; \alpha \geq \beta\}$, be an (affine) faithfully flat filtered system of affine extensions of finite type, and assume that all the extensions are Chevalley decompositions. Consider the limit $\mathcal{S} := \lim \mathcal{S}_\alpha$*

$$\mathcal{S}: \qquad 1 \longrightarrow H \longrightarrow G \xrightarrow{\ q\ } A \longrightarrow 0.$$

Then H is minimal among the affine subgroups schemes $H' \subset G$ such that the quotient scheme G/H' exists and is proper.

Proof. Let $H' \subset H$ be an affine subgroup scheme of H such that G/H' exists and is proper. If $f_\alpha : G \to G_\alpha$ denotes as usual the mid morphism of ϕ_α (recall the notations in 2.48 and 2.19), then $f_\alpha(H')$, the scheme theoretic image of H' by f_α, is a closed affine subscheme of $G_{\alpha,\mathrm{aff}}$. Since f_α is faithfully flat for all α, then $f_\alpha|_H : H \to G_{\alpha,\mathrm{aff}}$ is faithfully flat (see Lemma 2.33), and we can factor f_α to a faithfully flat morphism $\overline{f}_\alpha : G/H' \to G_\alpha/f_\alpha(H')$.

Since G/H' is proper, it follows that $G_\alpha/f_\alpha(H')$ is also proper, and therefore, by minimality of $G_{\alpha,\mathrm{aff}}$, $f_\alpha(H') = G_{\alpha,\mathrm{aff}}$. It follows that $\{f_\alpha|_{H'} : H' \to G_{\alpha,\mathrm{aff}}\}$ is a compatible family of faithfully flat morphism, with $\mathrm{inc} : H' \to H = \lim G_{\alpha,\mathrm{aff}}$ as induced morphism. We deduce from that $H' = H$, since inc is a faithfully flat morphism (see Remark 2.47). $\square$

Lemma 2.57. *The affinization functor* $\mathrm{Aff} : \mathrm{Sch}|_{\mathrm{qc}}\Bbbk \to \mathrm{Sch}|_{\mathrm{aff}}\Bbbk$ *preserves limits of affine faithfully flat filtered systems of quasi-compact group schemes.*

Proof. Let $\{G_\alpha,\ f_{\alpha,\beta} : G_\alpha \to G_\beta, \alpha \geq \beta \in I\}$ be an affine faithfully flat filtered system of quasi-compact group schemes, with limit the quasi-compact group scheme $G = \lim G_\alpha$. Then the transition morphisms induce morphisms $\widetilde{f_{\alpha,\beta}} = \mathrm{Aff}(f_{\alpha,\beta}) : \mathrm{Aff}(G_\alpha) \to \mathrm{Aff}(G_\beta)$, such that the solid part of the following diagram is commutative:

$$\begin{tikzcd} G \arrow[r, dotted, "f_\alpha"] \arrow[d, dotted, "\lim \eta_{G_\alpha}"'] & G_\alpha \arrow[r, "f_{\alpha,\beta}"] \arrow[d, "\eta_{G_\alpha}"] & G_\beta \arrow[d, "\eta_{G_\beta}"] \\ \lim \mathrm{Aff}(G_\alpha) \arrow[r, dotted, "\widetilde{f_\alpha}"'] & \mathrm{Aff}(G_\alpha) \arrow[r, "\widetilde{f_{\alpha,\beta}}"'] & \mathrm{Aff}(G_\beta) \end{tikzcd} \tag{2.4}$$

Since the morphisms $\eta_{G_\alpha}, \eta_{G_\beta}$ and $f_{\alpha,\beta}$ are faithfully flat (see Prop. 2.40), it follows that $\widetilde{f_{\alpha,\beta}}$ is faithfully flat. Thus, the limit $\lim \mathrm{Aff}(G_\alpha)$ exists and it is an affine group scheme fitting into the left part of diagram (2.4). Moreover, the (faithfully flat) morphisms $\eta_{G_\alpha}\circ f_\alpha : G \to \mathrm{Aff}(G_\alpha)$ induce a faithfully flat morphism $f : G \to L$ (see Remark 2.47), that factorizes through η_G, by the universal property of the affinization morphism as shown in the diagram below.

$$\begin{tikzcd} & G \arrow[dl, "\eta_G"'] \arrow[r, "f_\alpha"] \arrow[d, "\lim \eta_{G_\alpha}"] & G_\alpha \arrow[d, "\eta_{G_\alpha}"] \\ \mathrm{Aff}(G) \arrow[r, "f"'] & \lim \mathrm{Aff}(G_\alpha) \arrow[r, "\widetilde{f_\alpha}"'] & \mathrm{Aff}(G_\alpha) \end{tikzcd} \tag{2.5}$$

It is then clear that $f_\alpha\big(\mathrm{Ker}(f\circ\eta_G)\big) \subset G_{\alpha,\mathrm{ant}}$. Therefore, $\mathrm{Ker}(f\circ\eta_G) \subseteq G_{\mathrm{ant}}$ and as the other inclusion is evident, it follows that $\lim \mathrm{Aff}(G_\alpha) = G/G_{\mathrm{ant}} = \mathrm{Aff}(G)$. $\square$

Theorem 2.58. *Let $\mathcal{S} = \lim \mathcal{S}_\alpha$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension, with $\mathcal{S}_\alpha$: $1 \longrightarrow H_\alpha \longrightarrow G_\alpha \xrightarrow{q_\alpha} A \longrightarrow 0$ an (affine) faithfully flat filtered system of finite type in $\mathrm{GE}|_{\mathrm{aff}} A$, with transition morphisms $\phi_{\alpha,\beta} : \mathcal{S}_\alpha \to \mathcal{S}_\beta$. Let us call $G_{\alpha,\mathrm{ant}} = \mathrm{Ker}(\eta_{G_\alpha}) \subset G_\alpha$ (see Remark 2.39 and Theorem 2.42). Then the morphisms $f_{\alpha,\beta}|_{G_{\alpha,\mathrm{ant}}}$ define an affine faithfully flat filtered system for the family $\{\mathcal{S}_{\alpha,\mathrm{ant}}\}_{\alpha\in I}$*

$$\begin{tikzcd} \mathcal{S}_{\alpha,\mathrm{ant}}: \arrow[d, "\widetilde{\phi}_{\alpha,\beta}"] & 1 \arrow[r] & G_{\alpha,\mathrm{ant}}\cap H_\alpha \arrow[r] \arrow[d, "f_{\alpha,\beta}|_{G_{\alpha,\mathrm{ant}}\cap H}"'] & G_{\alpha,\mathrm{ant}} \arrow[r, "q_\alpha|_{G_{\alpha,\mathrm{ant}}}"] \arrow[d, "f_{\alpha,\beta}|_{G_{\alpha,\mathrm{ant}}}"] & A \arrow[r] \arrow[d, equal] & 0 \\ \mathcal{S}_{\beta,\mathrm{ant}}: & 1 \arrow[r] & G_{\beta,\mathrm{ant}}\cap H_\beta \arrow[r] & G_{\beta,\mathrm{ant}} \arrow[r, "q_\beta|_{G_{\beta,\mathrm{ant}}}"'] & A \arrow[r] & 0 \end{tikzcd}$$

with limit $\mathcal{S}_{\mathrm{ant}} = \lim \mathcal{S}_{\alpha,\mathrm{ant}}$: $1 \longrightarrow H\cap G_{\mathrm{ant}} \longrightarrow G_{\mathrm{ant}} \xrightarrow{q|_{G_{\mathrm{ant}}}} A \longrightarrow 0$.

Proof. First we prove that in the above context, $\lim G_{\alpha,\mathrm{ant}} = G_{\mathrm{ant}}$. By Lemma 2.46, $f_{\alpha,\beta}(G_{\alpha,\mathrm{ant}}) \subset G_{\beta,\mathrm{ant}}$ for all $\alpha \geq \beta$ and $f_{\alpha,\beta}|_{G_{\alpha,\mathrm{ant}}}$ is an affine faithfully flat morphism; thus the limit $\lim G_{\alpha,\mathrm{ant}}$ exists (see Remark 2.47); let $L = \lim G_{\alpha,\mathrm{ant}}$ and $\widetilde{f_\alpha} : L \to G_{\alpha,\mathrm{ant}}$ be the canonical morphisms. Then the family $\widetilde{f_\alpha}$ induces a morphism $\widetilde{f} : L \to G$.

On the other hand, since $f_\alpha(G_{\mathrm{ant}}) \subset G_{\alpha,\mathrm{ant}}$, it follows that there exists a morphism $h : G_{\mathrm{ant}} \to L$, such that $\widetilde{f_\alpha}\circ h = f_\alpha|_{G_{\mathrm{ant}}}$ for all α. By the universal property of the limit $G = \lim G_\alpha$, we deduce that $\widetilde{f}\circ h = \mathrm{inc} : G_{\mathrm{ant}} \subseteq G$. Hence, it suffices to prove that L is anti-affine, since if this is the case then $\widetilde{f}(L)$ is anti-affine and therefore $\widetilde{f}(L) \subset G_{\mathrm{ant}}$. Then $G_{\mathrm{ant}} = L = \lim G_{\alpha,\mathrm{ant}}$. But applying Lemma 2.57 we see that $\mathrm{Aff}(\lim G_{\alpha,\mathrm{ant}}) = \lim \mathrm{Aff}(G_{\alpha,\mathrm{ant}}) = \mathrm{Spec}(\Bbbk)$.

Now we prove that $G_{\mathrm{ant}}\cap H = \lim G_{\alpha,\mathrm{ant}}\cap H_\alpha$. Since $f_{\alpha,\beta}|_{H_\alpha} : H_\alpha \to H_\beta$ is an affine morphism (see Remark 2.49), it follows that the family $\{G_{\alpha,\mathrm{ant}}\cap H_\alpha, f_{\alpha,\beta}|_{G_{\alpha,\mathrm{ant}}\cap H_\alpha} : G_{\alpha,\mathrm{ant}}\cap H_\alpha \to G_{\beta,\mathrm{ant}}\cap H_\beta : \alpha,\beta \in I\}$ is an affine filtered system of group schemes. Then the limit $N = \lim G_{\alpha,\mathrm{ant}}\cap H_\alpha$ is a group scheme, and the restriction morphisms $f|_{G_{\mathrm{ant}}\cap H} : G_{\mathrm{ant}}\cap H \to G_{\alpha,\mathrm{ant}}\cap H_\alpha$ induce a morphism

$\ell : G_{\mathrm{ant}} \cap H \to N$. But it is clear that $N \subset G_{\mathrm{ant}} \cap H$ — since $0 = q_\alpha \circ f_\alpha : N \to A$ for all α and that $q_\alpha(N) \subset G_{\alpha,\mathrm{ant}}$ —; therefore $N = G_{\mathrm{ant}} \cap H$. □

As a consequence of Theorem 2.58, we have the following result.

Lemma 2.59. *Let $\mathcal{S} = \lim \mathcal{S}_\alpha$, $\mathcal{S}' = \lim \mathcal{S}'_\alpha$ be two affine extensions, where $\mathcal{S}_\alpha, \mathcal{S}'_\alpha$ are of finite type, and $\phi : \mathcal{S} \to \mathcal{S}'$ a morphism of affine extensions:*

$$\begin{tikzcd}
\mathcal{S} : \arrow[d, "\phi"'] & 0 \arrow[r] & H \arrow[r] \arrow[d] & G \arrow[r, "q"] \arrow[d, "f"] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{S}' : & 0 \arrow[r] & H' \arrow[r] & G' \arrow[r, "q'"'] & A \arrow[r] & 0
\end{tikzcd}$$

Then f induces by restriction a morphism of affine extensions

$$\begin{tikzcd}
\mathcal{S}_{\mathrm{ant}} : \arrow[d] & 0 \arrow[r] & G_{\mathrm{ant}} \cap H \arrow[r] \arrow[d, "f|_{G_{\mathrm{ant}} \cap H}"'] & G_{\mathrm{ant}} \arrow[r, "q|_{G_{\mathrm{ant}}}"] \arrow[d, "f|_{G_{\mathrm{ant}}}"] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{S}'_{\mathrm{ant}} : & 0 \arrow[r] & G'_{\mathrm{ant}} \cap H' \arrow[r] & G'_{\mathrm{ant}} \arrow[r, "q'|_{G_{\mathrm{ant}}}"'] & A \arrow[r] & 0
\end{tikzcd}$$

Moreover, if f is faithfully flat then $f|_{G_{\mathrm{ant}}} : G_{\mathrm{ant}} \to G'_{\mathrm{ant}}$ is faithfully flat.

Proof. By Lemma 2.46, we have that $f(G_{\mathrm{ant}} \subset G'_{\mathrm{ant}}$ and that $f|_{G_{\mathrm{ant}}}; G_{\mathrm{ant}} \to G_{\mathrm{ant}}$ is a faithfully flat morphism. □

Theorem 2.60 (Rosenlicht decomposition of affine extensions, revisited)**.**
Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension, with G connected. Let $\{\mathcal{S}_\alpha, \phi_{\alpha,\beta} : \alpha, \beta \in I\}$ be an (affine) faithfully flat filtered system of affine extensions of finite type, with $\mathcal{S}_\alpha$: $1 \longrightarrow H_\alpha \longrightarrow G_\alpha \xrightarrow{q_\alpha} A \longrightarrow 0$. Then:

(1) Let $\mathcal{G}_{\alpha,\mathrm{ant}}$: $1 \longrightarrow (G_{\alpha,\mathrm{ant}})_{\mathrm{aff}} \longrightarrow G_{\alpha,\mathrm{ant}} \xrightarrow{\widetilde{q_\alpha}} A \longrightarrow 0$ be the Chevalley decomposition of $G_{\alpha,\mathrm{ant}}$, for $\alpha \in I$. Then $G_{\mathrm{ant}} \cap H$ contains $K = \lim (G_{\alpha,\mathrm{ant}})_{\mathrm{aff}}$ as a closed subgroup scheme, and $(G_{\mathrm{ant}} \cap H)/K$ is finite.

(2) The induced space $G' = G_{\mathrm{ant}} \times^K H$ is a quasi-compact group scheme, and the canonical morphism (induced by the multiplication) $f : G' \to G$ is an isogeny, and $\mathrm{Ker}(f) = (G_{\mathrm{ant}} \cap H)/K$.

Proof. First, observe that since $G_{\mathrm{ant}} = \lim G_{\alpha,\mathrm{ant}}$ (by Theorem 2.58) and $H = \lim H_\alpha$ (by Proposition 2.50), it follows that $G_{\mathrm{ant}} \cap H = \lim G_{\alpha,\mathrm{ant}} \cap H_\alpha$. Let $\nu_\alpha : K \to (G_{\alpha,\mathrm{ant}})_{\mathrm{aff}}$ be the canonical morphisms. Then, by Theorem 2.42 the family $\{\nu_\alpha\}$ induces a compatible family $\{\widetilde{\nu}_\alpha : K \to G_{\alpha,\mathrm{ant}} \cap H_\alpha\}$, with in turn induces a morphism $\nu : K \to G_{\mathrm{ant}} \cap H$. In order to prove that $K \subset G_{\mathrm{ant}} \cap H$ is a closed subgroup of finite type, observe that $\mathrm{Ker}(\nu) = \nu^{-1}\big(\cap_\alpha \mathrm{Ker}(f_\alpha|_{H \cap G_{\mathrm{ant}}})\big)$ (again by Theorem 2.58 and Proposition 2.50). It follows from the compatibility conditions that $\mathrm{Ker}(\nu) = \bigcap_\alpha \big(\mathrm{Ker}((\nu_\alpha|_{H \cap G_{\mathrm{ant}}}) = \mathrm{Spec}(\Bbbk)$.

In order to prove that K is of finite index in $G_{\mathrm{ant}} \cap H$, we follow the procedure presented in [14, Theorem 5.1.1].

Since the filtered system $\big\{(G_{\alpha,\mathrm{ant}})_{\mathrm{aff}}, f_{\alpha,\beta}|_{G_{\alpha,\mathrm{ant}})_{\mathrm{aff}}}\big\}$ is affine and faithfully flat (see Lemma 2.46), we have the following commutative diagrams of groups schemes, with exact sequences of group schemes as rows and affine faithfully flat vertical arrows, for any $\beta \geq \alpha$ — notice that by Theorem 2.42 $\overline{G_\alpha}$ is a finite group scheme —:

$$\begin{tikzcd}
1 \arrow[r] & (G_{\alpha,\mathrm{ant}})_{\mathrm{aff}} \arrow[r] \arrow[d, "f_{\alpha,\beta}|_{(G_{\alpha,\mathrm{ant}})_{\mathrm{aff}}}"'] & G_\alpha \arrow[r, "\overline{q_\alpha}"] \arrow[d, "f_{\alpha,\beta}"] & \overline{G_\alpha} = G_\alpha/(G_{\alpha,\mathrm{ant}})_{\mathrm{aff}} \arrow[r] \arrow[d, "\overline{f_{\alpha,\beta}}"] & 1 \\
1 \arrow[r] & (G_{\beta,\mathrm{ant}})_{\mathrm{aff}} \arrow[r] & G_\beta \arrow[r, "\overline{q_\beta}"] & \overline{G_\beta} = G_\beta/(G_{\beta,\mathrm{ant}})_{\mathrm{aff}} \arrow[r] & 1
\end{tikzcd}$$

Taking limits, we deduce that $G/K \cong \lim \overline{G_\alpha}$ — since the compatible morphisms $\widetilde{q_\alpha} \circ f_\alpha : G \to \overline{G_\alpha}$ induce a faithfully flat morphism $G \to \lim \overline{G_\alpha}$ with Kernel equal to K. Thus we have the following commutative diagram of groups schemes, with exact sequences as rows and faithfully flat vertical arrows:

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, two heads, "f_\alpha|_K"'] & G \arrow[r] \arrow[d, two heads, "f_\alpha"] & G/K \arrow[r] \arrow[d, two heads, "\overline{f_\alpha}"] & 1 \\
1 \arrow[r] & (G_{\alpha,\mathrm{ant}})_{\mathrm{aff}} \arrow[r] & G_\alpha \arrow[r] & \overline{G_\alpha} = G_\alpha/(G_{\alpha,\mathrm{ant}})_{\mathrm{aff}} \arrow[r] & 1
\end{tikzcd}$$

Sine $G = G_{\mathrm{ant}}H$ (see Theorem 2.42), it follows that $G/K = (G_{\mathrm{ant}}H)/K = (G_{\mathrm{ant}}/K)(H/K)$. Since H/K is the limit of the affine group schemes of finite type $H_\alpha/(G_{\alpha,\mathrm{ant}})_{\mathrm{aff}}$, it is an affine group scheme. On the other hand, $G_{\mathrm{ant}}/K \cong \big(G_{\mathrm{ant}}/(G_{\mathrm{ant}} \cap H)\big)/\big((G_{\mathrm{ant}} \cap H)/K\big) = A/\big((G_{\mathrm{ant}} \cap H)/H\big)$ is an abelian variety (see Theorem 2.58). Therefore, $(H \cap G_{\mathrm{ant}})/K \cong (H/K) \cap (G_{\mathrm{ant}}/K)$ is finite.

The remaining assertions follow easily. $\square$

2.6. H–torsors and induced spaces.

Let G be a smooth group scheme of finite type over an algebraically closed field $\Bbbk$, $H \subset G$ a closed subgroup scheme and X a quasi-projective scheme equipped with an H–action. Serre proved in [55] that the diagonal action $H \times (G \times X) \to G \times X$, $h \cdot (g, x) = (gh^{-1}, h \cdot x)$ has a geometric quotient, that we denote as $G \times^H X$. If $(g, x) \in G \times X$, then we denote by $[g, x]$ the class of (g, x) in the quotient. Then $G \times^H X$ is a G–scheme (with action given by $g' \cdot [g, x] = [g'g, x]$), and the canonical projection $G \times^H X \to G/H$, induced by $[g, x] \mapsto gH$, is a fiber bundle, with fibers isomorphic to X. We call $G \times^H X$ the *induced space*.

Later on, Serre's result was generalized in several directions: let H be a group scheme of finite type and Y an H–scheme (for a right H–action), such that the geometric quotient $Y \to Y/H$ exists (in the sense of GIT, [44, pages 3,4]). If X is an H–scheme, we are concerned with the existence of the quotient for the diagonal action, that we denote as $\pi_{Y \times X} : Y \times X \to Y \times^H X := (Y \times X)/H$. In [44, Proposition 7.1], Mumford gives sufficient conditions in terms of the existence of an ample H–linearized line bundle on $H \times X$ (see Definition 7.24 below) in order to guarantee the existence of $Y \times^H X$ (by means of "fppf descent" techniques) — see [16, § 3.3] for a detailed proof of how to apply Mumford's result in order to prove the existence of $Y \times^H X$. In [36, Chapter I.5], Jantzen studies this problem in the context of schemes over a commutative ring R. It is also worth noting that in [6] Bialynicki-Birula studied the existence of the induced space $Y \times^H X$ for locally isotrivial (in the finite étale topology) H–torsors $Y \to Y/H$, in the context of algebraic spaces — of course, some additional hypothesis must be made on Y.

Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension and V a finite dimensional H–module. Then $q : G \to A$ is affine and faithfully flat. Thus, we are in the setting of fpqc descent (see [29, Exposé VIII]), and we can guarantee the existence of the quotient $\pi_{G\times V} : G \times V \to G \times^H V$, as in Theorem 2.61 below — notice that this result follows from the works cited above, but we couldn't find it as a precise statement.

Theorem 2.61. *Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be a affine extension of the abelian variety A, V be a finite dimensional H–module, and consider the diagonal H–action $a_{G\times V} := \big(m\circ(p_2, i\circ p_1), a_V\circ p_{13}\big) : H \times (G \times V) \to G \times V$, where p_{13} is the projection in the first and third coordinates and $a_V : H \times V \to V$ is the action associated to the representation. Then the scheme $G \times V$ endowed with the H–action $a_{G\times V}$ admits a geometric quotient $(G \times^H V, \pi_{G\times V} : G \times V \to G \times^H V)$ in the category of schemes over $\Bbbk$, in the sense of GIT, [44, pages 3,4]. Moreover, $E_V := G \times^H V$ is a G–*linearized *vector bundle with fibers isomorphic to V — that is, E_V admits a left G–action, linear on the fibers, such that the canonical projection $\pi_V : E_V \to A$ is a G–equivariant morphism.*

Proof. The existence of the quotient E_V, as well as the fact that the fibers of $\pi_V : E_V \to A$ are isomorphic to V, follow directly from fpqc descent (see [29, Exposé VIII, Theorem 2.1]). Moreover, the affine morphism $G \times V \to G$ can be seen as the bundle associated to the free sheaf $\mathcal{O}_G^{\oplus \dim V}$ and the local triviality of $E_V \to A$ follows from *loc.cit.* Exposé VIII, Theorem 1.1 and Corollary 1.2. Finally, it is clear that $G \times (G \times^H V) \to G \times^H V$, (induced by $g' \cdot (g, v) = [g'g, v]$), is an action linear on the fibers, and that π_V is a G–equivariant fibration. □

Notation 2.62. Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension and V an H–module. If $f : G \times V \to Y$ is H–invariant, then by the universal property of the quotient there exists a unique morphism $\widetilde{f} : E_V \to Y$ such that $\widetilde{f}\circ \pi_{G\times V} = f$. We will abuse notations denote $\widetilde{f}\big([g, v]\big) = f(g, y)$.

Remark 2.63. (1) Notice that $\pi_{G\times V} : G \times V \to G \times^H V$ is an H–torsor.

(2) By definition of geometric quotient, $G \times^H V$ represents the quotient of the fpqc sheaf $G \times V$ by the pre-relation $j = (a_{G\times V}, p_{23}) : H \times (G \times V) \to (G \times V) \times (G \times V)$ (see [57, Tags 022O and 02VE]). It follows in particular that $\pi_{G\times V} : G \times V \to G \times^H V$ is a categorical quotient in the category of fpqc sheaves (see [44, pages 3,4]).

(3) In this context, recall that a morphism (of schemes, resp. fpqc sheaves) $f : G \times V \to Z$ is H–invariant if $f \circ a_{G\times V} = f \circ p_{23} : H \times (G \times V) \to Z$.

3. A finite dimensional representation theory for affine extensions

3.1. Homogeneous vector bundles over an abelian variety.

In this section we recall some basic facts on the category of *homogeneous vector bundles* over an abelian variety (see Definition 3.9 below). The study of homogeneous vector bundles over an abelian variety was initiated by Atiyah in 1956 (see [3], [4], [5]). Later on, Miyanishi, Mukai and others generalized Atiyah's original results (for homogeneous vector bundles over elliptic curves over $\mathbb{C}$) to a more general setting — for homogeneous vector bundles over an abelian variety A, over an arbitrary algebraically closed field $\Bbbk$ —, giving a nice description of the corresponding category and its main properties (see for example [42], [43] and [8]). Recently, Brion in [18] introduced the definition and first properties of the category of homogeneous vector bundles over an arbitrary field $\Bbbk$. In what follows we take Brion's definition as departure point in order to enlarge the category of homogeneous vector bundles by introducing new morphisms (see Definition 3.18 below). For this, we adapt the approach taken in [8] (where the authors dealt with homogeneous vector bundles over an algebraically closed field) to this more general context.

Definition 3.1. If T is an scheme, then the *category of vector bundles with base* T, denoted as $\mathrm{VB}_0(T)$, is defined as follows:

(1) *Objects*: the family of vector bundles with base T. Recall that a vector bundle is a pair (E,π) with $\pi : E \to T$ a morphism that is locally trivial in the Zariski topology: there exists an open covering $\{U_i\}_{i\in I}$ of T and isomorphisms $\psi_i : \mathbb{A}^n_{U_i} \to \pi^{-1}(U_i)$ compatible with π — called the trivializations of the bundle — such that for any affine open subset $V = \mathrm{Spec}(R) \subset U_i \cap V_j$, the "transition morphisms" $\psi_j^{-1} \circ \psi_i|_{\mathbb{A}^n_V} : \mathbb{A}^n_V \to \mathbb{A}^n_V$ are given by linear automorphisms of $R[x_1,\dots,x_n]$. If $t \in T$, the n-dimensional $\Bbbk(t)$–vector space $\pi^{-1}(t) := E \times_T \mathrm{Spec}\big(\Bbbk(t)\big)$ is called the fiber of E at $t \in T$. Notice that in particular the morphism π is affine.

(2) *Arrows*: if $\pi : E \to T, \pi' : E' \to T$ are vector bundles over T, a *morphism of vector bundles* is a morphism $f : \pi \to \pi'$ of schemes over T, i.e. a morphism of schemes $f : E \to E'$ such that $\pi' \circ f = \pi$ i.e. the diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ T \arrow[r, "\mathrm{id}_T"'] & T \end{tikzcd}$$

is commutative, and for any pair of trivializations $\psi'_j : \mathbb{A}^m_{V_j} \to {\pi'}^{-1}(V_j)$ and $\psi_i : \mathbb{A}^n_{U_i} \to \pi^{-1}(U_i)$ and any open affine subset $V = \mathrm{Spec}(R) \subset V_j \cap U_i$, the morphism ${\psi'_j}^{-1} \circ f \circ \psi_i|_{\mathbb{A}^n_V} : \mathbb{A}^n_V \to \mathbb{A}^m_V$ is given by a linear endomorphism $R[x_1,\dots,x_n] \to R[y_1,\dots,y_m]$.

Notation 3.2. (1) In the literature, the vector bundles (defined as above) are sometimes called *geometric vector bundles*, see for example [28, Definition 11.5] or [34, Exer. 5.18]. Later in the consideration of *Hopf sheaves*, it is convenient to view the vector bundles from a more general perspective (see Section 7.4). We regard them in terms of the relative spectrum of a locally free sheaf (see Section 6.3 and Remark 3.4).

(2) The pair (E,π) is abbreviated as E and if $t \in T$, the fiber $\pi^{-1}(t)$ is denoted as E_t. For further compatibility we denote the $\Bbbk$-vector space of arrows between two vector bundles E, E' as $\mathrm{Hom}_0(E,E')$ (see Definition 3.18 and Lemma 3.24). A morphism $f \in \mathrm{Hom}_0(E,E')$ restricts to the fibers defining a $\Bbbk(t)$–linear map written as $f_t : E_t \to E'_t$.

(3) If (E,π) is a vector bundle, we denote $\mathrm{End}_0(E) = \mathrm{Hom}_0(E,E)$, and $\mathrm{Aut}_0(E) \subset \mathrm{End}_0(E)$ the group of automorphisms of E.

Definition 3.3. Given a vector bundle $\pi : E \to T$ then, using the local characterization of a vector bundle, one can define a morphism of schemes, called the *zero section*, $\sigma_E : T \to E$ such that: i) $\pi \circ \sigma = \mathrm{id}_T$, ii) if $f : (\pi : E \to T) \to (\pi' : E' \to T)$ is a morphism in $\mathrm{Hom}_0(E, E')$, then $f \circ \sigma_E = \sigma_{E'}$. In other words, $\sigma_E(t) = 0 \in E_t$ for all $t \in T$.

Remark 3.4. (1) It is standard knowledge that the category $\mathrm{VB}_0(T)$ is op-equivalent with the category of locally free finitely generated sheaves of $\mathcal{O}_T$–modules (see e.g. [31, Def. 1.7.8] and the basic general constructions of Section 6.3 below).

(2) It is also well known that $\mathrm{VB}_0(T)$ is a monoidal, rigid, $\Bbbk$–linear category, with unit object $p_2 : \mathbb{A}^1_{\Bbbk} \times T = \Bbbk \times T \to T$ and final object the trivial bundle $\mathrm{Spec}(\Bbbk) \times T$.

(3) In particular, given the vector bundles E, E', $\mathrm{Hom}_0(E, E')$ is a $\Bbbk$–vector space, and naturally it supports a $\Bbbk$–scheme. Thus, $\mathrm{VB}_0(T)$ can be seen as a category enriched over $\mathrm{Sch}\,|\Bbbk$ in a canonical way (compare with Definition 3.14 and Remark 3.15).

If $\pi : E \to A$ is a vector bundle over an abelian variety, it is well known that $\mathrm{Aut}_0(E)$ can be endowed with a structure of a group scheme of finite type (see for example [41] and [18, §2.3]). Moreover, it is it is possible to define a group scheme $\mathrm{Aut}_{\mathrm{gr}}(E)$ (of *graded automorphisms*) as follows.

Notation 3.5. (1) If X, T are $\Bbbk$–schemes, then the canonical projection $p_2 : X \times T \to T$ endows $X \times T$ with a structure of T-scheme, that we denote as $X_T = (X \times T, p_2)$.

(2) If $\pi : E \to S$ is a vector bundle, then $\pi_T = \pi \times \mathrm{id}_T : E_T \to S_T$ is a vector bundle.

(3) If A is an abelian variety and $\ell \in A(T)$ is a T–point, then we define the *translation by* ℓ as the morphism of T–schemes $t_\ell = (s \times \mathrm{id}_T) \circ (\ell \circ p_2, \mathrm{id}_{A \times T}) : A_T \to A_T$ — notice that if $(b, l) \in A_T$, then $t_\ell(b, t) = (\ell + b, t)$.

Definition 3.6. If A is an abelian variety and $\pi : E \to A$ is a vector bundle, define a functor $\mathrm{Aut}_{\mathrm{gr}}(E) : (\mathrm{Sch}\,|\Bbbk)^{\mathrm{op}} \to \mathrm{Groups}$ as follows:

(1) *Objects.* If $T \in \mathrm{Obj}(\mathrm{Sch}\,|\Bbbk)$, then $\mathrm{Aut}_{\mathrm{gr}}(E)(T)$ is the group of pairs (f, ℓ), where $f : E_T \to E_T$ is a T–automorphism and $\ell : T \to A$ is a T–point. such that:

(i) the diagram below commutes

$$\begin{tikzcd} E_T = E \times T \arrow[r, "f"] \arrow[d, "\pi \times \mathrm{id}_T"'] & E_T = E \times T \arrow[d, "\pi \times \mathrm{id}_T"] \\ A_T = A \times T \arrow[r, "t_\ell"'] & A_T = A \times T \end{tikzcd}$$

(ii) the morphism of A_T-schemes $\widehat{f} : E_T \to t_\ell^* E_T$, defined by the pullback diagram given by base change, is an isomorphism of A_T–vector bundles in

$\mathrm{Sch}\,|T$ — in other words $\widehat{f}$ is an isomorphism in $\mathrm{VB}_0(A_T)$.

$$\begin{tikzcd}
E_T \arrow[dr, "\widehat{f}"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\pi\times\mathrm{id}_T"'] & & \\
& t_\ell^*(E_T) \arrow[r] \arrow[d] & E_T \arrow[d, "\pi\times\mathrm{id}_T"] \\
& A_T \arrow[r, "t_\ell"'] & A_T
\end{tikzcd}$$

(2) *hom-objects.* The functor $\mathrm{Aut}_{\mathrm{gr}}(E)$ at the level of hom-objects is defined as follows: if $g : T' \to T$ is a morphism of $\Bbbk$–schemes then the map $\mathrm{Aut}_{\mathrm{gr}}(E)(g) : \mathrm{Aut}_{\mathrm{gr}}(E)(T) \to \mathrm{Aut}_{\mathrm{gr}}(E)(T')$ is given by

$$\mathrm{Aut}_{\mathrm{gr}}(E)(g)(f,\ell) = \big(p_1\circ(f\circ(\mathrm{id}_E\times g), p_2), \ell\circ g\big) \in \mathrm{Aut}_{\mathrm{gr}}(E)(T').$$

Remark 3.7. (1) The product in $\mathrm{Aut}_{\mathrm{gr}}(E)(T)$ (*the composition of graded automorphisms*) is defined by the following rule: $(f',\ell')(f,\ell) = \big(f'\circ f, s\circ(\ell,\ell')\big)$.

(2) Notice that by construction, if $(f,\ell) \in \mathrm{Aut}_{\mathrm{gr}}(E)(T)$, then $f = (\widetilde{f}, p_2)$, where $\widetilde{f} : E\times T \to E$ is a morphism of $\Bbbk$–schemes. It is equivalent to give f or $\widetilde{f}$ provided that the new map makes the diagram below commutative,

$$\begin{tikzcd}
E\times T \arrow[rr, "\widetilde{f}"] \arrow[d, "\pi\times\mathrm{id}_T"'] & & E \arrow[d, "\pi\times\mathrm{id}_T"] \\
A\times T \arrow[r, "\mathrm{id}_A\times\ell"'] & A\times A \arrow[r, "s"'] & A.
\end{tikzcd}$$

To simplify notations we write indistinctly $f : E_T \to E_T$ or $f : E\times T \to E$ in accordance with the context.

(3) Let $(f,\ell) \in \mathrm{Aut}_{\mathrm{gr}}(E)(T)$ and $g : T' \to T$. If $\mathrm{Aut}_{\mathrm{gr}}(E)(g)(f,\ell) = (h, \ell\circ g) \in \mathrm{Aut}_{\mathrm{gr}}(E)(T')$, then $h = (\widetilde{h}, p_2) : E_{T'} \to E_{T'}$, with

$$\widetilde{h}(e,t') = \big(p_1\circ(f\circ(\mathrm{id}_E\times g)\big)(e,t') = \widetilde{f}\big(e, g(t')\big) \in E.$$

Remark 3.8. (1) In [18, Lemma 2.8] it is proved that the functor $T \to \mathrm{Aut}_{\mathrm{gr}}(E)(T)$ is representable by a group scheme of finite type, denoted as $\mathrm{Aut}_{\mathrm{gr}}(E)$. It follows that the projection $d : \mathrm{Aut}_{\mathrm{gr}}(E) \to A$, given by $d(T)(f,\ell) = \ell \in A(T)$ for any $(f,\ell) \in \mathrm{Aut}_{\mathrm{gr}}(E)(T)$, is a morphism of group schemes. It is clear that $\mathrm{Ker}(d) = \mathrm{Aut}_0(E)$, i.e. the smooth, affine and connected group scheme of finite type consisting of all the automorphisms of the vector bundle E.

(2) In the particular case where $A = \mathrm{Spec}(\Bbbk)$ and $T = \mathrm{Spec}(R)$ for R a commutative $\Bbbk$–algebra, it is clear that E is a $\Bbbk$–vector space and $f \in \mathrm{Aut}_{\mathrm{gr}}(E)\big(\mathrm{Spec}(R)\big)$ is determined by a morphism $f_R : E\times_\Bbbk \mathrm{Spec}(R) \to E\times_\Bbbk \mathrm{Spec}(R)$, linear on the fibers, which is equivalent to give an $\Bbbk$–linear automorphism $\widehat{f} : E \to E$.

(3) Consider the canonical action $a : \mathrm{Aut}_{\mathrm{gr}}(E)\times E \to E$ of the group scheme $\mathrm{Aut}_{\mathrm{gr}}(E)$ on E as described in Remark 2.4. If T is a $\Bbbk$–scheme and $\big(g, d(g)\big) \in \mathrm{Aut}_{\mathrm{gr}}(E)(T)$, then $g = (\widetilde{g}, p_2) : E_T \to E_T$. If $e \in E(T)$, then $a(g,e) = \widetilde{g}\circ(e,\mathrm{id}_T) : T \to E\times T \to E \in E(T)$. A direct computation shows that the degree $d\big(e \to \widetilde{g}\circ(e,\mathrm{id})\big) = d(e \to a(g,e)) = d(g)$.

Therefore, the diagram in Definition 3.6, (i) in this context reads as:

$$\begin{tikzcd} \operatorname{Aut}_{\mathrm{gr}}(E)\times E \arrow[r, "a"] \arrow[d, "d\times\pi"'] & E \arrow[d, "\pi"] \\ A\times A \arrow[r, "s"] & A. \end{tikzcd}$$

Definition 3.9. Let A be an abelian variety. A vector bundle $\pi : E \to A$ is called *homogeneous* if the induced morphism of group schemes $d : \operatorname{Aut}_{\mathrm{gr}}(E) \to A$ is faithfully flat — i.e. if d is surjective, in view of Theorem 2.9.

The *category* $\mathrm{HVB}_0(A)$ is defined as the full subcategory of $\mathrm{VB}_0(A)$ that has as objects the homogeneous vector bundles.

Remark 3.10. (1) In view of Corollary 2.10, a vector bundle $\pi : E \to A$ is homogeneous if and only if for any geometric point $b \in A\big(\overline{\Bbbk}\big)$, there exists an isomorphism of $\overline{\Bbbk}$–vector bundles $E_{\overline{\Bbbk}} \to t_b^* E_{\overline{\Bbbk}}$.
(2) Since A is an abelian variety, it follows that if $\pi : E \to A$ is a homogeneous vector bundle, then the short exact sequence:

$$Aut_{\mathrm{gr}}(E): \qquad 1 \longrightarrow \operatorname{Aut}_0(E) \longrightarrow \operatorname{Aut}_{\mathrm{gr}}(E) \xrightarrow{\ d\ } A \longrightarrow 0$$

is a smooth affine extension of A, of finite type. In particular, $\operatorname{Aut}_{\mathrm{gr}}(E)$ is a smooth group scheme of finite type.
(3) It follows from Remark 2.44 that $\operatorname{Aut}_{\mathrm{gr}}(E) = \operatorname{Aut}_0(E)\operatorname{Aut}_{\mathrm{gr}}(E)^0$ and therefore $\operatorname{Aut}_{\mathrm{gr}}(E)$ is a connected group scheme.

Lemma 3.11. *Let* $\mathcal{S}$*:* $1 \longrightarrow H \longrightarrow G \xrightarrow{\ q\ } A \longrightarrow 0$ *be an affine extension of the abelian variety* A*, and let* V *be a finite dimensional* H*–module. Then the vector bundle* $\pi_V : E_V = G \times^H V \to A$ *is homogeneous. Conversely, if* $\pi_E : E \to A$ *is an arbitrary homogeneous vector bundle, then* $E \cong \operatorname{Aut}_{\mathrm{gr}}(E) \times^{\operatorname{Aut}_0(E)} E_0$*, where* $E_0 = \pi^{-1}(0)$ *is as usual the fiber over* $0 \in A$*.*

Proof. Indeed, it follows from Theorem 2.61 that $\pi_V : E_V \to A$ is a vector bundle and that G acts linearly on E_V. Since the G–action of E_V induces a morphism of affine extensions $\mathcal{S} \to Aut_{\mathrm{gr}}(E)$, it follows (for example from Remark 3.10) that the vector bundle E_V is homogeneous.

Conversely if $\pi_E : E \to A$ is a homogeneous vector bundle, by the first part of the lemma $\operatorname{Aut}_{\mathrm{gr}}(E) \times^{\operatorname{Aut}_0(E)} E_0$ is a homogeneous vector bundle, and clearly the restriction of the action $\operatorname{Aut}_{\mathrm{gr}}(E) \times E \to E$ to $\operatorname{Aut}_{\mathrm{gr}}(E) \times E_0 \to E$ induces the required isomorphism (see Remark 3.10). □

Definition 3.12. The vector bundle E_V is called the *homogeneous vector bundle associated to the* H*–module* V.

Remark 3.13. It is clear from the definition that if $E, E' \in \mathrm{HVB}_0(A)$ are homogeneous vector bundles, then $E \oplus E'$, $E \otimes E'$ and $E^\vee$ also are objects of $\mathrm{HVB}_0(A)$ and these operations (and the corresponding morphisms) endow this category with a $\Bbbk$–linear and monoidal rigid structure — this was proved in [42] and [43]) when $\Bbbk$ is an algebraically closed field, then by Remark 3.10 we deduce the general case. Moreover, in [18, Theorem 2.9 and Corollary 2.10], Brion proved that $\mathrm{HVB}_0(A)$ is also an abelian category, stable by direct summands.

Next we enlarge the family of arrows in the categories $\mathrm{VB}_0(A)$ and $\mathrm{HVB}_0(A)$ taking instead of arrows of degree zero, arrows of arbitrary degree $a \in A$. We use the same notations and abbreviations than in Definition 3.6, and remarks 3.7 and 3.8.

Definition 3.14. Let $\pi : E \to A$ and $\pi' : E' \to A$ be two vector bundles. We define the *graded homomorphisms functor* $\mathrm{Hom}_{\mathrm{gr}}(E, E') : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$ as follows.
(1) If $T \in \mathrm{Sch}$, then $\mathrm{Hom}_{\mathrm{gr}}(E, E')(T)$ is the set of pairs (f, ℓ), where $f : E_T \to E'_T$ is a T–morphism and $\ell \in A(T)$ such that:

(i) the diagram below commutes

$$\begin{tikzcd}
E_T = E \times T \arrow[r, "f"] \arrow[d, "\pi \times \mathrm{id}_T"'] & E'_T = E' \times T \arrow[d, "\pi' \times \mathrm{id}_T"] \\
A_T = A \times T \arrow[r, "t_\ell"'] & A_T = A \times T
\end{tikzcd}$$

(ii) The induced morphism of A_T-schemes $E_T \to t_\ell^* E'_T$ (see diagram below) is a morphism in $\mathrm{VB}_0(A_T)$.

$$\begin{tikzcd}
E_T \arrow[drr, bend left, "f"] \arrow[dr, "\widehat{f}"] \arrow[ddr, bend right, "\pi \times \mathrm{id}_T"'] & & \\
& t_\ell^*(E'_T) \arrow[r] \arrow[d] & E'_T \arrow[d, "\pi' \times \mathrm{id}_T"] \\
& A_T \arrow[r, "t_\ell"'] & A_T .
\end{tikzcd}$$

(2) If $g : T' \to T$ is a morphism of schemes and $(f, \ell) \in \mathrm{Hom}_{\mathrm{gr}}(E, E')(T)$, with $f = (\widetilde{f}, p_2) : E \times T \to E \times T$, then

$$\mathrm{Hom}_{\mathrm{gr}}(E, E')(g)(f) = \big((\widetilde{f} \circ (\mathrm{id}_E \times g), p_2), \ell \circ g\big).$$

Notice that $\big(\widetilde{f} \circ (\mathrm{id}_E \times g), p_2\big) : E_{T'} \to E'_{T'}$ is a morphism of $A_{T'}$–schemes.

Remark 3.15. (1) By construction (and descent theory, see [23, I.2.2.7] of [57, Tag 0238]) $\mathrm{Hom}_{\mathrm{gr}}(E, E')$ is a fpqc sheaf.
(2) Let $\pi : E \to A$, $\pi' : E' \to A$ and $\pi'' : E'' \to A$ be three vector bundles. Then the composition of morphisms induces a natural composition morphism (a natural transformation between the functors) $\mathrm{Hom}_{\mathrm{gr}}(E, E') \times \mathrm{Hom}_{\mathrm{gr}}(E', E'') \to \mathrm{Hom}_{\mathrm{gr}}(E, E'')$.
(3) Notice that the family $d(T) : \mathrm{Hom}_{\mathrm{gr}}(E, E')(T) \to A(T)$ produces also a morphism (natural transformation) $d : \mathrm{Hom}_{\mathrm{gr}}(E, E') \to A$.

Example 3.16. Given two homogeneous vector bundles E, E', the *zero morphism* $0 : E \to E'$ is given by $0 = \sigma_{E'} \circ \pi_E$ (see Definition 3.3). We generalize this construction to arbitrary degrees as follows.

If $\ell \in A(T)$, it is clear that the pair (ζ_ℓ, ℓ) with $\zeta_\ell = \sigma_{E'} \circ p_1 \circ t_\ell \circ (\pi \times \mathrm{id}_T) : E \times T \to E'$ yields a graded morphism of degree ℓ. The T–point $(\zeta_\ell, \ell) \in \mathrm{Hom}_{\mathrm{gr}}(E, E')(T)$ is called *the pseudo-zero of degree ℓ.*

Notice that (ζ_ℓ, ℓ) induces the zero morphism $0 : E_T \to t_\ell^* E_T$.

Remark 3.17. Let $\mathcal{V} = \operatorname{Func}\big((\operatorname{Sch}|\Bbbk)^{\rm op}, \operatorname{Sets}\big)$; then we can endow $\mathcal{V}$ with the product induced by the cartesian product in Sets, which final object is the constant functor equal to the final object in Sets.

Clearly, the Yoneda embedding $Y : \operatorname{Sch}|\Bbbk \to \mathcal{V} = \operatorname{Func}\big((\operatorname{Sch}|\Bbbk)^{\rm op}, \operatorname{Sets}\big)$ preserves finite products.

Definition 3.18. (1) For $\mathcal{V}$ as above, we define the $\mathcal{V}$–category $\operatorname{VB}_{\rm gr}(A)$ (i.e. $\operatorname{VB}_{\rm gr}(A)$ is enriched over $\mathcal{V}$) as follows:

The *objects* of $\operatorname{VB}_{\rm gr}(A)$ are the same than $\operatorname{VB}_0(A)$.

Given (E,π), $(E',\pi') \in \operatorname{VB}_{\rm gr}(A)$, the *hom-object* with domain (E,π) and codomain (E',π') is $\operatorname{Hom}_{\rm gr}(E,E') \in \mathcal{V}$, i.e. the *functor of graded homomorphisms of vector bundles*, with compositions as defined before (Remark 3.15).

(2) The $\mathcal{V}$–category $\operatorname{HVB}_{\rm gr}(A)$ is the full $\mathcal{V}$–subcategory of $\operatorname{VB}_{\rm gr}(A)$ with objects the homogeneous vector bundles (compare with Definition 3.9).

Similarly than Definition 3.9, the category $\operatorname{HVB}_{\rm gr}(A)$ is the full subcategory of $\operatorname{VB}_{\rm gr}(A)$ with objects the homogeneous vector bundles.

Remark 3.19. Let $\pi : E \to A$ and $\pi' : E' \to A$ be two vector bundles and $f : E \to E' \in \operatorname{Hom}_0(E,E')$ be a morphism of vector bundles. Then $f_T = f \times \operatorname{id}_T : E_T \to E'_T \in \operatorname{Hom}_{\rm gr}(E,E')(T)$. Thus $\operatorname{Hom}_0(E,E')$ represents a subfunctor of $\operatorname{Hom}_{\rm gr}(E,E')$, with $\operatorname{Hom}_0(E,E')(T) = \big\{f \in \operatorname{Hom}_{\rm gr}(E,E')(T) : d(T)(f) = 0\big\}$. Thus, $\operatorname{VB}_0(A) \subseteq \operatorname{VB}_{\rm gr}(A)$ is a *wide* ($\mathcal{V}$–enriched) subcategory — in the sense that has the same objects but less morphisms. Similarly for the homogeneous situation.

Notation 3.20. If $E = E'$, then $\operatorname{End}_{\rm gr}(E) := \operatorname{Hom}_{\rm gr}(E,E)$ and $\operatorname{End}_0(E) := \operatorname{Hom}_0(E,E)$.

Remark 3.21. It is clear that $\operatorname{End}_{\rm gr}(E)$ and $\operatorname{End}_0(E)$ are functors on monoids, and that the group $\operatorname{Aut}_{\rm gr}(E)$ (resp. $\operatorname{Aut}_0(E)$) is a subfunctor on monoids of $\operatorname{End}_{\rm gr}(E)$ (resp. $\operatorname{End}_0(E)$).

The relationships between the (enriched) categories we just defined is illustrated in the diagram below, where the vertical arrows are full subcategories and the horizontal are wide subcategories.

$$\begin{array}{ccc} \operatorname{VB}_0(A) & \subseteq & \operatorname{VB}_{\rm gr}(A) \\ \subseteq & & \subseteq \\ \operatorname{HVB}_0(A) & \subseteq & \operatorname{HVB}_{\rm gr}(A) \end{array}$$

Example 3.22. It is clear that if $A = \operatorname{Spec}(\Bbbk)$, then all these categories collapse into $\operatorname{Vect}_\Bbbk = \operatorname{VB}_0\big(\operatorname{Spec}(\Bbbk)\big) = \operatorname{HVB}_0\big(\operatorname{Spec}(\Bbbk)\big) = \operatorname{HVB}_{\rm gr}\big(\operatorname{Spec}(\Bbbk)\big) = \operatorname{VB}_{\rm gr}\big(\operatorname{Spec}(\Bbbk)\big)$.

Indeed, if $T = \operatorname{Spec}(R) \in \operatorname{Sch}|_{\rm aff}\Bbbk$, and $V, W \in \operatorname{Vect}_\Bbbk$, then $V_T = (V \times T \to T)$ and $\operatorname{Hom}_{\rm gr}(V,W)(T) \cong \operatorname{Hom}_\Bbbk(V,W) \otimes_\Bbbk R$ — in other words, the functor $\operatorname{Hom}_{\rm gr}(V,W)$ is represented by the vector space $\operatorname{Hom}_\Bbbk(V,W)$.

Remark 3.23. (1) In the category $\operatorname{VB}_0(A_{\overline{\Bbbk}})$, for $\ell \in A$ we denote as T_ℓ the "pullback by the translation t_ℓ" functor (compare with Definition 3.6,(ii)). Thus,

$T_\ell : \mathrm{VB}_0(A_{\overline{\Bbbk}}) \to \mathrm{VB}_0(A_{\overline{\Bbbk}})$ is given at the level of objects by:

$$\begin{tikzcd} T_\ell(E) \arrow[r, "p_E"] \arrow[d, "\widehat{\pi_\ell}"'] & E \arrow[d, "\pi"] \\ A \arrow[r, "t_\ell"'] & A. \end{tikzcd}$$

It is clear that the vector bundle $\big(T_\ell(E), \widehat{\pi_\ell}\big) \cong (E, t_{-\ell} \circ \pi)$; when there is no danger of confusion, the structure map $\widehat{\pi_\ell}$ is denoted simply as π_ℓ.

If (E, π) and (E', π') are objects in $\mathrm{VB}_0(A_{\overline{\Bbbk}})$ and $f : (E, \pi) \to (E', \pi')$ is an arrow in $\mathrm{VB}_0(A_{\overline{\Bbbk}})$, then $T_\ell(f) = f : \big(T_\ell(E), \pi_\ell\big) \to \big(T_\ell(E'), \pi'_\ell\big)$ is an arrow in $\mathrm{VB}_0(A_{\overline{\Bbbk}})$ as shown in the diagram below.

$$\begin{tikzcd} E \arrow[rr, "T_\ell(f)=f"] \arrow[dr, "\pi"] \arrow[ddr, "\pi_e ll"'] & & E' \arrow[dl, "\pi'"'] \arrow[ddl, "\pi'_\ell"] \\ & A \arrow[d, dashed, "t_{-\ell}"] & \\ & A & \end{tikzcd}$$

The map $\ell \to T_\ell : A \to \mathrm{Fun}\big(\mathrm{VB}_0(A_{\overline{\Bbbk}})\big)$ is a morphism of the monoid $(A, +)$ to $\big(\mathrm{Fun}(\mathrm{VB}_0(A_{\overline{\Bbbk}})), \circ\big)$ ($\circ$ denotes the composition of functors). In particular for each $\ell \in A$ the functor T_ℓ is invertible and its inverse is $T_{-\ell}$.

(2) Let $\ell \in A$, and (E, π), (E', π') be two objects in $\mathrm{VB}_0(A_{\overline{\Bbbk}})$ and $f : E \to E'$ a morphism of the underlying schemes. The diagram (whose rightmost triangle is commutative):

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \arrow[dl, "\pi'_\ell"'] \\ A \arrow[r, "t_\ell"'] & A \end{tikzcd}$$

proves that $\mathrm{Hom}_0\big(E, T_\ell(E')\big) = \big\{ f : E \to E' : \pi' f = t_\ell \pi, f|_{E_b} : E_b \to E_{\ell+b} \text{ linear} \big\}$.

In view of the preceding remark, if E, E' are homogeneous vector bundles over an algebraically closed field one can work with *sets of graded morphisms*, i.e. to consider the set of morphisms $f : E \to E'$ such that $\pi' \circ f = t_\ell \circ \pi$ for some $\ell \in A$, rather than with the functor $\mathrm{End}_{\mathrm{gr}}(E, E')$. This is the approach taken by L. Brambila-Paz and A. Rittatore in [8], for examination of the geometry and algebraic structure of $\mathrm{End}_{\mathrm{gr}}(E)$ and $\mathrm{Hom}_{\mathrm{gr}}(E, E')$.

If $\Bbbk$ is an arbitrary field, and $E, E' \in \mathrm{HVB}_{\mathrm{gr}}(A)$, we present below the proof that $\mathrm{Hom}_{\mathrm{gr}}(E, E')$ is representable by the vector bundle $L_{\mathrm{Hom}_0(E,E')} = \mathrm{Aut}_{\mathrm{gr}}(E') \times^{\mathrm{Aut}_0(E')} \mathrm{Hom}_0(E, E')$ (see Lemma 3.11). It is similar to the proof in [8] (in the hypothesis that $\Bbbk = \overline{\Bbbk}$), with the necessary adaptations to the general situation.

Lemma 3.24. *Let $\pi : E \to A$, $\pi' : E' \to A$ be two homogeneous vector bundles over the abelian variety A. Then the homogeneous vector bundle $L_{\mathrm{Hom}_0(E,E')} = \mathrm{Aut}_{\mathrm{gr}}(E') \times^{\mathrm{Aut}_0(E')} \mathrm{Hom}_0(E, E')$ (see Lemma 3.11) represents $\mathrm{Hom}_{\mathrm{gr}}(E, E')$.*

Moreover, $\mathrm{Hom}_{\mathrm{gr}}(E,E') \cong R_{\mathrm{Hom}_0(E,E')} = \mathrm{Aut}_{\mathrm{gr}}(E) \times^{\mathrm{Aut}_0(E)} \mathrm{Hom}_0(E,E') \in \mathrm{HVB}_0(A)$.

Proof. We adapt the strategy used in [8] for the algebraically closed field case to this general case.

Let $\varphi : \mathrm{Aut}_{\mathrm{gr}}(E') \times \mathrm{Hom}_0(E,E') \to \mathrm{Hom}_{\mathrm{gr}}(E,E')$ the morphism of fpqc sheaves given by composition. Then clearly φ is $\mathrm{Aut}_0(E)$–invariant (see Remark 2.63), and therefore induces a morphism of fpqc sheaves $\phi : L_{\mathrm{Hom}_0(E,E')} \to \mathrm{Hom}_{\mathrm{gr}}(E,E')$.

We prove now that φ is a monomorphism. Let $y_1 : T \to L_{\mathrm{Hom}_0(E,E')}$, $y_2 : T \to L_{\mathrm{Hom}_0(E,E')}$ be two points in $L_{\mathrm{Hom}_0(E,E')}(T)$ such that $\phi(T)(y_1) = \phi(T)(y_2) \in \mathrm{Hom}_{\mathrm{gr}}(E,E')(T)$. Let $\sigma_i : T_i \to T$, $i = 1,2$, be fpqc morphisms and $x_1 = (g_1, f_1), x_2 = (g_2, f_2) \in \mathrm{Aut}_{\mathrm{gr}}(E') \times \mathrm{Hom}_0(E,E')(T_i)$ be such that $\pi(x_i) = y_i \circ \sigma_i$. Then as points in $\mathrm{Hom}_{\mathrm{gr}}(E,E')(T_1 \times_T T_2)$, we have that

$$g_1 \circ f_1 = \phi(T_1 \times_T T_2)(x_1) = \phi(y_1) = \phi(y_2) = \phi(T_1 \times_T T_2)(x_2) = g_2 \circ f_2.$$

It follows that $f_2 = g_2^{-1} \circ g_1 \circ f_1 \in \mathrm{Hom}_0(E,E')(T_1 \times_T T_2)$, with $g_2^{-1} \circ g_1 \in \mathrm{Aut}_0(E')$. Thus, $y_1 = y_2 \in L_{\mathrm{Hom}_0(E,E')}(T_1 \times_T T_2)$ and it follows that $y_1 = y_2 \in L_{\mathrm{Hom}_0(E,E')}(T)$.

In order to prove that $\varphi(T)$ is surjective for all T, let $(f,\ell) \in \mathrm{Hom}_{\mathrm{gr}}(E,E')(T)$. Let $\sigma : T' \to T$ a fpqc morphism and $g \in \mathrm{Aut}_{\mathrm{gr}}(E')(T')$ such that $q(T)(g) = \ell \circ \sigma \in A(T')$. Then $h = g^{-1} \circ \big(f \circ (\mathrm{id}_E \times \sigma), \ell \circ \sigma\big) \in \mathrm{Hom}_0(E,E')(T')$. It follows that $\varphi(T')\big(g,h\big) == \big((f \circ (\mathrm{id}_E \times \sigma), \ell \circ \sigma) \in \mathrm{Hom}_{\mathrm{gr}}(E,E')(T')$. From the commutative diagram:

$$\begin{tikzcd}
L_{\mathrm{Hom}_0(E,E')}(T) \arrow[r, "\phi(T)"] \arrow[d, "\sigma^*"'] & \mathrm{Hom}_{\mathrm{gr}}(E,E')(T) \arrow[d, "\sigma^*"] \\
L_{\mathrm{Hom}_0(E,E')}(T') \arrow[r, "\phi(T')"] & \mathrm{Hom}_{\mathrm{gr}}(E,E')(T')
\end{tikzcd}$$

we deduce that there exists $y \in L_{\mathrm{Hom}_0(E,E')}(T)$ such that $\phi(y) = (f,\ell)$ by descent.

The last assertion can be proved by a similar argument. $\square$

Remark 3.25. Let $E, E' \in \mathrm{HVB}_{\mathrm{gr}}(A)$. Since $\mathrm{Hom}_{\mathrm{gr}}(E,E') = \mathrm{Aut}_{\mathrm{gr}}(E') \times^{\mathrm{Aut}_0(E')} \mathrm{Hom}_0(E,E')$ is a homogeneous vector bundle, it follows that $\mathrm{HVB}_{\mathrm{gr}}(A)$ is a closed category.

Corollary 3.26. *Let $\pi : E \to A$ be a homogeneous vector bundle. Then* $\mathrm{End}_{\mathrm{gr}}(E)$ *is a smooth monoid scheme of finite type, such that the following diagram is commutative, where the vertical arrows are open immersions.*

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{End}_0(E) \arrow[r] & \mathrm{End}_{\mathrm{gr}}(E) \arrow[r, "d"] & A \arrow[r] & 0 \\
1 \arrow[r] & \mathrm{Aut}_0(E) \arrow[r] \arrow[u, hook] & \mathrm{Aut}_{\mathrm{gr}}(E) \arrow[r, "d"'] \arrow[u, hook] & A \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Proof. Once we know that $\mathrm{End}_{\mathrm{gr}}(E)$ is a smooth scheme of finite type, and taking into account [13, Theorem 1], the result follows easily. $\square$

Remark 3.27. We observed in Remark 3.13 that $\mathrm{HVB}_0(A)$ is a abelian, monoidal, rigid category. Nevertheless, these structures cannot be defined in the (wide) extension of the category $\mathrm{HVB}_0(A)$ that we denoted as $\mathrm{HVB}_{\mathrm{gr}}(A)$.

However, for homogeneous morphisms of the *same* degree it is clear that the following holds.

Lemma 3.28. *Let $E, E', F, F' \in \mathrm{HVB}_{\mathrm{gr}}(A)$ and $(f,\ell) \in \mathrm{Hom}_{\mathrm{gr}}(E,F)(T)$, $(f',\ell) \in \mathrm{Hom}_{\mathrm{gr}}(E',F')(T)$ be graded morphisms. Then the following maps are graded morphisms in $\mathrm{HVB}_{\mathrm{gr}}(A)$:*

(i) $(f\oplus f',\ell)$, where $f\oplus f' : (E\oplus E')_T \cong E_T \oplus E'_T \to (F\oplus F')_T$ is given by $(f\oplus f')(e+e') = f(e)+f'(e')$;

(ii) $(f\otimes f',\ell)$; where $f\otimes f' : (E\otimes E')_T \cong E_T \otimes E'_T \to (F\otimes F')_T$ is given by $(f\otimes f')(e\otimes e') = f(e)\otimes f(e')$.

(iii) $(f^\vee, -\ell)$, where $f^\vee : (E'_T)^\vee \cong ((E')^\vee)_T \to E_T$. □

Remark 3.29. Let $E \to A$ be a homogeneous vector bundle and assume that $Aut_{\mathrm{gr}}(E)$ admits a section $\sigma : A \to \mathrm{Aut}_{\mathrm{gr}}(E)$, $d\circ\sigma = \mathrm{id}_A$. Then $(\sigma, \mathrm{id}_{E_0}) : A\times E_0 \to \mathrm{Aut}_{\mathrm{gr}}(E)\times E_0$ clearly induces a morphism of vector bundles $A\times E_0 \to E = \mathrm{Aut}_{\mathrm{gr}}(E)\times^{\mathrm{Aut}_0(E)} E_0$. Thus, we have proved that a homogeneous vector bundle is trivial if and only if $Aut_{\mathrm{gr}}(E)$ admits a section. This is a well known result when $\Bbbk$ is algebraically closed field (see [42] and [8]).

Definition 3.30. Given an object E in the category $\mathrm{HVB}_0(A)$, we call $\mathrm{HVB}_0(A)_E$ the full abelian monoidal rigid category generated by E. We call $\mathrm{HVB}_{\mathrm{gr}}(A)_E$ the full subcategory of $\mathrm{HVB}_{\mathrm{gr}}(A)$ that has the same objects that $\mathrm{HVB}_0(A)_E$.

Remark 3.31. (1) By definition the category $\mathrm{HVB}_0(A)_E$ is characterized by the following universal property: for every abelian monoidal rigid category $\mathcal{C}$ and any object $c\in\mathcal{C}$ there is one and only one additive monoidal functor $F_c : \mathrm{HVB}_0(A)_E \to \mathcal{C}$ such that $F_c(E) = c$.

(2) The relations between the above categories is depicted in the diagram below:

$$\begin{array}{ccc} \mathrm{HVB}_0(A) & \subseteq & \mathrm{HVB}_{\mathrm{gr}}(A) \\ \subseteq & & \subseteq \\ \mathrm{HVB}_0(A)_E & \subseteq & \mathrm{HVB}_{\mathrm{gr}}(A)_E \end{array}$$

where the horizontal maps are wide inclusions and the vertical ones are full.

Remark 3.32. Let $\pi_E : E \to A$, $\pi_{E'} : E' \to A \in \mathrm{HVB}_0(A)$. Then we have a morphism of schemes $a : \mathrm{Hom}_{\mathrm{gr}}(E,E')\times E \to E'$ as follows:

If T is a $\Bbbk$–scheme, then $a(T) : \mathrm{Hom}_{\mathrm{gr}}(E,E')(T)\times E(T) \to E'(T)$ is given by $a(T)\big((f,\ell),e\big) = f\circ(e\times\mathrm{id}_T)$. Notice that $\pi_{E'}\big(f(e\times\mathrm{id}_T)\big) = d(f)+\pi_E(e)$.

In other words, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{gr}}(E,E')\times E \arrow[r, "a"] \arrow[d, "d\times\pi_E"'] & E' \arrow[d, "\pi_{E'}"] \\ A\times A \arrow[r, "s"] & A \end{tikzcd}$$

If $E = E'$, then $a : \mathrm{End}_{\mathrm{gr}}(E)\times E \to E$ is an action of the smooth monoid $\mathrm{End}_{\mathrm{gr}}(E)$ (see Corollary 3.26), we say that the action is *linear on the fibers*.

3.2. **Representations of affine extensions.**

Definition 3.33. Let $\mathcal{S}: \; 1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension of the abelian variety A. A *representation of* $\mathcal{S}$ or $\mathcal{S}$*–module*, is a homogeneous vector bundle $\pi_E : E \to A$ equipped with a morphism of affine extensions $\varrho : \mathcal{S} \to \mathcal{A}ut_{\mathrm{gr}}(E)$

$$\begin{tikzcd}
\mathcal{S}: \arrow[d, "\varrho"'] & 1 \arrow[r] & H \arrow[r] \arrow[d] & G \arrow[r, "q"] \arrow[d, "\rho"'] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{A}ut_{\mathrm{gr}}(E) & 1 \arrow[r] & \mathrm{Aut}_0(E) \arrow[r] & \mathrm{Aut}_{\mathrm{gr}}(E) \arrow[r, "d_E"] & A \arrow[r] & 0
\end{tikzcd}$$

Remark 3.34. (1) To give a representation of $\mathcal{S}$ on an homogeneous vector bundle $\pi_E : E \to A$ is equivalent to give an action of $a : G \times E \to E$, linear on the fibers (see Remark 3.8), such that the following diagram is commutative

$$\begin{tikzcd}
G \times E \arrow[r, "a"] \arrow[d, "q \times \pi_E"'] & E \arrow[d, "\pi_E"] \\
A \times A \arrow[r, "s"'] & A
\end{tikzcd}$$

Therefore, when we talk about a representation of $\mathcal{S}$ we mean either a morphism of affine extensions schemes $\varrho : \mathcal{S} \to \mathcal{A}ut_{\mathrm{gr}}(E)$ or a vector bundle E together with the action a_ϱ of G associated to ϱ.

In particular in the above perspective, if $g \in G(T)$ and $\rho(T)(g) = \big(f_g, \ell\big)$, then $\ell = d\big(\rho(T)(g)\big) = q(g) \in A(T)$ and $\big(a_\varrho(g,-), p_2\big) = f_g : E_T \to E_T$ is such that the induced morphism $E_T \to t^*_{q(g)} E_T$ is an automorphism of vector bundles (i.e. an isomorphism in the category HVB_0).

(2) By construction, if $\rho(G)$ is the scheme theoretic image of $\rho : G \to \mathrm{Aut}_{\mathrm{gr}}(E)$, then

$\varrho(\mathcal{S})$: $1 \longrightarrow \rho(H) \longrightarrow \rho(G) \xrightarrow{d_E|_{\rho(G)}} A \longrightarrow 0$ is a closed sub-extension of $\mathcal{A}ut_{\mathrm{gr}}(E)$.

Example 3.35. Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension and $\mathbb{I} := (p_2 : \Bbbk \times A \to A)$ be the trivial bundle. Then $\mathcal{A}ut_{\mathrm{gr}}(\mathbb{I})$ is the extension $0 \longrightarrow G_m \xrightarrow{(\mathrm{id}, 0_A \circ\, \mathrm{st})} G_m \times A \xrightarrow{p_2} A \longrightarrow 0$, where $\mathrm{st} : G \to \mathrm{Spec}(\Bbbk)$ is the structure morphism of G as a $\Bbbk$–scheme.

It is clear that the representations $\varrho : \mathcal{S} \to \mathcal{A}ut_{\mathrm{gr}}(\mathbb{I})$ are in bijective correspondence with the *characters* of G, i.e. with the group scheme homomorphisms $\chi : G \to G_m$, this identification is given by $\chi \mapsto (\chi, q) : G \to G_m \times A$ for χ a character as above.

The *trivial character* i.e. the morphism $\chi_0 = 1_{G_m} \circ \mathrm{st}$, induces the representation $\varrho_0 = (\chi_0, q) : \mathcal{S} \to \mathcal{A}ut_{\mathrm{gr}}(\mathbb{I})$, and the associated action $a_{\varrho_0} : G \times \mathbb{I} \to \mathbb{I}$ is called the *trivial representation* or *trivial* $\mathcal{S}$*–module*.

Remark 3.36. Let $(E, \varrho_E), (E', \varrho_{E'})$ be two $\mathcal{S}$–modules. Then G acts on the vector bundle $\mathrm{Hom}_{\mathrm{gr}}(E, E')$ as follows.

If $g : T \to G \in G(T)$ and $(f, \ell) \in \mathrm{Hom}_{\mathrm{gr}}(E, E')(T)$, then $a_\varrho\big(g, (f, \ell)\big) = \varrho'_E(g) \circ (f, \ell) \circ \varrho_E(g^{-1}) \in \mathrm{Hom}_{\mathrm{gr}}(E, E')(T)$.

Notice that $a_\varrho(g,-) : \operatorname{Hom}_{\mathrm{gr}}(E,E')(T) \to \operatorname{Hom}_{\mathrm{gr}}(E,E')(T)$ is a morphism of A_T–vector bundles and that the diagram below is commutative

$$\begin{tikzcd} G \times \operatorname{Hom}_{\mathrm{gr}}(E,E') \arrow[rr, "a_\varrho"] \arrow[dr, "d\circ p_2"'] & & \operatorname{Hom}_{\mathrm{gr}}(E,E') \arrow[dl, "d"] \\ & A & \end{tikzcd}$$

In particular, a_ϱ induces a morphism of group schemes $G \to \operatorname{Aut}_0\big(\operatorname{Hom}_{\mathrm{gr}}(E,E')\big) \subset \operatorname{Aut}_{\mathrm{gr}}\big(\operatorname{Hom}_{\mathrm{gr}}(E,E')\big)$.

Lemma 3.37. *Let $(E,\varrho_E),(E',\varrho_{E'})$ be two $\mathcal{S}$–modules and consider the action $a_\varrho : G\times\operatorname{Hom}_{\mathrm{gr}}(E,E') \to \operatorname{Hom}_{\mathrm{gr}}(E,E')$ defined in Remark 3.36. Then ${}^G\operatorname{Hom}_{\mathrm{gr}}(E,E') \cong G_{\mathrm{ant}} \times^{G_{\mathrm{ant}}\cap H} {}^G\operatorname{Hom}_0(E,E')$, where ${}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')$ denotes as usual the fixed points subscheme. In particular, ${}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')$ is an $\mathcal{S}_{\mathrm{ant}}$–module and hence it is a homogeneous vector sub-bundle of $\operatorname{Hom}_{\mathrm{gr}}(E,E')$.*

Proof. Consider the action of G given by post-composition by $\rho_{E'}(g)$, that is $a_{\rho_{E'}} = \rho_{E'}\circ - : G \times \operatorname{Hom}_{\mathrm{gr}}(E,E') \to \operatorname{Hom}_{\mathrm{gr}}(E,E')$.

Let $\mathcal{S}_{\mathrm{ant}}$ be the closed subextension associated to the Rosenlicht decomposition of $\mathcal{S}$ (see Theorem 2.42), and notice that ${}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')$ is stable by the action of G_{ant}, since G_{ant} is central in G. In particular, ${}^G\operatorname{Hom}_0(E,E')$ is $G_{\mathrm{ant}}\cap H$–submodule (clearly ${}^G\operatorname{Hom}_0(E,E')$ is a vector space), and $R_{{}^G\operatorname{Hom}_0(E,E')} = G_{\mathrm{ant}} \times^{G_{\mathrm{ant}}\cap H} {}^G\operatorname{Hom}_0(E,E') \to A$ is a homogeneous vector bundle (see Lemma 3.11). The action $a_{\rho_{E'}}$ clearly induces an injective morphism of vector bundles $\widetilde{a_{\rho_{E'}}} : R_{{}^G\operatorname{Hom}_0(E,E')} \to \operatorname{Hom}_{\mathrm{gr}}(E,E')$, with image contained in ${}^G\operatorname{Hom}_0(E,E')$. Since $\big(R_{{}^G\operatorname{Hom}_0(E,E')}\big)_0 = {}^G\operatorname{Hom}_0(E,E')$, it follows that $R_{{}^G\operatorname{Hom}_0(E,E')} \cong {}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')$. □

Definition 3.38. In the notations of Lemma 3.37, the sub-bundle ${}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')$ is called the *(homogeneous) vector bundle of G–equivariant morphisms*.

Remark 3.39. Let (E,ϱ_E) and $(E,\varrho_{E'})$ be $\mathcal{S}$–modules and denote by $a_E : G\times E \to E$ and $a_{E'} : G \times E' \to E'$ the corresponding linear actions. Let us call $a := a_{{}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')} : {}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')\times E \to E'$ the morphism associated to ${}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')$ (see remarks 3.34 and 3.32). Then we have the following commutative diagram:

$$\begin{tikzcd} {}^G\operatorname{Hom}_{\mathrm{gr}}(E,E') \times G \times E \arrow[rr, "id_{{}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')}\times a_E"] \arrow[d, "\sigma_{12}"'] & & {}^G\operatorname{Hom}_{\mathrm{gr}}(E,E') \times E \arrow[dd, "a"] \\ G \times {}^G\operatorname{Hom}_{\mathrm{gr}}(E,E') \times E \arrow[d, "\mathrm{id}_G\times a"'] & & \\ G \times E' \arrow[rr, "a_{E'}"] & & E' \end{tikzcd}$$

where $\sigma_{12} : {}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')\times G\times E \to G\times {}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')\times E$ is the isomorphism given by the permutation of the first two coordinates.

Corollary 3.40. *Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension and $E,E' \in \operatorname{Rep}(\mathcal{S})$. Then a morphism $f \in \operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E,E')$ is determined by its restriction to $E_0 = \pi^{-1}(0)$, the fiber over $0\in A$.*

Proof. Indeed, $f\big([g,e]\big) = f\big(g\cdot[1,e]\big) = g\cdot f\big([1,e]\big)$ for all $(g,e)\in G\times E_0$. □

Definition 3.41. Let $\mathcal{S}:\ 1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension of the abelian variety A. We define the category (enriched over $\mathrm{Sch}\,|\Bbbk)$ of *representations of* $\mathcal{S}$ or $\mathcal{S}$*–modules*, denoted as $\operatorname{Rep}(\mathcal{S})$, as follows:

The *objects* are the *representations* of $\mathcal{S}$.

If $(E,\varrho_E),(E',\varrho_{E'}) \in \operatorname{Rep}(\mathcal{S})$, then $\operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E,E') := {}^G\operatorname{Hom}_{\mathrm{gr}}(E,E')$, the (homogeneous) vector bundle of G–equivariant morphisms.

We define $\operatorname{Rep}_0(\mathcal{S})$ as the wide subcategory of $\operatorname{Rep}(\mathcal{S})$ that has as morphisms $\operatorname{Hom}_{\operatorname{Rep}_0(\mathcal{S})}(E,E') := \operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E,E')_0$. Notice that $\operatorname{Hom}_{\operatorname{Rep}_0(\mathcal{S})}(E,E')$ can be identified with the vector space of morphisms of vector bundles $f: E \to E'$ that commute with the action (compare with remarks 3.19 and 3.39).

Remark 3.42. Later — for reasons of notational uniformity — we represent an affine extension such as $\mathcal{S}:\ 1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ simply as $q: G \to A$ and $\operatorname{Rep}(\mathcal{S})$ simply as $\operatorname{Rep}(q)$ and the same for $\operatorname{Rep}_0(\mathcal{S}) = \operatorname{Rep}_0(q)$. See the comments at the introduction of Section 6 and also Example 6.25.

The following theorem exhibits the relationship between $\operatorname{Rep}(\mathcal{S})$ and $\operatorname{Rep}(H)$ — it can be seen as a generalization of [18, Theorem 2.9].

Theorem 3.43. *Let* $\mathcal{S}:\ 1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ *be an affine extension and* $V \in \operatorname{Rep}_{\mathrm{fin}}(H)$ *a finite dimensional (rational)* H*–module. Then* $\pi_V: E_V = G \times^H V = (G\times V)/H \to A$ *is a representation of* $\mathcal{S}$ *— recall from Theorem 2.61 that the quotient* $\pi_{G\times V}: G\times V \to E_V$ *exists and that* $\pi_V: E_V \to A$ *is a vector bundle of fiber isomorphic to* V*.*

Conversely, if the vector bundle $\pi: E \to A$ *is a representation of* $\mathcal{S}$*, then* E *and* $G\times^H E_0$ *are isomorphic in the category* $\mathrm{HVB}_0(A)$*, where the action* $H\times E_0 \to E_0$ *is given by restriction.*

Moreover, the category $\operatorname{Rep}_0(\mathcal{S})$ *is equivalent to* $\operatorname{Rep}_{\mathrm{fin}}(H)$*. In particular,* $\operatorname{Rep}_0(\mathcal{S})$ *is an abelian, monoidal, rigid, category.*

Proof. The first assertion is the content of Theorem 2.61.

Conversely, if $E \to A$ is a $\mathcal{S}$–module, then E_0 is an H–module and therefore, again by Theorem 2.61, the induced space $E_V = G\times^H E_0$ is a representation of $\mathcal{S}$. Moreover, the morphism $f: G\times E_0 \to E$, $(g,v)\mapsto g\cdot v$ is H–invariant and therefore induces a morphism $\widetilde{f}: E_V \to E$ (given by $\widetilde{f}\big([g,v]\big) = g\cdot v$). Since $\widetilde{f}$ is clearly a bijective morphism of vector bundles, it follows that $\widetilde{f}$ is an isomorphism.

It is now an easy exercise to verify that a morphism of H–modules $f: V \to W$ induces the morphism of $\mathcal{S}$–modules $\widetilde{f}: G\times^H V \to G\times^H W$ $\widetilde{f}\big([g,v]\big) = \big[g, f(v)\big]$. Therefore, we have just constructed a functor $\operatorname{Rep}_{\mathrm{fin}}(H) \to \operatorname{Rep}_0(\mathcal{S})$ such that $V \mapsto G\times^H V$ and $\operatorname{Hom}_{\operatorname{Rep}_{\mathrm{fin}}(H)}(V,W) \ni f \mapsto \widetilde{f} \in \operatorname{Hom}_0(G\times^H V, G\times^H W)$. This functor is clearly the inverse functor of the “restriction to the fiber” functor $\operatorname{Rep}_0(\mathcal{S}) \to \operatorname{Rep}_{\mathrm{fin}}(H)$. $\square$

Corollary 3.44. *Let* $\mathcal{S}$*:* $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ *be an affine extension and* $E,E' \in \operatorname{Rep}(\mathcal{S})$*. Then* $E \cong E'$ *as* $\mathcal{S}$*–modules if and only if* $E_0 \cong E'_0$ *as* H*–modules.*

Proof. Since the H–action of E_0, E'_0 is by restriction, it is clear that if $E\cong E'$, then $E_0 \cong E'_0$. Assume now that $f: E_0 \to E'_0$ is an isomorphism of H–modules.

Then the canonical morphism $G \times E_0 \to E' = G \times^H E'_0$ given by $(g, v) \mapsto \big[g, f(v)\big]$ induces a morphism $f : E = G \times^H E_0 \to E'$, that is clearly a G–equivariant isomorphism of vector bundles. □

Corollary 3.45. *Let* $\mathcal{S}$*:* $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ *be an affine extension and* $E \in \operatorname{Rep}(\mathcal{S})$*. Then* $E \cong G_{\text{ant}} \times^{G_{\text{ant}} \cap H} E_0$*.*

Proof. Let $(G_{\text{ant}} \cap H) \times E_0 \to E_0$ be the $(G_{\text{ant}} \cap H)$–module obtained by restriction of the H–action. Since $\mathcal{S}$ has a Rosenlicht decomposition, $G/(G_{\text{ant}} \cap H) = A$ and $G_{\text{ant}} \times^{G_{\text{ant}} \cap H} E_0 \to A$ is a vector bundle. The canonical inclusion $G_{\text{ant}} \times E_0 \hookrightarrow G \times E_0$ induces a morphism of vector bundles $G_{\text{ant}} \times^{G_{\text{ant}} \cap H} E_0 \to G \times^H E_0 = E$ (of the same dimension) that is clearly an isomorphism. □

Corollary 3.46. *Let* $\mathcal{S}$ *:* $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ *be an affine extension, and* $\big(\pi : E \to A, \varrho_E : G \to Aut_{\text{gr}}(E)\big)$ *an* $\mathcal{S}$*–module. Then* $E \cong \rho(G) \times^{\rho(H)} E_0$*, where, as usual,* $\rho : G \to \operatorname{Aut}_{\text{gr}}(E)$ *is the mid morphism of* ϱ_E *and* $E_0 = \pi^{-1}(0)$*.*

Proof. Immediate. □

Combining Theorem 3.43 with Corollary 3.46 we obtain the following characterization of an homogeneous vector bundle.

Proposition 3.47. *Let* $\pi : E \to A$ *be vector bundle. Then* E *is homogeneous if and only if there exists an affine extension* $\mathcal{S}$*:* $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ *and an action* $a : G \times E \to E$*, such that the following diagram is commutative.*

$$\begin{array}{ccc} G \times E & \xrightarrow{a} & E \\ {\scriptstyle q\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ A \times A & \xrightarrow{s} & A \end{array}$$

and the restriction $a|_{H \times E_0} H \times E_0 \to E_0$ *is a linear representation of* H*.*

Proof. Let E, $\mathcal{S}$ and a be as in the hypothesis. Then, as in the proof of Theorem 3.43, $a|_{G \times E_0} : G \times E_0 \to E$ induces an isomorphisms $G \times^H E_0 \cong E$. Conversely, if If $\pi : E \to A$ is homogeneous, then E is a $Aut_{\text{gr}}(E)$–module by Corollary 3.46. □

Examples 3.48. (1) Let $\mathcal{G}_{\text{aff}}$: $1 \longrightarrow G \xrightarrow{\text{id}} G \longrightarrow \operatorname{Spec}(\Bbbk) \longrightarrow 0$ be an affine group scheme viewed as an affine extension. Then $\operatorname{Rep}(\mathcal{G}_{\text{aff}}) = \operatorname{Rep}(G)$, the "classical" category of representations of an affine group scheme (see Example 3.22).

(2) Let $\mathcal{A}$ be the trivial extension $0 \longrightarrow 0 \longrightarrow A \xrightarrow{\text{id}} A \longrightarrow 0$. Since a homogeneous vector bundle E is trivial if and only if there exists a section $A \hookrightarrow \operatorname{Aut}_{\text{gr}}(E)$ (see Remark 3.29), it follows that $\operatorname{Rep}(\mathcal{A})$ has as objects the trivial bundles $A \times V$, with action $a : A \times (A \times V) \to A \times V$, $b \cdot (c, v) = (b + c, v)$. On the other hand $\operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E, E') = \operatorname{Hom}_{\text{gr}}(E, E') = \operatorname{Hom}_{\Bbbk}(E_0, E'_0) \times A$.

(3) Consider an isogeny $g : A \to A$ and the corresponding affine extension $\mathcal{S}_N$: $1 \longrightarrow N \longrightarrow A \xrightarrow{g} A \cong A/N \longrightarrow 0$, where N is a normal finite subgroup scheme. If $E \in \operatorname{Rep}(\mathcal{S}_N)$, then $E = A \times^N V$, where $V \in \operatorname{Rep}(N)$.

It follows that $\operatorname{Rep}(\mathcal{S}_N)$ can be obtained as follows. Let $\mathcal{N}$ be the category of the trivial homogeneous vector bundles built on $\operatorname{Rep}(N)$: (i) $E \in \operatorname{Obj}(\mathcal{N})$ if

$E = A \times V$, with $V \in \operatorname{Rep}(N)$; (ii) $(f,\ell) \in \operatorname{Hom}_{\mathcal{N}}(A \times V, A \times V')(T)$ if and only if $f = (t_\ell \times h) : A \times T \times V \to A \times T \times V'$, with $h \in \operatorname{Hom}_{\operatorname{Rep}(N)}(V, V')$. Consider the functor $Q : \mathcal{N} \to \operatorname{HVB}_{\mathrm{gr}}(A)$ given by the quotient by the diagonal action $n \cdot (a, v) = (an^{-1}, nv)$. Then the $\operatorname{Rep}(\mathcal{S}_N)$ is the image of $\mathcal{N}$ by Q.

(4) Assume that $\Bbbk = \overline{\Bbbk}$ and let $L \in \operatorname{Pic}(A)$ be an invertible homogeneous vector bundle. Then $L^\times = L \setminus \theta(L)$, where $\theta : A \to L$ is the trivial section, is a smooth group scheme, with Chevalley decomposition induced by the canonical projection $\pi : L \to A$ (see [47, Theorem 2] and [8, Corollary 6]): $\mathcal{L}^\times$: $1 \longrightarrow \Bbbk^* \longrightarrow L^\times \xrightarrow{\pi|_{L^\times}} A \longrightarrow 0$.

It follows from Theorem 3.43 that E is an $\mathcal{L}^\times$–module if and only if $E \cong L^\times \times^{\Bbbk^*} V$, where V is a $\Bbbk^*$–module. On the other hand, it is clear that $L^{\otimes n}$ is an $\mathcal{L}^\times$–module, with action $L^\times \times L^{\otimes n} \to L^{\otimes n}$ given by $a \cdot (l_1 \otimes \cdots \otimes l_n) = (a \cdot l_1) \otimes \cdots \otimes (a \cdot l_n)$. It follows that if $V \cong \oplus V_i$, where $a \cdot v = a^i v$ for $v \in V_i$, then $E \cong \bigoplus_i \bigoplus_{j=1}^{\dim V_i} L^{\otimes i}$.

For further use in the next section, we introduce now the definition of T–*morphisms of homogeneous vector bundles* and some related notions.

Definition 3.49. Let $\mathcal{C}$ be a $\operatorname{Sch}|\Bbbk$–category and $T \in \operatorname{Sch}|\Bbbk$. The *category* $\mathcal{C}(T)$ is defined as follows:

(1) its *objects* are the objects of $\mathcal{C}$.

(2) given two objects $x, y \in \mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}(T)}(x,y) = \operatorname{Hom}_{\mathcal{C}}(x,y)(T)$.

Remark 3.50. (1) Notice that if $\mathcal{C}$ is a $\operatorname{Sch}|\Bbbk$–category, $T \in \operatorname{Sch}|\Bbbk$ and $x \in \mathcal{C}$, then the structure morphism $\mathrm{st} : T \to \operatorname{Spec}(\Bbbk)$ induces an identity morphism in $\operatorname{End}_{\mathcal{C}(T)}(x)$, by post-composition with the identity morphism $\operatorname{Spec}(\Bbbk) \to \operatorname{End}_{\mathcal{C}}(x)$.

(2) Let $\mathcal{C}, \mathcal{D}$ be two $\operatorname{Sch}|\Bbbk$–categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor. It is clear that F induces a functor $F(T) : \mathcal{C}(T) \to \mathcal{D}(T)$.

Definition 3.51. Let $\mathcal{S}$ be an affine extension and $T \in \operatorname{Sch}|\Bbbk$. We define the *category of $\mathcal{S}$–modules with T–morphisms*, as the category $\operatorname{Rep}(\mathcal{S})(T)$.

Remark 3.52. Notice that the degree morphism $\operatorname{Hom}_{\mathrm{gr}}(E, E') \to A$ induces a degree morphism $\operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})(T)}(E, E') = \operatorname{Hom}_{\mathrm{gr}}(E, E')(T) \to A(T)$.

Definition 3.53. Let $\mathcal{C}, \mathcal{D}$ be two $\operatorname{Sch}|\Bbbk$–categories, and $F, G : \mathcal{C} \to \mathcal{D}$ two functors. If $T \in \operatorname{Sch}|\Bbbk$, a *$T$–natural transformation* is a natural transformation $\lambda : F(T) \Rightarrow G(T)$.

A *functor on natural transformations* $\lambda : F \Rightarrow G$ is a functor $\lambda : \operatorname{Sch}|\Bbbk^{\mathrm{op}} \to \operatorname{Sets}$, such that $\lambda(T) : F(T) \Rightarrow G(T)$ is a T–natural transformation.

3.3. **The category** $\operatorname{Rep}(\mathcal{S})$**.**

In this paragraph we collect some basic properties of the categories of representations $\operatorname{Rep}(\mathcal{S})$ and $\operatorname{Rep}_0(\mathcal{S})$ of an affine extension $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$.

Remark 3.54. Let $\mathcal{S}$ be an affine extension. Even though the category $\operatorname{Rep}(\mathcal{S})$ is not monoidal, a situation similar to the one described in Lemma 3.28 holds, since $\operatorname{Rep}_0(\mathcal{S})$ is an abelian monoidal rigid category.

Indeed, once Theorem 3.43 is established, the assertion for $\operatorname{Rep}_0(\mathcal{S})$ is proved by transplanting the corresponding structure from $\operatorname{Rep}_{\mathrm{fin}}(H)$. In that manner we obtain for E, E' objects in $\operatorname{Rep}_0(\mathcal{S})$ (that has the same objects than $\operatorname{Rep}(\mathcal{S})$), other objects in the same categories called $E^\vee$, $E \oplus E'$ and $E \otimes E'$ and for arrows in $\operatorname{Rep}_0(\mathcal{S})$ we can define the arrows — also in $\operatorname{Rep}_0(\mathcal{S})$ —: $f^\vee$, $f \otimes g$, $f + g$ as well as the functors Ker and Coker in $\operatorname{Rep}_0(\mathcal{S})$. This construction at the level of arrows can be defined directly or by transporting them from $\operatorname{Rep}_{\mathrm{fin}}(H)$.

On the other hand, Lemma 3.28 implies the following weaker version of the universal properties.

Lemma 3.55. *Let $\mathcal{S}$ be an affine extension and E, E', F, F' objects in $\operatorname{Rep}(\mathcal{S})$. Consider the graded morphisms $(f, \ell) \in \operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E, F)(T)$, $(f', \ell) \in \operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E', F')(T)$, and $(g, \ell) \in \operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E', F)(T)$. Then $(f \otimes f', \ell) \in \operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E \otimes E', F \otimes F')(T)$, $(f + g, \ell) \in \operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E \oplus E', F)(T)$ and $(f^\vee, -\ell) \in \operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(F^\vee, E^\vee)(T)$.*

Proof. Immediate. $\square$

Definition 3.56. Let $\mathcal{S}$: $1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension of the abelian variety A. We call $\omega_{\mathrm{gr}} : \operatorname{Rep}(\mathcal{S}) \to \mathrm{HVB}_{\mathrm{gr}}(A)$ the forgetful functor in the category of homogeneous vector bundles over A; and $\omega_0 : \operatorname{Rep}_0(\mathcal{S}) \to \mathrm{HVB}_0(A)$ is the functor induced by restriction of ω_{gr} — notice that ω_0 is a monoidal functor.

$$\begin{tikzcd}
\operatorname{Rep}_0(\mathcal{S}) \arrow[r, hook] \arrow[d, "\omega_0"'] & \operatorname{Rep} \mathcal{S} \arrow[d, "\omega_{\mathrm{gr}}"] \\
\mathrm{HVB}_0(A) \arrow[r, hook] & \mathrm{HVB}_{\mathrm{gr}}(A).
\end{tikzcd}$$

Notation 3.57. In the future and in order to simplify the notations, if $(E, \varrho_E) \in \operatorname{Rep}(\mathcal{S})$ we often omit the morphism ϱ_E and write that E *is an $\mathcal{S}$–module*. The forgetful functor $\omega_{\mathrm{gr}} : \operatorname{Rep}(\mathcal{S}) \to \mathrm{HVB}_{\mathrm{gr}}(A)$ is given at the level of objects by $(E, \varrho_E) \mapsto E$. Occasionally and when it does not produce confusions, the forgetful functor applied to objects might be omitted and we write $\omega_{\mathrm{gr}}(E) := E$, and similarly for the hom-objects.

Remark 3.58. Consider the functor $\omega_{\mathrm{gr}} : \operatorname{Rep}(\mathcal{S}) \to \mathrm{HVB}_{\mathrm{gr}}(A)$ and let $T \in \mathrm{Sch}\,|\Bbbk$. Then a T–natural transformation $\lambda : \omega_{\mathrm{gr}}(T) \Rightarrow \omega_{\mathrm{gr}}(T)$ is given by a family of graded morphisms $\lambda_E = (f_E, \ell_E) \in \operatorname{End}_{\mathrm{gr}}\big(\omega_{\mathrm{gr}}(E)\big)(T) = \operatorname{End}_{\mathrm{gr}}(E)(T)$, $E \in \operatorname{Rep}(\mathcal{S})$, such that the graded morphisms λ_E satisfy the following compatibility condition:

For all $E, F \in \operatorname{Rep}(\mathcal{S})$ and $(g, a) \in \operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E, F)(T) = {}^G\operatorname{Hom}_{\mathrm{gr}}(E, F)(T)$ the diagram below, that is a diagram in $\mathrm{Sch}\,|T$, commutes:

$$\begin{tikzcd}
\omega_{\mathrm{gr}}(E)_T \arrow[r, "f_E"] \arrow[d, "g"'] & \omega_{\mathrm{gr}}(E)_T \arrow[d, "g"] \\
\omega_{\mathrm{gr}}(F)_T \arrow[r, "f_F"'] & \omega_{\mathrm{gr}}(F)_T
\end{tikzcd}$$

Notice that in the diagram above we use the fact that $\omega_{\mathrm{gr}}(T)(g, a) = (g, a) \in \operatorname{Hom}_{\mathrm{gr}}(E, F)$.

In particular, given $E_1, E_2 \in \operatorname{Rep}(\mathcal{S})$, then the canonical inclusions $\operatorname{inc}_i : (E_i)_T \to (E_1 \oplus E_2)_T \cong (E_1 \oplus E_2)_T$ induce the following commutative diagram:

$$\begin{tikzcd} \omega_{\mathrm{gr}}(E_i)_T \arrow[r, "\lambda_{E_i}"] \arrow[d, "\omega_{\mathrm{gr}}(\mathrm{inc}_i,0)"'] & \omega_{\mathrm{gr}}(E_i)_T \arrow[d, "\omega_{\mathrm{gr}}(\mathrm{inc}_i,0)"] \\ \omega_{\mathrm{gr}}(E_1 \oplus E_2)_T \arrow[r, "\lambda_{E_1 \oplus E_2}"'] & \omega_{\mathrm{gr}}(E_1 \oplus E_2)_T \end{tikzcd}$$

It follows that $d(\lambda_{E_1}) = d(\lambda_{E_1 \oplus E_2}) = d(\lambda_{E_2})$. In other words, the degree of the morphisms λ_E is constant.

Definition 3.59. In view of Remark 3.58, if $\lambda : \omega_{\mathrm{gr}}(T) \Rightarrow \omega_{\mathrm{gr}}(T)$ is a T–natural transformation, the *degree* of Λ is defined as $d(\lambda_{\mathbb{I}})$, where $\mathbb{I} = \Bbbk \times A \to A$ is the trivial representation (see Example 3.35).

Definition 3.60. Given the functor $\omega_{\mathrm{gr}} : \operatorname{Rep}(\mathcal{S}) \to \mathrm{HVB}_{\mathrm{gr}}(A)$ we consider the functor on natural transformations $\operatorname{End}(\omega_{\mathrm{gr}}) : \mathrm{Sch}\,|\Bbbk \to \mathrm{Mon}$, defined as

$$\operatorname{End}(\omega_{\mathrm{gr}})(T) = \big\{\lambda : \omega_{\mathrm{gr}}(T) \Rightarrow \omega_{\mathrm{gr}}(T) : \lambda \text{ is } T\text{-natural transformation}\big\}.$$

If $g : T' \to T$, then $\operatorname{End}(\omega_{\mathrm{gr}})(f) : \operatorname{End}(\omega_{\mathrm{gr}})(T) \to \operatorname{End}(\omega_{\mathrm{gr}})(T')$ is given by $\operatorname{End}(\omega_{\mathrm{gr}})(f)\big(\{\lambda_E\}\big) = \{\lambda'_E\}$, where $\lambda'_E = \operatorname{End}_{\mathrm{gr}}\big(\omega_{\mathrm{gr}}(E)\big)(g) = \operatorname{End}_{\mathrm{gr}}(E)(g)$ (see Definition 3.14).

Remark 3.61. (1) By definition, an element of $\operatorname{End}(\omega_{\mathrm{gr}})(T)$ is a family $\big\{\lambda_E = (f_E, \ell) \in \operatorname{End}_{\mathrm{gr}}(E)(T) : E \in \operatorname{Rep}(\mathcal{S})\big\}$ (see Remark 3.58).

(2) The monoid structure on $\operatorname{End}(\omega_{\mathrm{gr}})(T)$ is given by vertical composition of the families: $\{\lambda_E\}\circ\{\mu_E\} = \{\lambda_E \circ \mu_E\}$. The unit of the monoid is the family $\big\{(\mathrm{id}_{E\times T}, 0) \in \operatorname{End}_{\mathrm{gr}}(E)(T)\big\}$. Notice that $d(\lambda_E \circ \mu_E) = d(\lambda_E) + d(\mu_E) \in A(T)$.

Definition 3.62. The *degree map* $d_{\omega_{\mathrm{gr}}} : \operatorname{End}(\omega_{\mathrm{gr}}) \to A$ is given by $d_{\omega_{\mathrm{gr}}}(T)\big(\lambda\big) = d\big(\lambda_E\big) = d\big(\lambda_{\mathbb{I}}\big)$.

We denote by $\operatorname{End}_0(\omega_{\mathrm{gr}}) \subset \operatorname{End}(\omega_{\mathrm{gr}})$ the subfunctor of the families of degree 0 $\operatorname{End}_0(\omega_{\mathrm{gr}})(T) = \big\{\lambda \in \operatorname{End}(\omega_{\mathrm{gr}})(T) : d_{\omega_{\mathrm{gr}}}(\lambda) = 0\big\}$.

Remark 3.63. It is clear the degree map d is a morphism of functors on monoids, and that $\operatorname{End}_0(\omega_{\mathrm{gr}}) = \operatorname{Ker}(d)$.

Definition 3.64. Define the subfunctor on monoids $\operatorname{Aut}(\omega_{\mathrm{gr}}) \subset \operatorname{End}(\omega_{\mathrm{gr}})$ by

$$\operatorname{Aut}(\omega_{\mathrm{gr}})(T) = \big\{\lambda = \{\lambda_E\} : \lambda_E \in \operatorname{Aut}_{\mathrm{gr}}(E)(T)\big\} \subset \operatorname{End}(\omega_{\mathrm{gr}})(T)$$

and the corresponding subfunctor $\operatorname{Aut}_0(\omega_{\mathrm{gr}}) \subset \operatorname{End}_0(\omega_{\mathrm{gr}})$ by $\operatorname{Aut}_0(\omega_{\mathrm{gr}}) = \operatorname{Ker}\big(d_{\omega_{\mathrm{gr}}}|_{\operatorname{Aut}(\omega_{\mathrm{gr}})}\big) : \operatorname{Aut}(\omega_{\mathrm{gr}}) \to A$:

$$\operatorname{Aut}_0(\omega_{\mathrm{gr}})(T) = \{\lambda \in \operatorname{Aut}(\omega_{\mathrm{gr}})(T) : d_{\omega_{\mathrm{gr}}}(\lambda) = 0\}.$$

Remark 3.65. In accordance with Corollary 3.26 and Lemma 3.37, if $E := (\pi : E \to A) \in \operatorname{Rep}(\mathcal{S})$, then $\operatorname{End}_{\mathrm{gr}}\big(\omega_{\mathrm{gr}}(E)\big) = \operatorname{End}_{\mathrm{gr}}(E)$ and $\omega_{\mathrm{gr}}\big(\operatorname{End}_{\operatorname{Rep}(\mathcal{S})}$ are a smooth monoid scheme of finite type, and $\operatorname{Aut}_{\mathrm{gr}}\big(\omega_{\mathrm{gr}}(E)\big) \hookrightarrow \operatorname{End}_{\mathrm{gr}}\big(\omega_{\mathrm{gr}}(E)\big)$ and $\omega_{\mathrm{gr}}\big(\operatorname{Aut}_{\operatorname{Rep}(\mathcal{S})}(E)\big) \hookrightarrow \omega_{\mathrm{gr}}\big(\operatorname{End}_{\operatorname{Rep}(\mathcal{S})}(E)\big)$ are open immersions.

On the other hand, is not clear that the functor on monoids $\operatorname{End}(\omega_{\mathrm{gr}})$ and $\operatorname{Aut}(\omega_{\mathrm{gr}})$ are representable, since the situation much more complex as one must take into account the complete natural transformation — i.e. the family $\lambda = \{\lambda_E \in \operatorname{End}_{\mathrm{gr}}(E) : E \in \operatorname{Rep}(\mathcal{S})\}$ — as well as all the consistency conditions (see Remark 3.58).

Next we define a subfunctor on monoids of $\operatorname{End}(\omega_{\mathrm{gr}})$ and the corresponding subfunctor on groups, that will be crucial for the reconstruction process.

Definition 3.66. (1) In the context above, we call $\operatorname{End}^{\otimes}(\omega_{\mathrm{gr}})$ the subfunctor on monoids of $\operatorname{End}(\omega_{\mathrm{gr}})$ given by the natural transformations $\lambda \in \operatorname{End}(\omega_{\mathrm{gr}})(T)$, $T \in \operatorname{Sch}|\Bbbk$, such that:

(i) $\lambda_{E_1 \otimes E_2} = \lambda_{E_1} \otimes \lambda_{E_2}$ for all $E_1, E_2 \in \operatorname{Rep}(\mathcal{S})$ (see Lemma 3.28);
(ii) If $\mathbb{I} = (p_2 : \Bbbk \times A \to A)$ is the trivial representation (see Example 3.35), then $(\lambda_{\mathbb{I}}, \ell) = (\operatorname{id}_{\Bbbk} \times t_\ell, \ell) \in \operatorname{End}_{\mathrm{gr}}(\Bbbk \times A)(T)$.

(2) We define the subfunctor on monoids $\operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})(T) = \big\{\lambda_E : \lambda_E \text{ is an isomorphism}\big\} \subset \operatorname{End}^{\otimes}(\omega_{\mathrm{gr}})(T)$, for $T \in \operatorname{Sch}|\Bbbk$ — notice that $\operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})$ is a functor on groups.

Remark 3.67. It is clear that $\operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})$ can also be seen as a subfunctor of $\operatorname{Aut}(\omega_{\mathrm{gr}})$.

Example 3.68. *G be an affine group and let $\mathcal{S}$ be the associated affine extension of $\operatorname{Spec}(\Bbbk)$, (see examples 3.48 and 3.22).* In this case, $\omega_{\mathrm{gr}} : \operatorname{Rep}(\mathcal{S}) \to \operatorname{HVB}_{\mathrm{gr}}\big(\operatorname{Spec}(\Bbbk)\big)$ is the forgetful functor $\omega : \operatorname{Rep}(G) \to \operatorname{Vect}_{\Bbbk}$. If $T = \operatorname{Spec}(R) \in \operatorname{Sch}|_{\mathrm{aff}}\Bbbk$, since $\operatorname{Hom}_{\operatorname{Rep}(S)}(V,W)(T) \cong \operatorname{Hom}_{\operatorname{Rep}(G)}(V,W) \otimes R$, we deduce that a T-natural transformation $\omega_{\mathrm{gr}}(T) \Rightarrow \omega_{\mathrm{gr}}(T)$ is a family $\lambda_V : V \otimes R \to V \otimes R$ of R–linear morphisms, such that the following diagram is commutative for all $f \in \operatorname{Hom}_{\operatorname{Rep}(G)}(V,W)$

$$\begin{tikzcd} V \otimes R \arrow[r, "\lambda_V"] \arrow[d, "f\otimes \mathrm{id}_R"'] & V \otimes R \arrow[d, "f\otimes \mathrm{id}_R"] \\ W \otimes R \arrow[r, "\lambda_W"] & W \otimes R \end{tikzcd}$$

Moreover, in the notation of [21, page 20] $\{\lambda_R\} \in \operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})(T)$ if and and only if $\{\lambda_V\} \in \operatorname{Aut}^{\otimes}(\omega)$.

Definition 3.69. Adapting Definition 3.60 we can consider the forgetful functor $\omega_0 : \operatorname{Rep}_0(\mathcal{S}) \to \operatorname{HVB}_0(A)$ and define $\operatorname{End}(\omega_0)$ as

$$\operatorname{End}(\omega_0) = \big\{\zeta : \omega_0 \Rightarrow \omega_0 : \zeta \text{ is a natural transformation}\big\}.$$

We can proceed similarly and define: $\operatorname{Aut}(\omega_0), \operatorname{End}^{\otimes}(\omega_0), \operatorname{Aut}^{\otimes}(\omega_0)$.

Remark 3.70. It is easy to see that the functor on monoids $\operatorname{End}_0(\omega_{\mathrm{gr}})$ and $\operatorname{End}(\omega_0)$; $\operatorname{End}_0^{\otimes}(\omega_{\mathrm{gr}})$ and $\operatorname{End}^{\otimes}(\omega_0)$; $\operatorname{Aut}_0(\omega_{\mathrm{gr}})$ and $\operatorname{Aut}(\omega_0)$ as well as $\operatorname{Aut}_0^{\otimes}(\omega_{\mathrm{gr}})$ and $\operatorname{Aut}^{\otimes}(\omega_0)$ are isomorphic. For example, it is clear that $d(\lambda_E, \ell) = 0$ if and only if $\ell = 0$ and $\lambda_E \in \operatorname{End}(\omega_0)$.

Definition 3.71. (1) Given E an object in $\operatorname{Rep}_0(\mathcal{S})$ (or in $\operatorname{Rep}(\mathcal{S})$) we call $\operatorname{Rep}_0(\mathcal{S})_E$ the abelian (monoidal) subcategory of $\operatorname{Rep}_0(\mathcal{S})$ generated by E and $\operatorname{Rep}(\mathcal{S})_E$ the wide extension (in $\operatorname{Rep}(\mathcal{S})$) obtained by taking the graded morphisms.

(2) For $E \in \operatorname{Rep}(\mathcal{S})$ we call $\omega_{\mathrm{gr}}| : \operatorname{Rep}(\mathcal{S})_E \to \operatorname{HVB}_{\mathrm{gr}}(A)$ the restriction of the forgetful functor $\omega_{\mathrm{gr}} : \operatorname{Rep}(\mathcal{S}) \to \operatorname{HVB}_{\mathrm{gr}}(A)$ to the subcategory $\operatorname{Rep}(\mathcal{S})_E$ and similarly for $\omega_0| : \operatorname{Rep}_0(\mathcal{S})_E \to \operatorname{HVB}_{\mathrm{gr}}(A)$ the restriction of $\omega_0 : \operatorname{Rep}_0(\mathcal{S}) \to \operatorname{HVB}_{\mathrm{gr}}(A)$

Remark 3.72. (1) The two categories $\mathrm{Rep}_0(\mathcal{S})_E$ and $\mathrm{Rep}(\mathcal{S})_E$ are defined along the same lines than the constructions of Definition 3.30.

(2) The structures just defined are illustrated in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Rep}_0(\mathcal{S}) \arrow[rr, hook] \arrow[d, "\omega_0"'] & & \mathrm{Rep}\,\mathcal{S} \arrow[d, "\omega_{\mathrm{gr}}"] \\
& \mathrm{Rep}_0(\mathcal{S})_E \subseteq \mathrm{Rep}(\mathcal{S})_E \arrow[ul, hook] \arrow[ur, hook] \arrow[dl, "\omega_0|"'] \arrow[dr, "\omega_{\mathrm{gr}}|"] & \\
\mathrm{HVB}_0(A) \arrow[rr, hook] & & \mathrm{HVB}_{\mathrm{gr}}(A)
\end{tikzcd}$$

(3) If $\mu \in \mathrm{End}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\mathrm{Rep}(\mathcal{S})_E}\big)$, from the conditions on the family μ it follows that μ_E determines μ. Moreover, the universal property of the category $\mathrm{Rep}(\mathcal{S})_E$ guarantees that $\mathrm{End}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\mathrm{Rep}(\mathcal{S})_E}\big)$ is isomorphic with a closed submonoid scheme of $\mathrm{End}_{\mathrm{gr}}\big(\omega_{\mathrm{gr}}(E)\big)$ and hence it is a monoid scheme of finite type.

Example 3.73 (The universal extension of an abelian variety)**.** In [17] and [18], Brion constructs the projective cover of A in the category of commutative proalgebraic group schemes. This cover has associated an affine extension $\mathcal{G}_A$ of anti-affine type, called the *universal extension of the abelian variety A*. We prove in this example that $\mathrm{Rep}(\mathcal{G}_A) \cong \mathrm{HVB}_{\mathrm{gr}}(A)$.

Given a homogeneous vector bundle $E \to A$, consider the smooth affine extension $Aut_{\mathrm{gr}}(E)$ (see Remark 3.10), and let $Aut_{\mathrm{gr}}(E)_{\mathrm{ant}}$ be the associated closed subextension of anti-affine type (see Theorem 2.42). Then, by Corollary 3.45, $E \cong \mathrm{Aut}_{\mathrm{gr}}(E)_{\mathrm{ant}} \times^{\mathrm{Aut}_{\mathrm{gr}}(E)_{\mathrm{ant}} \cap \mathrm{Aut}_0(E)} E_0$.

Consider an affine faithfully flat filtered system within the family of the affine extensions $Aut_{\mathrm{gr}}(E)_{\mathrm{ant}}$, $E \in \mathrm{HVB}_{\mathrm{gr}}(A)$ — for example, such a family can by constructed using the partial order $E \leq E'$ if $E \cong E' \oplus E''$ for some homogeneous vector bundle E'', see the proof of Lemma 4.4 —. Then, taking limit on E we get a (commutative) affine extension $\mathcal{G}_A$ together with morphisms $\varrho_E : \mathcal{G}_A \to Aut_{\mathrm{gr}}(E)_{\mathrm{ant}}$:

$$\begin{tikzcd}
1 \arrow[r] & H_A \arrow[r] \arrow[d, "\rho_E|_{H_A}"] & G_A \arrow[r, "q"] \arrow[d, "\rho_E"] & A \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & \mathrm{Aut}_{\mathrm{gr}}(E)_{\mathrm{ant}} \cap \mathrm{Aut}_0(E) \arrow[r] & \mathrm{Aut}_{\mathrm{gr}}(E)_{\mathrm{ant}} \arrow[r, "q_E"'] & A \arrow[r] & 0
\end{tikzcd}$$

The affine extension $\mathcal{G}_A$ is called the *universal (anti-affine) extension of the abelian variety A*.

The equivalence of Brion's construction and the construction of $\mathcal{G}_A$ as an limit, is a direct consequence of the Tannaka Duality Theorem 4.6, see Example 5.3 below.

Observe that the affine extension $\mathcal{G}_A$, being the limit of extensions of anti-affine type, is also an extension of anti-affine type, by Theorem 2.58.

Next, we prove that $\mathrm{Rep}(\mathcal{G}_A) \cong \mathrm{HVB}_{\mathrm{gr}}(A)$.

If $E \to A$ is a homogeneous vector bundle, then the morphism $\varrho_A : \mathcal{G}_A \to Aut_{\mathrm{gr}}(E)_{\mathrm{ant}} \subset Aut_{\mathrm{gr}}(E)$ is a representation for $\mathcal{G}_A$. Consider the restricted action $H_A \times E_0 \to E_0$; by Theorem 3.43 $G_A \times^{H_A} E_0 \to A$ exists and is a $\mathcal{G}_A$–module. Clearly, $E \cong G_A \times^{H_A} E_0$ in $\mathrm{HVB}_0(A)$, and therefore the vector bundles are isomorphic in $\mathrm{HVB}_{\mathrm{gr}}(A)$.

Moreover, let $E, E' \in \mathrm{HVB}_{\mathrm{gr}}(A)$ be two vector bundles and consider the structures of $\mathcal{G}_A$–modules defined above. Then ${}^{G_A}\operatorname{Hom}_{\mathrm{gr}}(E,E') = \operatorname{Hom}_{\mathrm{gr}}(E,E')$. Indeed, the action $G_A \times \operatorname{Hom}_{\mathrm{gr}}(E,E') \to \operatorname{Hom}_{\mathrm{gr}}(E,E')$ (given as in Remark 3.36) is such that $\operatorname{Hom}_{\mathrm{gr}}(E,E')_a = \operatorname{Hom}_0(E,E') \otimes \Bbbk(a)$ is $(G_A)_{\Bbbk(a)}$–stable for all $a \in A$. Thus the anti-affine group $(G_A)_{\Bbbk(a)}$ acts trivially on $\operatorname{Hom}_{\mathrm{gr}}(E,E')_a$, since $\operatorname{Hom}_0(E,E')_a$ is an affine $\Bbbk(a)$–space. It follows that G_A acts trivially on $\operatorname{Hom}_{\mathrm{gr}}(E,E')$ and $\operatorname{Hom}_{\mathcal{G}_A}(E,E') = \operatorname{Hom}_{\mathrm{gr}}(E,E')$.

The remarks above clearly show that the category $\operatorname{Rep}(\mathcal{G}_A)$ is equivalent to $\mathrm{HVB}_{\mathrm{gr}}(A)$.

Example 3.74. Recall that any affine group scheme G can be interpreted as an affine extension of the trivial abelian variety $A = \operatorname{Spec}(\Bbbk)$ (see Example 2.23); in particular, the trivial group $\operatorname{Spec}(\Bbbk)$ corresponds to the sequence

$$\mathcal{E}: \qquad 1 \longrightarrow \operatorname{Spec}(\Bbbk) \longrightarrow \operatorname{Spec}(\Bbbk) \longrightarrow \operatorname{Spec}(\Bbbk) \longrightarrow 0\ .$$

Analogously, the category $\mathrm{HVB}_0\big(\operatorname{Spec}(\Bbbk)\big)$ is equivalent to $\mathrm{Vect}_{\Bbbk}$.

Moreover, $\operatorname{Rep}(\mathcal{E}) = \mathrm{HVB}_0\big(\operatorname{Spec}(\Bbbk)\big) \cong \mathrm{Vect}_{\Bbbk} = \operatorname{Rep}\big(\operatorname{Spec}(\Bbbk)\big)$. On the other hand, since $\operatorname{Aut}_{\mathrm{gr}}(V) = \mathrm{GL}(V)$ and that $\mathrm{GL}(V)_{\mathrm{ant}} = \operatorname{Spec}(\Bbbk)$, it follows that $\mathcal{G}_{\operatorname{Spec}(\Bbbk)}$ is the limit of the constant trivial extension $\mathcal{E}$. Hence, $\mathcal{G}_{\operatorname{Spec}(\Bbbk)} = \mathcal{E}$ and in particular $G_{\operatorname{Spec}(\Bbbk)} = \operatorname{Spec}(\Bbbk)$ — as expected from the Tannaka Duality Theorem for affine group schemes applied to the category $\mathrm{Vect}_{\Bbbk}$ with the identity as forgetful functor.

The definition that follows is the natural generalization of the one referred to in the affine case.

Definition 3.75. An $\mathcal{S}$–module $E \in \operatorname{Rep}(\mathcal{S})$ is *faithful* is the corresponding morphism $\mathcal{S} \to \mathcal{A}ut_{\mathrm{gr}}(E)$ is a closed immersion of affine extensions.

Remark 3.76. Let $\mathcal{S}:\ 1 \longrightarrow H \longrightarrow G \xrightarrow{\ q\ } A \longrightarrow 0$ be an affine extension and $\varrho : \mathcal{S} \to \mathcal{A}ut_{\mathrm{gr}}(E)$ be a representation. Since $H \hookrightarrow G$ is a closed immersion, it follows that ϱ is faithful if and only if $\rho : G \to \operatorname{Aut}_{\mathrm{gr}}(E)$ is a closed immersion, if and only if ρ is an immersion (since G is a quasi-compact group scheme, see Theorem 2.9).

Theorem 3.77. *Let $\mathcal{S}:\ 1 \longrightarrow H \longrightarrow G \xrightarrow{\ q\ } A \longrightarrow 0$ be an affine extension. Then $\mathcal{S}$ is of finite type if and only if there exists a faithful $\mathcal{S}$–module $E \in \operatorname{Rep}(\mathcal{S})$.*

Proof. Recall that G is of finite type if and only if H is so (see Remark 2.17). If H is of finite type, then there exists a faithful representation $\rho_V : H \hookrightarrow \mathrm{GL}(V)$. Consider the induced $\mathcal{S}$–module $E_V = G \times^H V$ (see Theorem 3.43). Then we have a morphism of affine extensions

$$\begin{tikzcd}
\mathcal{S}: \arrow[d, "\varrho"'] & 1 \arrow[r] & H \arrow[r] \arrow[d, "\rho|_H"'] & G \arrow[r, "q"] \arrow[d, "\rho"'] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{A}ut_{\mathrm{gr}}(E_V): & 1 \arrow[r] & \operatorname{Aut}_0(E) \arrow[r] & \operatorname{Aut}_{\mathrm{gr}}(E) \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

where $\rho|_H : H \to \operatorname{Aut}_0(E)$ is a closed immersion. It follows that ϱ is a closed immersion (since $\operatorname{Ker}(\rho) \subset H$).

On the other hand, if there exists a faithful representation $\varrho : \mathcal{S} \to \mathrm{Aut}_{\mathrm{gr}}(E)$, then the restriction $\rho|_{H} : H \to \mathrm{Aut}_0(E)$ is a closed immersion. It follows that the restriction $\overline{\varrho}|_{H\times E_0} : H \times E_0 \to E_0$ is a faithful representation of H. Therefore, $\mathcal{S}$ is of finite type. □

Lemma 3.78. *Let* $\mathcal{S} :\ 1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ *be an affine extension, and* $\mathcal{S}' :\ 1 \longrightarrow H' \longrightarrow G' \xrightarrow{q|_{G'}} A \longrightarrow 0$ *a closed sub-extension of* $\mathcal{S}$*. Then there exists a homogeneous vector bundle* $E \in \mathrm{Rep}(\mathcal{S})$ *and a homogeneous line sub-bundle* $L \subset E$*, such that* G' *is the stabilizer of* L*, that is for all schemes* T*,*

$$G'(T) = \big\{ g \in G(T) : g \text{ induces a } T\text{–automorphism in } L \times T \big\}$$

(see for example [14, § 2.2]).

Proof. It is well known that given the pair $H' \subset H$ as above, there exists a finite dimensional H–module V and a one dimensional subspace $W \subset V$ such that H' is the stabilizer of W, i.e. H' is the largest closed group subscheme of H such that $H' \cdot W \subset W$ (see for example [26, Chapter 8, Theorem 2.3]). Since $\mathcal{S}'$ is an affine extension, it follows from Theorem 3.43 that the quotients $E_V = G \times^H V$ and $E_W = G' \times^{H'} W$ exist and are representations of the extensions $\mathcal{S}$ and $\mathcal{S}'$ respectively. We affirm that $\varphi : E_W \to E_V$, the morphism induced by the canonical morphism $G' \times W \to E_V$, $(g, w) \mapsto [g, w]$; is an immersion of vector bundles. Indeed, if $\xi_i = [g_i, w_i] \in E_W = G' \times^{H'} W$, $i = 1, 2$, such that $[g_1, w_1] = [g_2, w_2] \in E_V = G \times^H V$, then there exists $h \in H$ such that $g_2 h = g_1$ and $w_1 = h \cdot w_2$. It follows that $h \in G'$ and therefore $h \in H'$; hence, $\xi_1 = \xi_2$.

Let $L = \varphi(E_W) \subset E_V$ be the subvector bundle image of φ; we prove that $L \subset E_V$ does the required job for G and G'. Let $g \in G$ be such that $g \cdot L = L$; we want to prove that $g \in G'$. Since g stabilizes L, it follows that $g \cdot [g_1, w_1] = [gg_1, w_1] \in L$ for all $[g_1, w_1] \in L$; therefore there exist $g_2 \in G'$, $w_2 \in W$ such that $[gg_1, w_1] = [g_2, w_2]$.

Assume that $g \in H$. If moreover $g_1 = 1$, then $[g_2, w_2] = [g, w_1] = [1, gw_1]$, and there exists $t \in H$ such that $t = g_2$ and $tw_2 = gw_1$. It follows that $t \in H \cap G'$, and thus $gw_1 \in W$ for all $w_1 \in W$. Therefore, $g \in H'$.

If $g \in G(T)$ is arbitrary, let $f : T' \to T$ a fpqc morphism and $c \in G'\big(T'\big)$, $q \circ c = q \circ g \circ f \in A(T')$ (such a pair (f, c) exists because $\mathcal{S}'$ is a short exact sequence). Then $(g \circ f)c^{-1} \in H(T')$ stabilizes $L\big(T'\big)$ and therefore $(g \circ f)c^{-1} \in H'(T')$. It follows that $g \circ f \in G'(T')$, and hence $g \in G'(T)$ (indeed f is a faithfully flat morphism and hence we can apply Lemma 2.41). □

4. Recovering an affine extension from its representations

In this section we fix an affine extension $\mathcal{S} :\ 1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$, $\mathcal{S} = \lim \mathcal{S}_\alpha$, where $\big\{\mathcal{S}_\alpha :\ 1 \longrightarrow H_\alpha \longrightarrow G_\alpha \xrightarrow{q_\alpha} A \longrightarrow 0\, ; \phi_{\alpha,\beta}\big\}_{\alpha,\beta \in I}$ is an (affine) faithfully flat filtered system of affine extensions of finite type. Call $\phi_\alpha : \mathcal{S} \to \mathcal{S}_\alpha$ the canonical maps depicted in the diagram below:

$$\begin{tikzcd}
\mathcal{S} : \arrow[d, "\phi_\alpha"] & 1 \arrow[r] & H \arrow[r] \arrow[d, "f_\alpha|_H"'] & G \arrow[r, "q"] \arrow[d, "f_\alpha"] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{S}_\alpha : & 1 \arrow[r] & H_\alpha \arrow[r] & G_\alpha \arrow[r, "q_\alpha"] & A \arrow[r] & 0.
\end{tikzcd}$$

As in the classical case of Tannaka Duality for affine group schemes, given now the more general situation of an affine extension $\mathcal{S}$ and the category $\operatorname{Rep}(\mathcal{S})$, we characterize G as the group scheme consisting of all the (families of) automorphisms of the objects $E \in \operatorname{Rep}(\mathcal{S})$ that commute with all the morphisms of the category $f \in \operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E, E')$ and that satisfy additional compatibility conditions related to the abelian and monoidal properties of $\operatorname{Rep}(\mathcal{S})$. In order to formalize this idea, we will make use of the forgetful functors $\omega_{\mathrm{gr}} : \operatorname{Rep}(\mathcal{S}) \to \mathrm{HVB}_{\mathrm{gr}}(A)$ and $\omega_0 : \operatorname{Rep}_0(\mathcal{S}) \to \mathrm{HVB}_0(A)$, as well as the associated functors on groups $\operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})$ and $\operatorname{Aut}^{\otimes}_0(\omega_{\mathrm{gr}}) \cong \operatorname{Aut}^{\otimes}(\omega_0)$ (see definitions 3.56 and 3.66).

Following the usual pattern and similarly to the classical case, we first treat the problem in the "finite type" setting, and then take limits.

Remark 4.1. (1) By definition, $(\lambda_E, \ell)_E \in \operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})(T)$ if

(i) The morphisms λ_E fit in the commutative diagram of T–schemes

$$\begin{tikzcd} E_T = E \times T \arrow[r, "\lambda_E"] \arrow[d, "\pi_E \times \mathrm{id}_T"'] & E \times T \arrow[d, "\pi_E \times \mathrm{id}_T"] \\ A_T = A \times T \arrow[r, "t_\ell"'] & A \times T \end{tikzcd}$$

for all $E \in \operatorname{Rep}(\mathcal{S})$, and the induced morphisms $\widehat{\lambda_E} : E_T \to t_\ell^*(E_T)$ are isomorphisms of A_T–vector bundles (recall that $\ell \in A(T)$);

(ii) for all $E, E' \in \operatorname{Rep}(\mathcal{S})$ we have equalities of morphisms of A_T–vector bundles $\widehat{\lambda_{E\otimes E'}} = \widehat{\lambda_E} \otimes \widehat{\lambda_{E'}} : (E_T \otimes E'_T) \to t_\ell^*(E_T \otimes E'_T)$;

(iii) $\lambda_{\mathbb{I}} = \big(\mathrm{id}_{\Bbbk} \times t_\ell, \ell\big) : (\Bbbk \times A) \times T \to (\Bbbk \times A) \times T$, where $\mathbb{I}$ is the trivial representation, and

(iv) for every G-equivariant morphism $(f, b) \in \operatorname{Hom}_{\operatorname{Rep}(\mathcal{S})}(E, E')(T)$ the following diagram of morphisms of T–schemes is commutative:

$$\begin{tikzcd} E \times T \arrow[r, "f"] \arrow[d, "\lambda_E"'] & E' \times T \arrow[d, "\lambda_{E'}"] \\ E \times T \arrow[r, "f"'] & E' \times T \end{tikzcd}$$

(2) There exists a canonical morphism (natural transformation) from the group functor G into $\operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})$, given as follows. If T is a scheme, we consider the morphism of groups $(g : T \to G) \mapsto \overline{g} = \big(\rho_E(T)(g)\big)_E : G(T) \to \operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})(T)$, where $\varrho_E : \mathcal{S} \to Aut_{\mathrm{gr}}(E)$ is as usual the morphism of affine extensions associated to the representation E.

Observe that if $\rho_E(T)(g) = (\rho_g, b)$, then the morphisms of T–schemes $\rho_g : E_T \to E_T$ satisfy the following commutative diagram.

$$\begin{tikzcd} E \times T \arrow[r, "\rho_g"] \arrow[d, "\pi_E \times \mathrm{id}_T"'] & E \times T \arrow[d, "\pi_E \times \mathrm{id}_T"] \\ A \times T \arrow[r, "t_b"'] & A \times T, \end{tikzcd}$$

and induce morphisms of A_T–vector bundles $\widehat{\rho_E(T)}(g) : E_T \to t_b^* E_T$. Moreover, by definition of $\operatorname{Rep}(\mathcal{S})$, the commutativity of the maps $\overline{g}$ with the maps that come from applying the forgetful functor (condition stated in Remark 4.1, (iv)) follows directly. Regarding the other requirements in the remarks just mentioned we have that condition (i) was already checked, and conditions (ii) and (iii) are direct.

(3) Let $E \in \operatorname{Rep}(\mathcal{S})$ and consider the restriction of the forgetful functor $\omega_{\mathrm{gr}} : \operatorname{Rep}(\mathcal{S}) \to \mathrm{HVB}_{\mathrm{gr}}(A)$ to the subcategory $\operatorname{Rep}(\mathcal{S})_E$ (see Definition 3.71 and Remark 3.72) and construct the corresponding group functor $\operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big)$. Then the map $\lambda \mapsto \lambda_E$ identifies $\operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big)$ as a group subfunctor with its image in $\operatorname{Aut}_{\mathrm{gr}}(E) \subset \operatorname{Aut}(E)$. Moreover, it follows (in a similar manner than in the mentioned remark) that $\operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big)$ can be identified with a closed subgroup scheme of the smooth group scheme of finite type $\operatorname{Aut}_{\mathrm{gr}}(E)$ and therefore $\operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big)$ is of finite type — $\operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big)$ is the unit group of the algebraic monoid scheme $\operatorname{End}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big)$.

Remark 4.2. Let $(E, \varrho_E) \in \operatorname{Rep}(\mathcal{S})$. We denote the scheme theoretic image $\rho_E(G)$ by G_E. Since $\operatorname{Aut}_{\mathrm{gr}}(E)$ of finite type, it follows that G_E is a group scheme of finite type, and the morphism ϱ_E factors through an affine subextension $\mathcal{S}_E$ as follows

$$
\begin{tikzcd}
\mathcal{S}: \arrow[d, "\varrho_E"] & 1 \arrow[r] & H \arrow[r] \arrow[d] & G \arrow[r, "q"] \arrow[d, "\rho_E"] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{S}_E: \arrow[d, hook] & 1 \arrow[r] & (G_E)_0 = \rho_E(H) \arrow[r] \arrow[d, hook] & G_E \arrow[r, "d_E|_{G_E}"] \arrow[d, hook] & A \arrow[r] \arrow[d, equal] & 0 \\
Aut_{\mathrm{gr}}(E): & 1 \arrow[r] & \operatorname{Aut}_0(E) \arrow[r] & \operatorname{Aut}_{\mathrm{gr}}(E) \arrow[r, "d_E"'] & A \arrow[r] & 0
\end{tikzcd}
$$

Lemma 4.3. *Let $E \in \operatorname{Rep}(\mathcal{S})$. Then $\operatorname{Rep}(\mathcal{S})_E \cong \operatorname{Rep}(\mathcal{S}_E)$. Moreover, the canonical inclusion $G_E \hookrightarrow \operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big)$ is an isomorphism. In particular, the corresponding affine extensions are isomorphic.*

Proof. Recall that $G_E \subset \operatorname{Aut}_{\mathrm{gr}}(E)$ is a closed subgroup scheme, and hence of finite type. Since any representation of G_E (resp. $\operatorname{Aut}_{\mathrm{gr}}(E)$) is a G-homogeneous vector bundle, and that E is a faithful representation of G_E (resp. $\operatorname{Aut}_{\mathrm{gr}}(E)$), it follows that any representation of G_E (resp. $\operatorname{Aut}_{\mathrm{gr}}(E)$) belongs to $\operatorname{Rep}(\mathcal{S})_E$. Indeed, it follows from Theorem 3.43 that E_0 is a faithful representation of $(G_E)_0$ and $\operatorname{Aut}_0(E)$; therefore, any $(G_E)_0$–module (resp. $\operatorname{Aut}_0(E)$–module) belongs to $(\mathrm{Vect}_{\Bbbk})_{E_0}$ (see for example [60, § 3.5]). Applying again Theorem 3.43 we deduce that $\operatorname{Obj}\big(\operatorname{Rep}(\mathcal{S}_E)\big) = \operatorname{Obj}\big(\operatorname{Rep}_0(\mathcal{S}_E)\big)$ and $\operatorname{Obj}\big(\operatorname{Rep}\big(\operatorname{Aut}_{\mathrm{gr}}(E)\big)\big) = \operatorname{Obj}\big(\operatorname{Rep}_0\big(\operatorname{Aut}_{\mathrm{gr}}(E)\big)\big)$ are contained in $\operatorname{Obj}\big(\operatorname{Rep}(\mathcal{S})_E\big)$.

Let $F \in \operatorname{Rep}(\mathcal{S})_E$ be a $\operatorname{Aut}_{\mathrm{gr}}(E)$–homogeneous vector bundle and $L \subset F$ a G_E–line sub-bundle. We affirm that $\operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big)$ stabilizes L. If this is the case, since $\operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big) \subset \operatorname{Aut}_{\mathrm{gr}}(E)$ (by the assignment $(\lambda_{E'}, \ell) \mapsto (\lambda_E, \ell)$) is an closed subgroup scheme, it follows from Lemma 3.78 applied to $G_E \subset \operatorname{Aut}_{\mathrm{gr}}(E)$ that $\operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big) = G_E$; in particular, notice that $\operatorname{Aut}_0(E) = (G_E)_0$.

Let $L \subset F$ as before; then the morphism $\rho_E : G \to \operatorname{Aut}_{\mathrm{gr}}(E)$ induces G–linearizations on L and F. Since the inclusion $\iota : L \hookrightarrow F$ is G_E–equivariant, it is also G–equivariant, and it follows that if T is a $\Bbbk$–scheme, $g \in G(T)$, $(\lambda_{E'}, b)_{E'} \in \operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big)(T)$ and $(\ell, t) \in L \times T$, then

$$\lambda_E(\ell,t) = \big(\lambda_E \circ (\iota \times \mathrm{id}_T)\big)(\ell,t) = \big((\iota \times \mathrm{id}_T)\circ \lambda_L\big)(\ell,t) \in L \times T.$$

In other words, (λ_E, b) stabilizes L, and therefore $(\lambda_{E'}, b)_{E'} \in \operatorname{Aut}_{\mathrm{gr}}(E)$ stabilizes L. $\square$

Lemma 4.4. *Let $\mathcal{S}$ be an affine extension. Let $\operatorname{Aut}_0^{\otimes}(\omega_{\mathrm{gr}}) \subset \operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})$ be the subgroup functor constructed in Definition 3.66. Then the sequence*

$$Aut^{\otimes}(\omega_{\mathrm{gr}}): \quad 1 \longrightarrow \operatorname{Aut}_0^{\otimes}(\omega_{\mathrm{gr}}) = \operatorname{Aut}^{\otimes}(\omega_0) \longrightarrow \operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}}) \xrightarrow{q_{\omega_{\mathrm{gr}}}} A \longrightarrow 0$$

is the limit of the affine faithfully flat filtered system of affine extensions of finite type $Aut^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S}_E)}\big)$:

$$1 \to \operatorname{Aut}_0^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big) = \operatorname{Aut}^{\otimes}\big(\omega_0|_{\operatorname{Rep}(\mathcal{S})_E}\big) \to \operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big) \xrightarrow{q_{\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}}} A \to 0$$

where the system is directed as follows: if $E, E' \in \operatorname{Rep}(\mathcal{S})$, then $E' \geq E$ if and only $E = E' \oplus F$ for some $F \in \operatorname{Rep}(\mathcal{S})$, with transition morphisms given by restriction.

In particular, $Aut^{\otimes}(\omega_{\mathrm{gr}})$ is an affine extension.

Proof. Ii is clear that if $E' \geq E$, then $\operatorname{Rep}(\mathcal{S})_{E'} \subset \operatorname{Rep}(\mathcal{S})_E$, and the system defined above is filtered, with transition morphisms given by restriction.

$$\begin{tikzcd}
1 \arrow[r] & (G_E)_0 \arrow[r] \arrow[d, equal] & G_E \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & \operatorname{Aut}_0^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big) \arrow[r] \arrow[d] & \operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big) \arrow[r, "q_{\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}}"] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & \operatorname{Aut}_0^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_{E'}}\big) \arrow[r] & \operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_{E'}}\big) \arrow[r, "q_{\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_{E'}}}"'] & A \arrow[r] & 0
\end{tikzcd}$$

Moreover, by the very definition of $\operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})$ and $\operatorname{Aut}_0^{\otimes}(\omega_{\mathrm{gr}})$ as group functors, it follows that

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{Aut}_0^{\otimes}(\omega_{\mathrm{gr}}) \arrow[r] \arrow[d, equal] & \operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}}) \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & \lim \operatorname{Aut}_0^{\otimes}(\omega_{\mathrm{gr}}) \arrow[r] & \lim \operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\operatorname{Rep}(\mathcal{S})_E}\big) \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

$\square$

Notation 4.5. In what follows, $\mathcal{K}$ denotes the (affine, faithfully flat) filtered system defined in Lemma 4.4 above.

Theorem 4.6 (Reconstruction of affine extensions)**.** *Let $\mathcal{S}$ be an affine extension. Then the natural map $\varphi : G \to \operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})$ is an isomorphism of functors $G \cong \operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}}) : Sch^{op} \to \mathrm{Groups}$. Moreover, this isomorphism induces an isomorphism* *of affine extensions*

$$\begin{tikzcd}
\mathcal{S}: \arrow[d, "\phi"', "\cong"] & 1 \arrow[r] & H \arrow[r] \arrow[d, "f|_H"', "\cong"] & G \arrow[r, "q"] \arrow[d, "f"', "\cong"] & A \arrow[r] \arrow[d, equal] & 0 \\
Aut^{\otimes}(\omega_{\mathrm{gr}})): & 1 \arrow[r] & \mathrm{Aut}_0^{\otimes}(\omega_{\mathrm{gr}}) = \mathrm{Aut}^{\otimes}(\omega_0) \arrow[r] & \mathrm{Aut}^{\otimes}(\omega_{\mathrm{gr}}) \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

In particular, two affine extensions $\mathcal{S}$ and $\mathcal{S}'$ of the abelian variety A are isomorphic if and only if there exists an equivalence of categories $F : \mathrm{Rep}(\mathcal{S}) \to \mathrm{Rep}(\mathcal{S}')$ such that $F|_{\mathrm{Rep}_0(\mathcal{S})} : \mathrm{Rep}_0(\mathcal{S}) \to \mathrm{Rep}_0(\mathcal{S}')$ is a monoidal functor and the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Rep}(\mathcal{S}) \arrow[rr, "F"] \arrow[dr, "\omega_{\mathrm{gr},\mathrm{Rep}(\mathcal{S})}"'] & & \mathrm{Rep}(\mathcal{S}') \arrow[dl, "\omega_{\mathrm{gr},\mathrm{Rep}(\mathcal{S}')}"] \\
& \mathrm{HVB}_{\mathrm{gr}}(A) &
\end{tikzcd}$$

Proof. Let $E \in \mathrm{Rep}(\mathcal{S})$ and $G_E \subset \mathrm{Aut}_{\mathrm{gr}}(E)$ be the scheme theoretic image of $\rho_E : G \to \mathrm{Aut}_{\mathrm{gr}}(E)$. The group G_E is by definition a closed subgroup scheme of $\mathrm{Aut}_{\mathrm{gr}}(E)$, and fits into the affine extension $\mathcal{S}_E = \varrho_E(\mathcal{S})$ (see Remark 4.2). Moreover, $G_E = \mathrm{Aut}(\omega_{\mathrm{gr}}|_{\mathrm{Rep}(\mathcal{S})_E}) \subset \mathrm{Aut}_{\mathrm{gr}}(E)$ by Lemma 4.3.

We direct the system of affine extensions $\{\mathcal{S}_E\}_{E \in \mathrm{Rep}(\mathcal{S})}$ by $E' \geq E$ if and only if the representation E' factorizes trough G_E — i.e. there exists a morphism of group schemes $\rho_{E,E'} : G_E \to G_{E'}$, with $\rho_{E'} = \rho_{E,E'} \circ \rho_E$. In particular, if $E' \geq E$, then $E' \in \mathrm{Rep}(\mathcal{S}_E)$; it follows that $\mathrm{Rep}(\mathcal{S})_{E'} \subset \mathrm{Rep}(\mathcal{S})_E$. Hence, we have the following commutative diagram of group schemes (of finite type)

$$\begin{tikzcd}
G_E \arrow[r, "\cong"] \arrow[d, "\rho_{E,E'}"'] & \mathrm{Aut}^{\otimes}(\omega_{\mathrm{gr}}|_{\mathrm{Rep}(\mathcal{S})_E}) \arrow[d, "f_{E,E'}"] \\
G_{E'} \arrow[r, "\cong"'] & \mathrm{Aut}^{\otimes}(\omega_{\mathrm{gr}}|_{\mathrm{Rep}(\mathcal{S})_{E'}})
\end{tikzcd}$$

that fits in a commutative diagram of affine extensions. In particular, one has that $f_{E,E'}$ is an epimorphism if and only if $\rho_{E,E'}$ is so. It is clear that these morphisms induce an affine faithfully flat filtered system indexed by $\mathrm{Rep}(\mathcal{S})$, that we call $\mathcal{J}$.

Since $\mathcal{S}$ is an affine extension, we deduce from Theorem 3.77 that $\mathcal{S}$ is the limit of a subsystem of affine extensions $\{\mathcal{S}_E\}_{E \in I}$, $I \subset \mathcal{J}$, and therefore $\lim_{\mathcal{J}} \mathcal{S}_E = \lim_I \mathcal{S}_E = \mathcal{S}$.

On the other hand, it follows from Lemma 4.4 that the systems of affine extensions $\{\mathcal{S}_E\}_{\mathcal{J}}$ and $\{ Aut^{\otimes}(\omega_{\mathrm{gr}}|_{\mathrm{Rep}(\mathcal{S}_E)})\}_{\mathcal{K}}$ (see Notation 4.5) have the same limit $\lim_{\mathcal{J}} \mathcal{S}_E = \lim_{\mathcal{K}} Aut^{\otimes}(\omega_{\mathrm{gr}}|_{\mathrm{Rep}(\mathcal{S}_E)}) = Aut^{\otimes}(\omega_{\mathrm{gr}})$.

The last assertion is clear. □

Definition 4.7. Let $\mathcal{S}: \ 1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension and E an object in $\mathrm{Rep}(\mathcal{S})$. Call $\langle E \rangle$ the full subcategory of $\mathrm{Rep}(\mathcal{S})$ generated by the objects of the form E^n and its subquotients.

Proposition 4.8. *Let* $\mathcal{S}:\ 1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ *be an affine extension. Then*

(1) H is a finite group if and only if there exists a representation $E \in \operatorname{Rep}(\mathcal{S})$ such that any object in $\operatorname{Rep}(\mathcal{S})$ is isomorphic to an object of $\langle E\rangle$. In particular, the extension $\mathcal{S}$ is of finite type.

(2) G is a group scheme of finite type if and only if there exist $E \in \operatorname{Rep}(\mathcal{S})$ such that $\operatorname{Rep}(\mathcal{S}) = \operatorname{Rep}(\mathcal{S})_E$ (see Definition 3.71).

(3) $\mathcal{S}$: $1 \longrightarrow H \longrightarrow H \times A \longrightarrow A \longrightarrow 0$ is a trivial extension of A, if and only if any representation of $\mathcal{S}$ is constructed over a trivial bundle $\Bbbk^n \times A$ (compare with Example 3.48 (3)).

Proof. (1) It is enough to prove the corresponding result for $\operatorname{Rep}_0(S)$ (see Theorem 3.43). For the proof in this situation of the classical representation theory of affine groups, see for example [21, Prop. 2.20].

(2) Just combine Theorem 3.77 and Lemma 4.3, together with the fact that if $E \in \operatorname{Obj}\big(\operatorname{Rep}(\mathcal{S})\big)$ is such that $\operatorname{Rep}(\mathcal{S})_E = \operatorname{Rep}(\mathcal{S})$, then $G \cong G_E$ by the Reconstruction Theorem.

(3) If $G = H \times A$, and E is a representation, then we clearly have a section $A \to \operatorname{Aut}_{\mathrm{gr}}(E)$ of the corresponding affine extension. It follows that E is a trivial homogeneous vector bundle (see Remark 3.29 above).

Assume now that any $\mathcal{S}$–representation is trivial. Since $\operatorname{Aut}_{\mathrm{gr}}(\Bbbk^n \times A) = \mathrm{GL}_n(\Bbbk) \times A$, it follows that $G_E = K_E \times A$ for some closed subgroup scheme $K_E \subset \mathrm{GL}_n(\Bbbk)$. Therefore, $G \cong \lim G_E = \lim K_E \times A = K \times A$, where K is the affine group scheme $K = \lim K_E$. □

5. The Recognition Theorem

Once that the Reconstruction Theorem 4.6 has been proved, its combination with the structure Theorem 3.43 and with the Recognition Theorem for affine group schemes, yields the Recognition Theorem for affine extensions.

Theorem 5.1 (Recognition Theorem)**.** *Let $(\mathcal{C}, \omega_{\mathrm{gr}})$ be a category $\mathcal{C}$, enriched over $\mathrm{Sch}\,|\Bbbk$, together with a fully faithful functor $\omega_{\mathrm{gr}} : \mathcal{C} \to \mathrm{HVB}_{\mathrm{gr}}(A)$, such that:*

(1) $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ is a homogeneous vector bundle over A.

(2) For any pair of objects $X, Y \in \mathcal{C}$,

$$\omega_{\mathrm{gr}}\big(\operatorname{Hom}_{\mathcal{C}}(X,Y)\big) = \operatorname{Hom}_{\omega_{\mathrm{gr}}(\mathcal{C})}\big(\omega_{\mathrm{gr}}(X), \omega_{\mathrm{gr}}(Y)\big) \subset \operatorname{Hom}_{\mathrm{gr}}\big(\omega_{\mathrm{gr}}(X), \omega_{\mathrm{gr}}(Y)\big)$$

is a subvector bundle.

(3) The category $\mathcal{C}_0$ with objects $\operatorname{Obj}(\mathcal{C}_0) = \operatorname{Obj}(\mathcal{C})$ and morphisms

$$\operatorname{Hom}_{\mathcal{C}_0}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(X,Y)_0 = {\omega_{\mathrm{gr}}}^{-1}\Big(\operatorname{Hom}_0\big(\omega_{\mathrm{gr}}(X), \omega_{\mathrm{gr}}(Y)\big)\Big)$$

is abelian, monoidal, rigid.

(4) $\operatorname{End}_{\mathcal{C}_0}(\mathbb{I}) \cong \Bbbk$.

(5) The restriction of the forgetful functor $\omega_0 = \omega_{\mathrm{gr}}|_{\mathcal{C}_0} : \mathcal{C}_0 \to \mathrm{HVB}_0(A)$ is a monoidal functor.

(6) The functor ω_0 remains fully faithful after taking restriction to the fiber over $0 \in A$. In other words, the functor $\widetilde{\omega} : \mathcal{C}_0 \to \mathrm{Vect}_{\Bbbk}$, $\widetilde{\omega}(X) = \big(\omega_0(X)\big)_0$, $\widetilde{\omega}(f :$

$X \to X') = f|_{(\omega_0(X))_0} : \big(\omega_0(X)\big)_0 \to \big(\omega_0(X')\big)_0$ is a fully faithful abelian, monoidal functor.

Then there exists an affine extension $\mathcal{S}_\mathcal{C}$ and an equivalence of categories $F : \mathcal{C} \to \operatorname{Rep}(\mathcal{S}_\mathcal{C})$ such that the following diagrams are commutative

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "\omega_{\mathrm{gr}}"'] & & \operatorname{Rep}(\mathcal{S}_\mathcal{C}) \arrow[dl, "\omega_{\mathrm{gr}}"] \\
& \operatorname{HVB}_{\mathrm{gr}}(A) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_0 \arrow[rr, "F|_{\mathcal{C}_0}"] \arrow[dr, "\omega_0"'] & & \operatorname{Rep}_0(\mathcal{S}_\mathcal{C}) \arrow[dl, "\omega_0"] \\
& \operatorname{HVB}_0(A) &
\end{tikzcd}
$$

where the restriction $F|_{\mathcal{C}_0}$ is a monoidal functor.

Proof. Since the pair $\big(\mathcal{C}_0, \widetilde{\omega} : \mathcal{C}_0 \to \operatorname{Vect}_\Bbbk\big)$ satisfies the hypothesis of the Recognition Theorem for affine group schemes (see [21, Proposition 2.8]), it follows that there exists an affine group scheme H such that $\mathcal{C}_0 \cong \operatorname{Rep}_{\mathrm{fin}}(H)$.

Let $Aut^{\otimes}(\omega_{\mathrm{gr}})$ be as presented in Definition 3.66 (for the category $\mathcal{C}$ instead of $\operatorname{Rep}(\mathcal{S})$), and for X an object of $\mathcal{C}$ define $\mathcal{C}_X \subset \mathcal{C}$ as in definitions 3.71 and 3.30. Then, as in Remark 4.2 and Lemma 4.4, it follows that we have a limit of affine extensions of finite type

$$
\begin{tikzcd}
Aut^{\otimes}(\omega_{\mathrm{gr}}) : \arrow[d] & 1 \arrow[r] & \operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})_0 = H \arrow[r] \arrow[d] & \operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}}) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
Aut^{\otimes}\big(\omega_{\mathrm{gr}}|_{\mathcal{C}_X}\big) : & 1 \arrow[r] & \operatorname{Aut}_0^{\otimes}\big(\omega_{\mathrm{gr}}|_{\mathcal{C}_X}\big) \arrow[r] & \operatorname{Aut}^{\otimes}\big(\omega_{\mathrm{gr}}|_{\mathcal{C}_X}\big) \arrow[r] & A \arrow[r] & 0
\end{tikzcd}
$$

Indeed, since the functor $\omega_{\mathrm{gr}}|_{\mathcal{C}_0}$ is monoidal, the same calculations hold — recall that $\operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}})_0 = H$ by the Reconstruction Theorem for affine group schemes.

Next, we show that $\mathcal{C}$ (or equivalently $\omega_{\mathrm{gr}}(\mathcal{C})$) is equivalent to the representation theory of $Aut^{\otimes}(\omega_{\mathrm{gr}})$. For this, let $X \in \mathcal{C}$; then $\omega_{\mathrm{gr}}(X)$ is a $Aut^{\otimes}(\omega_{\mathrm{gr}})$–module.

Conversely, if E is a $Aut^{\otimes}(\omega_{\mathrm{gr}})$–module, then E_0 is a H–module, and $E \cong \operatorname{Aut}^{\otimes}(\omega_{\mathrm{gr}}) \times^H E_0$ by Theorem 3.43. Let $X \in \mathcal{C}$ be such that $\omega_{\mathrm{gr}}(X)_0 \cong E_0$ as H–modules — recall that $\mathcal{C}_0$ is the representation theory of H. Since $\omega_{\mathrm{gr}}(X)$ is an $Aut^{\otimes}(\omega_{\mathrm{gr}})$–module, it follows that $\omega_{\mathrm{gr}}(X) \cong E$, by Corollary 3.44.

Let $X, Y \in \mathcal{C}$, be two objects. Since $\omega_{\mathrm{gr}}(\mathcal{C})_0 = \omega_{\mathrm{gr}}(\mathcal{C}_0) \cong \operatorname{Rep}(H) = \operatorname{Rep}\big(Aut^{\otimes}(\omega_{\mathrm{gr}})_0\big)$, it follows that

$$
\begin{aligned}
&\omega_{\mathrm{gr}}\big(\operatorname{Hom}_{\mathcal{C}_0}(X,Y)\big) = \operatorname{Hom}_{\omega_{\mathrm{gr}}(\mathcal{C})_0}\big(\omega_{\mathrm{gr}}(X), \omega_{\mathrm{gr}}(Y)\big) \cong \\
&\operatorname{Hom}_{\operatorname{Rep}(H)}(X_0, Y_0) = \operatorname{Hom}_{\operatorname{Rep}\left(Aut^{\otimes}(\omega_{\mathrm{gr}})_0\right)}\big(\omega_{\mathrm{gr}}(X)_0, \omega_{\mathrm{gr}}(Y)_0\big)
\end{aligned}
$$

Recall that $\operatorname{Hom}_{\omega_{\mathrm{gr}}(\mathcal{C})}\big(\omega_{\mathrm{gr}}(X), \omega_{\mathrm{gr}}(Y)\big) = \omega_{\mathrm{gr}}\big(\operatorname{Hom}_{\mathcal{C}}(X,Y)\big) \in \operatorname{HVB}_{\mathrm{gr}}(A)$ is a vector bundle, with fiber $\omega_{\mathrm{gr}}\big(\operatorname{Hom}_{\mathcal{C}_0}(X,Y)\big) = \operatorname{Hom}_{\operatorname{Rep}(Aut^{\otimes}(\omega_{\mathrm{gr}})_0)}\big(\omega_{\mathrm{gr}}(X)_0, \omega_{\mathrm{gr}}(Y)_0\big)$. On the other hand, by construction we have that

$$\operatorname{Hom}_{\omega_{\mathrm{gr}}(\mathcal{C})}\big(\omega_{\mathrm{gr}}(X), \omega_{\mathrm{gr}}(Y)\big) \subset \operatorname{Hom}_{\operatorname{Rep}(Aut^{\otimes}(\omega_{\mathrm{gr}}))}\big(\omega_{\mathrm{gr}}(X), \omega_{\mathrm{gr}}(Y)\big);$$

the later being also a vector bundle of fiber $\operatorname{Hom}_{\operatorname{Rep}(Aut^{\otimes}(\omega_{\mathrm{gr}})_0)}\big(\omega_{\mathrm{gr}}(X)_0, \omega_{\mathrm{gr}}(Y)_0\big)$ by definition. It follows that these vector bundles coincide. In other words, $\omega_{\mathrm{gr}}(\mathcal{C})$ is the category of representations of $Aut^{\otimes}(\omega_{\mathrm{gr}})$. $\square$

Remark 5.2. Condition (5) in Theorem 5.1 states that any morphism $f \in \operatorname{Hom}_{\mathcal{C}_0}(X,Y)$ is determined (after taking the forgetful functor) by its value in the fiber over $0 \in A$. *A fortiori*, by Corollary 3.40, this implies that this condition holds for any morphism in $\operatorname{Hom}_{\mathcal{C}}(X,Y)$.

We finish this section by describing $\mathrm{HVB}_{\mathrm{gr}}(A)$ as the category or representations of $\mathcal{G}_A$, the universal extension of the abelian variety A (see Example 3.73).

Example 5.3. The identity functor $\mathrm{Id} : \mathrm{HVB}_{\mathrm{gr}}(A) \to \mathrm{HVB}_{\mathrm{gr}}(A)$ can be though as a forgetful functor. Therefore, $Aut^{\otimes}(\mathrm{Id})$ is an affine extension, such that $\operatorname{Rep}\big(Aut^{\otimes}(\mathrm{Id})\big)$ is equivalent as a category with the forgetful functor (in the sense of Theorem 4.6) with $\mathrm{HVB}_{\mathrm{gr}}(A)$ with the identity functor.

Since $\operatorname{Rep}(\mathcal{G}_A)$, the representation theory of the universal extension of A (see Example 3.73), is also equivalent to $\mathrm{HVB}_{\mathrm{gr}}(A)$, it follows by the Reconstruction Theorem 4.6 that $\mathcal{G}_A \cong Aut^{\otimes}(\mathrm{Id})$.

6. Affine extensions and Hopf sheaves

The well known op–equivalence between the category of affine group schemes over a field $\Bbbk$ and the category of Hopf algebras over $\Bbbk$ has been generalized in [24, Exposé I, Section 4.2] to the context of *affine group schemes over a scheme S* — that is, group objects in the category of affine schemes over S with respect to the monoidal structure given by the fibered product over S. The algebraic counterpart of the group object is in this case a sheaf of bialgebras in S-alg. In this section we go one step further and establish an op–equivalence between the category $\mathrm{GE}\left|_{\mathrm{aff}} A\right.$ of *affine extensions of the abelian variety A* (see Definitions 2.18 and Notation 6.2 as well as Definition 6.4) and a category of $\mathcal{O}_A$–algebras with additional structure that we call *faithful (commutative) Hopf sheaves* — named as HQA_{f}-alg in Definition 6.60.

In our situation $\mathrm{GE}\left|_{\mathrm{aff}} A\right.$ will appear as a subcategory of the category of bimonoids (with an antipode) in a duoidal category based upon $\mathrm{Sch}\left|_{\mathrm{sqc}} A\right.$, the category of separated, quasi-compact schemes over A (see Definition 6.7). One of the two monoidal structures is the fibered product over A as in the classical case, but the other is defined taking into account the additive structure of A: it is the composition of the product over $\Bbbk$ with the base change by $s : A \times A \to A$, where s is the sum in A. This second structure is essential in order to capture in abstract terms the fact that the base scheme has the additional structure of an *abelian variety* and that $q : G \to A$ is a group homomorphism (see Definitions 6.4 and 6.42). See also remark 6.52.

For the following undertakings, as was mentioned before, it is better to view the affine extensions of an abelian variety A as a surjective (faithfully flat, separated) affine morphism of group schemes: $q : G \to A$ (see Remark 2.16 and Section 2.3).

To make the exposition clearer, we deal first with the monoid structure of G — and the bialgebra structure of the associated sheaf —, and after this is firmly secured, we present a formal treatment of the inversion morphism of G and the corresponding “antipode” in the associated sheaf. As it happens frequently when dealing with “generalized Hopf type objects”, it is harder to deal with the antipode than with the bialgebra structure.

To be in safe ground from the viewpoint of the categorical considerations, we will recall and use some definitions and concepts pertaining to the theory of *duoidal categories*, that are categories with two monoidal structures, that are related by an *interchange law* (see [1] and [27]). The reader should be aware that other names are used in the literature for this concept, see Definition 6.7 below.

Notation 6.1. In what follows we will deal with several monoidal structures, on different categories. A monoidal structure in a category $\mathcal{C}$ will be denoted as a 3-uple (*e.g.* $(\mathcal{C}, \otimes, \mathbb{I})$) — that is, we omit the associativity and unit constraints in the formulæ. We write $\mathcal{C}^{\mathrm{op}}$ for the opposite category with the same monoidal structure.

6.1. **Affine extensions as schemes over an abelian variety, revisited.**

Even though our final goal is to work in the category $\mathrm{Sch}\,|_{\mathrm{aff}} A$ (of affine schemes over A) we have to formulate our basic definitions in larger categories such as $\mathrm{Sch}\,|_{\mathrm{qc}} A$ (of quasi-compact schemes over A) and others. This is due to the fact that some of the basic ingredients — for example the construction of the new monoidal structure — do not live in the "affine universe" (see Remark 6.52).

It is convenient to begin by setting the notation of the different subcategories of $\mathrm{Sch}\,|S$ that we will use henceforth.

Notation 6.2. Let S be a $\Bbbk$–scheme.

(1) We denote the category of quasi-compact schemes over the $\Bbbk$–scheme S as $\mathrm{Sch}\,|_{\mathrm{qc}} S$: its objects are the quasi-compact morphisms of $\Bbbk$-schemes $x : X \to S$ and its morphisms $f : (x : X \to S) \to (y : Y \to S)$ are morphisms of schemes $f : X \to Y$ such that $y \circ f = x$.

We denote an object $(x : X \to S) \in \mathrm{Sch}\,|_{\mathrm{qc}} S$ as x, when no confusion arises.

(2) We denote as $\mathrm{Sch}\,|_{\mathrm{sqc}} S$ the full subcategory of separated, quasi-compact schemes over S; the full subcategory of affine schemes over S is denoted as $\mathrm{Sch}\,|_{\mathrm{aff}} S$. Since any affine morphism is separated and quasi-compact, we have that $\mathrm{Sch}\,|_{\mathrm{aff}} S$ is fully embedded in $\mathrm{Sch}\,|_{\mathrm{sqc}} S$.

(3) We also consider the categories $\mathrm{Sch}\,|_{\mathrm{pqc}} S$, $\mathrm{Sch}\,|_{\mathrm{fpqc}} S$ defined by the conditions that the map $x : X \to S$ is flat (*plate* in french) quasi-compact and faithfully flat (*fidèlement plate*) quasi-compact respectively.

(4) Also, we denote as $\mathrm{Sch}\,|_{\mathrm{psqc}} S$ and $\mathrm{Sch}\,|_{\mathrm{fpsqc}} S$ the categories defined by the conditions that the map $x : X \to S$ is flat separated and quasi-compact or faithfully flat separated and quasi-compact, respectively.

(5) Let $x : X \to S \in \mathrm{Sch}\,|_{\mathrm{sqc}} S$. If there exists a closed point $s : \mathrm{Spec}(\Bbbk) \to S \in S(\Bbbk)$, such that x factors through s, way say that x has constant structure morphism equal to s. See diagram below.

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{st}"'] \arrow[dd, "x"] \\ \mathrm{Spec}(\Bbbk) \arrow[dr, "s"'] & \\ & S \end{tikzcd}$$

(6) If $f : T \to S$ is a morphism of schemes, recall that the *pull-back functor* $f^* : \mathrm{Sch}\,|S \to \mathrm{Sch}\,|T$ has the *push forward functor* $f_* : \mathrm{Sch}\,|T \to \mathrm{Sch}\,|S$ as left adjoint — the functor f_* is defined as $f_*(x : X \to T) = f \circ x$ for $x \in \mathrm{Sch}\,|T$ and

$f^*(y : Y \to S) = (p_T : Y \times_S T \to T)$ is given by the pull-back diagram:

$$\begin{tikzcd} Y \times_S T \arrow[r, "p_T"] \arrow[d, "p_Y"'] & T \arrow[d, "f"] \\ Y \arrow[r, "y"'] & S. \end{tikzcd}$$

At the level of arrows the definitions are the standard ones. Also recall that if f is an isomorphism, then $f^* = (f^{-1})_* = (f_*)^{-1}$.

In the case that $S = A$ an abelian variety, we have additional elements to take into account.

Definition 6.3. (1) Let $\mathrm{op} : A \to A$ be the inversion morphism, $\mathrm{op}(a) = -a$ and denote $\mathrm{op}_*(x) = -x = x^-$, where $\mathrm{op}_* : \mathrm{Sch}\,|_{\mathrm{qc}} A \to \mathrm{Sch}\,|_{\mathrm{qc}} A$ is $\mathrm{op}_*(x : X \to A) = \mathrm{op} \circ x : X \to A$ and $\mathrm{op}_*(f) = f$.
(2) Let $c_0 = 0 \circ \mathrm{st} : A \to A$, where $0 = e_A : \mathrm{Spec}(\Bbbk) \to A$ and $\mathrm{st} : A \to \mathrm{Spec}(\Bbbk)$ is the structural morphism of the $\Bbbk$–scheme A — thus, c_0 is "the constant morphism equal to $0 \in A$".

Definition 6.4. (1) The *Cauchy monoidal structure in* $\mathrm{Sch}\,|_{\mathrm{qc}} A$ is defined as follows:

$$\widetilde{\times} := s_* \times :\ \mathrm{Sch}\,|_{\mathrm{qc}} A \times \mathrm{Sch}\,|_{\mathrm{qc}} A \xrightarrow{\times} \mathrm{Sch}\,|_{\mathrm{qc}}\, (A \times A) \xrightarrow{s_*} \mathrm{Sch}\,|_{\mathrm{qc}} A\ ,$$

where s denotes as usual the addition in A and the functor $\times$ is the product in the category $\mathrm{Sch}\,|_{\mathrm{qc}} \Bbbk$, i.e. if $x : X \to A$, $y : Y \to A \in \mathrm{Sch}\,|_{\mathrm{qc}} A$, then

$$(x : X \to A) \times (y : Y \to A) :=\ (X \times Y \xrightarrow{x \times y} A \times A).$$

The fact that the construction $\widetilde{\times}$ induces a monoidal structure on $\mathrm{Sch}\,|_{\mathrm{qc}} A$, with unit element $0 : \mathrm{Spec}(\Bbbk) \to A$, is a straightforward calculation that we omit. We denote its unit element as $\mathbb{I}_{\widetilde{\times}}$.
(2) Similarly the fibered product $\times_A$, that we call the *Hadamard monoidal structure*, induces a monoidal structure on $\mathrm{Sch}\,|_{\mathrm{qc}} A$, with unit element $\mathrm{id}_A : A \to A$ that we denote as $\mathbb{I}_{\times_A}$.[1]

Remark 6.5. Both monoidal structures presented in Definition 6.4 are symmetric braided. This fact is true in general for the fibered (Hadamard) product over any base, and in the case of the Cauchy product is due to the abelianity of the group structure in A.

Later, when working with the "group type objects" in the category $\mathrm{Sch}\,|_{\mathrm{qc}} A$ we will concentrate our attention mainly to the case of group extensions, i.e. morphisms of group schemes $q : G \to A$ with additional properties. As these morphisms are separated, it is natural to consider the restriction of the Cauchy and Hadamard monoidal structures to the category $\mathrm{Sch}\,|_{\mathrm{sqc}} A$ of separated, quasi-compact schemes over A. Similarly, one can consider the subcategory $\mathrm{Sch}\,|_{\mathrm{psqc}} A$ (see Notation 6.2). These restrictions will be necessary to deal with certain technical aspects such as the ones considered in Section 6.3.

[1]The names of Hadamard and Cauchy monoidal structure, are used in similar situations in other contexts, in particular in the theory of species (see [1, Sect. 6.1,Ex. 6.22,Ex. 8.13.5]).

Lemma 6.6. *The Cauchy and Hadamard monoidal structures restrict to* $\mathrm{Sch}\,|_{\mathrm{psqc}} A$ *and* $\mathrm{Sch}\,|_{\mathrm{sqc}} A$.

Proof. This is clear. $\square$

The Cauchy and the Hadamard monoidal structures endow $\mathrm{Sch}\,|_{\mathrm{qc}} A$ with a structure of *duoidal category*:

Definition 6.7. A *duoidal category* — also known as *2–monoidal category* or *two-fold monoidal category*, see [27, Sect. 4.9] or [1, Chapter 6] — is a quintuple $(\mathcal{C}, \diamond, \mathbb{I}_\diamond, \star, \mathbb{I}_\star)$ with the following properties — the quintuple will be abbreviated as $\mathcal{C}$ when there is no danger of confusion —:

(1) $(\mathcal{C}, \diamond, \mathbb{I}_\diamond)$ and $(\mathcal{C}, \star, \mathbb{I}_\star)$ are monoidal categories with their respective units.

(2) There is a natural transformation, called the *interchange law*,

$$\zeta_{a,b,c,d} : (a \star b) \diamond (c \star d) \Rightarrow (a \diamond c) \star (b \diamond d),$$

defined for all $a, b, c, d \in \mathcal{C}$.

(3) There are three morphisms:

$$\Delta_\diamond : \mathbb{I}_\diamond \to \mathbb{I}_\diamond \star \mathbb{I}_\diamond \,,\; \mu_\star : \mathbb{I}_\star \diamond \mathbb{I}_\star \to \mathbb{I}_\star \,,\; u_{\mathbb{I}_\star} = \varepsilon_{\mathbb{I}_\diamond} : \mathbb{I}_\diamond \to \mathbb{I}_\star.$$

(4) All the data above satisfy additional conditions:

(i) *compatibility of units*, that amounts to the following assertions:

$(\mathbb{I}_\star, \mu_\star, u_{\mathbb{I}_\star})$ is a monoid in $(\mathcal{C}, \diamond, \mathbb{I}_\diamond)$;

$(\mathbb{I}_\diamond, \Delta_\diamond, \varepsilon_{\mathbb{I}_\diamond})$ is a comonoid in $(\mathcal{C}, \star, \mathbb{I}_\star)$.

(ii) *associativity for* ζ, that is expressed as the commutativity of the following diagrams for all objects $a, b, c, d, e, f \in \mathcal{C}$ — for the sake of simplicity, all the diagrams are written omitting the associativity constrains, i.e. pretending that the monoidal structures are strict —:

$$\begin{tikzcd}
(a \star b) \diamond (c \star d) \diamond (e \star f) \arrow[r, "\mathrm{id}_{a\star b} \diamond \zeta_{c,d,e,f}"] \arrow[d, "\zeta_{a,b,c,d} \diamond \mathrm{id}_{e\star f}"'] & (a \star b) \diamond \big((c \diamond e) \star (d \diamond f)\big) \arrow[d, "\zeta_{a,b,c\diamond e,d\diamond f}"] \\
\big((a \diamond c) \star (b \diamond d)\big) \diamond (e \star f) \arrow[r, "\zeta_{a\diamond c,b\diamond d,e,f}"'] & (a \diamond c \diamond e) \star (b \diamond d \diamond f)
\end{tikzcd}$$

$$\begin{tikzcd}
((a \star b) \star c) \diamond ((d \star e) \star f) \arrow[r, "\zeta_{a,b\star c,d,e\star f}"] \arrow[d, "\zeta_{a\star b,c,d\star e,f}"'] & (a \diamond d) \star \big((b \star c) \diamond (e \star f)\big) \arrow[d, "\mathrm{id}_{a\diamond d} \star \zeta_{b,c,e,f}"] \\
\big((a \star b) \diamond (d \star e)\big) \star (c \diamond f) \arrow[r, "\zeta_{a,b,d,e} \star \mathrm{id}_{c\diamond f}"'] & (a \diamond d) \star (b \diamond e) \star (c \diamond f)
\end{tikzcd}$$

(iii) *unitality/counitality for* ζ, that is expressed as the commutativity of the following diagrams for all $a, b \in \mathcal{C}$ — again we omit the associativity constrains —:

$$\begin{tikzcd}
(a \star b) = (a \star b) \diamond \mathbb{I}_\diamond \arrow[d, "\mathrm{id}_{a\star b} \diamond \Delta_\diamond"'] & = \mathbb{I}_\diamond \diamond (a \star b) \arrow[r, "\Delta_\diamond \diamond \mathrm{id}_{a\star b}"] & (\mathbb{I}_\diamond \star \mathbb{I}_\diamond) \diamond (a \star b) \arrow[d, "\zeta_{\mathbb{I}_\diamond,\mathbb{I}_\diamond,a,b}"] \\
(a \star b) \diamond (\mathbb{I}_\diamond \star \mathbb{I}_\diamond) \arrow[rr, "\zeta_{a,b,\mathbb{I}_\diamond,\mathbb{I}_\diamond}"'] & & a \star b \arrow[ul, "\mathrm{id}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(a \diamond b) = (a \diamond b) \star \mathbb{I}_\star = \mathbb{I}_\star \star (a \diamond b) & (\mathbb{I}_\star \diamond \mathbb{I}_\star) \star (a \diamond b) \arrow[l, "\mu_\star \star \mathrm{id}_{a \diamond b}"'] \\
(a \diamond b) \star (\mathbb{I}_\star \diamond \mathbb{I}_\star) \arrow[u, "\mathrm{id}_{a \diamond b} \star \mu_\star"] & (a \star \mathbb{I}_\star) \diamond (b \star \mathbb{I}_\star) = a \diamond b = (\mathbb{I}_\star \star a) \diamond (\mathbb{I}_\star \star b) \arrow[l, "\zeta_{a,\mathbb{I}_\star,b,\mathbb{I}_\star}"] \arrow[u, "\zeta_{\mathbb{I}_\star,a,\mathbb{I}_\star,b}"'] \arrow[ul, leftrightarrow, "\mathrm{id}"']
\end{tikzcd}$$

It is clear that if $(\mathcal{C}, \diamond, \mathbb{I}_\diamond, \star, \mathbb{I}_\star)$ is a duoidal category, then $(\mathcal{C}^{\mathrm{op}}, \star, \mathbb{I}_\star, \diamond, \mathbb{I}_\diamond)$ is also a duoidal category that is written simply as $\mathcal{C}^{\mathrm{op}}$ and is called the opposite of the duoidal category $\mathcal{C}$ — the interchange law of $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ is the same morphism.

We refer the reader to [1, Paragraph 6.1.1] or [27, Section 4] for more information on the properties of duoidal categories.

In the context of duoidal categories, one can establish the notion of bimonoid as follows.

Definition 6.8. Let $(\mathcal{C}, \diamond, \mathbb{I}_\diamond, \star, \mathbb{I}_\star)$ be a duoidal category. A quintuple $(b, \mu_b, u_b, \Delta_b, \varepsilon_b)$ consisting of an object $b \in \mathcal{C}$, and morphisms $\mu_b : b \diamond b \to b$, $u_b : \mathbb{I}_\diamond \to b$, $\Delta_b : b \to b \star b$ and $\varepsilon_b : b \to \mathbb{I}_\star$ is a *bimonoid for the duoidal category* if:

(1) (b, μ_b, u_b) is a monoid in $(\mathcal{C}, \diamond, \mathbb{I}_\diamond)$;

(2) $(b, \Delta_b, \varepsilon_b)$ is a comonoid in $(\mathcal{C}, \star, \mathbb{I}_\star)$;

(3) The following conditions hold:

$\varepsilon_b : b \to \mathbb{I}_\star$ is a morphism of monoids in $(\mathcal{C}, \diamond, \mathbb{I}_\diamond)$;

$u_b : \mathbb{I}_\diamond \to b$ is a morphism of comonoids in $(\mathcal{C}, \star, \mathbb{I}_\star)$;

(4) The following diagram is commutative:

$$\begin{tikzcd}
(b \star b) \diamond (b \star b) \arrow[rr, "\zeta_{b,b,b,b}"] & & (b \diamond b) \star (b \diamond b) \arrow[d, "\mu_b \star \mu_b"] \\
b \diamond b \arrow[u, "\Delta_b \diamond \Delta_b"] \arrow[r, "\mu_b"'] & b \arrow[r, "\Delta_b"'] & b \star b
\end{tikzcd}$$

We call $\mathrm{Bimon}(\mathcal{C})$ the category whose objects are the bimonoids in $\mathcal{C}$ and its arrows the morphisms of $\mathcal{C}$ that preserve the bimonoid structure.

Remark 6.9. (1) In accordance with Definition 6.7,4.(i) $(\mathbb{I}_\star, \mu_\star, u_{\mathbb{I}_\star})$ is a monoid in $(\mathcal{C}, \diamond, \mathbb{I}_\diamond)$ and in a trivial manner $\mathbb{I}_\star$ is also a comonoid in $(\mathcal{C}, \star, \mathbb{I}_\star)$. It is clear that the compatibility conditions are satisfied and hence, $\mathbb{I}_\star$ is a bimonoid in the duoidal category $\mathcal{C}$.

(2) Similarly $(\mathbb{I}_\diamond, \Delta_\diamond, \varepsilon_{\mathbb{I}_\diamond})$ is a comonoid in $(\mathcal{C}, \star, \mathbb{I}_\star)$ and also has a natural structure of monoid in $(\mathcal{C}, \diamond, \mathbb{I}_\diamond)$ and also of bimonoid in the duoidal category $\mathcal{C}$.

(3) The map $\mathbb{I}_\star \to \mathbb{I}_\diamond$ in Definition 6.7, is a morphism of bimonoids.

We proceed now to show that if A is an abelian variety, then the Hadamard and Cauchy monoidal structures on $\mathrm{Sch}|_{\mathrm{qc}} A$ combine into a structure of duoidal category. In this duoidal category, bimonoids correspond to morphisms of monoid schemes $M \to A$, where M is a quasi-compact monoid scheme.

Lemma 6.10. *The quintuple* $(\mathrm{Sch}|_{\mathrm{qc}} A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$ *consisting of the category of quasi-compact schemes over A with the Cauchy and Hadamard monoidal structures,* *and their respective unit objects, together with the morphisms* $\Delta_{\widetilde{\times}} : \mathbb{I}_{\widetilde{\times}} \to \mathbb{I}_{\widetilde{\times}} \times_A \mathbb{I}_{\widetilde{\times}}$ *(the diagonal morphism);* $\mu_{\times_A} := s : \mathbb{I}_{\times_A} \widetilde{\times}\, \mathbb{I}_{\times_A} \to \mathbb{I}_{\times_A}$, $u_{\mathbb{I}_{\times_A}} = \varepsilon_{\mathbb{I}_{\widetilde{\times}}} := 0 : \mathbb{I}_{\widetilde{\times}} \to \mathbb{I}_{\times_A}$ *constitute a duoidal category. Moreover, with the restricted structures (see Lemma 6.6) the categories* $(\mathrm{Sch}\,|_{\mathrm{sqc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$ *and* $(\mathrm{Sch}\,|_{\mathrm{psqc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$ *are duoidal.*

Proof. We only give an sketch of the proof. First of all, notice that the morphisms $\Delta_{\widetilde{\times}}$, $\mu_{\times_A}$ and $u_{\mathbb{I}_{\times_A}} = \varepsilon_{\mathbb{I}_{\widetilde{\times}}}$ are morphisms in $\mathrm{Sch}\,|_{\mathrm{qc}}A$, since the following diagrams are commutative:

$$
\begin{tikzcd}
\mathrm{Spec}(\Bbbk) \arrow[r, "\Delta_{\widetilde{\times}}"] \arrow[d, "0"'] & \mathrm{Spec}(\Bbbk) \times_A \mathrm{Spec}(\Bbbk) \arrow[d, "0"] \\
A \arrow[r, "\mathrm{id}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times A \arrow[r, "\mu_{\times_A}"] \arrow[d, "\mathrm{id}\times\mathrm{id}"'] & A \arrow[dd, "\mathrm{id}"] \\
A \times A \arrow[d, "s"'] & \\
A \arrow[r, "\mathrm{id}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Spec}(\Bbbk) \arrow[rr, "u_{\mathbb{I}_{\times_A}} = \varepsilon_{\mathbb{I}_{\widetilde{\times}}}"] \arrow[dr, "0"'] & & A \arrow[dl, "\mathrm{id}"] \\
& A &
\end{tikzcd}
$$

It is clear that $(\mathbb{I}_{\widetilde{\times}}, \Delta_{\widetilde{\times}}, \varepsilon_{\widetilde{\times}} = 0)$ is a comonoid for the Hadamard monoidal structure, and $(\mathbb{I}_{\times_A}, \mu_{\times_A}, u_{\times_A} = 0)$ is a monoid for the Cauchy monoidal structure.

The interchange law is defined as follows: for $x : X \to A$, $y : Y \to A$, $z : Z \to A$ and $w : W \to A$, $\zeta_{x,y,z,w} : (x \times_A y)\, \widetilde{\times} (z \times_A w) \to (x \,\widetilde{\times}\, z) \times_A (y \,\widetilde{\times}\, w)$ is the unique morphism given by the universal property of the fibered product:

$$
\begin{tikzcd}
& (X \times_A Y) \times (Z \times_A W) \arrow[dl, "p_X \times p_Z"'] \arrow[d, dotted, "\zeta_{x,y,z,w}"] \arrow[dr, "p_Y \times p_W"] & \\
X \times Z \arrow[d, "x \times z"'] & (X \times Z) \times_A (Y \times W) \arrow[l, "p_{X\times Z}"] \arrow[r, "p_{Y\times W}"'] & Y \times W \arrow[d, "y \times w"] \\
A \times A \arrow[r, "s"'] & A & A \times A \arrow[l, "s"]
\end{tikzcd}
$$

Once the interchange law is established, the associativity and unitality of ζ follow easily.

The fact that all the duoidal structure can be restricted to the subcategories is clear. For example:

$$\Big(\mathrm{Sch}\,|_{\mathrm{qc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}} := \big(0 : \mathrm{Spec}(\Bbbk) \to A\big), \times_A, \mathbb{I}_{\times_A} := (\mathrm{id}_A : A \to A)\Big),$$

restricts to a duoidal structure $(\mathrm{Sch}\,|_{\mathrm{sqc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$ — if $x : X \to A$, $y : Y \to A \in \mathrm{Sch}\,|_{\mathrm{qc}}A$, then the morphisms $s \circ (x, y) : X \times Y \to A$ and $x \circ p_1 = y \circ p_2 : X \times_A Y \to A$ are separated, as well as $\mathbb{I}_{\widetilde{\times}} = 0 : \mathrm{Spec}(\Bbbk) \to A$ and $\mathbb{I}_{\times_A} = \mathrm{id}_A : A \to A$. $\square$

The following remark is of some relevance for future use.

Remark 6.11. (1) All the objects in the monoidal category $(\mathrm{Sch}\,|_{\mathrm{qc}}A, \times_A, \mathrm{id}_A)$ can naturally be endowed with a unique comonoid structure, given by the morphisms depicted in the diagrams below, where $\delta_X : X \to X \times_A X$ is the canonical diagonal morphism and the counit is $\varepsilon_X := q_X$.

$$
\begin{tikzcd}
X \arrow[rr, "\delta_X"] \arrow[dr, "q_X"'] & & X \times_A X \arrow[dl, "q_{X\times_A X}"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, "\varepsilon_X"] \arrow[dr, "q_X"'] & & A \arrow[dl, "\mathrm{id}"] \\
& A &
\end{tikzcd}
$$

Notice that $q_{X\times_A X} = q_X \circ p_1 = q_X \circ p_2 : X \times_A X \to A$.

(2) Similarly — in a somewhat redundant way — in the monoidal category of sheaves of $\mathcal{O}_A$–algebras $(A\text{-alg}, \otimes_{\mathcal{O}_A}, \mathcal{O}_A)$ all objects are monoids in a unique way (see Definition 6.42 below).

(3) A peculiarity of this duoidal category is related to the element $\mathbb{I}_\diamond = \mathbb{I}_{\widetilde{\times}}$, which is the unit of the monoidal half $(\mathrm{Sch}\,|_{\mathrm{qc}} A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}})$: $\mathbb{I}_{\widetilde{\times}}$ is idempotent with respect to the other half of the monoidal structure, i.e. $\mathbb{I}_{\widetilde{\times}} \times_A \mathbb{I}_{\widetilde{\times}} \cong \mathbb{I}_{\widetilde{\times}}$.

Also, $\mathbb{I}_{\times_A} \widetilde{\times}\, \mathbb{I}_{\times_A} = s : A \times A \to A$.

Proposition 6.12 below encompasses the main properties of quasi-compact morphisms of monoid schemes $M \to A$ in a categorical framework, and will translate — by the op-equivalence of categories mentioned before, once we take into consideration the inverse when M is a group scheme — to the notion of Hopf sheaf.

Proposition 6.12. *In the duoidal category* $(\mathrm{Sch}\,|_{\mathrm{qc}} A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$*, an object* $q_M : M \to A \in \mathrm{Sch}\,|_{\mathrm{qc}} A$ *is a bimonoid if and only if* M *is a monoid in* Sch *and the morphism* $q_M : M \to A$ *is a morphism of monoids; that is,* q_M *is multiplicative and* $q(1_M) = 0_A$.

Given two bimonoids $q_M : B \to A$ *and* $q_{M'} \to A$*, a morphism* $f : q_M \to q_{M'}$ *is of bimonoids if and only if* $f : M \to M'$ *is a morphism of monoid schemes.*

Proof. Indeed, a structure of monoid in $q_M : M \to A$ is given by two morphisms $\mu_M : M \times M \to M$ and $u_M : \mathrm{Spec}(\Bbbk) \to M$, such that μ_M and u_M satisfy the usual axioms of associativity and unitality, together with the diagrams depicted below.

$$
\begin{tikzcd}
M \times M \arrow[r, "\mu_M"] \arrow[d, "q_M \times q_M"'] & M \arrow[dd, "q_M"] \\
A \times A \arrow[d, "s"'] & \\
A \arrow[r, "\mathrm{id}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Spec}(\Bbbk) \arrow[rr, "u_M"] \arrow[dr, "0"'] & & M \arrow[dl, "q_M"] \\
& A &
\end{tikzcd}
$$

The last assertion follows easily. □

Notation 6.13. We denote $\mathrm{MM}\,|_{\mathrm{aff}} A \subset \mathrm{Bimon}(\mathrm{Sch}\,|_{\mathrm{qc}} A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$ the full subcategory with objects the affine morphisms of monoid schemes $q_M : M \to A$.

The main properties of the functor op_* with respect to the duoidal structure are expressed in the following easy proposition.

Proposition 6.14. *Consider the duoidal category* $(\mathrm{Sch}\,|_{\mathrm{qc}} A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$*. The functor* $\mathrm{op}_* : \mathrm{Sch}\,|_{\mathrm{qc}} A \to \mathrm{Sch}\,|_{\mathrm{qc}} A$ *satisfies the following properties:*

(1) It is a strict monoidal involution with respect to $\widetilde{\times}$ *and* $\times_A$*.*

(2) $\operatorname{op}_*(\mathbb{I}_{\widetilde{\times}}) = \mathbb{I}_{\widetilde{\times}}$ *and* $\operatorname{op}_*(\mathbb{I}_{\times_A}) \times_A \operatorname{op}_*(\mathbb{I}_{\times_A}) \cong \operatorname{op}_*(\mathbb{I}_{\times_A})$. □

Next, with the purpose of obtaining an adequate categorical formulation for the inverse morphism for a group object in $\mathrm{Sch}|_{\mathrm{qc}}A$, we need to establish some basic properties involving the functors

$$- \,\widetilde{\times}\, \mathbb{I}_{\times_A} : \mathrm{Sch}|_{\mathrm{qc}}A \to \mathrm{Sch}|_{\mathrm{qc}}A \quad , \quad - \times_A \mathbb{I}_{\widetilde{\times}} : \mathrm{Sch}|_{\mathrm{qc}}A \to \mathrm{Sch}|_{\mathrm{qc}}A,$$

and others associated to the duoidal structure.

Lemma 6.15. *Consider the duoidal category* $(\mathrm{Sch}|_{\mathrm{qc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$*. Then, with the above notations we have that:*

(1) *(a)* $0_*0^*x = x \times_A \mathbb{I}_{\widetilde{\times}}$ *for any* $x : X \to A$*;*
 (b) The map $\varepsilon_x = \mathrm{id}_x \times_A \varepsilon_{\mathbb{I}_{\widetilde{\times}}} : 0_*0^*x \to x$ *is the counit of the adjunction* $0_* \dashv 0^*$*;*
 (c) The unit $u_z : z \to 0^*0_*z$ *of the adjunction* $0_* \dashv 0^*$ *is an isomorphism.*

(2) *(a)* $\mathrm{st}^* z = 0_*z \,\widetilde{\times}\, \mathbb{I}_{\times_A}$ *for* $z : Z \to \mathrm{Spec}(\Bbbk) \in \mathrm{Sch}|\Bbbk$ *and* $\mathrm{st} : A \to \mathrm{Spec}(\Bbbk)$*;*
 (b) $0^*(c_0)_* = 0^*0_* \,\mathrm{st}_* = \mathrm{st}_*$*.*

(3) $c_0^*x = (x \times_A \mathbb{I}_{\widetilde{\times}}) \,\widetilde{\times}\, \mathbb{I}_{\times_A}$*. Equivalently, if* $x : X \to A$*,* $X_0 = x^{-1}(0)$ *and* $x_0 = x|_{X_0} : X_0 \to A$*, then* $c_0^*x = p_A : X_0 \times A \to A$*.*

(4) There is a natural transformation $\rho_x : (c_0)^*x \to x \,\widetilde{\times}\, \mathbb{I}_{\times_A}$*.*

Proof. (1) In the commutative diagram

$$\begin{tikzcd}
X \times_A \mathrm{Spec}(\Bbbk) \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, dotted] & \mathrm{Spec}(\Bbbk) \arrow[d, "0"] \\
X \arrow[r, "x"'] & A,
\end{tikzcd}$$

the upper horizontal arrow is 0^*x and its composition with the vertical 0 arrow yields 0_*0^*x that is the arrow: $0p_2 = xp_1 : X \times_A \mathrm{Spec}(\Bbbk) \to A$. It is clear that $0_*0^*x = x \times_A \mathbb{I}_{\widetilde{\times}}$. The remaining parts of the proof are direct.

(2) The proof of the first part is direct and for the second, the chain of equalities is guaranteed by 1(c).

(3) If $x : X \to A \in \mathrm{Sch}|_{\mathrm{qc}}A$, then $c_0^*x = \mathrm{st}^*\, 0^*x = 0_*0^*x \,\widetilde{\times}\, \mathbb{I}_{\times_A} = (x \times_A \mathbb{I}_{\widetilde{\times}}) \,\widetilde{\times}\, \mathbb{I}_{\times_A}$ the second equality follows from (2)(a) and the third from (1)(a). The proof of the second assertion is easy.

(4) The natural transformation ρ is obtained by considering the equality proved in (3), and then applying the unit morphism $u : \mathbb{I}_{\widetilde{\times}} \to \mathbb{I}_{\times_A}$ to the second factor. □

Next we define two natural transformations that are crucial for the definition of the antipode in Theorem 6.18 below.

Proposition 6.16. *Consider the duoidal category* $(\mathrm{Sch}|_{\mathrm{qc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$ *and let* $x : X \to A$*,* $y : Y \to A \in \mathrm{Sch}|_{\mathrm{qc}}A$*.*

(1) There is a natural transformation $\pi_{x,y} : (c_0)_*\big(x \times_A -y\big) \to x \,\widetilde{\times}\, y$*.*

(2) There is a natural transformation $\widetilde{\gamma}_{x,y} : x \times_A -y \to (x \,\widetilde{\times}\, y) \,\widetilde{\times}\, \mathbb{I}_{\times_A}$*.*

(3) There is a natural transformation $\overline{\gamma}_{x,y} : (-x) \times_A y \to \mathbb{I}_{\times_A} \,\widetilde{\times}\, (x \,\widetilde{\times}\, y)$*.*

Proof. For the proof of (1) we consider the commutative diagram that follows — in the next diagrams in order not to loose track of the structure morphism $-y$ we are using the slightly unorthodox notation $\mathrm{op}_*(y : Y \to A) = (-y : Y^- \to A)$ even though as schemes $Y^- = Y$.

$$\begin{tikzcd}
X \times_A Y^- \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y^- \arrow[d, "-y"] \arrow[ddr, bend left] & \\
X \arrow[r, "x"'] \arrow[drr, bend right] & A \arrow[dr, "\mathrm{st}"] & \\
& & \mathrm{Spec}(\Bbbk)
\end{tikzcd}$$

The canonical projections p_i induce a morphism $\tau_{x,y} : X \times_A Y^- \to X \times Y$ such that $s \circ (x \times y) \circ \tau_{x,y} = 0 \circ \mathrm{st}_{X \times_A Y}$. In other words, the following diagram is commutative:

(6.1)
$$\begin{tikzcd}
X \times_A Y^- \arrow[rr, "\tau_{x,y}"] \arrow[d, "\mathrm{st}\, x p_1 = \mathrm{st}(-y) p_2"'] & & X \times Y \arrow[d, "x \times y"] \\
\mathrm{Spec}(\Bbbk) \arrow[dr, "0"'] & & A \times A \arrow[dl, "s"] \\
& A &
\end{tikzcd}$$

Diagram (6.1) means that the map $\tau_{x,y}$ is a morphism $(c_0)_*\big(x \times_A -y\big) \xrightarrow{\tau_{x,y}} x \widetilde{\times} y$.
(2) It is clear that applying $(c_0)^*$ to the natural transformation $\tau_{x,y}$ we obtain a natural transformation $(c_0)^*(\tau_{x,y}) : (c_0)^*(c_0)_*\big(x \times_A -y\big) \to (c_0)^*(x \widetilde{\times} y)$. By composing with the unit of the adjunction $(c_0)_* \dashv (c_0)^*$ we obtain a natural transformation: $x \times_A -y \Rightarrow (c_0)^*(x \widetilde{\times} y)$. The conclusion of (2) follows by post composition of this natural transformation with the natural transformation $\rho_{x \widetilde{\times} y}$ defined in Lemma 6.15.

(3) To obtain $\overline{\gamma}$ one proceeds similarly. □

Remark 6.17. For a pair $x : X \to A, y : Y \to A$ the natural transformation $\widetilde{\gamma}_{x,y}$, is a morphism of schemes that has domain $x \times_A (-y) : X \times_A Y^- \to A$ and codomain $x + y + \mathrm{id}_A : X \times Y \times A \to A$. Tracking down the above construction it is easy to see that $\widetilde{\gamma}_{x,y} = \big\langle \pi, x \times_A (-y) \big\rangle : X \times_A Y^- \to X \times Y \times A$. It is clear that the diagram below is commutative:

$$\begin{tikzcd}
X \times_A Y^- \arrow[rr, "\langle \tau_{x,y}, x \times_A (-y) \rangle"] \arrow[dr, "x \times_A (-y)"'] & & X \times Y \times A \arrow[dl, "x + y + \mathrm{id}_A"] \\
& A &
\end{tikzcd}$$

Notice that if $(u, v) \in x \times_A (-y)$ and $a \in A$, then $\big\langle \tau_{x,y}, x \times_A (-y) \big\rangle (u, v) = \big(u, v, x(u)\big)$, $\big(x \times_A (-y)\big)(u, v) = x(u) = -y(v)$ and $(x + y + \mathrm{id}_A)(u, v, a) = x(u) + y(v) + a = a$.

We are ready to wrap up the duoidal perspective for a group extension by completing the result of Proposition 6.12.

Theorem 6.18. *In the duoidal category* $(\mathrm{Sch}\,|_{\mathrm{qc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$ *a bimonoid* $(b : M \to A, \mu_b, u_b, \Delta_b, \varepsilon_b)$ *is such that* M *is a quasi-compact group scheme and* b *a quasi-compact morphism of group schemes if and only if there is a morphism* $\iota_b : b \to -b$ *in* $\mathrm{Sch}\,|_{\mathrm{qc}}A$*, called an* antipode*, such that both diagrams below commute:*

$$\begin{tikzcd}
 & b \times_A b \arrow[r, "\mathrm{id} \times_A \iota_b"] & b \times_A (-b) \arrow[r, "\widetilde{\gamma}_{b,b}"] & (b \,\widetilde{\times}\, b) \,\widetilde{\times}\, \mathbb{I}_{\times_A} \arrow[dr, "\mu_b \,\widetilde{\times}\, \mathrm{id}"] & \\
b \arrow[ur, "\Delta_b"] \arrow[dr, "\varepsilon_b"'] & & & & b \,\widetilde{\times}\, \mathbb{I}_{\times_A} \\
 & \mathbb{I}_{\times_A} \arrow[rr, "\cong"'] & & \mathbb{I}_{\widetilde{\times}} \,\widetilde{\times}\, \mathbb{I}_{\times_A} \arrow[ur, "(u_b \,\widetilde{\times}\, \mathrm{id})"'] &
\end{tikzcd} \tag{6.2}$$

$$\begin{tikzcd}
 & b \times_A b \arrow[r, "\iota_b \times_A \mathrm{id}"] & (-b) \times_A b \arrow[r, "\overline{\gamma}_{b,b}"] & \mathbb{I}_{\times_A} \,\widetilde{\times}(b \,\widetilde{\times}\, b) \arrow[dr, "\mathrm{id} \,\widetilde{\times}\, \mu_b"] & \\
b \arrow[ur, "\Delta_b"] \arrow[dr, "\varepsilon_b"'] & & & & \mathbb{I}_{\times_A} \,\widetilde{\times}\, b \\
 & \mathbb{I}_{\times_A} \arrow[rr, "\cong"'] & & \mathbb{I}_{\times_A} \,\widetilde{\times}\, \mathbb{I}_{\widetilde{\times}} \arrow[ur, "(\mathrm{id} \,\widetilde{\times}\, u_b)"'] &
\end{tikzcd} \tag{6.3}$$

where $\widetilde{\gamma}$ *and* $\overline{\gamma}$ *are the natural transformations depicted in Proposition 6.16 (see also Remark 6.17) and the bottom maps* $\cong$ *are the natural identifications associated to the unit of the* $\widetilde{\times}$ *monoidal structure.*

Proof. If $(M, \mu, u, \mathrm{inv})$ is a group scheme over $\Bbbk$ and $b : M \to A \in \mathrm{Sch}\,|_{\mathrm{qc}}A$ is a quasi-compact group extension, in accordance with Proposition 6.12, b is a bimonoid in the duoidal category $\mathrm{Sch}\,|_{\mathrm{qc}}A$. Additionally, if we define $\iota_b := \mathrm{inv}$, it is clear that ι_b is a morphism in the category $\mathrm{Sch}\,|_{\mathrm{qc}}A$. Indeed, the diagram below is commutative

$$\begin{tikzcd}
M \arrow[rr, "\mathrm{inv}"] \arrow[dr, "b"'] & & M \arrow[dl, "-b"] \\
 & A &
\end{tikzcd}$$

A direct verification shows that diagrams (6.2) and (6.3) are commutative provided that $\mu_b = \mu, u_b = u, \iota_b = \mathrm{inv}$, Δ_b is the diagonal morphism and $\varepsilon_b = b : M \to A$.

Conversely, suppose we have a bimonoid $(b : M \to A, \mu_b, u_b, \Delta_b, \varepsilon_b)$ in the duoidal category, such that the bimonoid is equipped with a map $\iota_b : b \to -b$ satisfying diagrams such as (6.2), (6.3). A direct computation shows that the morphism associated to the upper path of the diagram (6.2) corresponds to the upper curved arrow of the diagram below, and similarly for the lower path and the lower curved arrow:

$$\begin{tikzcd} M \arrow[rr, bend left, "\langle d,b\rangle"] \arrow[rr, bend right, "\langle c_1,b\rangle"'] \arrow[dr, "b"'] & & M\times A \arrow[dl, "p_A"] \\ & A & \end{tikzcd}$$

with $d=\mu_b(\mathrm{id}\times\iota_b)$ and c_1 the 1–morphism of b (or in other words the constant morphism to the unit of M) — here we use the structure and conclusions considered in Proposition 6.12. The commutativity of the diagram implies the equality of both curved arrows. Then, ι_b is a right inverse of the identity. Similarly, interpreting the second diagram (6.3), we conclude the proof of the theorem. □

Definition 6.19. We denote $\mathrm{GM}\,|_{\mathrm{qc}}A\subset \mathrm{Bimon}(\mathrm{Sch}\,|_{\mathrm{qc}}A,\widetilde{\times},\mathbb{I}_{\widetilde{\times}},\times_A,\mathbb{I}_{\times_A})$ the full subcategory with objects the bimonoids $b:M\to A$ such that M is a (quasi-compact) group scheme and b a morphisms of group schemes.

We denote $\mathrm{GM}\,|_{\mathrm{aff}}A\subset \mathrm{GM}\,|_{\mathrm{qc}}A$ the full subcategory with objects the affine morphisms of group schemes — notice that $\mathrm{GM}\,|_{\mathrm{aff}}A\subset \mathrm{MM}\,|_{\mathrm{aff}}A$ in a canonical way (see Notation 6.13).

Notice that $\mathrm{GE}\,|_{\mathrm{qc}}A$, the category of quasi-compact group extensions of A, is a full subcategory of $\mathrm{GM}\,|_{\mathrm{qc}}A$ and that the category of affine extensions $\mathrm{GE}\,|_{\mathrm{aff}}A$ is a full subcategory of $\mathrm{GM}\,|_{\mathrm{aff}}A$ (see Definition 2.18).

Remark 6.20. The procedure of Theorem 6.18 is related to the work of Böhm and Lack [7], where the authors construct a certain duoidal category associated to a so-called Frobenius map–monoidale and introduce an antipode in that context.

Notice also that our construction of an antipode can be generalized as follows: let $\mathcal{C}=(\mathcal{C},\diamond,\mathbb{I}_\diamond,\star,\mathbb{I}_\star)$ be a duoidal category and assume that it can be equipped with a functor $\mathrm{op}:\mathcal{C}\to\mathcal{C}$ and natural transformations similar to $\widetilde{\gamma},\overline{\gamma}$ satisfying conditions such as the ones appearing in 6.14 and 6.15. For a bimonoid $b\in\mathcal{C}$ a morphism $\iota_b:b\to\mathrm{op}(b)$ that satisfies diagrams and properties such as the ones appearing in (6.2) and (6.3), is called an *antipode* in b. We intend to explore this construction in further work.

6.2. **Bimonoids in duoidal categories.**

We briefly recall the behavior of bilax functor between duoidal categories, as well as other monoidal structures related to duoidality, such as the constructions of modules and comodules over bialgebras in this context (see [1, Chap. 6, §6]).

Definition 6.21. (1) Given a functor $F:(\mathcal{C},\otimes_{\mathcal{C}},\mathbb{I}_{\mathcal{C}})\to(\mathcal{D},\otimes_{\mathcal{D}},\mathbb{I}_{\mathcal{D}})$ between two monoidal categories, a *lax monoidal* structure consists of a map $\ell_0:\mathbb{I}_{\mathcal{D}}\to F(\mathbb{I}_{\mathcal{C}})$ and a natural transformation $\ell_{c,c'}:F(c)\otimes_{\mathcal{D}}F(c')\to F(c\otimes_{\mathcal{C}}c')$ subject to associative and unitality axioms. If such an structure exists, we say that F is a *lax functor*. A monoidal functor $F:\mathcal{C}\to\mathcal{D}$ is called *op-lax* (or *colax*) if the induced functor $\mathcal{C}^{\mathrm{op}}\to\mathcal{D}^{\mathrm{op}}$ is lax (monoidal).

If the maps $\ell_0,\ell_{c,c'}$ are isomorphisms we call F a *strong monoidal functor*.

(2) Given the above situation and two lax monoidal functors $(F,\ell_0,\ell),(G,\ell'_0,\ell')$ a *monoidal natural transformation* is a natural transformation $\sigma:F\Rightarrow G:\mathcal{C}\to\mathcal{D}$, that satisfies the commutativity of the diagrams below:

$$\begin{tikzcd}
F(c) \otimes_{\mathcal{D}} F(c') \arrow[r, "\sigma_x \otimes_{\mathcal{D}} \sigma_y"] \arrow[d, "\ell_{c,c'}"'] & G(c) \otimes_{\mathcal{D}} G(c') \arrow[d, "\ell'_{c,c'}"'] \\
F(c \otimes_{\mathcal{C}} c') \arrow[r, "\sigma_{c \otimes_{\mathcal{C}} c'}"'] & G(c \otimes_{\mathcal{C}} c')
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{I}_{\mathcal{D}} \arrow[dl, "\ell_0"] \arrow[dr, "\ell'_0"] & \\
F(\mathbb{I}_{\mathcal{C}}) \arrow[rr, "\sigma_{\mathbb{I}_{\mathcal{C}}}"'] & & G(\mathbb{I}_{\mathcal{C}})
\end{tikzcd}$$

(3) Suppose that $\mathcal{C}, \mathcal{D}$ are two monoidal categories and let L and R be lax monoidal functors. An adjunction $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$ is *monoidal* if the unit η and the counit ε are monoidal natural transformations. In that case we say that the *abstract* adjunction "lifts" to a monoidal adjunction.

Suppose now that $L : \mathcal{C} \to \mathcal{D}$ is colax with associated maps $c\ell_0 : L(\mathbb{I}_{\mathcal{C}}) \to \mathbb{I}_{\mathcal{D}}$ and $c\ell_{c,c'} : L(c \otimes c') \to L(c) \otimes L(c')$. Then, the maps defined by the diagrams below define a lax monoidal structure for R and vice versa (see [1, Prop (3.84)] for the proof).

$$\begin{tikzcd}
\mathbb{I}_{\mathcal{C}} \arrow[rr, "\ell'_0"] \arrow[dr, "\eta_{\mathbb{I}_{\mathcal{C}}}"'] & & R(\mathbb{I}_{\mathcal{D}}) \\
& RL(\mathbb{I}_{\mathcal{C}}) \arrow[ur, "R(c\ell_0)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
R(d) \otimes R(d') \arrow[r, "\ell'_{d,d'}"] \arrow[d, "\eta_{R(d) \otimes R(d')}"'] & R(d \otimes d') \\
RL(R(d) \otimes R(d')) \arrow[r, "R(c\ell_{R(d),R(d')})"'] & R(LR(d) \otimes LR(d')) \arrow[u, "R(\varepsilon_d \otimes \varepsilon'_d)"']
\end{tikzcd}$$

Definition 6.22. Let $\mathcal{C} = (\mathcal{C}, \diamond, \mathbb{I}_\diamond, \star, \mathbb{I}_\star)$ and $\mathcal{D} = (\mathcal{D}, \diamond', \mathbb{I}_{\diamond'}, \star', \mathbb{I}_{\star'})$ be duoidal categories.

(1) A functor $F : \mathcal{C} \to \mathcal{D}$ that is lax monoidal with respect to $(\diamond, \diamond')$ with structure $\tau_{c,d} : F(c) \diamond' F(d) \to F(c \diamond d)$, $\nu : \mathbb{I}_{\diamond'} \to F(\mathbb{I}_\diamond)$, and colax monoidal with respect to $(\star, \star')$ with structure $\rho_{c,d} : F(c \star d) \to F(c) \star' F(d)$, $\lambda : F(\mathbb{I}_\star) \to \mathbb{I}_{\star'}$, is called a *bilax functor* from $\mathcal{C}$ into $\mathcal{D}$, provided the following diagrams are commutative.

(1) `Interchange.`

$$\begin{tikzcd}
& F(a \star b) \diamond' F(c \star d) \arrow[dl, "\tau_{a\star b, c\star d}"'] \arrow[dr, "\rho_{a,b} \diamond' \rho_{c,d}"] & \\
F((a \star b) \diamond (c \star d)) \arrow[d, "F(\zeta_{a,b,c,d})"'] & & (F(a) \star' F(b)) \diamond' (F(c) \star' F(d)) \arrow[d, "\zeta'_{F(a),F(b),F(c),F(d)}"] \\
F((a \diamond c) \star (b \diamond d)) \arrow[dr, "\rho_{a\diamond c, b\diamond d}"'] & & (F(a) \diamond' F(c)) \star' (F(b) \diamond' F(d)) \arrow[dl, "\tau_{a,c} \diamond' \tau_{b,d}"] \\
& F(a \diamond c) \star' F(b \diamond d) &
\end{tikzcd}$$

(2) `Unitality.`

$$\begin{tikzcd}
\mathbb{I}_{\diamond'} \arrow[r, "\nu"] \arrow[d, "\Delta_{\diamond'}"'] & F(\mathbb{I}_\diamond) \arrow[r, "F(\Delta_\diamond)"] & F(\mathbb{I}_\diamond \star \mathbb{I}_\diamond) \arrow[d, "\rho_{\mathbb{I}_\diamond,\mathbb{I}_\diamond}"] \\
\mathbb{I}_{\diamond'} \star' \mathbb{I}_{\diamond'} \arrow[rr, "\nu \star' \nu"'] & & F(\mathbb{I}_\diamond) \star' F(\mathbb{I}_\diamond)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}_{\star'} & F(\mathbb{I}_\star) \arrow[l, "\lambda"'] & F(\mathbb{I}_\star \diamond \mathbb{I}_\star) \arrow[l, "F(\mu_\star)"'] \\
\mathbb{I}_{\star'} \diamond' \mathbb{I}_{\star'} \arrow[u, "\mu_{\star'}"] & & F(\mathbb{I}_\star) \diamond' F(\mathbb{I}_\star) \arrow[ll, "\lambda \star' \lambda"] \arrow[u, "\tau_{\mathbb{I}_\star,\mathbb{I}_\star}"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(\mathbb{I}_\star) \arrow[r, "F(u_{\mathbb{I}_\star})=F(\varepsilon_{\mathbb{I}_\star})"] & F(\mathbb{I}_\star) \arrow[d, "\lambda"] \\
\mathbb{I}_{\star'} \arrow[u, "\nu"] \arrow[r, "u_{\mathbb{I}_{\diamond'}}=\varepsilon_{\mathbb{I}_{\star'}}"] & \mathbb{I}_{\star'}
\end{tikzcd}$$

(2) Let $F, G : (\mathcal{C}, \diamond, \star) \to (\mathcal{D}, \diamond', \star')$ be bilax functors. Call $(\tau, \nu), (\tau', \nu')$ the lax $(\diamond, \diamond')$–monoidal structures for F and G respectively and call $(\rho, \lambda), (\rho', \lambda')$ the colax $(\star, \star')$–monoidal structures for F and G respectively.

A natural transformation $\sigma : F \Rightarrow G$ is a *morphism of bilax functors* if it is at the same time a natural transformation of lax and colax functors. In other words if the following diagrams for the monoidal structures are commutative for all $c, c' \in C$.

$$\begin{tikzcd}
F(c) \diamond F(c') \arrow[r, "\tau_{c,c'}"] \arrow[d, "\sigma_c \diamond \sigma_{c'}"'] & F(c \diamond c') \arrow[d, "\sigma_{c \diamond c'}"] \\
G(c) \diamond G(c') \arrow[r, "\tau'_{c,c'}"'] & G(c \diamond c')
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{I}_{\diamond'} \arrow[dl, "\nu"'] \arrow[dr, "\nu'"] & \\
F(\mathbb{I}_\diamond) \arrow[rr, "\sigma_{\mathbb{I}_\diamond}"'] & & G(\mathbb{I}_\diamond)
\end{tikzcd}$$

$$\begin{tikzcd}
F(c) \star F(c') & F(c \star c') \arrow[l, "\rho_{c,c'}"'] \arrow[d, "\sigma_{c \star c'}"] \\
G(c) \star G(c') \arrow[u, leftarrow, "\sigma_c \star \sigma_{c'}"] & G(c \star c') \arrow[l, "\rho'_{c,c'}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{I}_{\star'} & \\
F(\mathbb{I}_\star) \arrow[ur, "\lambda"] \arrow[rr, "\sigma_{\mathbb{I}_\star}"'] & & G(\mathbb{I}_\star) \arrow[ul, "\lambda'"']
\end{tikzcd}$$

The details of the proof of the result that follows (and some variations) can be found in [1, Chap. 6, §8].

Lemma 6.23. *Let $\mathcal{C} = (\mathcal{C}, \diamond, \mathbb{I}_\diamond, \star, \mathbb{I}_\star)$ and $\mathcal{D} = (\mathcal{D}, \diamond', \mathbb{I}_{\diamond'}, \star', \mathbb{I}_{\star'})$ be duoidal categories and $b = (b, \mu_b, u_b, \Delta_b, \varepsilon_b)$ a bimonoid in $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a lax $(\diamond, \diamond')$–monoidal functor and colax $(\star, \star')$–monoidal functor and τ, ν, ρ and λ as in Definition 6.22. Then $F(b) = \big(F(b), F(\mu_b) \circ \tau_{b,b}, F(u_b) \circ \nu, \rho_{b,b} \circ F(\Delta_b), \lambda \circ F(\varepsilon_b)\big)$ is at the same time a monoid and a comonoid with respect to $\diamond'$ and $\star'$ respectively. Moreover, if F is bilax monoidal, then $F(b)$ is a* bimonoid *in $\mathcal{D}$.*

Moreover, if f is a morphism of bimonoids so is $F(f)$, hence F restricts to a functor in the categories of bimonoids: $F : \mathrm{Bimon}(\mathcal{C}) \to \mathrm{Bimon}(\mathcal{D})$. $\square$

Definition 6.24. (1) Assume that $(\mathcal{C}, \odot, \mathbb{I}_\odot)$ is a monoidal category and let $m = (m, \mu_m, u_m)$, be a monoid in $(\mathcal{C}, \odot, \mathbb{I}_\odot)$. A (left) *$m$–module structure* on $x \in \mathcal{C}$ or a (left) *action* of m on x is a morphism $\alpha_x : m \odot x \to x$ such that the following diagrams are commutative:

$$\begin{tikzcd} m \odot m \odot x \arrow[r, "\mathrm{id}_m \odot \alpha_x"] \arrow[d, "\mu_m \odot \mathrm{id}_x"'] & m \odot x \arrow[d, "\alpha_x"] \\ m \odot x \arrow[r, "\alpha_x"'] & x \end{tikzcd} \qquad \begin{tikzcd} \mathbb{I}_\odot \odot x \arrow[r, "u_m \odot \mathrm{id}_x"] \arrow[dr, "\cong"'] & m \odot x \arrow[d, "\alpha_x"] \\ & x \end{tikzcd}$$

The pair (x, α) is called an *m–module* — when there is no ambiguity on the action, we will say that x is an m–module. If (x, α_x), (y, α_y) are left m–modules, a *morphism of left m–modules* $x \to y$ is a morphism $f \in \mathrm{Hom}(x, y)$ such that $\alpha_y \circ (\mathrm{id}_m \odot f) = f \circ \alpha_x : m \odot x \to y$.

We define ${}_m\mathsf{M}$, the *category of left m-modules* as the category with objects the left m–modules and morphism the morphisms the morphism of left m–modules.

In a similar way, we define M_m, the *category of right m-modules*.

(2) If $(c, \Delta_c, \varepsilon_c)$ is a comonoid in $(\mathcal{C}, \odot, \mathbb{I}_\odot)$, a right *$c$–comodule structure* on $y \in \mathcal{C}$ or a (right) *coaction* of c on y is a morphism $\chi_y : y \to y \odot c$ such that the following diagrams are commutative:

$$\begin{tikzcd} y \arrow[r, "\chi"] \arrow[d, "\chi"'] & y \odot c \arrow[d, "\chi \odot \mathrm{id}_c"] \\ y \odot c \arrow[r, "\mathrm{id} \odot \Delta_c"'] & y \odot c \odot c \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "\chi"] \arrow[dr, "\cong"'] & y \odot c \arrow[d, "\mathrm{id}_y \odot \varepsilon_c"] \\ & y \odot \mathbb{I}_\odot \end{tikzcd}$$

We say that the pair (y, χ) is a *right c–comodule* — we will often omit the coaction and say that y is a c–comodule.

If (x, χ_x) and (y, χ_y) are c–comodules, a *morphism of c–comodules* between x an y is a morphism $f \in \mathrm{Hom}(x, y)$ such that $(f \odot \mathrm{id}_c) \circ \chi_x = \chi_y \circ f : x \to y \odot c$.

We denote M^c the *category of right c–comodules*, defined in the usual way. Analogously, we define its left counter part ${}^c\mathsf{M}$, the *category of left c–comodules*,

Example 6.25. If $m : M \to A$ is a quasi-compact morphism of monoid (group) schemes, then to give an m–module in the monoidal category $(\mathrm{Sch}\,|_{\mathrm{qc}} A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}})$ is equivalent to give a pair $(x : X \to A, a)$, where $a : M \times X \to X$ is an action — in the usual sense — such that the following diagram is commutative:

$$\begin{tikzcd} M \times X \arrow[r, "a"] \arrow[d, "m \times x"'] & X \arrow[d, "x"] \\ A \times A \arrow[r, "s"'] & A \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}(\Bbbk) \times X \arrow[r, "u_M \times \mathrm{id}"] \arrow[dr, "\cong"'] & M \times X \arrow[d, "a"] \\ & X \end{tikzcd}$$

In particular, if $\mathcal{S} \in q : G \to A$ is an affine (or more generally quasi-compact) extension, then any representation $E \in \mathrm{Rep}(\mathcal{S})$ — in the nomenclature of Definition 3.41 — is a q–module in the monoidal category $\mathrm{Sch}\,|_{\mathrm{qc}} A$ and conversely.

In the case that the monoidal category $(\mathcal{C}, \odot, \mathbb{I}_\odot)$ is part of a duoidal category, more structure is available as shown in the next proposition, that admits — wherever it is possible — a left and a right version.

Proposition 6.26. *Let $(\mathcal{C},\diamond,\mathbb{I}_\diamond,\star,\mathbb{I}_\star)$ be a duoidal category and m_1,m_2 be monoids in $(\mathcal{C},\diamond,\mathbb{I}_\diamond)$, c_1,c_2 comonoids in $(\mathcal{C},\star,\mathbb{I}_\star)$ and $(b,\mu_b,u_b,\Delta_b,\varepsilon_b)$ a bimonoid in $(\mathcal{C},\diamond,\mathbb{I}_\diamond,\star,\mathbb{I}_\star)$. Then:*

(1) $m_1\star m_2$ is a monoid in $(\mathcal{C},\diamond,\mathbb{I}_\diamond)$ and $c_1\diamond c_2$ is a comonoid in $(\mathcal{C},\star,\mathbb{I}_\star)$;

(2) if x_1,x_2 are left modules for m_1 and m_2 with structures α_1,α_2 respectively, then $x_1\star x_2$ is a left $m_1\star m_2$ module with structure:

$$\alpha_1\underline{\star}\alpha_2:(m_1\star m_2)\diamond(x_1\star x_2)\xrightarrow{\zeta_{m_1,m_2,x_1,x_2}}(m_1\diamond x_1)\star(m_2\diamond x_2)\xrightarrow{\alpha_1\star\alpha_2}x_1\star x_2;$$

(3) if y_1,y_2 are right comodules for c_1 and c_2 with structures χ_1,χ_2 respectively, then $y_1\diamond y_2$ is a right $c_1\star c_2$ comodule with structure:

$$\chi_1\underline{\star}\chi_2:y_1\diamond y_2\xrightarrow{\chi_1\diamond\chi_2}(y_1\star c_1)\diamond(y_2\star c_2)\xrightarrow{\zeta_{y_1,c_1,y_2,c_2}}(y_1\diamond y_2)\star(c_1\diamond c_2);$$

(4) if x_1,x_2 are left modules for b with structures α_1,α_2 then, $x_1\star x_2$ is also a left module for b with structure:

$$\alpha_{1,2}:b\diamond(x_1\star x_2)\xrightarrow{\Delta_b\diamond\,\mathrm{id}}(b\star b)\diamond(x_1\star x_2)\xrightarrow{\alpha_1\underline{\star}\alpha_2}x_1\star x_2.$$

(5) if y_1,y_2 are right comodules for b with structures χ_1,χ_2 then, $y_1\diamond y_2$ is also a right comodule for b with structure:

$$\chi_{1,2}:y_1\diamond y_2\xrightarrow{\chi_1\underline{\diamond}\chi_2}(y_1\diamond y_2)\star(b\diamond b)\xrightarrow{\mathrm{id}\,\star\mu_b}(y_1\diamond y_2)\star b.$$

Proof. The detailed proof of this result can be found in [1, Prop.6.25]. For example, the morphism given by the composition of the interchange map and the $\star$ monoidal product of the multiplications of m_1 and m_2,

$$(m_1\star m_2)\diamond(m_1\star m_2)\xrightarrow{\zeta_{m_1,m_2,m_1,m_2}}(m_1\diamond m_1)\star(m_2\diamond m_2)\xrightarrow{\mu_1\star\mu_2}m_1\star m_2,$$

is the multiplication morphism of $m_1\star m_2$. □

Definition 6.27. If b is a bimonoid in $\mathcal{C}$, a *right b–comodule algebra* in the duoidal category $\mathcal{C}$ is a right b–comodule (y,χ) equipped also with a monoid structure (y,μ_y,u_y) in $(\mathcal{C},\diamond,\mathbb{I}_\diamond)$ such that the diagrams below commute:

$$\begin{tikzcd} y\diamond y \arrow[r, "\chi_{12}"] \arrow[d, "\mu_y"'] & (y\diamond y)\star b \arrow[d, "\mu_y\star\mathrm{id}"] \\ y \arrow[r, "\chi"] & y\star b \end{tikzcd} \qquad \begin{tikzcd} \mathbb{I}_\diamond \arrow[r, "\Delta_\diamond"] \arrow[d, "u_y"'] & \mathbb{I}_\diamond\star\mathbb{I}_\diamond \arrow[d, "u_y\star u_b"] \\ y \arrow[r, "\chi"] & y\star b \end{tikzcd}$$

As usual, we have also the notion of *left b–comodule algebra.*

The following Corollary follows easily from the proposition above and extends the results of Lemma 6.23 to modules and comodules.

Corollary 6.28. *If $\mathcal{C}=(\mathcal{C},\diamond,\mathbb{I}_\diamond,\star,\mathbb{I}_\star)$ be a duoidal category, then the subcategory* $\mathrm{Mon}(\mathcal{C},\diamond)\subset\mathcal{C}$ *is monoidal with respect to the structure $(\star,\mathbb{I}_\star)$ and the subcategory* $\mathrm{Comon}(\mathcal{C},\star)\subset\mathcal{C}$, *is monoidal with respect to the structure $(\diamond,\mathbb{I}_\diamond)$.*

Assume that $\mathcal{D}=(\mathcal{D},\diamond',\mathbb{I}_{\diamond'},\star',\mathbb{I}_{\star'})$ is another duoidal category and that $F:\mathcal{C}\to\mathcal{D}$ is a functor that is lax $(\diamond,\diamond')$–monoidal and colax $(\star,\star')$–monoidal.

Then F restricts to a colax–$(\star,\star')$ monoidal functor $F:\big(\mathrm{Mon}(\mathcal{C},\diamond,\mathbb{I}_\diamond),\star,\mathbb{I}_\star\big)\to\big(\mathrm{Mon}(\mathcal{D},\diamond',\mathbb{I}_{\diamond'}),\star',\mathbb{I}_{\star'}\big)$.

Dually, F restricts to a lax–$(\diamond,\diamond')$ monoidal functor $F:\big(\mathrm{Comon}(\mathcal{C},\star,\mathbb{I}_\star),\diamond,\mathbb{I}_\diamond\big)\to\big(\mathrm{Mon}(\mathcal{D},\star',\mathbb{I}_{\star'}),\diamond',\mathbb{I}_{\diamond'}\big)$.

Also, in the case that the functor F is a bilax functor and $b\in\mathrm{Bimod}(\mathcal{C})$, we call $\mathrm{Mod(b)}$ *and* $\mathrm{Comod(b)}$ *the categories of b–modules with respect to the $\diamond$–monoidal structure and of b–comodules with respect to the $\star$–monoidal structure. Then:*

(i) $\big(\mathrm{Mod(b)},\star\big)$ *and* $\big(\mathrm{Comod(b)},\diamond\big)$ *are monoidal categories.*

(ii) *If $F:\mathcal{C}\to\mathcal{D}$ is a bilax monoidal functor as above. Then F induces a monoidal functor* $F:\big(\mathrm{Mod}(b),\star\big)\to(\mathrm{Mod}(F(b)),\star')$ *and* $F:\big(\mathrm{Comod}(b),\diamond\big)\to\big(\mathrm{Comod}(F(b)),\diamond'\big)$.

Proof. The proof is direct using the definitions and results of Lemma 6.23 and Proposition 6.26. ☐

6.3. **Quasi-compact morphisms and their associated sheaves.**

In this section we collect some results and definitions on (separated) quasi-compact morphisms and their associated (quasi-coherent) sheaves, that will be used later. The basic definitions can be found in [34, Chap. II] or [30, Chap. 0, Chap. 1]. The original reference for the adjunction results is [31, §1.2,§1.3], but they also appear as a series of exercises in [34, Ex. II.5.17, II.5.18] as well as in many other references, e.g. [57], [59].

Notation 6.29. If S is a scheme, the category of $\mathcal{O}_S$–modules (algebras) will be denoted as S-mod (S-alg) and the category of quasi-coherent S–modules (algebras) as QS-mod (QS-alg).

Definition 6.30. We denote $\mathcal{P}:\mathrm{Sch}|_{\mathrm{qc}}S\to S\text{-alg}^{\mathrm{op}}$ the functor given as follows (see [31][Prop (1.3.1)] or [34, Proposition II.5.8]): If $x:X\to S\in\mathrm{Sch}|_{\mathrm{qc}}S$, then $\mathcal{P}(x):=x_*(\mathcal{O}_X)$; if $(f,f^\#):(x:X\to S)\to(x':X'\to S)$ is a morphism in $\mathrm{Sch}|_{\mathrm{qc}}S$ (recall that $f^\#:\mathcal{O}_{X'}\to f_*\mathcal{O}_X$) then $\mathcal{P}(f,f^\#):=x'_*(f^\#):\mathcal{P}(x')=x'_*(\mathcal{O}_{X'})\to\mathcal{P}(x)=x_*(\mathcal{O}_X)$.

It is well known that the restriction of $\mathcal{P}$ to $\mathrm{Sch}|_{\mathrm{sqc}}S$ induces a functor to $QS\text{-alg}^{\mathrm{op}}$, that we still call $\mathcal{P}:\mathrm{Sch}|_{\mathrm{sqc}}S\to QS\text{-alg}^{\mathrm{op}}$.

We recall now the well known construction of a right adjoint to $\mathcal{P}:\mathrm{Sch}|_{\mathrm{sqc}}S\to QS\text{-alg}^{\mathrm{op}}$.

Remark 6.31. (see [31, Prop. 1.3.1 and Prop. 1.2.7]) The functor $\mathcal{P}:\mathrm{Sch}|_{\mathrm{sqc}}S\to QS\text{-alg}^{\mathrm{op}}$ admits a right adjoint $\mathrm{Spec}:QA\text{-alg}^{\mathrm{op}}\to\mathrm{Sch}|_{\mathrm{sqc}}S$. Given $\mathcal{F}\in QS$-alg, then $\mathrm{Spec}(\mathcal{F})$ is the unique — up to isomorphisms of S-schemes — scheme over S such that $\mathrm{Spec}(\mathcal{F})\in\mathrm{Sch}|_{\mathrm{aff}}S$ and $\mathcal{P}\big(\mathrm{Spec}(\mathcal{F})\big)=\mathcal{F}$. We denote $\mathrm{Spec}(\mathcal{F})$ as $\pi_{\mathcal{F}}:\mathrm{Spec}(\mathcal{F})\to S$.

Recall that the adjunction $\mathrm{Sch}|_{\mathrm{sqc}}S \underset{\mathrm{Spec}}{\overset{\mathcal{P}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} QS\text{-alg}^{\mathrm{op}}$ is given by a natural transformation

$$\mathrm{Hom}_{QS\text{-alg}}\big(\mathcal{F},\mathcal{P}(x)\big)=\mathrm{Hom}_{(QS\text{-alg})^{\mathrm{op}}}\big(\mathcal{P}(x),\mathcal{F}\big)\cong\mathrm{Hom}_{\mathrm{Sch}|_{\mathrm{sqc}}S}(x,\mathrm{Spec}\,\mathcal{F}),$$

that has a unit η and counit ε the natural transformations:

$$\begin{cases} \text{(i) } \eta_x : x \to \operatorname{Aff}_S(x) := \operatorname{Spec}\big(\mathcal{P}(x)\big) \in \operatorname{Sch}|_{\text{sqc}} S; \\ \text{(ii) } \varepsilon_{\mathcal{F}} : \mathcal{P}\big(\operatorname{Spec}(\mathcal{F})\big) \to \mathcal{F} \in (QS\text{-alg})^{\text{op}} \text{ or} \\ \qquad \varepsilon_{\mathcal{F}} : \mathcal{F} \to \mathcal{P}\big(\operatorname{Spec}(\mathcal{F})\big) \in QS\text{-alg}, \end{cases}$$

In this particular case, we have that for all $\mathcal{F} \in QS\text{-alg}$, $\varepsilon_{\mathcal{F}}$ is an isomorphism (see [31, §1.2, §1.3, §1.4]).

Definition 6.32. By construction, for $x \in \operatorname{Sch}|_{\text{sqc}} S$ the morphism $\eta_x : x \to \operatorname{Aff}_S(x) \in \operatorname{Sch}|_{\text{sqc}} S$ is affine; η_x is called the *(relative) affinization map* and its codomain is called the *(S–)affinization* of x (frequently just called "affinization"). The functor $\operatorname{Aff}_S = \operatorname{Spec} \circ \mathcal{P} : \operatorname{Sch}|_{\text{sqc}} S \to \operatorname{Sch}|_{\text{aff}} S \subset \operatorname{Sch}|_{\text{sqc}} S$ is called the *affinization over S* or the *relative affinization (over S)*. Compare with Definition 2.38.

Remark 6.33. The affinization satisfies the following universal property (compare with Remark 2.39):

For any morphism $f : x \to y$ with $x \in \operatorname{Sch}|_{\text{sqc}} S, y \in \operatorname{Sch}|_{\text{aff}} S$ there is a unique morphism $\widehat{f}$ that makes commutative the diagram below:

$$\begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[dr, "f"'] & \operatorname{Aff}_S(x) \arrow[d, dotted, "\widehat{f}"] \\ & y \end{tikzcd}$$

In view of Remark 6.31, we have the following equivalence of categories:

Proposition 6.34. *The adjunction* $\operatorname{Sch}|_{\text{sqc}} S \underset{\operatorname{Spec}}{\overset{\mathcal{P}}{\underset{\perp}{\rightleftarrows}}} (QS\text{-alg})^{\text{op}}$ *restricts to functors* $\mathcal{P}|_{\operatorname{Sch}|_{\text{aff}} S} : \operatorname{Sch}|_{\text{aff}} S \to QS\text{-alg}^{\text{op}}$ *and* $\operatorname{Spec} : QS\text{-alg}^{\text{op}} \to \operatorname{Sch}|_{\text{aff}} S$ *that establish an adjoint op-equivalence between* $\operatorname{Sch}|_{\text{aff}} S$ *and* $QS\text{-alg}$.

In particular the counit ε of the original adjunction is an isomorphism (see Remark 6.31 and [31, Prop. 1.3.1] for a proof). □

We finish this section presenting (without proofs) some known results on flat (separated, quasi-compact) schemes over S that will be needed; we follow [32, §2.1 – 2.3].

Lemma 6.35. *Let $x : X \to S \in \operatorname{Sch}|_{\text{sqc}} S$. Then x is flat if and only if the pull-back functor $x^* : S\text{-mod} \to X\text{-mod}$ is exact.* □

Lemma 6.36. *Let $x : X \to S \in \operatorname{Sch}|_{\text{sqc}} S$ be a flat morphism. Then $\mathcal{P}(x) = x_*(\mathcal{O}_X)$ is a quasi-coherent flat sheaf of algebras in $S\text{-mod}$.*

Conversely, if $\mathcal{F}$ is a quasi-coherent flat sheaf of algebras in $S\text{-mod}$, then $\pi_{\mathcal{F}} : \operatorname{Spec}(\mathcal{F}) \to S$ is an affine (hence separated) flat morphism. □

Remark 6.37. Let $X, Y \in \operatorname{Sch}|S$, $\mathcal{F} \in X\text{-mod}$, $\mathcal{G} \in Y\text{-mod}$, and let $h : X \to Y$ $\ell : Y \to Y'$ be morphisms of schemes over S. Then the counits associated to the adjunction between $h^* \dashv h_*; \ell^* \dashv \ell_*$: $\varepsilon_{\mathcal{F}} : h^*h_*\mathcal{F} \to \mathcal{F}$ and $\varepsilon_{\mathcal{G}} : \ell^*\ell_*\mathcal{G} \to \mathcal{G}$, induce a homomorphism of sheaves

$$\varepsilon_{\mathcal{F}} \boxtimes \varepsilon_{\mathcal{G}} : h^*h_*\mathcal{F} \boxtimes \ell^*\ell_*\mathcal{G} = (h \times \ell)^*(h_*\mathcal{F} \boxtimes \ell_*\mathcal{G}) \to \mathcal{F} \boxtimes \mathcal{G}.$$

Using the standard adjunction again we obtain the map:

$$\Gamma_{\mathcal{F},\mathcal{G}} := (h \times \ell)_*(\varepsilon_{\mathcal{F}} \boxtimes \varepsilon_{\mathcal{G}})\nu_{h_*\mathcal{F} \boxtimes \ell_*\mathcal{G}} : h_*\mathcal{F} \boxtimes \ell_*\mathcal{G} \to (h \times \ell)_*(\mathcal{F} \boxtimes \mathcal{G}),$$

where $\nu_{_{h_*\mathcal{F}\boxtimes \ell_*\mathcal{G}}}$ is the unit of the adjunction.

We are interested in conditions on h,ℓ and $\mathcal{F},\mathcal{G}$ in order to guarantee that $\Gamma_{\mathcal{F},\mathcal{G}}$ is an isomorphism. A set of conditions was established by Brandenburg in response to a question in "stackexchange/math", in a more general setting. In our context, Brandenburg answer can be stated as follows:

Lemma 6.38. *Let h,ℓ, $\mathcal{F},\mathcal{G}$ as above and assume that either:*

(a) h,ℓ are quasi-compact and quasi-separated, or

(b) h,ℓ are affine morphisms.

Then $\Gamma_{\mathcal{F},\mathcal{G}}$ is an isomorphism.

Proof. See [9]. □

Remark 6.39. In the notations of Lemma 6.38, notice that since we are working with $\Bbbk$–schemes, it follows that $\mathcal{F},\mathcal{G}$ are S–flat quasi-coherent sheaves — that is, $h_*\mathcal{F}$ and $\ell_*\mathcal{G}$ are flat sheaves.

6.4. **A duoidal structure for QA-mod.**

In Section 6.1 we considered a duoidal structure on $\mathrm{Sch}\,|_{\mathrm{qc}}A$ together with an additional functor $\mathrm{op}_* : \mathrm{Sch}\,|_{\mathrm{qc}}A \to \mathrm{Sch}\,|_{\mathrm{qc}}A$ (denoted as $\mathrm{op}_*(x) = -x$) for which the subcategory $\mathrm{Bimon}(\mathrm{Sch}\,|_{\mathrm{qc}}A) \subseteq \mathrm{Sch}\,|_{\mathrm{qc}}A$ has as objects the quasi-compact morphisms of monoids $q_M : M \to A$ (and as morphisms the morphisms over A, that are of monoids). Thus, the category $\mathrm{GE}\,|_{\mathrm{qc}}A$ of quasi-compact group extensions of the abelian variety A can be interpreted as the full subcategory of $\mathrm{Bimon}(\mathrm{Sch}\,|_{\mathrm{qc}}A)$ that has objects the faithfully flat morphisms of group schemes. In terms of the category $\mathrm{Sch}\,|_{\mathrm{qc}}A$, a group extension is a bimonoid x in the category, equipped with an additional arrow $\iota_x : x \to (-x)$ satisfying the supplementary conditions of Theorem 6.18. Since the category $\mathrm{GE}\,|_{\mathrm{aff}}A$ of affine extensions of A is a full subcategory of $\mathrm{GE}\,|_{\mathrm{qc}}A$, it can viewed also as a full subcategory of $\mathrm{Bimon}(\mathrm{Sch}\,|_{\mathrm{qc}}A)$.

In order to dualize the above situation to the category of sheaves, we first restrict the above setting to the duoidal category based upon $\mathrm{Sch}\,|_{\mathrm{sqc}}A$ — recall that since a since a group scheme is separated $\mathrm{GE}\,|_{\mathrm{qc}}A \subset \mathrm{Bimon}(\mathrm{Sch}\,|_{\mathrm{sqc}}A)$. In this context, the functors $\mathcal{P}$ and Spec (see Remark 6.31) establish an adjunction between $\mathrm{Sch}\,|_{\mathrm{sqc}}A$ and $QA\text{-alg}^{\mathrm{op}}$ — our goal is to describe the behavior of $\mathrm{GE}\,|_{\mathrm{aff}}A$ under the mentioned adjunction. In order to describe the correlate of $\mathrm{Bimon}(\mathrm{Sch}\,|_{\mathrm{sqc}}A)$ in QA-alg, we introduce a duoidal structure on QA-mod such that the corresponding subcategory of bimonoids $\mathrm{Bimon}(QA\text{-mod})$ is the seeked correlate. Finally, we introduce an additional structure that corresponds under the adjunction to the antipode, and construct the category of *commutative Hopf sheaves*. In this we end up with a monoidal adjunction from the category $\mathrm{GM}\,|_{\mathrm{qc}}A$ (see Definition 6.19) and the category of schemes *commutative Hopf sheaves*. If moreover we restrict ourselves the subcategory $\mathrm{GE}\,|_{\mathrm{qc}}A$, we obtain an adjunction with the subcategory of *flat commutative Hopf sheaves*, that restrict in turn to an equivalence between the category of affine extensions of A and the category of flat commutative Hopf sheaves.

Be begin by recalling the definition of the external tensor product of sheaves over a scheme S.

Definition 6.40. Let S be a scheme and $X, Y \in \mathrm{Sch}|S$. If $\mathcal{F} \in QX$-mod, and $\mathcal{G} \in QY$-mod, we define the sheaf $\mathcal{F} \boxtimes_S \mathcal{G} := p_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} p_2^*\mathcal{G} \in Q(X \times_S Y)$-mod, where $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ are the canonical projections. This correspondence can be extended to a functor $\boxtimes_S : QX\text{-mod} \times QY\text{-mod} \to Q(X \times_S Y)\text{-mod}$. This functor is called in [30, Section 9] the *tensor product over* $\mathcal{O}_S$ or the *tensor product over* S, but currently it is called the *external tensor product* (over S). In the particular case that $S = \Bbbk$, we usually write $\mathcal{F} \boxtimes_{\mathrm{Spec}(\Bbbk)} \mathcal{G} = \mathcal{F} \boxtimes \mathcal{G}$.

Remark 6.41. In the situation that $X = Y = S$ and $\mathcal{F}, \mathcal{G} \in QS$-mod, $\mathcal{F} \boxtimes_S \mathcal{G} = \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ the usual monoidal structure in the category of the sheaves of $\mathcal{O}_S$–mod.

Definition 6.42. Let A be an abelian variety. (1) Along this section and to be consistent with Definition 6.4 we call the usual monoidal structure $\otimes_A : QA\text{-mod} \times QA\text{-mod} \to QA\text{-mod}$ the *Hadamard monoidal structure* — the unit $\mathbb{I}_{\otimes_A}$ is $\mathcal{O}_A$.
(2) We define the *Cauchy monoidal structure in* QA-mod as follows:

$$\widetilde{\boxtimes} = s_* \circ \boxtimes_{\mathrm{Spec}(\Bbbk)} : \; QA\text{-mod} \times QA\text{-mod} \xrightarrow{\boxtimes_{\mathrm{Spec}(\Bbbk)}} Q(A \times_\Bbbk A)\text{-mod} \xrightarrow{s_*} QA\text{-mod},$$

where s_* is the push-forward functor by the addition morphism $s : A \times A \to A$.

It is easy to show that $\big(QA\text{-mod}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}} = \mathrm{skysc}_0(\Bbbk)\big)$, where $\mathrm{skysc}_0(\Bbbk) \in QA$-mod denotes the skyscraper sheaf at $0 \in A$ with stalk $\Bbbk$, is a monoidal structure.

Indeed, if $\mathcal{F}, \mathcal{G}$ are quasi-coherent sheaves on A, then $\mathcal{F} \boxtimes_{\mathrm{Spec}(\Bbbk)} \mathcal{G}$ is also quasi-coherent. Moreover, since $\mathcal{F} \,\widetilde{\boxtimes}\, \mathcal{G} = s_*(\mathcal{F} \boxtimes_{\mathrm{Spec}(\Bbbk)} \mathcal{G})$ is the push-forward of a quasi-coherent sheaf by the proper morphism $s : A \times A \to A$, the sheaf $\mathcal{F} \,\widetilde{\boxtimes}\, \mathcal{G}$ is also quasi-coherent (see for example [34, Proposition 5.8]).

On the other hand, if $\iota = (\mathrm{id}, 0) : A \to A \times_\Bbbk A$ is the closed immersion $\iota(a) = (a, 0)$ and $\mathcal{F} \in QA$-mod, then $\iota_*(\mathcal{F}) \cong \mathcal{F} \boxtimes_{\mathrm{Spec}(\Bbbk)} \mathrm{skysc}_0(\Bbbk)$. Applying s_* to the above isomorphism (and using that $s\iota = \mathrm{id}_A$), we have that $\mathcal{F} = s_*\big(\mathcal{F} \boxtimes_{\mathrm{Spec}(\Bbbk)} \mathrm{skysc}_0(\Bbbk)\big) = \mathcal{F} \,\widetilde{\boxtimes}\, \mathrm{skysc}_0(\Bbbk)$. Therefore, $\mathrm{skysc}_0(\Bbbk)$ is a right side unit, and similarly one proves that $\mathrm{skysc}_0(\Bbbk)$ is also a left side unit.

The next elementary constructions and notations will be useful in what follows.

Definition 6.43. If $\mathrm{op} : A \to A$ is the inversion morphism of A we consider the functor $\mathrm{op}_* : QA\text{-mod} \to QA\text{-mod}$ and define $-\mathcal{F} := \mathrm{op}_*(\mathcal{F})$.

Remark 6.44. Let $0 : \mathrm{Spec}(\Bbbk) \to A$, $\mathrm{st} : A \to \mathrm{Spec}(\Bbbk)$ and $c_0 = 0 \circ \mathrm{st} : A \to A$ (see Definition 6.3). Then:

(1) If $\mathcal{F} \in A$-mod, then the sheaf $\mathrm{st}_*(\mathcal{F})$ has stalk $\mathcal{F}(A)$ at the only point of $\mathrm{Spec}(\Bbbk)$. If $\mathcal{V}$ a sheaf on $\mathrm{Spec}(\Bbbk)$ of stalk V, then $0_*(\mathcal{V}) = \mathrm{skysc}_0(V)$ and $(c_0)_*(\mathcal{F}) = \mathrm{skysc}_0\big(\mathcal{F}(A)\big)$.

(2) On the other hand, for $\mathcal{V}$ as above, $\mathrm{st}^{-1}(\mathcal{V})(U) = V$ for all U open in A. Hence $\mathrm{st}^*(\mathcal{V}) = \mathcal{O}_A \otimes_\Bbbk V$.

(3) If $\mathcal{F} \in A$-mod, then $0^{-1}\mathcal{F}$ is the sheaf of $\mathcal{O}_{A,0}$–modules on $\mathrm{Spec}(\Bbbk)$ with stalk $\mathcal{F}_0$. Hence, $0^*\mathcal{F}$ is the sheaf of $\mathcal{O}_{\mathrm{Spec}(\Bbbk)}$–modules (i.e. $\Bbbk$–spaces) with stalk $\Bbbk \otimes_{\mathcal{O}_{A,0}} \mathcal{F}_0 = \mathcal{F}_0/\mathcal{M}_{A,0}\mathcal{F}_0$, where $\mathcal{M}_{A,0} \subseteq \mathcal{O}_{A,0}$ is the maximal ideal of the local ring $\mathcal{O}_{A,0}$.

(4) Combining (2) and (3), we deduce that $c_0^*(\mathcal{F}) = \mathcal{O}_A \otimes_\Bbbk \mathcal{F}_0/\mathcal{M}_{A,0}\mathcal{F}_0$. In particular, $c_0^*\big(\mathrm{skysc}_0(\Bbbk)\big) = \mathcal{O}_A$.

Proposition 6.45. *Let A be an abelian variety. Then $(QA\text{-mod}, \otimes_A, \mathbb{I}_{\otimes_A}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}})$ can be completed to a duoidal structure (see Definitions 6.7, 6.40, 6.42). Moreover, the subcategory QA-alg inherits this duoidal category, that is $(QA\text{-alg}, \otimes_A|_{QA\text{-alg}\times QA\text{-alg}}, \mathbb{I}_{\otimes_A}, \widetilde{\boxtimes}\,|_{QA\text{-alg}\times QA\text{-alg}}, \mathbb{I}_{\widetilde{\boxtimes}})$ is a duoidal category.*

Proof. We have already shown that $(QA\text{-mod}, \otimes_A, \mathbb{I}_{\otimes_A})$ and $(QA\text{-mod}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}})$ are monoidal structures. We concentrate now our attention in the description of the interchange law as presented in Definition 6.7: for sheaves $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in QA$-mod we need to define:

$$\zeta_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}} : (\mathcal{A} \mathbin{\widetilde{\boxtimes}} \mathcal{B}) \otimes_A (\mathcal{C} \mathbin{\widetilde{\boxtimes}} \mathcal{D}) \to (\mathcal{A} \otimes_A \mathcal{C}) \mathbin{\widetilde{\boxtimes}} (\mathcal{B} \otimes_A \mathcal{D}).$$

Since $p_1^*(\mathcal{A} \otimes_{\mathcal{O}_A} \mathcal{C}) = p_1^*\mathcal{A} \otimes_{\mathcal{O}_{A\times A}} p_1^*\mathcal{C}$ and $p_2^*(\mathcal{B} \otimes_{\mathcal{O}_A} \mathcal{D}) = p_2^*\mathcal{B} \otimes_{\mathcal{O}_{A\times A}} p_2^*\mathcal{D}$, if we write $p_1^*\mathcal{A} = \mathcal{L}$, $p_2^*\mathcal{B} = \mathcal{R}$, $p_1^*\mathcal{C} = \mathcal{M}$, $p_2^*\mathcal{D} = \mathcal{N}$, we need to check that there is a natural morphism

$$\begin{tikzcd}
\zeta : s_*(\mathcal{L} \otimes_{\mathcal{O}_{A\times A}} \mathcal{R}) \otimes_{\mathcal{O}_A} s_*(\mathcal{M} \otimes_{\mathcal{O}_{A\times A}} \mathcal{N}) \arrow[r] \arrow[dr] & s_*(\mathcal{L} \otimes_{\mathcal{O}_{A\times A}} \mathcal{M} \otimes_{\mathcal{O}_{A\times A}} \mathcal{R} \otimes_{\mathcal{O}_{A\times A}} \mathcal{N}) \arrow[d, "\cong"] \\
 & s_*(\mathcal{L} \otimes_{\mathcal{O}_{A\times A}} \mathcal{R} \otimes_{\mathcal{O}_{A\times A}} \mathcal{M} \otimes_{\mathcal{O}_{A\times A}} \mathcal{N})
\end{tikzcd}$$

The existence of this map follows from the general fact that in the context above if $\mathcal{X}, \mathcal{Y}$ are sheaves on $A \times A$ then, due to the existence of the morphism of algebras $s^\sharp : \mathcal{O}_A \to s_*\mathcal{O}_{A\times A}$, there is a natural map $s_*\mathcal{X} \otimes_{\mathcal{O}_A} s_*\mathcal{Y} \to s_*\mathcal{X} \otimes_{s_*\mathcal{O}_{A\times A}} s_*\mathcal{Y} = s_*(\mathcal{X} \otimes_{\mathcal{O}_{A\times A}} \mathcal{Y})$.

The $\widetilde{\boxtimes}$–comonoidal structure $\Delta_{\otimes_A} : \mathbb{I}_{\otimes_A} \to \mathbb{I}_{\otimes_A} \mathbin{\widetilde{\boxtimes}} \mathbb{I}_{\otimes_A}$ is $\Delta_{\otimes_A} := s^\sharp : \mathcal{O}_A \to s_*\mathcal{O}_{A\times A} = \mathcal{O}_A \mathbin{\widetilde{\boxtimes}} \mathcal{O}_A$ and the $\otimes_A$–monoidal structure $\mu_{\widetilde{\boxtimes}} : \mathbb{I}_{\widetilde{\boxtimes}} \otimes_A \mathbb{I}_{\widetilde{\boxtimes}} \to \mathbb{I}_{\widetilde{\boxtimes}}$ is the map associated to the structure of $\mathcal{O}_A$–algebra in $\operatorname{skysc}_0(\Bbbk)$. Finally, the map $\varepsilon_{\mathbb{I}_{\times_A}} = u_{\mathbb{I}_{\widetilde{\boxtimes}}} : \mathcal{O}_A \to \operatorname{skysc}_0(\Bbbk)$ is defined by the multiplication of an element of $\mathcal{O}_A$ by the unit element of $\operatorname{skysc}_0(\Bbbk)$.

The proofs of the associativity, unitality and counitality of ζ are easy exercises and therefore are omitted.

The fact that $(QA\text{-alg}, \otimes_A, \mathbb{I}_{\otimes_A}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}})$ is also a duoidal category follows easily from Proposition 6.26. □

The definitions of the duoidal structures on the categories $\mathrm{Sch}\,|_{\mathrm{sqc}}A \subset \mathrm{Sch}\,|_{\mathrm{qc}}A$ and QA-alg $\subset QA$-mod are tailored to give the following result.

Theorem 6.46. *Let A be an abelian variety. Then:*

(1) The functor $\mathcal{P} : (\mathrm{Sch}\,|_{\mathrm{sqc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A}) \to (QA\text{-alg}^{\mathrm{op}}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}}, \otimes_A, \mathbb{I}_{\otimes_A})$ is bilax monoidal. More precisely, $\mathcal{P} : (\mathrm{Sch}\,|_{\mathrm{sqc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}) \to (QA\text{-alg}^{\mathrm{op}}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}})$ is strong monoidal, $\mathcal{P} : (\mathrm{Sch}\,|_{\mathrm{sqc}}A, \times_A, \mathbb{I}_{\times_A}) \to (QA\text{-alg}^{\mathrm{op}}, \otimes_A, \mathbb{I}_{\otimes_A})$ is colax monoidal, and the conditions of interchange and unitality in the definition 6.22 are satisfied.

(2) The functor $\operatorname{Spec} : (QA\text{-mod}^{\mathrm{op}}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}}, \otimes_A, \mathbb{I}_{\otimes_A}) \to (\mathrm{Sch}\,|_{\mathrm{sqc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$ is bilax monoidal — in this case, $\operatorname{Spec} : (QA\text{-alg}^{\mathrm{op}}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}}) \to (\mathrm{Sch}\,|_{\mathrm{sqc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}})$ is lax monoidal and $\operatorname{Spec} : (QA\text{-alg}^{\mathrm{op}}, \otimes_A, \mathbb{I}_{\otimes_A}) \to (\mathrm{Sch}\,|_{\mathrm{sqc}}A, \times_A, \mathbb{I}_{\times_A})$ is strong monoidal.

(3) The adjunction $\mathrm{Sch}|_{\mathrm{sqc}}S \underset{\mathrm{Spec}}{\overset{\mathcal{P}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (QS\text{-alg})^{\mathrm{op}}$ $\mathcal{P} \dashv \mathrm{Spec}$ *is bimonoidal — that is the unit and counit with respect to both structures are (lax) monoidal morphisms.*

PROOF. Let $x : X \to A$, $y : Y \to A \in \mathrm{Sch}|_{\mathrm{sqc}}A$ and $x \times y : X \times Y \to A \times A$. In this situation and in accordance with Lemma 6.38, there is an isomorphism $\Gamma^{-1}_{\mathcal{O}_X,\mathcal{O}_Y} : (x\times y)_*(\mathcal{O}_{X\times Y}) = (x\times y)_*(\mathcal{O}_X \boxtimes \mathcal{O}_Y) \to x_*(\mathcal{O}_X) \boxtimes y_*(\mathcal{O}_Y) = \mathcal{P}(x) \boxtimes \mathcal{P}(y)$ — the existence of such an isomorphism is due to the fact that $\mathcal{O}_X, \mathcal{O}_Y$ are quasi-coherent, flat sheaves over $\mathrm{Spec}(\Bbbk)$ and x, y are separated, quasi-compact morphisms (see Remark 6.37). Then pushing forward the above isomorphism by $s : A \times A \to A$ we have $s_*(\Gamma^{-1}_{\mathcal{O}_X,\mathcal{O}_Y}) := s_*(x\times y)_*(\mathcal{O}_{X\times Y}) = (x \mathbin{\widetilde{\times}} y)_*(\mathcal{O}_{X\times Y}) \longrightarrow s_*\big(\mathcal{P}(x) \boxtimes \mathcal{P}(y)\big)$.

In other words we have a natural isomorphism:

$$s_*(\Gamma^{-1}_{\mathcal{O}_X,\mathcal{O}_Y}) : \mathcal{P}(x \mathbin{\widetilde{\times}} y) \longrightarrow \mathcal{P}(x) \mathbin{\widetilde{\boxtimes}} \mathcal{P}(y).$$

This guarantees that $\mathcal{P}$ is strong (and hence lax) monoidal in the category $QA\text{-alg}^{\mathrm{op}}$ with respect to $\widetilde{\times}$ and $\widetilde{\boxtimes}$. For the second assertion take $p : X \times_A Y \to X$ that induces a morphism $p^\sharp : \mathcal{O}_X \to p_*(\mathcal{O}_{X\times_A Y})$. Then, $x_*(p^\sharp) : x_*(\mathcal{O}_X) = \mathcal{P}(x) \to x_*p_*(\mathcal{O}_{X\times_A Y}) = (x \times_A y)_*(\mathcal{O}_{X\times_A Y}) = \mathcal{P}(x \times_A y)$. This map together with the map corresponding to y yields a morphism

$$\mathcal{P}(x) \otimes_A \mathcal{P}(y) \to \mathcal{P}(x \times_A y) \in QA\text{-alg},$$

the required condition of colax in the category $QA\text{-alg}^{\mathrm{op}}$ with respect to $(\times_A, \otimes_A)$. The proofs of the conditions in Definition 6.22[(1)(a),(b)] for $\mathcal{P}$ is direct and left to the reader.

It is well known that the functor Spec is strong monoidal with respect to $\otimes_A$ and $\times_A$ (see [31][Prop. 1.4.6]). The lax monoidality with respect to $\widetilde{\boxtimes}$ and $\widetilde{\times}$ follows by "doctrinal adjunction" from the fact that it is the right adjoint of the colax monoidal functor $\mathcal{P}$ (See Remark 6.47).

Assertion (3) is clearly a consequence of (1) and (2). ☐

Remark 6.47. (1) The version of "doctrinal adjunction" that we are using can be stated as: let $L \dashv R : \mathcal{C} \xrightarrow{L} \mathcal{D} \xleftarrow{R} \mathcal{C}$ be an adjunction with $\mathcal{C}$ and $\mathcal{D}$ monoidal categories. Then $L : \mathcal{C} \to \mathcal{D}$ can be endowed with a structure of colax monoidal functor if and only if $R : \mathcal{D} \to \mathcal{C}$ can be endowed with a structure of lax monoidal functor that makes (L, R) a monoidal adjunction. In any case each structure can be uniquely determined from the other. See [40] for the basic results on this subject and for example [1][Prop. 3.84] for a direct proof.

(2) In explicit terms, if $\mathcal{F}, \mathcal{F}' \in QA\text{-alg}$, then the lax monoidality of the functor Spec (see Theorem 6.46) is implemented *via* the natural transformation

$$\eta_{\mathrm{Spec}(\mathcal{F}) \mathbin{\widetilde{\times}} \mathrm{Spec}(\mathcal{F}')} : \mathrm{Spec}(\mathcal{F}) \mathbin{\widetilde{\times}} \mathrm{Spec}(\mathcal{F}') \to \mathrm{Aff}_A\big(\mathrm{Spec}(\mathcal{F}) \mathbin{\widetilde{\times}} \mathrm{Spec}(\mathcal{F}')\big) = \mathrm{Spec}\big(\mathcal{F} \mathbin{\widetilde{\boxtimes}} \mathcal{F}'\big).$$

See also Proposition 6.12.

6.5. **Bimonoid sheaves and schemes of monoids over** A.

Once we have established in Theorem 6.46 the adjunction between $\mathrm{Sch}|_{\mathrm{sqc}}A$ and $QA\text{-mod}^{\mathrm{op}}$, the notion of *sheaf of bimonoids* as a bimonoid in the duoidal category of separable, quasi-coherent A-alg, will appear as the natural counterpart of the notion of affine — or quasi-compact separable — bimonoid extension of A. In this section we set the basic result for bimonoids, namely that $\mathcal{P}$ and Spec establish an adjunction between the categories of quasi-compact separable bimonoids over A (i.e. bimonoids in $\mathrm{Sch}\,|_{\mathrm{sqc}}A$) and bimonoids in QA-alg. The additional structure of the inversion morphism for a "group extension", can also be added in a compatible fashion in order to extend the result to group extensions.

We begin by displaying in explicit terms, the definition of bimonoid in the duoidal category $QA\text{-mod}^{\mathrm{op}}$ (see Theorem 6.46) — we use the notations of Proposition 6.45.

Definition 6.48. A *sheaf of bimonoids* or a *bimonoid sheaf* on A is a sheaf $\mathcal{B} \in QA$-mod equipped with four sheaf morphisms $\Delta_{\mathcal{B}} : \mathcal{B} \to \mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{B}$, $\mu_{\mathcal{B}} : \mathcal{B} \otimes_A \mathcal{B} \to \mathcal{B}$, $\varepsilon_{\mathcal{B}} : \mathcal{B} \to \mathbb{I}_{\widetilde{\boxtimes}}$, $u_{\mathcal{B}} : \mathbb{I}_{\otimes_A} \to \mathcal{B}$ that make commutative the diagrams below:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\Delta_{\mathcal{B}}"] \arrow[d, "\Delta_{\mathcal{B}}"'] & \mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{B} \arrow[d, "\Delta_{\mathcal{B}} \,\widetilde{\boxtimes}\, \mathrm{id}"] \\
\mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{B} \arrow[r, "\mathrm{id} \,\widetilde{\boxtimes}\, \Delta_{\mathcal{B}}"] & \mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B} & \mathcal{B} \otimes_A \mathcal{B} \arrow[l, "\mu_{\mathcal{B}}"'] \\
\mathcal{B} \otimes_A \mathcal{B} \arrow[u, "\mu_{\mathcal{B}}"] & \mathcal{B} \otimes_A \mathcal{B} \otimes_A \mathcal{B} \arrow[l, "\mathrm{id} \otimes_A \mu_{\mathcal{B}}"'] \arrow[u, "\mu_{\mathcal{B}} \otimes_A \mathrm{id}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dl, leftrightarrow, "\cong"'] \arrow[d, "\Delta_{\mathcal{B}}"] \arrow[dr, leftrightarrow, "\cong"] & \\
\mathcal{B} \,\widetilde{\boxtimes}\, \mathbb{I}_{\widetilde{\boxtimes}} & \mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{B} \arrow[l, "\mathrm{id} \otimes \varepsilon_{\mathcal{B}}"] \arrow[r, "\varepsilon_{\mathcal{B}} \otimes \mathrm{id}"'] & \mathbb{I}_{\widetilde{\boxtimes}} \,\widetilde{\boxtimes}\, \mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{B} \arrow[dl, leftrightarrow, "\cong"'] \arrow[dr, leftrightarrow, "\cong"] & \\
\mathcal{B} \otimes_A \mathbb{I}_{\otimes_A} \arrow[r, "\mathrm{id} \otimes u_{\mathcal{B}}"'] & \mathcal{B} \otimes_A \mathcal{B} \arrow[u, "\mu_{\mathcal{B}}"] & \mathbb{I}_{\otimes_A} \otimes \mathcal{B} \arrow[l, "u_{\mathcal{B}} \otimes \mathrm{id}"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dr, "\Delta_{\mathcal{B}}"] & \\
\mathcal{B} \otimes_A \mathcal{B} \arrow[ur, "\mu_{\mathcal{B}}"] \arrow[d, "\Delta_{\mathcal{B}} \otimes_A \Delta_{\mathcal{B}}"'] & & \mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{B} \\
(\mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{B}) \otimes_A (\mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{B}) \arrow[rr, "\zeta_{\mathcal{B},\mathcal{B},\mathcal{B},\mathcal{B}}"] & & (\mathcal{B} \otimes_A \mathcal{B}) \,\widetilde{\boxtimes}\, (\mathcal{B} \otimes_A \mathcal{B}) \arrow[u, "\mu_{\mathcal{B}} \,\widetilde{\boxtimes}\, \mu_{\mathcal{B}}"']
\end{tikzcd}$$

Putting together the above results (see Proposition 6.12, Lemma 6.23, Remark 6.31 and Theorem 6.46) we obtain the following consequence.

Proposition 6.49. *The functor* $\mathcal{P} : \mathrm{Sch}\,|_{\mathrm{sqc}}A \to QA\text{-alg}^{\mathrm{op}}$*, takes bimonoids in* $(\mathrm{Sch}\,|_{\mathrm{sqc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$ *into bimonoids in* $(QA\text{-mod}^{\mathrm{op}}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}}, \otimes_A, \mathbb{I}_{\otimes_A})$*. Similarly, the functor* Spec *takes bimonoids in* $(QA\text{-mod}^{\mathrm{op}}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}}, \otimes_A, \mathbb{I}_{\otimes_A})$ *into affine bimonoids in the category of* $\mathrm{Sch}\,|_{\mathrm{sqc}}A$.

Moreover, the adjunction $\begin{tikzcd} \mathrm{Sch}\,|_{\mathrm{sqc}}S \arrow[r, shift left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & (QS - alg)^{\mathrm{op}} \arrow[l, shift left, "\mathrm{Spec}"] \end{tikzcd}$ *restricts to an adjunction as below:*

$$\begin{tikzcd}
\mathrm{Bimon}(\mathrm{Sch}\,|_{\mathrm{sqc}}A) \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \mathrm{Bimon}\big((QA\text{-alg})^{\mathrm{op}}\big) \arrow[l, bend left, "\mathrm{Spec}"]
\end{tikzcd}.$$

In particular, the relative affinization over A (see Definition 6.32) takes bimonoids in $\mathrm{Sch}\,|_{\mathrm{sqc}}A$ *into bimonoids that are affine schemes over A.* □

Remark 6.50. In view of Remark 6.47, if $(\mathcal{B}, \Delta_{\mathcal{B}}, \mu_{\mathcal{B}}, \varepsilon_{\mathcal{B}}, u_{\mathcal{B}})$ is a bimonoid in QA-alg, then the product $m : \operatorname{Spec}(\mathcal{B}) \mathbin{\widetilde{\times}} \operatorname{Spec}(\mathcal{B}) \to \operatorname{Spec}(\mathcal{B})$ is obtained as

$$\begin{tikzcd} \operatorname{Spec}(\mathcal{B}) \mathbin{\widetilde{\times}} \operatorname{Spec}(\mathcal{B}) \arrow[r, "\eta_{\operatorname{Spec}(\mathcal{B}) \widetilde{\times} \operatorname{Spec}(\mathcal{B})}"] \arrow[dr, "m"'] & \operatorname{Aff}_A\big(\operatorname{Spec}(\mathcal{B}) \mathbin{\widetilde{\times}} \operatorname{Spec}(\mathcal{B})\big) = \operatorname{Spec}(\mathcal{B} \boxtimes \mathcal{B}) \arrow[d, "\operatorname{Spec}(\Delta_B)"] \\ & \operatorname{Spec}(\mathcal{B}) \end{tikzcd}$$

As an immediate consequence of Proposition 6.49, we have the following.

Theorem 6.51. *Let A be an abelian variety. Then the functors $\mathcal{P}$ and* Spec *establish a contravariant isomorphism between $\mathrm{MM}|_{\mathrm{aff}} A$ and the category of sheaves of bimonoids on A (see Notation 6.13 and Definition 6.48) with arrows the sheaf morphisms of bimonoids.*

Proof. By Proposition 6.49, we have that if $q_M : M \to A$ is a morphism of monoid schemes — that is, a bimonoid in $\mathrm{Sch}|_{\mathrm{sqc}} A$, see Proposition 6.12 —, then $\mathcal{P}(q_M)$ is a sheaf of bimonoids, and if $f : (q_M : M \to A) \to (q_N : N \to A)$ is a morphism of bimonoids, then $\mathcal{P}(f) : \mathcal{P}(q_M) \to \mathcal{P}(q_N)$ is a morphism of sheaves of bimonoids. Conversely, if $\mathcal{H}$ is a sheaf of bimonoids then $\operatorname{Spec} H \to A$ is an affine morphism of monoid schemes, and if $f : \mathcal{H} \to \mathcal{H}'$ is a morphism of sheaves of bimonoids, then $\operatorname{Spec}(f) : \operatorname{Spec}(\mathcal{H}) \to \operatorname{Spec}(\mathcal{H}')$ is a morphism of bimonoids in $\mathrm{Sch}|_{\mathrm{qc}} A$. Notice that $\operatorname{Spec}(f)$ is in particular a morphism of affine schemes over A.

On the other hand, since the objects of $\mathrm{MM}|_{\mathrm{aff}} A$ are affine morphisms $q_M : M \to A$, clearly $\mathrm{MM}|_{\mathrm{aff}} A$ is isomorphic to a subcategory of $\mathrm{Sch}|_{\mathrm{aff}} A$, that we also denote $\mathrm{MM}|_{\mathrm{aff}} A$. Since $\mathcal{P}$ and Spec induce a contravariant isomorphism between $\mathrm{Sch}|_{\mathrm{aff}} A$ and QA-alg, in order to prove that $\mathcal{P}|_{\mathrm{MM}|_{\mathrm{aff}} A} : \mathrm{MM}|_{\mathrm{aff}} A \to \mathrm{Bimon}(QA\text{-alg})^{\mathrm{op}}$ we can use an elementary result on reflections of monoidal categories, that we added below for lack of an adequate reference (see Remark 6.53). ☐

Remark 6.52. The reader should be aware that, as we pointed out in the beginning of Section 6.1, in order to define the subcategory $\mathrm{MM}|_{\mathrm{aff}} A \subset \mathrm{Sch}|_{\mathrm{aff}} A$ (see Theorem 6.51), we need to work in the category $\mathrm{Sch}|_{\mathrm{qc}} A$, since $s \circ (q_M, q_M) : M \times M \to A$ is *not* an affine scheme over A.

Remark 6.53. (1) Assume that $\mathcal{C}_0 \subseteq \mathcal{C}$ is a pair of categories with the following additional conditions:

(a) $\mathcal{C}_0$ is a full and replete subcategory of $\mathcal{C}$;

(b) $\mathcal{C}$ is monoidal with structure $(\times, \mathbb{I})$ and $\mathcal{C}_0$ is monoidal with structure $(\times_0, \mathbb{I}_0)$;

(c) The inclusion functor $\mathrm{inc} : \mathcal{C}_0 \to \mathcal{C}$ has a left adjoint $\mathcal{A} : \mathcal{C} \to \mathcal{C}_0$ with counit $\varepsilon : \mathcal{A} \circ \mathrm{inc} \Rightarrow \mathrm{id}_{\mathcal{C}_0}$ and unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{inc} \circ \mathcal{A}$ such that: ε is an isomorphism and η is strong monoidal, i.e. $\mathcal{A}x \times_0 \mathcal{A}y \cong \mathcal{A}(x \times y)$ for all $x, y \in \mathcal{C}$ and $\mathbb{I}_0 \cong \mathcal{A}(\mathbb{I})$.

Then, in the above situation $\mathrm{Mon}(\mathcal{C}) \cap \mathcal{C}_0 \cong \mathrm{Mon}(\mathcal{C}_0)$.

The proof of the assertion above is easy: for $(x, m) \in \mathrm{Mon}(\mathcal{C}) \cap \mathcal{C}_0$, we consider the counit $\eta_{x \times x} : x \times x \to \mathcal{A}(x \times x) = \mathcal{A}(x) \times_0 \mathcal{A}(x) = x \times_0 x$. Then, the structure morphism $m : x \times x \to x$ can be uniquely extended as in the diagram below:

$$\begin{tikzcd} x \times x \arrow[rr, "\eta_{x\times x}"] \arrow[dr, "m"'] & & x \otimes_0 x \arrow[dl, "\widehat{m}"] \\ & x & \end{tikzcd}$$

It is clear that given a monoid structure in $\mathcal{C}_0$ such as $\widehat{m}$ the monoid in $\mathcal{C}$ is obtained by composition with the unit.

(2) Thus, we complete the proof of Theorem 6.51 by considering in (1) above the categories $\mathcal{C} = \mathrm{Sch}\,|_{\mathrm{sqc}}A$ and $\mathcal{C}_0 = \mathrm{MM}\,|_{\mathrm{aff}}A$ (or $\mathrm{Sch}\,|_{\mathrm{aff}}S$) and $\mathcal{A}$ the affinization functor, we complete the.

(3) In particular, if $\mathcal{B}$ is a sheaf of bimonoids in A with coproduct $\Delta_{\mathcal{B}}$, then the product in $\mathrm{Spec}\big(\mathcal{P}(m)\big)$ is obtained as

$$m = \mathrm{Aff}_A(m) = \mathrm{Spec}\big(\mathcal{P}(m)\big) \circ \eta_{M \,\widetilde{\times}\, M} = \mathrm{Spec}(\Delta_{\mathcal{B}_M}) \circ \eta_{\mathrm{Spec}(\mathcal{B}_M) \,\widetilde{\times}\, \mathrm{Spec}(\mathcal{B}_M)}.$$

6.6. **Affine extensions of abelian varieties and Hopf sheaves.**

To finish our considerations on this topic, we define — given an abelian variety A — the concept of Hopf sheaf on A and show the category of commutative Hopf sheaves and its morphisms is op–equivalent with the category $\mathrm{GM}\,|_{\mathrm{aff}}A$ of affine morphisms of group schemes (see Definition 6.19).

Recall that if we call $\mathrm{op} : A \to A$ the map given by the inverse morphism in A, the antipode of $x : X \to A$ in the duoidal category $\mathrm{Sch}\,|_{\mathrm{sqc}}A$ is a morphism $\iota_x : x \to \mathrm{op}_*(x)$ that fits in the commutative diagrams (6.2), (6.3) (see Theorem 6.18 and Remark 6.20). The situation is analogue in QA-alg.

Notation 6.54. Let $\mathrm{op} : A \to A$ be the morphism given by the inversion map in the abelian variety A and consider the push-forward functor $\mathrm{op}_* : QA\text{-alg} \to QA\text{-alg}$. We denote $\mathrm{op}_*(\mathcal{F}) = -\mathcal{F}$ and similarly for an arrow $F : \mathcal{F} \to \mathcal{G}$ we denote $\mathrm{op}_*(F : \mathcal{F} \to \mathcal{G}) = (-F : -\mathcal{F} \to -\mathcal{G})$.

Notice that since the inversion map is an involution, then $-(-\mathcal{F}) = \mathcal{F}$.

Remark 6.55. In order to fix notation, we recall the following easy properties of the functor op:

(1) $\mathrm{op}_* = \mathrm{op}^* : QA\text{-alg} \to QA\text{-alg}$;

(2) The diagrams below are commutative:

$$\begin{tikzcd} \mathrm{Sch}\,|_{\mathrm{sqc}}A \arrow[r, "\mathcal{P}"] \arrow[d, "\mathrm{op}_*"'] & QA\text{-alg} \arrow[d, "\mathrm{op}_*"] \\ \mathrm{Sch}\,|_{\mathrm{sqc}}A \arrow[r, "\mathcal{P}"'] & A\text{-alg} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sch}\,|_{\mathrm{sqc}}A \arrow[d, "\mathrm{op}_*"'] & QA\text{-alg} \arrow[l, "\mathrm{Spec}"'] \arrow[d, "\mathrm{op}_*"] \\ \mathrm{Sch}\,|_{\mathrm{sqc}}A & A\text{-alg}\,. \arrow[l, "\mathrm{Spec}"] \end{tikzcd}$$

(3) In the situation above we consider the morphisms $A \xrightarrow{\mathrm{st}} \mathrm{Spec}(\Bbbk) \xrightarrow{0} A$ (see Definition 6.3) and the associated adjunctions $\begin{tikzcd} \Bbbk-\mathrm{alg} \arrow[r, shift left, "\mathrm{st}^*"] \arrow[r, phantom, "\perp"] & (QA\text{-alg})^{\mathrm{op}} \arrow[l, shift left, "\mathrm{st}_*"] \end{tikzcd}$ and

$$\begin{tikzcd} (QA\text{-alg})^{\mathrm{op}} \arrow[r, shift left, "0^*"] \arrow[r, phantom, "\perp"] & \Bbbk-\mathrm{alg} \arrow[l, shift left, "0_*"] \end{tikzcd}.$$

The adjunctions defined in Remark 6.55 have the following properties, analogous to the situation in Lemma 6.15 and Proposition 6.16.

Remark 6.56. (1) Consider the following pull back diagram and the corresponding diagram of functors:

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\mathrm{st}"'] & A\times A \arrow[d, "s(\mathrm{id}\times\mathrm{op})"] \\ \mathrm{Spec}(\Bbbk) \arrow[r, "0"] & A \end{tikzcd} \qquad \begin{tikzcd} A\text{-mod} \arrow[r, "0^*"] \arrow[d, "(s(\mathrm{id}\times\mathrm{op}))^*"'] & \mathrm{Spec}(\Bbbk)-\mathrm{mod} \arrow[d, "\mathrm{st}^*"] \\ A\times A-\mathrm{mod} \arrow[r, "\delta^*"] & A\text{-mod}\,. \end{tikzcd}$$

Evaluating at $\mathcal{F}\,\widetilde{\boxtimes}\,-\mathcal{G} := \big(s(\mathrm{id}\times\mathrm{op})\big)_*(\mathcal{F}\boxtimes\mathcal{G})$ we obtain a natural transformation in $\mathcal{F},\mathcal{G}$: $\mathrm{st}^*\,0^*(\mathcal{F}\,\widetilde{\boxtimes}\,-\mathcal{G}) = \delta^*\big(s(\mathrm{id}\times\mathrm{op})\big)^*\big(s(\mathrm{id}\times\mathrm{op})\big)_*(\mathcal{F}\boxtimes\mathcal{G}) \to \delta^*(\mathcal{F}\boxtimes\mathcal{G}) \cong \mathcal{F}\otimes_A\mathcal{G}$, the penultimate arrow coming from the unit of the corresponding adjunction and the last equality follows from general properties of the external tensor product (see Definition 6.40). Indeed, it is well known that in the case of a morphism $f: X\to Y$ and a pair of sheaves $\mathcal{F},\mathcal{G}\in QY-\mathrm{mod}$ we have: $f^*(\mathcal{F}\otimes_Y\mathcal{G}) \cong f^*(\mathcal{F})\otimes_X f^*(\mathcal{G})$ (see [59, Theorem 16.3.7]). In the case that we are dealing with the situation of $\delta: A\to A\otimes A$ and $\mathcal{F},\mathcal{G}\in QA-\mathrm{mod}$, $\delta^*(\mathcal{F}\,\widetilde{\boxtimes}\,\mathcal{G}) = \delta^*(p_1^*\mathcal{F}\otimes_{A\times A}p_2^*\mathcal{G}) = \delta^*p_1^*\mathcal{F}\otimes_A\delta^*p_2^*\mathcal{G} = \mathcal{F}\otimes_A\mathcal{G}$.
(2) For $R\in\Bbbk-\mathrm{alg}$ we have that: $\mathrm{st}^*R = 0_*R\,\widetilde{\boxtimes}\,\mathbb{I}_{\otimes_A}$.

Proposition 6.57. *Assume that $\mathcal{F},\mathcal{G}$ are sheaves in QA-alg and recall the notation $\mathrm{op}_*\mathcal{G} = -\mathcal{G}$. Then we can define two natural transformations as below:*
(1) $\widetilde{\gamma}_{\mathcal{F},\mathcal{G}} : (\mathcal{F}\,\widetilde{\boxtimes}\,\mathcal{G})\,\widetilde{\boxtimes}\,\mathbb{I}_{\otimes_A} \to \mathcal{F}\otimes_A -\mathcal{G}$;
(2) $\overline{\gamma}_{\mathcal{F},\mathcal{G}} : \mathbb{I}_{\otimes_A}\,\widetilde{\boxtimes}\,\mathcal{F}\,\widetilde{\boxtimes}\,\mathcal{G} \to -\mathcal{F}\otimes_A\mathcal{G}$.

Proof. We sketch the proof of (1), the proof of (2) being similar. Using the first result of the Remark 6.56 we deduce the existence of a natural transformation $\mathrm{st}^*\,0^*(\mathcal{F}\,\widetilde{\boxtimes}\,-\mathcal{G}) \to \mathcal{F}\otimes_A\mathcal{G}$ — we are using that $-(-\mathcal{G}) = \mathcal{G}$. Using now the second result of the mentioned remark we transform the above to: $0_*0^*(\mathcal{F}\,\widetilde{\boxtimes}\,-\mathcal{G})\,\widetilde{\boxtimes}\,\mathbb{I}_{\otimes_A} \to \mathcal{F}\otimes_A\mathcal{G}$, and then using the adjunction $0^*\dashv 0_*$ we obtain a natural transformation $\gamma_{\mathcal{F},\mathcal{G}} : (\mathcal{F}\,\widetilde{\boxtimes}\,-\mathcal{G})\,\widetilde{\boxtimes}\,\mathbb{I}_{\otimes_A} \to \mathcal{F}\otimes_A\mathcal{G}$. □

We are ready to define *Hopf sheaf on the abelian variety A*. We will use the nomenclature summarized in Definition 6.48.

Definition 6.58. Assume that $\mathcal{H}$ is a sheaf of bimonoids on A (see Definition 6.48). We say that $\mathcal{H}$ is a *Hopf sheaf* if there is a sheaf homomorphism $\sigma_{\mathcal{H}} : -\mathcal{H}\to\mathcal{H}$ — called *the antipode* — such the diagrams below are commutative.

$$\begin{tikzcd}[column sep=small] & \mathcal{H}\otimes_A\mathcal{H} \arrow[dl, "\mu_{\mathcal{H}}"'] & \mathcal{H}\otimes_A -\mathcal{H} \arrow[l, "\mathrm{id}\otimes_A\sigma_{\mathcal{H}}"'] & (\mathcal{H}\,\widetilde{\boxtimes}\,\mathcal{H})\,\widetilde{\boxtimes}\,\mathbb{I}_{\otimes_A} \arrow[l, "\widetilde{\gamma}_{\mathcal{H},\mathcal{H}}"'] & \\ \mathcal{H} & & & & \mathcal{H}\,\widetilde{\boxtimes}\,\mathbb{I}_{\otimes_A} \arrow[ul, "\Delta_{\mathcal{H}}\,\widetilde{\boxtimes}\,\mathrm{id}"'] \arrow[dl, "\varepsilon_{\mathcal{H}}\,\widetilde{\boxtimes}\,\mathrm{id}"] \\ & \mathbb{I}_{\otimes_A} \arrow[ul, "u_{\mathcal{H}}"] \arrow[rr, dash, "\cong"] & & \mathbb{I}_{\widetilde{\boxtimes}}\,\widetilde{\boxtimes}\,\mathbb{I}_{\otimes_A} & \end{tikzcd} \tag{6.4}$$

$$
\begin{tikzcd}
 & \mathcal{H} \otimes_A \mathcal{H} \arrow[dl, "\mu_{\mathcal{H}}"'] & -\mathcal{H} \otimes_A \mathcal{H} \arrow[l, "\sigma_{\mathcal{H}} \otimes_A \mathrm{id}"'] & \mathbb{I}_{\otimes_A} \widetilde{\boxtimes} (\mathcal{H} \widetilde{\boxtimes} \mathcal{H}) \arrow[l, "\overline{\gamma}_{\mathcal{H},\mathcal{H}}"'] & \\
\mathcal{H} & & & & \mathbb{I}_{\otimes_A} \widetilde{\boxtimes} \mathcal{H} \arrow[ul, "\mathrm{id} \widetilde{\boxtimes} \Delta_{\mathcal{H}}"'] \arrow[dl, "\mathrm{id} \widetilde{\boxtimes} \varepsilon_{\mathcal{H}}"] \\
 & \mathbb{I}_{\otimes_A} \arrow[ul, "u_{\mathcal{H}}"] \arrow[rr, dash, "\cong"] & & \mathbb{I}_{\otimes_{\mathbb{A}}} \widetilde{\boxtimes} \mathbb{I}_{\widetilde{\boxtimes}} &
\end{tikzcd}
\tag{6.5}
$$

where $\widetilde{\gamma}$ and $\overline{\gamma}$ are the natural transformations defined in Proposition 6.57 and the bottom maps $\cong$ are the natural identifications associated to the unit of the $\widetilde{\boxtimes}$ monoidal structure.

A Hopf sheaf $\mathcal{H}$ is *commutative* if $(\mathcal{H}, \mu_{\mathcal{H}}, u_{\mathcal{H}})$ is a sheaf of commutative $\mathcal{O}_A$–algebras, and a *flat Hopf sheaf* is a Hopf sheaf that is flat as sheaf of $\mathcal{O}_A$–modules — that is, the stalks $\mathcal{H}_a$ are $\mathcal{O}_{a,A}$–flat modules for all $a \in A$. A Hopf sheaf $\mathcal{H}$ is *faithful* if the canonical morphism $\mathcal{O}_A \to \mathcal{H}$ is injective — in other words, $\mathcal{H}(U)$ is a faithful representation of $\mathcal{O}_A(U)$.

As a summary we write down explicitly the conditions of a *Hopf sheaf* on an abelian variety A.

Summary 6.59. Let A be an abelian variety. A *commutative Hopf sheaf* on A is a sextuple $(\mathcal{H}, \Delta_{\mathcal{H}}, \varepsilon_{\mathcal{H}}, \mu_{\mathcal{H}}, u_{\mathcal{H}}, \sigma_{\mathcal{H}})$, where $(\mathcal{H}, \mu_{\mathcal{H}}, u_{\mathcal{H}})$ is a sheaf of quasi-coherent commutative $\mathcal{O}_A$–algebras (i.e. $\mathcal{H} \in QA$-alg) with multiplication $\mu_{\mathcal{H}}$ unit $u_{\mathcal{H}}$, and $\Delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \widetilde{\boxtimes} \mathcal{H}$, $\varepsilon_{\mathcal{H}} : \mathcal{H} \to \mathrm{skysc}_0(\Bbbk)$, $\sigma_{\mathcal{H}} : -\mathcal{H} \to \mathcal{H}$ are morphisms of sheaves satisfying the following additional conditions:

(1) The triple $(\mathcal{H}, \Delta_{\mathcal{H}}, \varepsilon_{\mathcal{H}})$ is a comonoid in $\big(QA\text{-alg}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}} = \mathrm{skysc}_0(\Bbbk)\big)$;

(2) $\Delta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \widetilde{\boxtimes} \mathcal{H}$ and $\varepsilon_{\mathcal{H}} : \mathcal{H} \to \mathrm{skysc}_0(\Bbbk)$ are morphisms of QA-alg, that is:

  (a) The morphism $\Delta_{\mathcal{H}}$ is such that the following diagrams are commutative:

$$
\begin{tikzcd}
 & \mathcal{H} \arrow[dr, "\Delta_{\mathcal{H}}"] & \\
\mathcal{H} \otimes_A \mathcal{H} \arrow[ur, "\mu_{\mathcal{H}}"] \arrow[d, "\Delta_{\mathcal{H}} \otimes_A \Delta_{\mathcal{H}}"'] & & \mathcal{H} \widetilde{\boxtimes} \mathcal{H} \\
(\mathcal{H} \widetilde{\boxtimes} \mathcal{H}) \otimes_A (\mathcal{H} \widetilde{\boxtimes} \mathcal{H}) \arrow[rr, "\zeta_{\mathcal{H},\mathcal{H},\mathcal{H},\mathcal{H}}"] & & (\mathcal{H} \otimes_A \mathcal{H}) \widetilde{\boxtimes} (\mathcal{H} \otimes_A \mathcal{H}) \arrow[u, "\mu_{\mathcal{H}} \widetilde{\boxtimes} \mu_{\mathcal{H}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}_{\otimes_A} \arrow[r, "\Delta_{\otimes_A}"] \arrow[d, "u_{\mathcal{H}}"'] & \mathbb{I}_{\otimes_A} \widetilde{\boxtimes} \mathbb{I}_{\otimes_A} \arrow[d, "u_{\mathcal{H}} \widetilde{\boxtimes} u_{\mathcal{H}}"] \\
\mathcal{H} \arrow[r, "\Delta_{\mathcal{H}}"] & \mathcal{H} \widetilde{\boxtimes} \mathcal{H}
\end{tikzcd}
$$

  (b) The morphism $\varepsilon_{\mathcal{H}}$ is such that the following diagrams are commutative:

$$
\begin{tikzcd}
\mathcal{H} \otimes_A \mathcal{H} \arrow[r, "\mu_{\mathcal{H}}"] \arrow[d, "\varepsilon_{\mathcal{H}}"'] & \mathcal{H} \arrow[d, "\varepsilon_{\mathcal{H}}"] \\
\mathbb{I}_{\widetilde{\boxtimes}} \otimes_A \mathbb{I}_{\widetilde{\boxtimes}} \arrow[r, "\mu_{\mathbb{I}_{\widetilde{\boxtimes}}}"'] & \mathbb{I}_{\widetilde{\boxtimes}}
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbb{I}_{\otimes_A} \arrow[r, "\varepsilon_{\mathbb{I}_{\otimes_A}}"] \arrow[d, "u_{\mathcal{H}}"'] & \mathbb{I}_{\widetilde{\boxtimes}} \arrow[d, "\mathrm{id}"] \\
\mathcal{H} \arrow[r, "\varepsilon_{\mathcal{H}}"'] & \mathbb{I}_{\widetilde{\boxtimes}}.
\end{tikzcd}
$$

(3) The antipode $\sigma_{\mathcal{H}} : -\mathcal{H} \to \mathcal{H}$ is a morphism in QA-mod — recall that $-\mathcal{H} = \mathrm{op}_*(\mathcal{H})$ where op_* is the functor in A-mod given by push-forward (or pull-back) by $a \mapsto -a : A \overset{\mathrm{op}}{\to} A$. Moreover, the antipode map, fits in the commutative diagrams (6.4), (6.5).

If moreover $\mathcal{H}$ is a flat $\mathcal{O}_A$–module, then we say that the sextuple is a *flat commutative Hopf sheaf*; if $\mathcal{O}_A \to \mathcal{H}$ is an injective morphism, then the sextuple is a *faithful commutative Hopf sheaf.*

Given the abelian variety A we define the category of Hopf sheaves in the natural manner.

Definition 6.60. If A is a given abelian variety and $\mathcal{H}, \mathcal{K}$ are flat commutative Hopf sheaves, a *morphism from $\mathcal{H}$ into $\mathcal{K}$* is simply a morphism of bimonoids in the duoidal category $(QA\text{-mod}, \otimes_A, \mathbb{I}_{\otimes_A}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}})$. Explicitly it is a morphism of sheaves $F : \mathcal{H} \to \mathcal{K}$ of $\mathcal{O}_A$-algebras, with the additional property that the diagrams below commute:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "F"] \arrow[d, "\Delta_{\mathcal{H}}"'] & \mathcal{K} \arrow[d, "\Delta_{\mathcal{K}}"] \\ \mathcal{H} \widetilde{\boxtimes} \mathcal{H} \arrow[r, "F \widetilde{\boxtimes} F"] & \mathcal{K} \widetilde{\boxtimes} \mathcal{K} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H} \arrow[rr, "F"] \arrow[dr, "\varepsilon_{\mathcal{H}}"'] & & \mathcal{K} \arrow[dl, "\varepsilon_{\mathcal{K}}"] \\ & \mathbb{I}_{\widetilde{\boxtimes}} & \end{tikzcd}$$

We call HQA-alg (resp. HQA_{f}-alg) the category whose objects are the commutative Hopf sheaves (resp. faithful commutative Hopf sheaves) on A and whose arrows are the morphisms of Hopf sheaves.

Remark 6.61. In the context considered above, the following two assertions can be proved.

(1) In the case that the antipode $\sigma_{\mathcal{H}}$ exists for the bimonoid $\mathcal{H}$, then it is unique — for example, this can be proved using the equivalence given by Theorem 6.62 below and the fact that the inverse morphism of a group scheme is unique.

(2) If $F : \mathcal{H} \to \mathcal{K}$ is a morphism of Hopf sheaves, then $\sigma_{\mathcal{K}} \circ (-F) = F \circ \sigma_{\mathcal{H}}$. In other words, a morphism of sheaves that are Hopf sheaves and that preserve the bimonoid structure, automatically preserves the antipode. The proof of this assertion is a consequence of (1).

The close relationship between the affine extensions of an abelian variety A and the commutative Hopf sheaves on A is expressed in the theorem that follows.

Theorem 6.62. *Let A be an abelian variety, and $\mathrm{GE}|_{\mathrm{aff}}A$ and HQA_{f}-alg the categories of affine extensions of A and faithful commutative Hopf sheaves of A respectively. Then, $\mathcal{P} : \mathrm{GE}|_{\mathrm{aff}}A \to (HQA_{\mathrm{f}}\text{-alg})^{\mathrm{op}}$ and $\mathrm{Spec} : (HQA_{\mathrm{f}}\text{-alg})^{\mathrm{op}} \to \mathrm{GE}|_{\mathrm{aff}}A$ constitute an adjoint equivalence between $\mathrm{GE}|_{\mathrm{aff}}A$ and HQA_{f}-alg.*

Proof. If $q : G \to A$ is an affine extension, then q is a surjective morphism and therefore the sheaf $\mathcal{P}(q)$ (see Definition 6.30) is a faithful sheaf of commutative $\mathcal{O}_A$–algebras. On the other hand, by Theorem 6.18 the inverse morphism $\iota_G : G \to G$ verifies the commutative diagrams (6.2) and (6.3). It follows by construction that $\sigma_{\mathcal{H}} = \mathcal{P}(\iota_G)$ satisfies commutative diagrams (6.4) and (6.5) for $\mathcal{H} = \mathcal{P}(q)$. Indeed, it is easy to check that $\mathcal{P}(\widetilde{\gamma}_{q,q}) = \widetilde{\gamma}_{\mathcal{H},\mathcal{H}}$ and $\mathcal{P}(\overline{\gamma}_{q,q}) = \overline{\gamma}_{\mathcal{H},\mathcal{H}}$ (see Remark 6.17 and Proposition 6.57), thus applying the functor $\mathcal{P}$ to the diagrams (6.2) and (6.3) we obtain the diagrams (6.4) and (6.5). Since $\mathcal{P}$ takes affine morphisms of monoids to sheaves of bimonoids, it follows that $\mathcal{P}(q)$ is a faithful commutative Hopf sheaf.

Conversely, if $\mathcal{H} \in HQA_{\mathrm{f}}$-alg, with antipode $\sigma_{\mathcal{H}}$, then $\operatorname{Spec}\mathcal{H}$: $q : M \to A$ is a bimonoid in $\mathrm{Sch}\big|_{\mathrm{sqc}} A$, with q a faithful affine morphism (of monoid schemes), by Theorem 6.51. Moreover, applying Spec to the commutative diagrams (6.4) and (6.5), we deduce that $\iota_q = \operatorname{Spec}(\sigma_{\mathcal{H}}) : q \to -q$ satisfies the commutative diagrams (6.2) and (6.3). In other words, M is a group scheme and q an affine extension of A (see Remark 2.16). $\square$

Notation 6.63. The following notation will be used in the future. Assume that $q : G \to A$ is an affine extension, then $\mathcal{P}(q)$ the associated Hopf sheaf of A-alg will be denoted as $\mathcal{H}_q := \mathcal{P}(q)$.

Notice that Theorem 6.62 implies in particular the following result.

Corollary 6.64. *Let $\mathcal{H}$ be a commutative Hopf sheaf on the abelian variety A. Then $\mathcal{H}$ is a flat sheaf if and only if $\mathcal{H}$ is faithful, if and only if the unit morphism $u_{\mathcal{H}}$ is monic.*

Proof. Indeed, since a flat morphism of schemes $f : X \to Y$, with Y Nœtherian is dominant, it follows from Proposition 6.34 that a flat commutative Hopf sheaf $\mathcal{H}$ is faithful. Conversely, if $\mathcal{H}$ is faithful, Theorem 6.62 implies that $\operatorname{Spec}(\mathcal{H})$: $q : G \to A$ is an affine extension, and therefore a flat morphism by Theorem 2.9.

Finally, notice that the unit morphism $u_{\mathcal{H}}$ is monic if and only if $\mathcal{O}_A(U) \to \mathcal{H}(U)$ is an inclusion for any (affine) open subset $U \subset A$. $\square$

Examples 6.65. (1) Let H be an affine group scheme, and consider the corresponding affine extension $1 \longrightarrow H \xrightarrow{\mathrm{id}} H \longrightarrow 0 \longrightarrow 0$. Then $\mathcal{H}$ is the Hopf algebra $\Bbbk[H]$ seen as a sheaf on $\{*\} = \operatorname{Spec}(\Bbbk)$.

Conversely, given a Hopf algebra R, then R can be seen as a Hopf sheaf on $\{*\} = \operatorname{Spec}(\Bbbk)$, and the affine group scheme $\operatorname{Spec}(R)$ induces the affine extension $1 \longrightarrow \operatorname{Spec}(R) \xrightarrow{\mathrm{id}} \operatorname{Spec}(R) \longrightarrow 0 \longrightarrow 0$.

(2) If A is an abelian variety, then the structure sheaf $\mathcal{O}_A$ is a faithful commutative Hopf sheaf on A; it corresponds to the trivial extension $0 \longrightarrow 0 \longrightarrow A \xrightarrow{\mathrm{id}} A \longrightarrow 0$.

(3) More generally, if R is a Hopf algebra then $\mathcal{R} = R \otimes_{\Bbbk} \mathcal{O}_A$ is a flat Hopf sheaf; $\mathcal{R}$ corresponds to the direct product:

$$\operatorname{Spec}(R) \times A : \ 1 \longrightarrow \operatorname{Spec}(R) \longrightarrow \operatorname{Spec}(R) \times A \xrightarrow{p_2} A \longrightarrow 0 \, .$$

Remark 6.66. (1) Since an affine extension $\mathcal{S} : \ 1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ is of finite type if and only if H is of finite type (as follows from descent theory, see [14, Proposition 2.6.5] and [32, Prop. 2.7.1]), it follows that G if of finite type if and only if $\mathcal{H}(U)$ is a finitely generated $\mathcal{O}_A(U)$–algebra for any affine open subset $U \subset A$.

(2) If $\mathcal{H}$ is a faithful commutative Hopf sheaf on A, then $G = \operatorname{Spec}(\mathcal{H})$ is an anti-affine group scheme if and only if $\mathcal{H}(A) = \Bbbk$. Indeed, if $q : G \to A$ is the associated morphism of quasi-compact group schemes, then $\mathcal{H}(A) = q_*(\mathcal{O}_G)(A) = \mathcal{O}_G(G)$.

6.7. Hopf ideals and affine subextensions.

In this section we present the expected generalizations on the relationship between ideals of a Hopf algebra H and closed subgroups of the affine algebraic group $\operatorname{Spec}(H)$, to the context of Hopf sheaves and affine extensions.

We begin by recalling some definitions and known results concerning the correspondence between closed subschemes of X and quasi-coherent sheaves of ideals on $\mathcal{O}_X$ (see for example [34, Proposition II.5.9] or [30, §4]).

Remark 6.67. Let X be a $\Bbbk$–scheme and $(i, i^{\#}) : Y \subseteq X$, a closed subscheme. Then we have a short exact sequence of quasi-coherent sheaves of $\Bbbk$–algebras

$$0 \longrightarrow \mathcal{I}_{X|Y} \longrightarrow \mathcal{O}_X \xrightarrow{i^{\#}} i_*(\mathcal{O}_Y) \longrightarrow 0, \tag{6.6}$$

were $\mathcal{I}_{X|Y}$ is a sheaf of ideals in $\mathcal{O}_X$. In this manner we obtain a bijective correspondence between quasi-coherent sheaves of ideals of $\mathcal{O}_X$ and closed subschemes of X. The inverse map — that we call $\mathfrak{V}$ —, sends an ideal $\mathcal{I} \subseteq \mathcal{O}_X$ into the closed subscheme of X given by $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$.

In the case of schemes over a $\Bbbk$-scheme S, we can push forward the short exact sequence (6.6), provided that we impose additional conditions on Y and X.

Definition 6.68. Let $(x : X \to S) \in \operatorname{Sch}|_{\mathrm{psqc}} S$. We define $\mathcal{C}(x)$ as the poset of closed subschemes of X in $\operatorname{Sch}|_{\mathrm{psqc}} S$ — that is, we consider $y : Y \to S \in \operatorname{Sch}|_{\mathrm{psqc}} S$ with $(i, i^{\#}) : y \to x$ a closed subscheme.

If $\mathcal{F} \in QS\text{-alg}$, we define $\mathcal{II}(\mathcal{F})$ as the poset of quasi-coherent sheaves of ideals of $\mathcal{F}$.

Lemma 6.69. *Let $(x : X \to S) \in \operatorname{Sch}|_{\mathrm{psqc}} S$ and $\mathcal{F} \in QS\text{-alg}$. Then:*
(1) If $(y : Y \to S) \in \mathcal{C}(x)$, then the sequence in the category $QS\text{-mod}$:

$$0 \longrightarrow x_*\big(\mathcal{I}_{X|Y}\big) \longrightarrow \mathcal{P}(x) \xrightarrow{\mathcal{P}(i)} \mathcal{P}(y) \longrightarrow 0,$$

is exact.
(2) The map $\mathfrak{I} : \mathcal{C}(x) \to \mathcal{II}\big(\mathcal{P}(x)\big)$ given by $\mathfrak{I}(y) = x_*(\mathcal{I}_{X|Y})$ is a contravariant functor between the domain and codomain posets.
(3) The map $\mathfrak{V} : \mathcal{II}(\mathcal{F}) \to \mathcal{C}\big(\operatorname{Spec}(\mathcal{F})\big)$ given by $\mathfrak{V}(\mathcal{I}) = \operatorname{Spec}(\mathcal{F}/\mathcal{I}) \subset \operatorname{Spec}(\mathcal{F})$ is a contravariant functor between the domain and codomain posets — recall that in this context $\operatorname{Spec}(\mathcal{F})$ is a $\Bbbk$-scheme that is affine over S, with $\mathcal{P}\big(\operatorname{Spec}(\mathcal{F})\big) \cong \mathcal{F}$.
(4) If $(y : Y \to S) \in \mathcal{C}(x)$, then $y \cong \mathfrak{V}\mathfrak{I}(y) = \operatorname{Spec}\big(\mathcal{P}(x)/x_*(\mathcal{I}_{X|Y})\big)$. If $\mathcal{I} \in \mathcal{II}(\mathcal{F})$, then $\mathcal{P}(\mathfrak{V}(\mathcal{I})) \cong \mathcal{F}/\mathcal{I}$.

Proof. This is an easy exercise and its proof is therefore omitted. □

Definition 6.70. Let $\mathcal{H}$ be a commutative flat Hopf sheaf on A. A subsheaf $\mathcal{I} \subset \mathcal{H}$ is a *sheaf of Hopf ideals* if there exists a pair $(\mathcal{K}, F)$ where $\mathcal{K}$ is a Hopf sheaf and $F : \mathcal{H} \to \mathcal{K}$ is a surjective morphism of Hopf sheaves with $\operatorname{Ker}(F) = \mathcal{I}$ — recall that in this case $\mathcal{K} \cong \mathcal{H}/\mathcal{I}$.

We say that a sheaf of Hopf ideals $\mathcal{I} \subset \mathcal{H}$ is *faithful* if $\mathcal{K} = \mathcal{H}/\mathcal{I}$ is a faithful Hopf sheaf.

Remark 6.71. By definition, a sheaf of Hopf ideals is faithful if and only if $\mathcal{I}(U) \cap \mathcal{O}(U) = \{0\}$ for all open subset $U \subset A$.

Proposition 6.72. *If $\mathcal{H}$ is a commutative flat Hopf sheaf of A, then a subsheaf $\mathcal{I} \subset \mathcal{H}$ in A-mod is a sheaf of Hopf ideals if and only if the following conditions hold:*

(i) The subsheaf $\mathcal{I} \subset \mathcal{H}$ is a quasi-coherent sheaf of ideals;
(ii) Let $\mathrm{inc} : \mathcal{I} \to \mathcal{H}$ be the inclusion morphism and consider $\mathrm{inc} \,\widetilde{\boxtimes}\, \mathrm{id} + \mathrm{id} \,\widetilde{\boxtimes}\, \mathrm{inc} : \mathcal{I} \,\widetilde{\boxtimes}\, \mathcal{H} + \mathcal{H} \,\widetilde{\boxtimes}\, \mathcal{I} \to \mathcal{H} \,\widetilde{\boxtimes}\, \mathcal{H}$. Then the morphism $\Delta \circ \mathrm{inc} : \mathcal{I} \to \mathcal{H} \,\widetilde{\boxtimes}\, \mathcal{H}$ factors as in the diagram below:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\Delta"] & \mathcal{H} \,\widetilde{\boxtimes}\, \mathcal{H} \\ \mathcal{I} \arrow[u, "\mathrm{inc}"] \arrow[r, "\Delta|_{\mathcal{I}}"] & \mathcal{I} \,\widetilde{\boxtimes}\, \mathcal{H} + \mathcal{H} \,\widetilde{\boxtimes}\, \mathcal{I}, \arrow[u, "\mathrm{inc} \,\widetilde{\boxtimes}\, \mathrm{id} + \mathrm{id} \,\widetilde{\boxtimes}\, \mathrm{inc}"'] \end{tikzcd}$$

(iii) $\mathcal{I} \subset \mathrm{Ker}(\varepsilon_{\mathcal{H}})$.

Proof. Assume that $\mathcal{I}$ is a sheaf of Hopf ideals, i.e. $\mathcal{I} = \mathrm{Ker}(F)$ for some morphism of Hopf sheaves $F : \mathcal{H} \to \mathcal{K}$. From the flatness hypothesis it follows that $\mathcal{I}$ is an ideal of the sheaf of algebras $\mathcal{H}$. Also from the flatness it follows that in the monoidal abelian category $(QA\text{-alg}, \widetilde{\boxtimes})$,

$$\mathrm{Ker}(F \,\widetilde{\boxtimes}\, F) = \mathcal{I} \,\widetilde{\boxtimes}\, \mathcal{H} + \mathcal{H} \,\widetilde{\boxtimes}\, \mathcal{I}, \tag{6.7}$$

and therefore (ii) is verified (since F is a morphism of Hopf sheaves). The proof of assertion (iii) is trivial.

For the converse, assume that $\mathcal{I}$ satisfies conditions (i)–(iii) and call $F : \mathcal{H} \to \mathcal{H}/\mathcal{I} = \mathcal{K}$. Then $\mathcal{K}$ is a sheaf of algebras, with product $\mu_{\mathcal{K}}$ and unit $u_{\mathcal{K}}$ induced by $\mu_{\mathcal{H}}$ and $u_{\mathcal{H}}$.

It follows from the equality (6.7) that the map $(F \,\widetilde{\boxtimes}\, F) \circ \Delta$ factors through $\mathcal{K}$ and induces a morphism of sheaves $\Delta_{\mathcal{K}} : \mathcal{K} \to (\mathcal{H} \,\widetilde{\boxtimes}\, \mathcal{H})/(\mathcal{I} \,\widetilde{\boxtimes}\, \mathcal{H} + \mathcal{H} \,\widetilde{\boxtimes}\, \mathcal{I}) \cong \mathcal{K} \,\widetilde{\boxtimes}\, \mathcal{K}$.

From condition (iii) we deduce that $\varepsilon_{\mathcal{H}} : \mathcal{H} \to \mathrm{skysc}_0(\Bbbk)$ induces a morphism $\varepsilon_{\mathcal{K}} : \mathcal{K} \to \mathrm{skysc}_0(\Bbbk)$. On the other hand, since $-\mathcal{K} = (-\mathcal{H})/(-\mathcal{I})$, it is clear that $\sigma_{\mathcal{H}} : -\mathcal{H} \to \mathcal{H}$ induces a morphism $\sigma_{\mathcal{K}} : -\mathcal{K} \to \mathcal{K}$.

Finally, it is clear that, by construction, the morphisms $\Delta_{\mathcal{K}}$, $\varepsilon_{\mathcal{K}}$ and $\sigma_{\mathcal{K}}$ satisfy the required commutative diagrams for $(\mathcal{K}, \Delta_{\mathcal{K}}, \varepsilon_{\mathcal{K}}, \mu_{\mathcal{K}}, u_{\mathcal{K}}, \sigma_{\mathcal{K}})$ to be a commutative Hopf sheaf. $\square$

Proposition 6.73. *Let $\mathcal{S} :\ 1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 0$ be an affine extension and let $\mathcal{H} = q_*\big(\mathcal{O}_G\big)$ be the (faithful, commutative) Hopf sheaf associated to $\mathcal{S}$. Then the poset of closed sub-extensions of $\mathcal{S}$ is op-equivalent to the poset of faithful Hopf ideals of $\mathcal{H}$.*

PROOF. Let $\mathcal{T} :\ 1 \longrightarrow H' \longrightarrow G' \xrightarrow{q} A \longrightarrow 0$ be a closed sub-extension of $\mathcal{S}$ and consider $\mathcal{I}_{G'} \subset \mathcal{O}_G$, the subsheaf of ideals associated to G'. Clearly $q_*(\mathcal{I}_{G'}) \subset \mathcal{H}$ is a subsheaf of ideals. On the other hand, if we denote by $\mathrm{inc} : G' \to G$ the canonical inclusion, then we have a commutative diagram of sheaves of $\mathcal{O}_G$–modules

$$\begin{tikzcd}
\mathcal{O}_G \arrow[r, "m^{\#}"] \arrow[d] & m_*(\mathcal{O}_{G\times G}) = m_*(\mathcal{O}_G \boxtimes \mathcal{O}_G) \arrow[d] \\
\operatorname{inc}_*(\mathcal{O}_{G'}) \arrow[r, "\operatorname{inc}_* m^{\#}_{G'}"] & \operatorname{inc}_*\big((m_{G'})_*(\mathcal{O}_{G'\times G'})\big) = \operatorname{inc}_*\big((m_{G'})_*(\mathcal{O}_{G'} \boxtimes \mathcal{O}_{G'})\big)
\end{tikzcd}$$

where the vertical arrows are the canonical projections induced by the inclusions $G' \hookrightarrow G$ and $G'\times G' \hookrightarrow G\times G$. Since $\mathcal{I}_{G'\times G'} = \mathcal{I}_{G'} \boxtimes \mathcal{O}_G + \mathcal{O}_G \boxtimes \mathcal{I}_{G'}$, it follows that $\mathcal{O}_{G'} \boxtimes \mathcal{O}_{G'} = (\mathcal{O}_G \boxtimes \mathcal{O}_G)/\big(\mathcal{I}_{G'} \boxtimes \mathcal{O}_G + \mathcal{O}_G \boxtimes \mathcal{I}_{G'}\big)$. Hence, $m_*(\mathcal{I}_{G'}) \subset \mathcal{I}_{G'} \boxtimes \mathcal{O}_G + \mathcal{O}_G \boxtimes \mathcal{I}_{G'}$.

Also, it is easy to show that $\mathcal{I}_{G'} \subset \operatorname{Ker}(\varepsilon_G)$. From the functorial properties of q_*, it follows that $q_*(\mathcal{I}_{G'})$ is a sheaf of Hopf ideals and, $\mathcal{T}$ being a sub-extension, $q_*(\mathcal{I}_{G'})$ is faithful.

Conversely, given a faithful sheaf of Hopf ideals $\mathcal{I} \subset \mathcal{H}$, let

$$\mathcal{T}: \qquad 1 \longrightarrow \operatorname{Spec}\big(\mathcal{H}/\mathcal{I}\big)_0 \longrightarrow \operatorname{Spec}\big(\mathcal{H}/\mathcal{I}\big) \longrightarrow A \longrightarrow 0$$

be the affine extension associated to the Hopf sheaf $\mathcal{H}/\mathcal{I}$. If $U \subset A$ is an affine open subset, then the canonical projection $\mathcal{H}(U) \to (\mathcal{H}/\mathcal{I})(U)$ induces a closed immersion $\operatorname{Spec}\big((\mathcal{H}/\mathcal{I})(U)\big) \to \operatorname{Spec}\big(\mathcal{H}(U)\big)$. Therefore, $\mathcal{T}$ is a closed sub-extension of $\mathcal{S}$. $\square$

7. The category $\operatorname{Rep}(\mathcal{S})$ as a category of sheaves

If G is an affine group scheme, it is well known that the category of its (left) rational representations and the category of (right) $\Bbbk[G]$–comodules are equivalent — in the usual notations: $\operatorname{Rep}(G) \cong {}^{\Bbbk[G]}\mathsf{M}$. One can also consider the anti-equivalence between the category of vector spaces and the category of symmetric algebras (given by $V \mapsto \mathcal{O}_V(V) = S(V^\vee)$, the symmetric algebra generated by the dual $V^\vee$) in order to produce an anti-equivalence between $\operatorname{Rep}(G)$ and the category of $\Bbbk G$–comodule symmetric algebras. On the other hand, it is also well known that the category of vector bundles over a scheme T is equivalent to the category of locally free, coherent, sheaves of $\mathcal{O}_T$–modules, see for example [59, Chapter 13] (see also Proposition 7.5).

In view of the previous remarks, the objective of this section is two-fold:

In the light of Theorem 6.62 (and in the nomenclature of Definition 3.41 and Remark 3.42), given an affine extension $q : G \to A$ we want to establish an equivalence $\operatorname{Rep}_0(q) \cong \big({}^{\mathcal{H}_q}\mathsf{M}\big)_{\text{fin}}$, the category of locally free, coherent sheaves that are $\mathcal{H}_q$–comodules, were $\mathcal{H}_q = q_*\big(\mathcal{O}_G\big)$ is the Hopf sheaf associated to q. Moreover, we want to extend this equivalence to the graded setting — which involves the enlargement of the category ${}^{\mathcal{H}_q}\mathsf{M}$ to an (enriched) category with graded morphisms.

On the other hand, we also want to generalize Mumford's equivalence between G–linearized line bundles and G–linearized invertible sheaves to our context. Whereas the notion of G–linearized sheaf is well established (see [57, Tag 03LE] and [44, page 30]), we need (again) to develop the notions of graded morphisms between G–linearized sheaves and of homogeneous sheaves, in order to construct a replacement for $\operatorname{HVB}_{\text{gr}}(A)$ (see Definition 3.18).

7.1. **The category of comodules of a Hopf sheaf.**

In this section we consider the usual morphisms of sheaves in A-mod that correspond with $\mathrm{Rep}_0(q)$. In the next Section 7.2, we extend the equivalence given below in Proposition 7.5, to categories with graded morphisms (see Lemma 7.22) in order to obtain $\mathrm{Rep}(q)$.

We begin by writing down the definition of *comodule algebra* for a sheaf of bimonoids in the duoidal category $(QA\text{-mod}, \otimes_A, \mathbb{I}_{\otimes_A}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}})$ — or more particularly for a Hopf sheaf — as considered in Section 6.2, in particular definitions 6.24 and 6.27.

Remark 7.1. Consider a bimonoid $\mathcal{B} = (\mathcal{B}, \mu_{\mathcal{B}}, u_{\mathcal{B}}, \Delta_{\mathcal{B}}, \varepsilon_{\mathcal{B}})$ in the duoidal category $(QA\text{-mod}, \otimes_A, \mathbb{I}_{\otimes_A}, \widetilde{\boxtimes}, \mathbb{I}_{\widetilde{\boxtimes}})$.

(1) A *left $\mathcal{B}$–comodule* is a pair $(\mathcal{F}, \chi)$, with $\mathcal{F} \in QA$-mod and $\chi : \mathcal{F} \to \mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{F}$ a morphism of sheaves, that satisfies the corresponding commutative diagrams (as in Definition 6.24).

(2) A *morphism of left $\mathcal{B}$–comodules* is a morphism $\psi : \mathcal{M} \to \mathcal{M}' \in QA$-mod such that the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\psi"] \arrow[d, "\chi"'] & \mathcal{M}' \arrow[d, "\chi'"] \\
\mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{M} \arrow[r, "\mathrm{id}_{\mathcal{B}} \widetilde{\boxtimes} \psi"] & \mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{M}'
\end{tikzcd}$$

is commutative.

(3) The category ${}^{\mathcal{B}}\mathsf{M}$ *of left $\mathcal{B}$–comodules* has as objects the $\mathcal{B}$–comodules and as arrows $\mathrm{Hom}_{{}^{\mathcal{B}}\mathsf{M}}(\mathcal{M}, \mathcal{M}')$ the morphisms of $\mathcal{B}$–comodules.

(4) If we take two $\mathcal{B}$–comodules (with respect to the $\widetilde{\boxtimes}$ monoidal structure) $\mathcal{M}, \mathcal{M}'$ their product $\mathcal{M} \otimes_A \mathcal{M}'$ is also a $\mathcal{B}$–comodule. In other words ${}^{\mathcal{B}}\mathsf{M}_{A\text{-mod}}$ is a $\otimes_A$–monoidal category with unit $\mathcal{O}_A$, that is viewed as an object of ${}^{\mathcal{B}}\mathsf{M}$ via the structure

$\mathcal{O}_A \xrightarrow{s^{\#}} \mathcal{O}_A \,\widetilde{\boxtimes}\, \mathcal{O}_A \xrightarrow{u_{\mathcal{B}} \widetilde{\boxtimes} \mathrm{id}} \mathcal{B} \,\widetilde{\boxtimes}\, \mathcal{O}_A$ (see Proposition 6.26).

(5) A *right $\mathcal{B}$–comodule algebra* is a right $\mathcal{B}$–comodule $(\mathcal{F}, \chi_{\mathcal{F}})$ such that $(\mathcal{F}, \mu_{\mathcal{F}}, u_{\mathcal{F}}) \in QA$-alg and $\chi_{\mathcal{F}} \in \mathrm{Hom}_{A\text{-alg}}(\mathcal{F}, \mathcal{F} \,\widetilde{\boxtimes}\, \mathcal{B})$ with the adequate algebra structure in $\mathcal{F} \,\widetilde{\boxtimes}\, \mathcal{B}$ (see Proposition 6.26 and Definition 6.27).

In explicit terms, we ask the diagrams below to commute — we are using the notations of definitions 6.27 and 6.42:

$$\begin{tikzcd}
\mathcal{O}_A \arrow[r, "s^{\#}"] \arrow[d, "u_{\mathcal{F}}"'] & \mathcal{O}_A \,\widetilde{\boxtimes}\, \mathcal{O}_A \arrow[d, "u_{\mathcal{F}} \widetilde{\boxtimes} u_{\mathcal{B}}"] \\
\mathcal{F} \arrow[r, "\chi_{\mathcal{F}}"] & \mathcal{F} \,\widetilde{\boxtimes}\, \mathcal{B}.
\end{tikzcd}$$

$$
\begin{tikzcd}
& (\mathcal{F} \mathbin{\widetilde{\boxtimes}} \mathcal{B}) \otimes_A (\mathcal{F} \mathbin{\widetilde{\boxtimes}} \mathcal{B}) \arrow[r, "\zeta_{\mathcal{F},\mathcal{B},\mathcal{F},\mathcal{B}}"] & (\mathcal{F} \otimes_A \mathcal{F}) \mathbin{\widetilde{\boxtimes}} (\mathcal{B} \otimes_A \mathcal{B}) \arrow[dr, "\mathrm{id} \mathbin{\widetilde{\boxtimes}} \mu_{\mathcal{B}}"] & \\
\mathcal{F} \otimes_A \mathcal{F} \arrow[ur, "\chi_{\mathcal{F}} \otimes_A \chi_{\mathcal{F}}"] \arrow[rrr, "\chi_{\mathcal{F} \otimes_A \mathcal{F}}"] \arrow[d, "\mu_{\mathcal{F}}"'] & & & (\mathcal{F} \otimes_A \mathcal{F}) \mathbin{\widetilde{\boxtimes}} \mathcal{B} \arrow[d, "\mu_{\mathcal{F}} \mathbin{\widetilde{\boxtimes}} \mathrm{id}"] \\
\mathcal{F} \arrow[rrr, "\chi"] & & & \mathcal{F} \mathbin{\widetilde{\boxtimes}} \mathcal{B}
\end{tikzcd}
$$

(6) A *morphism of left $\mathcal{B}$–comodule algebras* from $\mathcal{F}$ to $\mathcal{F}'$ is a morphism $f \in \operatorname{Hom}_{{}^{\mathcal{B}}\mathsf{M}_{A\text{-mod}}}(\mathcal{F}, \mathcal{F}')$ that is also a morphism in $\operatorname{Hom}_{A\text{-alg}}(\mathcal{F}, \mathcal{F}')$.

(7) We denote the category of left (resp. right) $\mathcal{B}$–comodule algebras as ${}^{\mathcal{B}}\mathsf{M}_{QA\text{-alg}}$ (resp. $\mathsf{M}^{\mathcal{B}}_{QA\text{-alg}}$).

As expected, the adjunction between $\mathcal{P}$ and Spec gives a correspondence between actions of bimonoids $b : M \to A$ and structures of $\mathcal{P}(b)$-comodule algebras (see Theorem 6.46 and Proposition 6.49).

Proposition 7.2. *Let $b : M \to A \in \operatorname{Sch}|_{\mathrm{qc}}A$ be a bimonoid, $x : X \to A \in \operatorname{Sch}|_{\mathrm{qc}}A$ and a_X an action of b on x (see Definition 6.24 and Example 6.25). Then $\mathcal{P}(a_X)$ endows $\mathcal{P}(x)$ with a structure of $\mathcal{P}(b)$–comodule algebra.*

Conversely, let $\mathcal{B} \in QA$-alg be a bimonoid, $\mathcal{F} \in QA$-alg and $\chi : \mathcal{F} \to \mathcal{B} \mathbin{\widetilde{\boxtimes}} \mathcal{F}$ a $\mathcal{B}$–comodule algebra. Then $\operatorname{Spec}(\chi) \circ \eta_{\operatorname{Spec}(\mathcal{B}) \mathbin{\widetilde{\times}} \operatorname{Spec}(\mathcal{F})} : \operatorname{Spec}(\mathcal{B}) \mathbin{\widetilde{\times}} \operatorname{Spec}(\mathcal{F}) \to \operatorname{Spec}(\mathcal{F})$ is an $\operatorname{Spec}(\mathcal{B})$–action — recall that $\operatorname{Aff}\big(\operatorname{Spec}(\mathcal{B}) \mathbin{\widetilde{\times}} \operatorname{Spec}(\mathcal{F})\big) = \operatorname{Spec}(\mathcal{B} \mathbin{\widetilde{\boxtimes}} \mathcal{F})$.

In particular, $\mathcal{P}$ induces an (op)-equivalence between the following two categories:

(i) $b - \operatorname{Sch}|_{\mathrm{aff}}A$, with objects the pairs (x, a_X) where $x : X \to A \in \operatorname{Sch}|_{\mathrm{aff}}A$ and a_X is a b–action on x, and with arrows the b–equivariant morphisms;

(ii) ${}^{\mathcal{P}(b)}\mathsf{M}_{QA\text{-alg}}$, the category of quasi-coherent $\mathcal{P}(b)$–comodule algebras.

Under this equivalence, flat $\mathcal{P}(b)$–comodules correspond to flat b–objects.

Proof. The proof is straightforward and therefore, it is omitted. □

Our objective is to combine Proposition 7.2 with the well known (monoidal) equivalences between the category of vector bundles over A, the category of locally free sheaves of $\mathcal{O}_A$–mod, of finite rank, and the category of the symmetric algebras generated by these sheaves, in order to describe the category $\operatorname{Rep}_0(q)$, where $q : G \to A$ is an affine extension, as a category of sheaves with additional structure.

Remark 7.3. (1) Recall that if $\mathcal{F} \in QT$-mod, then one can consider the Hadamard monoidal structure $\otimes_T$ and construct $S(\mathcal{F})$, the *symmetric algebra generated by* $\mathcal{F}$.

(2) It is well known that if $\mathcal{F} \in QS$-mod is moreover a locally free sheaf of finite rank $\operatorname{rk}\mathcal{F} = n$, then $\mathcal{F}$ is a coherent sheaf and $\operatorname{Spec}\big(S(\mathcal{F})\big)$ is a vector bundle over T of rank n. Notice also that $S(\mathcal{F})$ is a flat sheaf.

(3) Conversely, if $\pi : E \to T$ is a vector bundle of rank n, then $\mathcal{P}(E)$ is a symmetric algebra generated by a locally free sheaf of $\mathcal{O}_T$–modules, of rank n. Namely, $\mathcal{P}(\pi) = S\big(\Gamma_E^\vee\big)$, where Γ_E is the *sheaf of sections* of the vector bundle E, $\Gamma_E(U) = \Gamma(E, U) = \{s : U \to E : \pi \circ u = \mathrm{id}_U\}$.

In this way, the restriction of the functors Spec and $\mathcal{P}$ gives a monoidal equivalence between $\mathrm{VB}_0(T)$ and ST-alg, the category of sheaves of $\mathcal{O}_T$–module symmetric algebras generated by locally free sheaves of finite rank.

(4) Taking into account the previous remark, we can produce another useful equivalence of categories: if we denote by $C_{\mathrm{lf}}T$-mod the category of (necessarily coherent) locally free sheaves of $\mathcal{O}_T$–modules, of finite rank, then the functor $\mathbb{VB}$: $C_{\mathrm{lf}}T\text{-mod} \to \mathrm{VB}_0(T)$, given by $\mathbb{VB}(\mathcal{F}) = \mathrm{Spec}\big(\mathcal{S}(\mathcal{F}^\vee)\big)$ and $\mathbb{VB}(f : \mathcal{F} \to \mathcal{F}') = \mathrm{Spec}\big(\mathcal{S}(f^\vee)\big) : \mathcal{S}(\mathcal{F}'^\vee) \to \mathcal{S}(\mathcal{F}^\vee)$ is an equivalence of categories.

Definition 7.4. Let A be an abelian variety. If $\mathcal{B}$ is a bimonoid in QA-mod, we denote ${}^{\mathcal{B}}\mathsf{M}_{\mathrm{fin}}$ the category of left $\mathcal{B}$–comodules with support on locally free sheaves of $\mathcal{O}_A$–modules, of finite rank.

We denote ${}^{\mathcal{B}}\mathsf{M}_{\text{SA-alg}} \subset {}^{\mathcal{B}}\mathsf{M}_{QA\text{-alg}}$ the full subcategory of symmetric algebras generated by the locally free $\mathcal{B}$–comodules of finite rank — notice that if $\mathcal{F} \in {}^{\mathcal{B}}\mathsf{M}_{\mathrm{fin}}$, then the $\mathcal{B}$–comodule structure on $\mathcal{F}$ induces a $\mathcal{B}$–comodule structure on $S(\mathcal{F})$ in a natural way.

We are now in condition to present our first result concerning the equivalence of the category of representations of an affine extension $q : G \to A$ with some categories of $\mathcal{H}_q$–comodules.

Proposition 7.5. *Let $\mathcal{S}$: $q : G \to A$ be an affine extension. Then:*

(1) the equivalence of Proposition 7.2 restricts to a monoidal (op)-equivalence between $\mathrm{Rep}_0(\mathcal{S})$ and ${}^{\mathcal{H}_q}\mathsf{M}_{\text{SA-alg}}$, where $\mathcal{H}_q$ denotes as usual the sheaf of Hopf algebras associated to q;

(2) the equivalence $\mathbb{VB} : C_{\mathrm{lf}}A\text{-mod} \to \mathrm{VB}_0(A)$ induces an equivalence of categories, that we also denote $\mathbb{VB} : {}^{\mathcal{H}_q}\mathsf{M}_{\mathrm{fin}} \to \mathrm{Rep}_0(\mathcal{S})$.

Proof. The proof of (1) is a straightforward consequence of propositions 3.47 and 7.2, and hence it is omitted.

In order to prove (2), we first observe that if $(\mathcal{F}, \chi_{\mathcal{F}}) \in {}^{\mathcal{H}_q}\mathsf{M}_{\mathrm{fin}}$, then $\mathrm{Spec}\big(\mathcal{S}(\mathcal{F})\big) = p_E : E \to A$ supports a $\mathcal{S}$–module structure $a : q \mathbin{\widetilde{\times}} p_E \to p_E$ — notice that we *are not* applying the functor $\mathbb{VB}$. It follows that $p_E^\vee : E^\vee \to A$ is a $\mathcal{S}$–module, and hence $\mathcal{P}(E^\vee)$ is a $\mathcal{H}_q$–comodule algebra. But by construction, $\mathcal{P}(E^\vee) = \mathcal{S}(\mathcal{F}^\vee)$ and the corresponding $\mathcal{H}_q$–coaction being linear, it restricts to a coaction $\chi_{\mathcal{H}_q} : \mathcal{F}^\vee \to \mathcal{H}_q \mathbin{\widetilde{\boxtimes}} \mathcal{F}^\vee$.

It is easy to see that the functor "take dual" induces an equivalence $\cdot^\vee : {}^{\mathcal{B}}\mathsf{M}_{\mathrm{fin}} \to {}^{\mathcal{B}}\mathsf{M}_{\mathrm{fin}}$. The proof of (2) is now straightforward, since $\mathbb{VB}(\mathcal{F}) = \mathrm{Spec}\big(S(\mathcal{F}^\vee)\big)$. □

If $\mathcal{S}$: $q : G \to A$ is an affine extension, in view of Proposition 7.5, it makes sense to define a category of "infinite dimensional" $\mathcal{S}$–modules either as a certain full subcategory of ${}^{\mathcal{H}_q}\mathsf{M}_{QA\text{-alg}}$ containing ${}^{\mathcal{H}_q}\mathsf{M}_{\text{SA-alg}}$ or as a full subcategory of ${}^{\mathcal{H}_q}\mathsf{M}_{QA\text{-mod}}$ containing ${}^{\mathcal{H}_q}\mathsf{M}_{\mathrm{fin}}$.

As proposed by V. Drinfeld in [25] (in the context of the definition of an infinite dimensional vector bundle"), we take the second approach and consider the full subcategory of ${}^{\mathcal{H}_q}\mathsf{M}_{QA\text{-mod}}$ with objects the quasi-coherent, flat sheaves of $\mathcal{H}_q$–comodules, that we denote ${}^{\mathcal{H}_q}\mathsf{M}_{QA_p\text{-mod}}$, as a replacement for $\mathrm{VB}_0(A)$ – recall that if $\mathcal{F}$ is a coherent flat sheaf of $\mathcal{O}_A$–modules, then $\mathcal{F}$ is locally free (see [56, Proposition 2]).

7.2. **Homogeneous sheaves on an abelian variety.**

In this section we define the category of homogeneous sheaves on an abelian variety A. We fix the following notation: if T is a $\Bbbk$–scheme and $\mathcal{F} \in A$-mod, we define $\mathcal{F}_T = \mathcal{F} \boxtimes \mathcal{O}_T \in A_T$-mod — recall that $A_T = A \times T$. Then $\mathcal{F}_T = p_1^*\mathcal{F} \otimes_{\mathcal{O}_{A\times T}} p_2^*(\mathcal{O}_T) = p_1^*\mathcal{F} \otimes_{\mathcal{O}_{A\times T}} \mathcal{O}_{A\times T} = p_1^*\mathcal{F}$, where $p_1 : A_T \to A$ and $p_2 : A_T \to T$ are the canonical projections.

Definition 7.6. (1) If A is an abelian variety and $\mathcal{F}, \mathcal{G} \in A$-mod, we define the *functor of graded morphisms* $\operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G}) : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$ as follows: for $T \in \mathrm{Sch}^{\mathrm{op}}$ an element of the set $\operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})(T)$ is a pair (f, ℓ), with $\ell \in A(T)$ and $f \in \operatorname{Hom}_{A_T\text{-mod}}(t_\ell^*\mathcal{F}_T, \mathcal{G}_T)$ — recall that the translation $t_\ell : A_T \to A_T$ is a morphism of T–schemes and that f is a morphism of sheaves of A_T–modules.

For an arrow $j : T' \to T$ the functor is defined as follows:

$$\operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})(j) : \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})(T) \to \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})(T')$$

is given by $\operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})(j)(f, \ell) = \big((\mathrm{id}_A \times j)^* f, \ell \circ j\big)$ — notice that $\ell \circ j \in A(T')$ and that since $p_1 \circ t_\ell \circ (\mathrm{id}_A \times j) = p_1' \circ t_{\ell\circ j}$, then $(\mathrm{id}_A \times j)^* f \in \operatorname{Hom}_{sATmod}(t_{\ell\circ j}^*\mathcal{F}_{T'}, \mathcal{G}_{T'})$.

(2) If $(f, \ell) \in \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})(T)$, then the element $\ell \in A(T)$ is called the *degree of* (f, ℓ).

The *degree maps* $d(T) : \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})(T) \to A(T)$, $d(f, \ell) = \ell$, conform — by definition — a natural transformation, that is also called the *degree map*.

(3) We call $\operatorname{Hom}_0(\mathcal{F}, \mathcal{G})$ the subfunctor of $\operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})$ given by the elements of degree zero: i.e. the set $\operatorname{Hom}_0(\mathcal{F}, \mathcal{G})(T) = \{(f, 0) : (f, 0) \in \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})(T)\}$ with $f \in \operatorname{Hom}_{A_T\text{-mod}}\big(\mathcal{F}_T, \mathcal{G}_T\big)$.

Definition 7.7. Let A be an abelian variety and $\mathcal{F}, \mathcal{G}, \mathcal{E} \in A$-mod. We define a natural transformation $\circ : \operatorname{Hom}_{\mathrm{gr}}(\mathcal{G}, \mathcal{E}) \times \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G}) \Rightarrow \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{E})$ as follows: given a $\Bbbk$–scheme T and $(f, \ell) \in \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})(T)$, $(g, b) \in \operatorname{Hom}_{\mathrm{gr}}(\mathcal{G}, \mathcal{E})$, then $f : t_\ell^*\mathcal{F}_T \to \mathcal{G}_T$ and $t_b^* f : t_{\ell+b}^*\mathcal{F}_T \to t_b^*\mathcal{G}_T$. Hence, we can define $(g, b) \circ (f, \ell) := (g \circ t_b^* f, \ell + b) \in \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{E})$.

Remark 7.8. It follows by definition that the degree map satisfies the following compatibility condition with the composition defined above.

$$\begin{tikzcd}
\operatorname{Hom}_{\mathrm{gr}}(\mathcal{G}, \mathcal{E}) \times \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G}) \arrow[r, "\circ"] \arrow[d, "d\times d"'] & \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{E}) \arrow[d, "d"] \\
A \times A \arrow[r, "s"] & A
\end{tikzcd}$$

Notation 7.9. (1) We denote $\operatorname{End}_{\mathrm{gr}}(\mathcal{F}) := \operatorname{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{F})$; notice that $\operatorname{End}_{\mathrm{gr}}(\mathcal{F})$ with the composition of graded morphisms is a monoid functor, with unit $e_{\operatorname{End}_{\mathrm{gr}}(\mathcal{F})}(T) = (\mathrm{id}_{\mathcal{F}_T}, 0)$.

(2) The group functor $\operatorname{Aut}_{\mathrm{gr}}(\mathcal{F})$ is the unit subfunctor of $\operatorname{End}_{\mathrm{gr}}(\mathcal{F})$; that is, the subgroup functor of $\operatorname{End}_{\mathrm{gr}}(\mathcal{F})$ given (for each scheme T) by the pairs (f, ℓ) such that $f : t_\ell^*\mathcal{F}_T \to \mathcal{F}_T$ is an isomorphism.

(3) We also consider the functor on monoids $\operatorname{End}_0(\mathcal{F}) = \operatorname{Hom}_0(\mathcal{F}, \mathcal{F})$, and the functor on groups $\operatorname{Aut}_0(\mathcal{F})$ given by the pull back of the inclusions $\operatorname{Aut}_{\mathrm{gr}}(\mathcal{F}) \subset \operatorname{End}_{\mathrm{gr}}(\mathcal{F})$ and $\operatorname{End}_0(\mathcal{F}) \subset \operatorname{End}_{\mathrm{gr}}(\mathcal{F})$ — therefore, $\operatorname{Aut}_0(\mathcal{F})$ is the subfunctor of $\operatorname{Aut}_{\mathrm{gr}}(\mathcal{F})$ given by the elements of degree zero.

As in the case of vector bundles, it is easy to see that the morphism of degree zero between two sheaves $\mathcal{F}, \mathcal{G}$ are in bijection with the morphisms of sheaves of $\mathcal{O}_A$–modules between $\mathcal{F}$ and $\mathcal{G}$.

Remark 7.10. Let $\mathcal{F}, \mathcal{G} \in A$-mod. Then the functor $\mathrm{Hom}_0(\mathcal{F}, \mathcal{G})$ is represented by the $\Bbbk$-vector space $\mathrm{Hom}_{A\text{-mod}}(\mathcal{F}, \mathcal{G})$. In particular, $\mathrm{End}_0(\mathcal{F})$ is a smooth scheme on monoids. Indeed, the $\Bbbk$–vector space $\mathrm{Hom}_{A\text{-mod}}(\mathcal{F}, \mathcal{G})$ represents the functor $T \mapsto \mathrm{Hom}_{A_T\text{-mod}}(\mathcal{F}_T, \mathcal{G}_T)$.

Definition 7.11. As in Definition 3.18, we consider the monoidal category $\mathcal{V} = \mathrm{Func}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Sets})$ and we define the $\mathcal{V}$–category A_{gr}-mod, that is called the *category of sheaves on A with graded morphisms*, with objects the sheaves of $\mathcal{O}_A$–modules and with hom-object $\mathrm{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G}) \in \mathcal{V}$, with composition defined as above. In other words, for $\mathcal{F}$ and $\mathcal{G}$ sheaves on A, $\mathrm{Hom}_{A_{\mathrm{gr}}\text{-mod}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})$.

We define the full subcategory QA_{gr}-mod $\subset A_{\mathrm{gr}}$-mod of *quasi-coherent sheaves of $\mathcal{O}_A$–modules with graded morphisms* with objects the quasi-coherent sheaves of $\mathcal{O}_A$–modules.

Similarly, we define the subcategory QA_{gr}-alg of *quasi-coherent sheaves of $\mathcal{O}_A$–algebras with graded morphisms* by taking as objects the quasi–coherent sheaves of $\mathcal{O}_A$–algebras, and if $\mathcal{F}, \mathcal{G}$ are two objects, then $\mathrm{Hom}_{A_{\mathrm{gr}}\text{-alg}}(\mathcal{F}, \mathcal{G})$ is the subfunctor of $\mathrm{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{G})$ given by $(f, \ell) \in \mathrm{Hom}_{A_{\mathrm{gr}}\text{-alg}}(\mathcal{F}, \mathcal{G})$ if $f : t_\ell^* \mathcal{F}_T \to \mathcal{G}_T$ is a morphism of $\mathcal{O}_{A_T}$–algebras (over the T–scheme A_T).

Remark 7.12. Call A_0-mod $\subseteq A_{\mathrm{gr}}$-mod the wide subcategory of A_{gr}-mod with morphisms between $\mathcal{F}, \mathcal{G}$ the functor $\mathrm{Hom}_0(\mathcal{F}, \mathcal{G})$. It is clear that the usual category A-mod $= \mathcal{O}_A -$ mod is equivalent to A_0-mod (see Remark 7.10). Similarly for the analogous situation but in QA-mod and QA-alg with respect to QA_0-mod and QA_0-alg.

Clearly, $\mathrm{End}_0(\mathcal{F})$ is the kernel of the morphism of functors on monoids $d : \mathrm{End}_{\mathrm{gr}}(\mathcal{F}) \to A$, and $\mathrm{Aut}_0(\mathcal{F})$ the kernel of $d : \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}) \to A$.

Remark 7.13. As in the case of vector bundles, if $\mathcal{F}, \mathcal{F}' \in A_{\mathrm{gr}}$-mod, then $\mathrm{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{F}')$ is a fpqc sheaf.

In particular, when we deal with locally free sheaves of finite rank we have the following equivalence result:

Lemma 7.14. *In the above situation, the equivalence $\mathbb{VB} : C_{\mathrm{lf}}A\text{-mod} \to \mathrm{VB}_0(A)$ extends to an equivalence $\mathbb{VB}_{\mathrm{gr}} : C_{\mathrm{lf}}A_{\mathrm{gr}}\text{-mod} \to \mathrm{VB}_{\mathrm{gr}}(A)$.*

Proof. Fist, we observe that if $\mathcal{F} \in C_{\mathrm{lf}}A$-mod and T is a $\Bbbk$–scheme, then $\mathbb{VB}(\mathcal{F}_T) \cong \mathbb{VB}(\mathcal{F})_T$. Indeed,

$$\mathrm{Spec}\big(\mathcal{S}\big((\mathcal{F}_T)^\vee\big)\big) = \mathrm{Spec}\big(\mathcal{S}\big((p_1^*\mathcal{F})^\vee\big)\big) = \mathrm{Spec}\big(\mathcal{S}\big(p_1^*(\mathcal{F}^\vee)\big)\big) = p_1^*\big(\mathrm{Spec}\big(\mathcal{S}(\mathcal{F}^\vee)\big)\big),$$

where the second of the above chain of equalities is due to the commutation of pullback with duals and the third by the commutation of base change with Spec (see [2] and [59][Thm. 17.1.3, Ex. 17.1.F)] respectively). In other words,

$$\mathbb{VB}(\mathcal{F}_T) \cong \mathbb{VB}(\mathcal{F})_T = \big(\pi_{\mathcal{F}} \times \mathrm{id} : \mathbb{VB}(\mathcal{F}) \times T \to A \times T\big).$$

Next we show that for all $\mathcal{F} \in C_{\mathrm{lf}}A$-mod and $b \in A(T)$ we have the commutation relation $\mathbb{VB}\big(t_b^*(\mathcal{F}_T)\big) \cong t_b^*\,\mathbb{VB}(\mathcal{F})_T$:

$$\begin{aligned}\mathbb{VB}(t_b^*\mathcal{F}_T) = \ & \operatorname{Spec}\big(t_b^*\mathcal{S}\big(\mathcal{F}_T^\vee\big)\big) = t_b^*\big(\operatorname{Spec}\big(\mathcal{S}\big((p_1^*\mathcal{F})^\vee\big)\big)\big) = \\ & t_b^*\,\mathbb{VB}(\mathcal{F}_T) = \big(t_b^*\,\mathbb{VB}(\mathcal{F})\big)_T,\end{aligned}$$

where we used again the commutation of duals and pullback for locally free shaves.

Once we have the identifications above, it is clear that the equivalence $\mathbb{VB}: C_{\mathrm{lf}}A\text{-mod} \to \mathrm{VB}_0(A)$ induces a natural isomorphism between the functors $\operatorname{Hom}_{\mathrm{gr}}(\mathcal{F},\mathcal{G})$ and $\operatorname{Hom}_{\mathrm{gr}}\big(\mathbb{VB}(\mathcal{F}),\mathbb{VB}(\mathcal{G})\big)$.

Indeed, if T is a $\Bbbk$–scheme, then the family of bijections (indexed on $\ell \in A(T)$) given by

$$\operatorname{Hom}_{A_T\text{-mod}}\big(t_\ell^*(\mathcal{F}_T),\mathcal{G}_T\big) \to \operatorname{Hom}_0\big(\mathbb{VB}\big(t_\ell^*\mathcal{F}_T\big),\mathbb{VB}(\mathcal{G}_T)\big) = \operatorname{Hom}_0\big(t_\ell^*\big(\mathbb{VB}(\mathcal{F})_T\big),\mathbb{VB}(\mathcal{G})_T\big),$$

combine into a bijection

$$\big\{(f,\ell) : f \in \operatorname{Hom}_{A_T\text{-mod}}\big(t_\ell^*(\mathcal{F}_T),\mathcal{G}_T\big)\,,\ \ell \in A(T)\big\} \to \operatorname{Hom}_{\mathrm{gr}}\big(\mathbb{VB}(\mathcal{F}),\mathbb{VB}(\mathcal{G})\big).$$

The family of bijections above (indexed on T) conform the seeked natural transformations. □

Remark 7.15. As an immediate consequence of Lemma 7.14, we have that if $\mathcal{F} \in C_{\mathrm{lf}}A$-mod, then $\operatorname{Aut}_{\mathrm{gr}}(\mathcal{F})$ is a smooth group scheme of finite type and the degree map $d : \operatorname{Aut}_{\mathrm{gr}}(\mathcal{F}) \to A$ is an affine morphism of group schemes, with kernel $\operatorname{Aut}_0(\mathcal{F})$, since $\operatorname{Aut}_{\mathrm{gr}}(\mathcal{F}) \cong \operatorname{Aut}_{\mathrm{gr}}\big(\mathbb{VB}(\mathcal{F})\big)$.

It is an open question whether $\operatorname{Aut}_{\mathrm{gr}}(\mathcal{F})$ is a group scheme or not if $\mathcal{F} \in QA$-mod. However, we have the following description.

Lemma 7.16. *Let A be an abelian variety and $\mathcal{F} \in A$-mod. Then $\operatorname{Aut}_0(\mathcal{F})$ is a smooth affine group scheme, and the degree map $d : \operatorname{Aut}_{\mathrm{gr}}(\mathcal{F}) \to A$ is a morphism of fpqc sheaves, with kernel $\operatorname{Aut}_0(\mathcal{F})$.*

Proof. Recall that $\operatorname{End}_0(\mathcal{F})$ is a smooth monoid (see Remark 7.10), and that $\operatorname{Aut}_0(\mathcal{F})$ is its unit group. It follows from [23, II.3.7] that $\operatorname{End}_0(\mathcal{F})$ is the limit in the category of monoid schemes of a family M_i of finite type; therefore, $\operatorname{Aut}_0(\mathcal{F})$ is the limit of the unit groups $G(M_i)$. Since $G(M_i) \subset M_i$ is open by [23, II.3.6], it follows that $\operatorname{Aut}_0(\mathcal{F})$ is an open subscheme of $\operatorname{End}_0(\mathcal{F})$.

It is clear that $\operatorname{Aut}_{\mathrm{gr}}(\mathcal{F})$ is a sheaf for the fpqc topology (e.g. by descent for morphisms). Moreover, $\operatorname{Aut}_0(\mathcal{F})$ is a subsheaf and the quotient morphism $\pi : \operatorname{Aut}_{\mathrm{gr}}(\mathcal{F}) \to \operatorname{Aut}_{\mathrm{gr}}(\mathcal{F})/\operatorname{Aut}_0(\mathcal{F})$ is a torsor under $\operatorname{Aut}_0(\mathcal{F})$, in view of [23, III.4.1.8]. Now is clear that d factorizes through π. □

Definition 7.17. Let A be an abelian variety. A sheaf $\mathcal{F} \in A_{\mathrm{gr}}$-mod is *homogeneous* if the degree map $d : \operatorname{Aut}_{\mathrm{gr}}(\mathcal{F}) \to A$ surjective in the fpqc topology.

In view of Lemma 7.16, it follows that the $\operatorname{Aut}_{\mathrm{gr}}(\mathcal{F})$ is a quasi-compact group scheme, the sequence

$$Aut_{\mathrm{gr}}(\mathcal{F}): \qquad 1 \longrightarrow \operatorname{Aut}_0(\mathcal{F}) \longrightarrow \operatorname{Aut}_{\mathrm{gr}}(\mathcal{F}) \xrightarrow{\ d\ } A \longrightarrow 0$$

is an affine extension. In particular d is a quasi-compact, faithfully flat morphism of group schemes.

Example 7.18. (1) Clearly, the structure sheaf $\mathcal{O}_A$ is homogeneous.

(2) It is clear that under the equivalence $\mathbb{VB}_{\mathrm{gr}} : C_{\mathrm{lf}}A_{\mathrm{gr}}\text{-mod} \to \mathrm{VB}_{\mathrm{gr}}(A)$, homogeneous sheaves correspond to homogeneous vector bundles (see Remark 7.15).

Definition 7.19. We define A_{h}-mod, *the category of homogeneous sheaves on* A, as the full subcategory of A-mod with objects the homogeneous sheaves on A. Similarly, we define the category QA_{h}-mod of *homogeneous quasi-coherent sheaves of* $\mathcal{O}_A$*–modules* as the full subcategory of QA-mod with objects the homogeneous, quasi-coherent sheaves on A, and $C_{lf}A_{\mathrm{h}}\text{-mod} \subset QA_{\mathrm{h}}$-mod the full subcategory of locally free, homogeneous, sheaves of finite rank. We denote QA_{h}-alg the category of *homogeneous quasi-coherent sheaves of* $\mathcal{O}_A$*–algebras*.

As in Definition 3.18 we define the $\mathcal{V}$–categories $A_{\mathrm{h,gr}}$-mod, $QA_{\mathrm{h,gr}}$-mod, $C_{\mathrm{lf}}A_{\mathrm{h,gr}}$-mod, $A_{\mathrm{h,gr}}$-alg and $QA_{\mathrm{h,gr}}$-alg as the full subcategories of A_{gr}-mod, QA_{gr}-mod, $C_{\mathrm{lf}}A_{\mathrm{gr}}$-mod, A_{gr}-alg and QA_{gr}-alg where on each situation the objects are limited to the homogeneous sheaves.

Lemma 7.20. *Let* $\mathcal{F}, \mathcal{F}' \in A_{\mathrm{h,gr}}$-mod *be two homogeneous sheaves. Then the homogeneous vector bundle* $R_{\mathrm{Hom}_0(\mathcal{F},\mathcal{F}')} = \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}') \times^{\mathrm{Aut}_0(\mathcal{F}')} \mathrm{Hom}_0(\mathcal{F},\mathcal{F}')$ *represents* $\mathrm{Hom}_{\mathrm{gr}}(\mathcal{F},\mathcal{F}')$.

Moreover, $R_{\mathrm{Hom}_0(\mathcal{F},\mathcal{F}')} \cong L_{\mathrm{Hom}_0(\mathcal{F},\mathcal{F}')} = \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}) \times^{\mathrm{Aut}_0(\mathcal{F})} \mathrm{Hom}_0(\mathcal{F},\mathcal{F}') \in \mathrm{HVB}_0(A)$.

Proof. We replicate the proof of Lemma 3.24.

Let $\varphi : \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}') \times \mathrm{Hom}_0(\mathcal{F},\mathcal{F}') \to \mathrm{Hom}_{\mathrm{gr}}(\mathcal{F},\mathcal{F}')$ the morphism of fpqc sheaves given by composition. Then clearly φ is $\mathrm{Aut}_0(\mathcal{F})$–invariant, and therefore induces a morphism of fpqc sheaves $\phi : R_{\mathrm{Hom}_0(\mathcal{F},\mathcal{F}')} \to \mathrm{Hom}_{\mathrm{gr}}(\mathcal{F},\mathcal{F}')$.

Let $y_1 : T \to \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}') \times \mathrm{Hom}_0(\mathcal{F},\mathcal{F}')$ and $y_2 : T \to \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}') \times \mathrm{Hom}_0(\mathcal{F},\mathcal{F}')$ be such that $\varphi(T)(y_1) = \varphi(T)(y_2) \in \mathrm{Hom}_{\mathrm{gr}}(\mathcal{F},\mathcal{F}')(T)$. Define $f_j := p_1(T)(y_j) \in \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}')(T)$, $\ell_j := d \circ p_1(T)(y_j) \in A(T)$, $g_j := p_2(T)(y_j) \in \mathrm{Hom}_0(\mathcal{F},\mathcal{F}')(T)$ and $\varphi(T)(y_j) = (f_j \circ g_j, \ell_j) \in \mathrm{Hom}_{\mathrm{gr}}(\mathcal{F},\mathcal{F}')(T)$, for $j = 1, 2$.

By hypothesis we have $\ell_1 = \ell_2$ and $f_1 \circ g_1 = f_2 \circ g_2$, but since the f_j's are invertible, we get $g_2 = f_2^{-1} \circ f_1 \circ g_1$, and obviously $f_2 = f_1 \circ (f_2^{-1} \circ f_1)^{-1}$ with $f_2^{-1} \circ f_1 \in \mathrm{Aut}_0(\mathcal{F}')(T)$.

On the other hand, given $(f, \ell) \in \mathrm{Hom}_{\mathrm{gr}}(\mathcal{F},\mathcal{F}')(T)$, since the sequence

$$1 \to \mathrm{Aut}_0(\mathcal{F}') \to \mathrm{Aut}_{\mathrm{gr}}(F') \to A \to 0$$

is exact there exist $h : T' \to T$ fpqc and $g : T' \to \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}')$ such that $d \circ g = \ell \circ h = \ell|_{T'}$. In particular, $g : t^*_{\ell|_{T'}} \mathcal{F}_{T'} \to \mathcal{F}_{T'}$ is invertible, therefore $f \circ g^{-1} \in \mathrm{Hom}_0(\mathcal{F},\mathcal{F}')(T')$, $g \in \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}')(T')$ and $\varphi(g, f \circ g^{-1}) = (f, \ell)|_{T'}$, showing that φ is locally surjective. Since both sides are sheaves on the fpqc topology, they turn out to be isomorphic and $R_{\mathrm{Hom}_0(\mathcal{F},\mathcal{F}')}$ represents the functor $\mathrm{Hom}_{\mathrm{gr}}(\mathcal{F},\mathcal{F}')$ (see Lemma 3.24). The claim for $L_{\mathrm{Hom}_0(\mathcal{F},\mathcal{F}')}$ follows in a similar way. The isomorphism of representing schemes is now obvious. $\square$

Remark 7.21. Let $\mathcal{F}A_{\mathrm{h}}$-mod be a homogeneous sheaf. Then $\mathrm{Aut}_{\mathrm{gr}}(\mathcal{F})$ is a quasi compact group scheme and form the proof of Lemma 7.20 above we deduce that $L_{\mathrm{Hom}_0(\mathcal{F},\mathcal{F}')}$ represents the functor $\mathrm{Hom}_{\mathrm{gr}}(\mathcal{F},\mathcal{F}')$, regardless on whether $\mathcal{F}'$ is homogeneous or not. Similarly if $\mathcal{F}'$ is homogeneous, then $R_{\mathrm{Hom}_0(\mathcal{F},\mathcal{F}')}$ represents the functor $\mathrm{Hom}_{\mathrm{gr}}(\mathcal{F},\mathcal{F}')$, regardless on whether $\mathcal{F}$ is homogeneous or not. The existence of a geometric structure for $\mathrm{Hom}_{\mathrm{gr}}(\mathcal{F},\mathcal{F}')$ when neither of the sheaves is homogeneous (i.e. the representability of this functor) remains open.

We finish this paragraph studying the relationship between homogeneous vector bundles and homogeneous sheaves — in particular, we present the correspondence between homogeneous vector bundles and homogeneous locally free sheaves of finite rank as an equivalence of categories.

Lemma 7.22. *In the above situation, the equivalence* $\mathbb{VB} : C_{\mathrm{lf}}A\text{-mod} \to \mathrm{VB}_0(A)$ *restricts to an equivalence* $\mathbb{VB}\,|_{C_{\mathrm{lf}}A_{\mathrm{h}}\text{-mod}} : C_{\mathrm{lf}}A_{\mathrm{h}}\text{-mod} \to \mathrm{HVB}_0(A)$*. Moreover,* $\mathbb{VB}\,|_{C_{\mathrm{lf}}A_{\mathrm{h}}\text{-mod}}$ *extends (as an equivalence) to* $\mathbb{VB}_{\mathrm{gr}} : C_{\mathrm{lf}}A_{\mathrm{h,gr}}\text{-mod} \to \mathrm{HVB}_{\mathrm{gr}}(A)$*.*

Proof. The result follows immediately from Lemma 7.16 and Remark 7.15. □

Corollary 7.23. *A vector bundle* $E \to A$ *is homogeneous if and only if* $\mathcal{P}(E) \in C\,\mathrm{SA\text{-}alg}$ *is homogeneous, and this happens if and only if its corresponding coherent, locally free sheaf* $\mathcal{F}_E$ *is homogeneous.*

In particular, if $\mathcal{H}$ *is a flat Hopf sheaf, then any coherent* $\mathcal{H}$*–comodule is homogeneous.*

Proof. This a direct consequence of Lemma 7.22 and Proposition 7.5. □

7.3. **Linearization of sheaves.**

We begin by recalling the definition of the category of G–linearized sheaves (see [44, page 30] and [57, Tag 03LE]). We work over schemes defined over $\Bbbk$ and the sheaves will be in general quasi–coherent $\mathcal{O}_X$–modules but the definitions can be performed for general $\mathcal{O}_X$–modules.

Let G be a group scheme with m, e_G the multiplication and the unit of the group, assume X a G–scheme with an action $a : G \times X \to X$.

In order to simplify notations, we display the following (rather obvious) commutation relations:

(1) $a\circ(\mathrm{id}_G \times a) = a\circ(m \times \mathrm{id}_X) : G \times G \times X \to X$ and $a\circ(e_G \times \mathrm{id}_X) = p_2 : \mathrm{Spec}(\Bbbk) \times X \to X$ (that is, a is an action).
(2) $a\circ p_{23} = p_2\circ(\mathrm{id}_G \times a) : G\times G\times X \to X$, where $p_{ij} : X_1\times X_2\times X_3 \to X_i\times X_j$ denotes the canonical projection.
(3) $p_2\circ(\mathrm{id}_G \times p_2) = p_2\circ(m \times \mathrm{id}_X) = p_3 : G \times G \times X \to X$ and $p_2\circ(e_G \times \mathrm{id}_X) = p_2 : \mathrm{Spec}(\Bbbk)\times X \to X$ (that is, the canonical projection p_2 is trivial action).

Definition 7.24. Let G be a group scheme, X a G–scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules; denote the G–action of G on X by $a : G \times X \to X$.

A *G–linearization of the sheaf* $\mathcal{F}$ (or a linearization *compatible with the action* $a : G\times X \to X$) is an isomorphism of sheaves of $\mathcal{O}_{G\times X}$–modules $\Phi : a^*(\mathcal{F}) \to p_2^*(\mathcal{F})$ such that:

(1) The diagram below is commutative:

$$\begin{tikzcd}
(\mathrm{id}_G \times a)^* a^*(\mathcal{F}) = (m \times \mathrm{id}_X)^* a^*(\mathcal{F}) \arrow[rr, "(m\times \mathrm{id}_X)^*(\Phi)"] \arrow[dr, "(\mathrm{id}_G \times a)^*(\Phi)"'] & & (m \times \mathrm{id}_X)^* p_2^*(\mathcal{F}) = p_3^*(\mathcal{F}) = p_{23}^* p_2^*(\mathcal{F}) \\
& (\mathrm{id}_G \times a)^* p_2^*(\mathcal{F}) = p_{23}^* a^*(\mathcal{F}) \arrow[ur, "p_{23}^*(\Phi)"'] &
\end{tikzcd}$$

(2) The pullback by the morphism of schemes $e_G \times \mathrm{id}_X : \mathrm{Spec}(\Bbbk) \times X \to G \times X$ of the diagram $\Phi : a^*(\mathcal{F}) \to p_2^*(\mathcal{F})$ yields $(e_G \times \mathrm{id}_X)^*(\Phi) = \mathrm{id} : \mathcal{F} \to \mathcal{F}$.

Remark 7.25. Conditions (1) and (2) of Definition 7.24 are the same than the "cocycle condition" stated originally in GIT [44, §1.3].

Next we define the category of G–linearized sheaves for a G–scheme X with a fixed action $a : G \times X \to X$.

Definition 7.26. If $a : G \times X \to X$ is an action of the group scheme G on the scheme X, define G-QX-mod, the *category of G–linearized sheaves for the G–scheme X*, as having:

(1) `as objects:` the pairs $(\mathcal{F}, \Phi)$, where $\mathcal{F}$ is a sheaf of quasi-coherent $\mathcal{O}_X$–modules and Φ a linearization of $\mathcal{F}$ as defined above;

(2) `as morphisms:` the morphisms from $(\mathcal{F}, \Phi)$ to $(\mathcal{F}', \Phi')$, are the morphisms $f \in \mathrm{Hom}_{QX\text{-mod}}(\mathcal{F}, \mathcal{F}')$ such that the following diagram is commutative

$$\begin{tikzcd}
a^*(\mathcal{F}) \arrow[r, "\Phi"] \arrow[d, "a^*(f)"'] & p_2^*(\mathcal{F}) \arrow[d, "p_2^*(f)"] \\
a^*(\mathcal{F}') \arrow[r, "\Phi'"] & p_2^*(\mathcal{F}').
\end{tikzcd}$$

Definition 7.27. Let $\mathcal{S}$: $q : G \to A$ be an affine extension. Then G acts on A by $a_q = s \circ (q \times id_A) : G \times A \to A$. An *$\mathcal{S}$–linearized sheaf* or *$q$–linearized sheaf* is a pair $(\mathcal{F}, \Phi)$, where $\mathcal{F}$ a quasi-coherent sheaf of $\mathcal{O}_A$–modules and $\Phi : a_q^*(\mathcal{F}) \to p_2^*(\mathcal{F})$ is a G–linearization compatible with a_q.

Given two q–linearized sheaves $\mathcal{F}, \mathcal{F}'$, then a *morphism of q–linearized sheaves* is a morphism $f : \mathcal{F} \to \mathcal{F}'$ of G–linearized sheaves with respect to a_q.

The *category* q-QA-mod *of quasi-coherent q–linearized sheaves* has as objects the quasi-coherent q–linearized sheaves, as morphisms $\mathrm{Hom}_{\text{q-}QA\text{-mod}}(\mathcal{F}, \mathcal{F}')$ the morphisms of q–linearized sheaves. We denote q-$C_{\mathrm{lf}}A$-mod $\subset$ q-QA-mod the full subcategory with objects the locally free sheaves of finite rank.

Remark 7.28. Let $\mathcal{S}$: $q : G \to A$ be an affine extension.

(1) It is easy to see that kernels, images, cokernels of homomorphisms of q-linearized sheaves as well as tensor products and symmetric powers (for the Hadamard monoidal structure $\otimes_A$) of q-linearized sheaves inherit q-linearizations in a natural way.

(2) Clearly, if $\mathcal{F} \in C_{\mathrm{lf}\,A\text{-mod}}$ is q–linearized, $\mathcal{F}^\vee$ inherits a q-linearization in a natural way.

(3) If X is a G–scheme of finite type and $f : X \to A$ is a G–equivariant morphism, then a q–linearization on $\mathcal{F} \in QA$-mod induces a G–linearization on $f^*\mathcal{F} \in QX$-mod (see [35, p. 94]).

(4) A direct generalization of the considerations in [44, §1.3] shows that a q–linearization in $\mathcal{F} \in Q\,\mathrm{SA}$-alg induces a G–action $a : G \times \mathrm{Spec}(\mathcal{F}) \to \mathrm{Spec}(\mathcal{F})$, such that the following diagram is commutative

$$\begin{tikzcd} G \times \mathrm{Spec}(\mathcal{F}) \arrow[r, "a"] \arrow[d, "q\times\pi"'] & \mathrm{Spec}(\mathcal{F}) \arrow[d, "\pi"] \\ A \times A \arrow[r, "s"] & A \end{tikzcd}$$

In other words, a q–linearization of $\mathcal{F}$ induces on $\mathrm{Spec}(\mathcal{F})$ a structure of q–module in the duoidal category $(\mathrm{Sch}\,|_{\mathrm{qc}}A, \widetilde{\times}, \mathbb{I}_{\widetilde{\times}}, \times_A, \mathbb{I}_{\times_A})$ — that is, an action $a : q \widetilde{\times} \mathrm{Spec}(\mathcal{F}) \to \mathrm{Spec}(\mathcal{F})$.

Conversely, if $x : X \to A$ is a q–module, then the action $a : q \widetilde{\times} x \to x$ induces a q–linearization on $\mathcal{P}(x)$.

In [35, p. 94] the reader will find a proof (where G is assumed of finite type) that is valid in our context.

(5) In particular, we deduce from (2) and (4) that if a sheaf $\mathcal{F} \in C_{\mathrm{lf}}A$-mod admits a q–linearization, then it induces a structure of q-module on $\mathbb{VB}(\mathcal{F})$. Conversely, if $(\pi : E \to A) \in \mathrm{Rep}_0(\mathcal{S})$, then the action $a : q \widetilde{\times} \pi \to \pi$ induces a q–linearization on $\mathcal{P}(\pi)$. It is easy to show that if $f \in \mathrm{Hom}_{\mathrm{q}\text{-}C_{\mathrm{lf}}A\text{-mod}}(\mathcal{F}, \mathcal{G})$, then $\mathbb{VB}(f) \in \mathrm{Hom}_0\big(\mathbb{VB}(\mathcal{F}), \mathbb{VB}(\mathcal{G})\big)$, and that we have an equivalence between q-$C_{\mathrm{lf}}A$-mod and $\mathrm{Rep}_0(\mathcal{S})$.

We proceed now to establish the notion of graded morphisms of $\mathcal{S}$–linearized sheaves. Before doing so, it is convenient to set some equalities between different pull-backs involved it the mentioned definition.

Remark 7.29. (1) Let $\mathcal{S}$: $q : G \to A$ be an affine extension and $\mathcal{F} \in QA$-mod a quasi-coherent sheaf of $\mathcal{O}_A$–modules. Let T be a $\Bbbk$–scheme and $\ell : T \to A$ a T–point. From the equalities of morphisms $G \times A \times T \to A$:

$$\begin{aligned} p_1\circ(a_q \times \mathrm{id}_T) &= a_q\circ p_{12} \\ p_1\circ p_{23} &= p_2\circ p_{12} \\ p_2\circ p_{12}(\mathrm{id}_G \times t_\ell) &= p_1\circ p_{23}\circ(\mathrm{id}_G \times t_\ell) = p_1\circ t_\ell\circ p_{23} \end{aligned}$$

and the equality

$$t_\ell\circ(a_q \times \mathrm{id}_T) = (a_q \times \mathrm{id}_T)\circ(\mathrm{id}_G \times t_\ell) : G \times A \times T \to A_T = A \times T$$

we deduce — in the same order — the following equalities of sheaves of $\mathcal{O}_{G\times A\times T}$–modules (recall that $\mathcal{F}_T = p_1^*\mathcal{F}$):

$$\begin{aligned} (a_q \times \mathrm{id}_T)^*(\mathcal{F}_T) &= p_{12}^*(a_q^*\mathcal{F}) = a_q^*(\mathcal{F})_T \\ p_{23}^*(\mathcal{F}_T) &= p_{12}^*\big(p_2^*(\mathcal{F})\big) \\ p_{23}^*(t_\ell)^*(\mathcal{F}_T) &= (\mathrm{id}_G \times t_\ell)^* p_{23}^*(\mathcal{F}_T) = (\mathrm{id}_G \times t_\ell)^* p_{12}^*\big(p_2^*(\mathcal{F})\big) \\ (a_q \times \mathrm{id}_T)^* t_\ell^*(\mathcal{F}_T) &= (\mathrm{id}_G \times t_\ell)^*\big((a_q^*\mathcal{F})_T\big) \end{aligned} \tag{7.1}$$

(2) In the situation above, suppose now that $\mathcal{F}$ is linearized with respect to a_q, and consider the action $a_q \times \mathrm{id}_T : G \times A \times T \to A \times T$. Then the linearization map $\Phi : a_q^*\mathcal{F} \to p_2^*\mathcal{F} \in \mathcal{O}_{G\times A}$ induces (by applying p_{12}^*) a linearization map on $\mathcal{F}_T \in QA_T$–mod given as: $p_{12}^*\Phi : (a_q \times \mathrm{id}_T)^*\mathcal{F}_T \to p_{23}^*\mathcal{F}_T$. In what follows we will sometimes write $\Phi_{\mathcal{F}}$ instead of just Φ, specially if there is more than one sheaf under consideration.

If we take a T–point $\ell : T \to A$, and apply to the above linearization for $\mathcal{F}_T$, the functor $(\mathrm{id}_G \times t_\ell)^*$ we obtain a linearization for $t_\ell^*\mathcal{F}_T \in QA_T$ -mod as:

$$(\mathrm{id}_G \times t_\ell)^* p_{12}^*(\Phi_{\mathcal{F}}) : (a_q \times \mathrm{id}_T)^*(t_\ell)^*(\mathcal{F}_T) \to p_{23}^*(t_\ell)^*(\mathcal{F}_T).$$

This is an easy consequence of the formulæ established just above.

Definition 7.30. Let $\mathcal{S}$: $q : G \to A$ be an affine extension and $\mathcal{F}, \mathcal{F}' \in QA_{\mathrm{gr}}$ -mod two q–linearized sheaves. The *functor of graded morphisms of q–linearized sheaves* is the subfunctor $\mathrm{Hom}_{\text{q-}QA_{\mathrm{gr}}\text{-mod}}(\mathcal{F}, \mathcal{F}') \subset \mathrm{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{F}')$ given as follows: $(f, \ell) \in \mathrm{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{F}')(T)$ belongs to $\mathrm{Hom}_{\text{q-}QA_{\mathrm{gr}}\text{-mod}}(\mathcal{F}, \mathcal{F}')(T)$ if the following diagram of sheaves on the T–scheme $G \times A \times T$ is commutative

$$\begin{tikzcd}
(a_q \times \mathrm{id}_T)^* t_\ell^*(\mathcal{F}_T) \arrow[r, "(\mathrm{id}_G \times t_\ell)^* p_{12}^*(\Phi_{\mathcal{F}})"] \arrow[d, "(a_q\times \mathrm{id}_T)^*(f)"'] & p_{23}^* t_\ell^*(\mathcal{F}_T) \arrow[d, "p_{23}^*(f)"] \\
(a_q \times \mathrm{id}_T)^*(\mathcal{F}'_T) \arrow[r, "p_{12}^*\Phi_{\mathcal{F}'}"] & p_{23}^*(\mathcal{F}'_T)
\end{tikzcd}$$

where we used the equalities (7.1).

Definition 7.31. Let $\mathcal{S}$: $q : G \to A$ be an affine extension. The *(enriched) category* q-QA_{gr} -mod *of q–linearized (or* $\mathcal{S}$–linearized*) sheaves with graded morphisms* has as objects the $\mathcal{S}$–linearized sheaves and as morphisms the functor of graded morphisms of $\mathcal{S}$–linearized sheaves.

We finish this section by showing the relationship between the concepts of homogeneous and $\mathcal{S}$–linearized shaves.

Lemma 7.32. *Let $\mathcal{S}$: $q : G \to A$ be an affine extension and $\mathcal{F}$ a $\mathcal{S}$–linearized sheaf. Then $\mathcal{F}$ is homogeneous.*

Conversely, let $\mathcal{F} \in A_{\mathrm{h}}$ -mod *be a homogeneous sheaf. Then $\mathcal{F}$ admits an $Aut_{\mathrm{gr}}(\mathcal{F})$–linearization.*

Proof. If $\ell \in A(T)$ then there exists a fpqc morphism $f : T' \to T$ and $g \in G(T')$ such that $q \circ g = \ell \circ f$. Let $a_T : G \times A \times T \to A \times T$ and $a_{T'} : G \times A \times T' \to A \times T'$ be the actions induced by a_q. Then the linearization $\Phi : a_q^*(\mathcal{F}) \cong p_2^*(\mathcal{F}) \in \mathcal{O}_{G\times A}$ induces a G–linearization

$$\Psi : a_{T'}^*(\mathcal{F}_{T'}) \cong p_{23}^*(\mathcal{F}_{T'}) \in \mathcal{O}_{G\times A\times T'}$$

via the G–equivariant morphism $p_2 \circ (\mathrm{id}_A \times f) : A \times T' \to A$ — here we use remarks 7.29 (2) and 7.28(3)).

Now consider the following commutative diagram of T'–schemes:

$$
\begin{tikzcd}
 & & A \times T' \arrow[dd, "t_{\ell\circ f}"] \\
A \times T' \arrow[r, "{(g\circ p_2, \mathrm{id}_{A\times T'})}"] \arrow[urr, bend left, "\mathrm{id}_{A\times T'}"] \arrow[drr, bend right, "t_{\ell\circ f}"'] & G \times A \times T' \arrow[ur, "p_{23}"] \arrow[dr, "a_{T'}"'] & \\
 & & A \times T'
\end{tikzcd}
\tag{7.2}
$$

It follows from diagram (7.2) that $\Psi : a_{T'}^*\mathcal{F}_{T'} \to p_{23}^*\mathcal{F}_{T'}$ induces an isomorphism

$$\psi : (g\circ p_2, \mathrm{id}_{A\times T'})^* a_{T'}^* \mathcal{F}_{T'} \cong t_{a\circ\sigma}^* \mathcal{F}_{T'} \to (g\circ p_2, \mathrm{id}_{A\times T'})^* p_{23}^* \mathcal{F}_{T'} \cong \mathcal{F}_{T'}.$$

It follows that $(\psi, \ell\circ f) \in \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F})(T')$ and is such that $d(\psi, \ell\circ f) = \ell\circ f$. We deduce that d is surjective in the fpqc topology, and therefore $\mathcal{F}$ is homogeneous.

In order to prove the converse, assume that $\mathcal{F} \in QA$-mod is homogeneous. Then $Aut_{\mathrm{gr}}(\mathcal{F})$ is an affine extension of A. In particular, $d : \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}) \to A$ is a fpqc morphism, and it follows from the homogeneity applied to the $\mathrm{Aut}_{\mathrm{gr}}(\mathcal{F})$–point d that there exists an isomorphism $\varphi : t_d^*\mathcal{F}_{\mathrm{Aut}_{\mathrm{gr}}(\mathcal{F})} \cong \mathcal{F}_{\mathrm{Aut}_{\mathrm{gr}}(\mathcal{F})}$.

Let $\sigma : \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}) \times A \to A \times \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F})$ be the switch morphism. Then $a_d = p_1\circ t_d\circ \sigma : \mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}) \times A \to A$. Since $\mathcal{F}_{\mathrm{Aut}_{\mathrm{gr}}(\mathcal{F})} = p_1^*\mathcal{F} \in QA_{\mathrm{Aut}_{\mathrm{gr}}(\mathcal{F})}$-mod, it follows that $\Psi = \sigma^*\varphi : a_d^*\mathcal{F} \to p_2^*\mathcal{F} \in Q\big(\mathrm{Aut}_{\mathrm{gr}}(\mathcal{F}) \times A\big)$-mod is the seeked d–linearization. $\square$

7.4. **The category of sheaf representations of an affine extension.**

Definition 7.33. Let $\mathcal{S}$: $q : G \to A$ be an affine extension of the abelian variety A.

(1) The *category* q- $QA_{\mathrm{p,gr}}$ -mod *(or* $\mathcal{S}$- $QA_{\mathrm{p,gr}}$ -mod*) of sheaf representations of* $\mathcal{S}$ is the full subcategory of $QA_{\mathrm{h,gr}}$-mod$(\mathcal{S})$ with objects the quasi-coherent, flat $\mathcal{S}$–linearized sheaves — recall that, by Lemma 7.32), the $\mathcal{S}$–linearized sheaves are homogeneous.

(2) We denote as q- $C_{\mathrm{lf}}A_{\mathrm{gr}}$ -mod $\subset \mathcal{S}$- $QA_{\mathrm{p,gr}}$ -mod the full subcategory of coherent, flat (necessarily locally free) sheaves — in other words, the objects of q- $C_{\mathrm{lf}}A_{\mathrm{gr}}$ -mod are the sheaf representations that are locally free, of finite rank. We will denote also $\mathcal{S}$- $C_{\mathrm{lf}}A_{\mathrm{gr}}$ -mod $=$ q- $C_{\mathrm{lf}}A_{\mathrm{gr}}$ -mod.

(3) The subcategory q- QA_{p} -mod $\subset \mathcal{S}$- $QA_{\mathrm{p,gr}}$ -mod is defined by taking as morphisms the pairs $(f, 0) \in \mathrm{Hom}_{\mathcal{S}\text{-}QA_{\mathrm{p,gr}}\text{-mod}}(\mathcal{S})(\mathcal{F}, \mathcal{G})$ belonging to $\mathrm{Hom}_0(\mathcal{F}, \mathcal{G})$. We will also denote $\mathcal{S}$- QA_{p} -mod $=$ q- QA_{p} -mod.

(4) The subcategory q- $C_{\mathrm{lf}}A$ -mod $= \mathcal{S}$- $C_{\mathrm{lf}}A$ -mod $\subset$ q- QA_{p} -mod is defined as the full subcategory of coherent sheaf representations (i.e. locally free of finite rank) of $\mathcal{S}$ with morphisms the pairs $(f, 0)$. Notice that q- $C_{\mathrm{lf}}A$ -mod can be seen also as a subcategory of q- $C_{\mathrm{lf}}A_{\mathrm{gr}}$ -mod.

Definition 7.34. A $\mathcal{S}$–sheaf representation $\mathcal{F}$ is *rational* if there exists a filtered system of coherent subrepresentations $\mathcal{F}_\alpha \subset \mathcal{F}_\beta \subset \mathcal{F}$, of finite rank n_α, such that $\mathcal{F} \cong \mathrm{colim}_\alpha \mathcal{F}_\alpha$. In relation with the categories considered in Definition 7.33 when we restrict the objects to the rational sheaves, we add the prefix r to the notations e.g. we write rq- $QA_{\mathrm{p,gr}}$ -mod, etc.

Theorem 7.35. *Let $\mathcal{S}$: $q : G \to A$ be an affine extension. Then the equivalence of categories $\mathbb{VB}_{\mathrm{gr}} : C_{\mathrm{lf}} A_{\mathrm{gr}}\text{-mod} \to \mathrm{HVB}_{\mathrm{gr}}(A)$ (see Lemma 7.22) induces an equivalence $\mathcal{S}\text{-} C_{\mathrm{lf}} A_{\mathrm{gr}}\text{-mod} \cong \mathrm{Rep}(\mathcal{S})$.*

PROOF. By Remark 7.28, we have that $\mathbb{VB}\, C_{\mathrm{lf}} A\text{-mod} \to \mathrm{HVB}_0(A)$ induces an equivalence $\mathcal{S}\text{-} C_{\mathrm{lf}} A\text{-mod} \cong \mathrm{Rep}_0(\mathcal{S})$. Hence, we only need to show that if $\mathcal{F}, \mathcal{F}' \in \mathcal{S}\text{-} C_{\mathrm{lf}} A\text{-mod}$ are q–linearized sheaves, then $\mathbb{VB}_{\mathrm{gr}} : \mathrm{Hom}_{\mathrm{gr}}(\mathcal{F}, \mathcal{F}') \to \mathrm{Hom}_{\mathrm{gr}}\big(\mathbb{VB}(\mathcal{F}), \mathbb{VB}(\mathcal{F}')\big)$ restricts to a bijection $\mathrm{Hom}_{\mathcal{S}\text{-} C_{\mathrm{lf}} A_{\mathrm{gr}}\text{-mod}}(\mathcal{F}, \mathcal{F}') \to \mathrm{Hom}_{\mathrm{Rep}(\mathcal{S})}\big(\mathbb{VB}(\mathcal{F}), \mathbb{VB}(\mathcal{F}')\big)$.

Denote $(\pi : F \to A) := \mathbb{VB}(\mathcal{F})$ and $(\pi' : F' \to A) := \mathbb{VB}(\mathcal{F}')$. Let T be a $\Bbbk$–scheme and $(f, \ell) \in \mathrm{Hom}_{\mathrm{gr}}(F, F')(T)$. Let $a_F : G \mathbin{\widetilde{\times}} F \to F$ and $a_{F'} : G \mathbin{\widetilde{\times}} F' \to F'$ be the q-actions on $\mathbb{VB}(\mathcal{F})$ and $\mathbb{VB}(\mathcal{F}')$ induced by $\Phi_{\mathcal{F}}$ and $\Phi_{\mathcal{F}'}$, the q–linearizations on $\mathcal{F}$ and $\mathcal{F}'$ respectively. Consider the morphisms $a_{q,T} = a_q \times \mathrm{id}_T : G \times A_T = G \times A \times T \to A_T = A \times T$, $a_{F,T} = a_1 \times \mathrm{id}_T : G \times F_T = G \times F \times T \to A_T$ and $\pi_T = \pi \times \mathrm{id}_T : F_T \to A_T$. Then, from the cartesian diagram of T–schemes

$$\begin{tikzcd} G \times F_T \arrow[r, "a_{F,T}"] \arrow[d, "\mathrm{id}_G \times \pi_T"'] & F_T \arrow[d, "\pi_T"] \\ G \times A_T \arrow[r, "a_{q,T}"] & A_T \end{tikzcd}$$

where $\pi_T = \pi \times \mathrm{id}_T$, we deduce that (f, ℓ) is G–equivariant if the following diagram of T-schemes is commutative:

$$\begin{tikzcd} G \times F_T \arrow[r, "\mathrm{id}_G \times f"] \arrow[d, "a_{F,T}"'] & G \times F'_T \arrow[d, "a_{F',T}"] \\ F_T \arrow[r, "f"] \arrow[d, "\pi_T"'] & F'_T \arrow[d, "\pi_T"] \\ A_T \arrow[r, "t_\ell"] & A_T \end{tikzcd} \tag{7.3}$$

where we consider the T-structure given by projection on the last coordinate.

Taking into account that $t_\ell \circ (\pi \times \mathrm{id}_T) = (\pi' \times \mathrm{id}_T) \circ f$, we deduce that the commutativity of Diagram (7.3) is equivalent to the commutativity of the diagram:

$$\begin{tikzcd} (a_q \times \mathrm{id}_T)^* t_\ell^*(\mathcal{F}_T) \arrow[r, "(\mathrm{id}_G \times t_\ell)^* p_{12}^*(\Phi_{\mathcal{F}}^\vee)"] \arrow[d, "(a_q \times \mathrm{id}_T)^*(\widetilde{f})"'] & p_{23}^* t_\ell^*(\mathcal{F}_T) \arrow[d, "p_{23}^*(\widetilde{f})"] \\ (a_q \times \mathrm{id}_T)^*(\mathcal{F}'_T) \arrow[r, "p_{12}^*(\Phi_{\mathcal{F}'}^\vee)"] & p_{23}^*(\mathcal{F}'_T) \end{tikzcd}$$

where $\mathbb{VB}(\widetilde{f}, \ell) = (f, \ell)$. In other words, $(\widetilde{f}, \ell) \in \mathrm{Hom}_{\mathcal{S}\text{-} C_{\mathrm{lf}} A_{\mathrm{gr}}\text{-mod}}(\mathcal{F}, \mathcal{F}')$. $\square$

Combining Proposition 7.5 and Theorem 7.35, we get the following:

Corollary 7.36. *Let $\mathcal{S}$: $q : G \to A$ be an affine extension and $\mathcal{H}_q$ its associated Hopf sheaf. Then the equivalences $\mathbb{VB} : {}^{\mathcal{H}_q}\mathsf{M}_{\mathrm{fin}} \to \mathrm{Rep}_0(\mathcal{S})$ and $\mathbb{VB}_{\mathrm{gr}} : C_{\mathrm{lf}} A_{\mathrm{h,gr}}\text{-mod} \to \mathrm{HVB}_{\mathrm{gr}}(A)$ induce equivalences*

$${}^{\mathcal{H}_q}\mathsf{M}_{\mathrm{fin}} \cong \mathrm{Rep}(\mathcal{S}) \cong \mathcal{S}\text{-} C_{\mathrm{lf}} A\text{-mod}. \quad \square$$

We would like to extend now the equivalence ${}^{\mathcal{H}_q}\mathsf{M}_{\mathrm{fin}} \cong$ q-$C_{\mathrm{lf}}A$-mod to the category of rational sheaf representations. First, we need to establish the notion of rational comodule.

Definition 7.37. Let$\mathcal{H}$ be a Hopf sheaf on the abelian variety A. A $\mathcal{H}$–comodule $\mathcal{F}$ is *rational* if there exists a filtered system of coherent flat sub-comodules $\mathcal{F}_\alpha \subset \mathcal{F}_\beta \subset \mathcal{F}$, of finite rank n_α, such that $\mathcal{F}$ is the filtered union of the subsheaves $\mathcal{F}_\alpha$; that is, $\mathcal{F} = \operatorname{colim}_\alpha \mathcal{F}_\alpha$.

Notice that since the limit of flat modules is flat, a rational $\mathcal{H}$–comodule is necessarily flat.

We denote by ${}^{\mathcal{H}}\mathsf{M}_{rA_{\mathrm{gr}}\text{-mod}} \subset {}^{\mathcal{H}}\mathsf{M}_{A_{\mathrm{gr}}\text{-mod}}$ the full category of rational $\mathcal{H}$–comodules, and by ${}^{\mathcal{H}}\mathsf{M}_{rA\text{-mod}} \subset {}^{\mathcal{H}}\mathsf{M}_{rA_{\mathrm{gr}}\text{-mod}}$ the wide subcategory with the same objects and morphisms $\operatorname{Ker} d$, where $d : {}^{\mathcal{H}}\mathsf{M}_{A_{\mathrm{gr}}\text{-mod}} \to A$ is the degree map. In other words,

$$\operatorname{Hom}_{{}^{\mathcal{H}}\mathsf{M}_{rA\text{-mod}}}(\mathcal{F},\mathcal{G})(T) = \big\{(f,0) : (f,0) \in \operatorname{Hom}_{{}^{\mathcal{H}}\mathsf{M}_{A_{\mathrm{gr}}\text{-mod}}}(\mathcal{F},\mathcal{G})(T)\big\}.$$

Proposition 7.38. *Let $\mathcal{S}$: $q : G \to A$ be an affine extension and $\mathcal{H}_q$ the associated Hopf sheaf. If $(\mathcal{F},\Phi_{\mathcal{F}}) \in$ q-QA_{p}-mod, then $\mathcal{F}$ admits a structure of (rational) $\mathcal{H}_q$-comodule.*

Conversely, if $(\mathcal{F},\chi_{\mathcal{F}}) \in {}^{\mathcal{H}_q}\mathsf{M}_{rA\text{-mod}}$ then $\mathcal{F}$ admits a structure of (rational) $\mathcal{S}$–sheaf representation.

Proof. Let $(\mathcal{F},\Phi_{\mathcal{F}}) \in$ q-QA_{p}-mod, and $(\mathcal{F}_\alpha, \phi_\alpha = \phi|_{a_q^*\mathcal{F}_\alpha})$ be a filtered system of coherent $\mathcal{S}$–sheaf subrepresentations with $\operatorname{colim}\mathcal{F}_\alpha = \cup\mathcal{F}_\alpha = \mathcal{F}$ and $(a_q)_\alpha : G \times \operatorname{Spec}\big(\mathcal{S}(\mathcal{F}_\alpha^\vee)\big) \to \operatorname{Spec}\big(\mathcal{S}(\mathcal{F}_\alpha^\vee)\big)$ the associated $\mathcal{S}$–action (see Theorem 7.35). By Proposition 7.5, $(a_q)_\alpha$ induces a structure of $\mathcal{H}_q$–comodule $\chi_{\mathcal{F}_\alpha} : F_\alpha \to \mathcal{H}_q \,\widetilde{\boxtimes}\, \mathcal{F}_\alpha \subset \mathcal{H}_q \,\widetilde{\boxtimes}\, \mathcal{F}$. Since this association is of functorial nature, it follows that is $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$, then $\chi_{\mathcal{F}_\alpha} = \chi_{\mathcal{F}_\beta}|_{\mathcal{F}_\alpha}$. It follows that the family $\chi_{\mathcal{F}_\alpha}$ induces a structure of $\mathcal{H}_q$–comodule $\chi_{\mathcal{F}} : \mathcal{F} = \operatorname{colim}\mathcal{F}_\alpha \to \mathcal{H}_q \,\widetilde{\boxtimes}\, \mathcal{F}$, such that $\chi_{\mathcal{F}}|_{\mathcal{F}_\alpha} = \chi_{\mathcal{F}_\alpha}$.

The proof of the converse is similar and therefore is omitted. □

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