

RC-positive metrics on rationally connected manifolds

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Abstract. In this paper, we prove that if a compact Kähler manifold X has a smooth Hermitian metric ω such that (T_X, ω) is uniformly RC-positive, then X is projective and rationally connected. Conversely, we show that, if a projective manifold X is rationally connected, then the dual tautological line bundle $\mathcal{O}_{T_X^*}(-1)$ is uniformly RC-positive (which is equivalent to the existence of some RC-positive complex Finsler metric on X). As an application, we prove that if (X, ω) is a compact Kähler manifold with *certain* quasi-positive holomorphic sectional curvature, then X is projective and rationally connected.

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1. Introduction

A projective manifold X is called *rationally connected* if any two points of X can be connected by some rational curves. It is easy to show that on a rationally connected projective manifold, one has

$$H^0(X, (T_X^*)^{\otimes m}) = 0, \quad \text{for every } m \geq 1.$$

A well-known conjecture of Mumford says that the converse is also true.

Conjecture. Let X be a projective manifold. If

$$H^0(X, (T_X^*)^{\otimes m}) = 0, \quad \text{for all } m \geq 1,$$

then X is rationally connected.

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This conjecture holds when $\dim X \leq 3$ ([KMM92]) and not much has been known in higher dimensions, and we refer to [KMM92, Cam92, DPS96B, Kol96, GHS03, Pet06, BDPP13, CDP14, CP14, BC15, Cam16, LP17, CH17, Yang18] and the reference therein.

In this paper, we obtain a differential geometric criterion for rational connectedness.

Definition 1.1. A Hermitian holomorphic vector bundle (\mathcal{E}, h) over a complex manifold X is called *uniformly RC-positive* (resp. *semi-positive*) at point $q \in X$, if there exists some nonzero vector $u \in T_q X$ such that for every nonzero vector $v \in \mathcal{E}_q$,

$$(1.1) \quad R^{\mathcal{E}}(u, \bar{u}, v, \bar{v}) > 0 \quad (\text{resp. } \geq 0).$$

(\mathcal{E}, h) is called *uniformly RC-positive* (resp. *semi-positive*) if it is uniformly RC-positive (resp. semi-positive) at each point of X . (\mathcal{E}, h) is called *uniformly RC-quasi-positive* if it is uniformly RC-semi-positive at all points of X , and uniformly RC-positive at some point of X .

The uniform RC-positivity in Definition 1.1 is slightly stronger than the notion ‘‘RC-positivity’’ introduced in [Yang18]. For instance, we show that Kähler manifolds with positive holomorphic sectional curvature have uniformly RC-positive tangent bundle (see Theorem 5.1 and Remark 5.2).

The first main result of this paper is on deformations of RC-quasi-positive Hermitian metrics, which demonstrates the flexibility of this new concept.

Theorem 1.2. *Let X be a compact Kähler manifold. Suppose there exists a smooth Hermitian metric h on T_X such (T_X, h) is uniformly RC-semi-positive over X . If there exist two open subsets S and U of X such that*

- (1) U is strongly pseudoconvex and $\bar{S} \subset U$;
- (2) (T_X, h) is uniformly RC-positive on $X \setminus S$.

Then X has a uniformly RC-positive Hermitian metric \tilde{h} . Moreover, X is a projective and rationally connected manifold.

The proof of Theorem 1.2 relies on some key ingredients in our previous paper [Yang18], a conformal perturbation method (Theorem 3.5) and the fundamental result of Campana-Demailly-Peternell on the criterion of rational connectedness. Recently, Lei Ni and Fangyang Zheng introduced in [NZ18a, NZ18b] some interesting curvature notions which can ensure the projectivity of compact Kähler manifolds. The following result is a special case of Theorem 1.2.

Corollary 1.3. *Let X be a compact Kähler manifold. If X admits a smooth Hermitian metric ω such that (T_X, ω) is uniformly RC-positive, then X is a projective and rationally connected manifold.*

Remark 1.4. Corollary 1.3 also holds if the uniformly RC-positive Hermitian metric is generalized to a uniformly RC-positive Finsler metric. See Theorem 6.7 for more details.

As an application of Corollary 1.3, we obtain the following Liouville type result.

Corollary 1.5. *Let X be a compact Kähler manifold. If X admits a smooth Hermitian metric ω such that (T_X, ω) is uniformly RC-positive, then there is no non-constant holomorphic map from X to a compact complex manifold Y where Y is Kobayashi hyperbolic, or Y has nef cotangent bundle.*

Corollary 1.5 may also hold if X is a compact complex manifold (see discussions in [Yang18a, Yang18c]).

It is well-known that a projective manifold X is rationally connected if and only if it has a very free rational curve C , i.e. $T_X|_C$ is ample. Recall that, on a smooth curve C , a vector bundle \mathcal{E} is uniformly RC-positive if and only if \mathcal{E} is ample. Hence, roughly speaking, the uniform RC-positivity of T_X is an *analytical analogue* of the existence of very free rational curves on X (see also [Pet12, Theorem 6.6]). Before giving a converse to Corollary 1.3, we fix the notations. Let \mathcal{E} be a holomorphic vector bundle over X and $\mathbb{P}(\mathcal{E}^*)$ be the projective bundle of \mathcal{E} . The tautological line bundle of $\mathbb{P}(\mathcal{E}^*)$ is denoted by $\mathcal{O}_{\mathcal{E}}(1)$. For instance, \mathcal{E} is called ample if $\mathcal{O}_{\mathcal{E}}(1)$ is an ample line bundle. The second main result of this paper is:

Theorem 1.6. *Let X be a projective manifold. If X is rationally connected, then the tautological dual line bundle $\mathcal{O}_{T_X^*}(-1)$ is uniformly RC-positive.*

We also conjecture that a vector bundle \mathcal{E} is (uniformly) RC-positive if and only if $\mathcal{O}_{\mathcal{E}^*}(-1)$ is uniformly RC-positive (e.g. Conjecture 4.9), which is analogous to a conjecture of Griffiths that a vector bundle \mathcal{E} is Griffiths positive if and only if $\mathcal{O}_{\mathcal{E}}(1)$ is Griffiths positive. Hence, it is reasonable to expect that rationally connected manifolds have (uniformly) RC-positive tangent bundles. On the other hand, it is well-known that there exists a one-to-one correspondence between the set of Hermitian metrics on $\mathcal{O}_{\mathcal{E}^*}(-1)$ and the set of Finsler metrics on \mathcal{E}^* . Hence, by Corollary 1.3, Remark 1.4 and Theorem 1.6, we can deduce that X is rationally connected if and only if X has certain “RC-positive” Finsler metric.

As an application of Theorem 1.2, we obtain

Theorem 1.7. *Let (X, ω) be a compact Kähler manifold with nonnegative holomorphic sectional curvature. If there exist two open subsets S and U of X such that U is strongly pseudoconvex, $\overline{S} \subset U$ and (X, ω) has positive holomorphic sectional curvature on $X \setminus S$. Then X has a uniformly RC-positive Hermitian metric. In particular, X is a projective and rationally connected manifold.*

In particular,

Corollary 1.8. *Let (X, ω) be a compact Kähler manifold with positive holomorphic sectional curvature. Then (T_X, ω) is uniformly RC-positive. Moreover, X is a projective and rationally connected manifold.*

Corollary 1.8 confirms a well-known conjecture ([Yau82, Problem 47]) of S.-T. Yau. It was firstly proved in our previous paper [Yang18, Theorem 1.7]. The proof here is slightly simpler than that in [Yang18]. We also conjecture that

Conjecture 1.9. *Let X be a compact Kähler manifold. If it has a smooth Hermitian metric ω with quasi-positive holomorphic sectional curvature (or, with uniformly RC-quasi-positive (T_X, ω)), then X is projective and rationally connected.*

The rest of the paper is organized as follows. In Section 2, we introduce the concept of uniform RC-positivity and investigate its geometric properties. In Section 3, we derive vanishing theorems for uniformly RC-positive vector bundles and prove Theorem 1.2 and Corollary 1.3. In Section 4, we prove Theorem 1.6 and propose more questions. In Section 5, we give a proof of Theorem 1.7.

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2. Uniformly RC-positive Hermitian vector bundles over complex manifolds

Let (\mathcal{E}, h) be a Hermitian holomorphic vector bundle over a complex manifold X with Chern connection ∇ . Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of \mathcal{E} . The curvature tensor $R^\mathcal{E} \in \Gamma(X, \Lambda^{1,1}T_X^* \otimes \text{End}(\mathcal{E}))$ has components

$$(2.1) \quad R_{i\bar{j}\alpha\bar{\beta}}^\mathcal{E} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}.$$

(Here and henceforth we sometimes adopt the Einstein convention for summation.)

If (X, ω_g) is a Hermitian manifold, then (T_X, g) has Chern curvature components

$$(2.2) \quad R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{\ell}}}{\partial \bar{z}^j}.$$

The Chern-Ricci curvature $\text{Ric}(\omega_g)$ of (X, ω_g) is represented by $R_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}$ and the second Chern-Ricci curvature $\text{Ric}^{(2)}(\omega_g)$ has components $R_{k\bar{\ell}}^{(2)} = g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}$.

Definition 2.1. A Hermitian holomorphic vector bundle (\mathcal{E}, h) over a complex manifold X is called *Griffiths positive* if at each point $q \in X$ and for any nonzero vector

$v \in \mathcal{E}_q$, and any nonzero vector $u \in T_q X$,

$$(2.3) \quad R^{\mathcal{E}}(u, \bar{u}, v, \bar{v}) > 0.$$

As analogous to Griffiths positivity, we introduced in [Yang18] the following concept.

Definition 2.2. A Hermitian holomorphic vector bundle (\mathcal{E}, h) over a complex manifold X is called *RC-positive at point $q \in X$* , if for each nonzero vector $v \in \mathcal{E}_q$, there exists **some** nonzero vector $u \in T_q X$ such that

$$(2.4) \quad R^{\mathcal{E}}(u, \bar{u}, v, \bar{v}) > 0.$$

(\mathcal{E}, h) is called *RC-positive* if it is RC-positive at every point of X .

Remark 2.3. Similarly, one can define semi-positivity, negativity and etc.. For a Hermitian line bundle $(\mathcal{L}, h^{\mathcal{L}})$, it is RC-positive if and only if its curvature $-\sqrt{-1}\partial\bar{\partial}\log h^{\mathcal{L}}$ has at least one positive eigenvalue at each point of X .

The following vanishing theorem is one of the key ingredients by introducing the terminology ‘‘RC-positivity’’.

Theorem 2.4. *Let X be a compact complex manifold. If (\mathcal{E}, h) is RC-positive, then*

$$(2.5) \quad H^0(X, \mathcal{E}^*) = 0.$$

Proof. An algebraic proof is included in [Yang18a, Lemma 2.10]. Here we use an alternative proof by a simple maximum principle. Since (\mathcal{E}, h) is RC-positive, the induced bundle (\mathcal{E}^*, g) is RC-negative, i.e., at point q , for any nonzero section v of \mathcal{E}^* , there exists a nonzero vector u such that

$$R^{\mathcal{E}^*}(u, \bar{u}, v, \bar{v}) < 0$$

For any $\sigma \in H^0(X, \mathcal{E}^*)$, we have

$$(2.6) \quad \partial\bar{\partial}|s|_g^2 = \langle \nabla s, \nabla s \rangle_g - R^{\mathcal{E}^*}(\bullet, \bullet, s, \bar{s}).$$

Suppose $|s|_g^2$ attains its maximum at some point p and $|s|_g^2(p) > 0$. By applying maximum principle to (2.6), we get a contradiction. Hence, we deduce $s = 0$ and $H^0(X, \mathcal{E}^*) = 0$. \square

In particular, we obtain a simple criterion for the projectiveness of compact Kähler manifolds.

Corollary 2.5. *Let X be a compact Kähler manifold. Suppose $\Lambda^2 T_X$ is RC-positive, then X is projective.*

Proof. By Theorem 2.4, we have $H_{\partial}^{2,0}(X) = H_{\bar{\partial}}^{0,2}(X) = 0$. Hence, by the Kodaira theorem ([Kod54, Theorem 1], see also [Huy05, Proposition 3.3.2 and Corollary 5.3.3]), the Kähler manifold X is projective. \square

We introduce a notion slightly stronger than RC-positivity.

Definition 2.6. A Hermitian holomorphic vector bundle (\mathcal{E}, h) over a complex manifold X is called *uniformly RC-positive at point $q \in X$* , if there exists some vector $u \in T_q X$ such that for any nonzero vector $v \in \mathcal{E}_q$, one has

$$(2.7) \quad R^\mathcal{E}(u, \bar{u}, v, \bar{v}) > 0.$$

(\mathcal{E}, h) is called *uniformly RC-positive* if it is uniformly RC-positive at every $q \in X$.

Remark 2.7. We can define uniform RC-negativity (resp. uniform RC-nonnegativity, uniform RC-nonpositivity) in a similar way. We can also define *uniformly RC-positive along k -linearly independent directions*, if there exist k linearly independent vectors $u_1, \dots, u_k \in T_q X$ such that for any nonzero vector $v \in \mathcal{E}_q$ and for each $i = 1, \dots, k$, one has

$$R^\mathcal{E}(u_i, \bar{u}_i, v, \bar{v}) > 0.$$

Remark 2.8. It is easy to see that, for a line bundle (\mathcal{L}, h) , RC-positivity and uniform RC-positivity are actually equivalent.

Proposition 2.9. *Let (\mathcal{E}, h) be a Hermitian holomorphic vector bundle over a compact complex manifold X . Then the following statements are equivalent:*

- (1) (\mathcal{E}, h) is uniformly RC-positive;
- (2) for any Hermitian metric ω on X , there exists a positive constant $C = C(\omega, h)$ such that: for any point $q \in X$, there exists a unit vector $u \in T_q X$ such that

$$(2.8) \quad R^\mathcal{E}(u, \bar{u}, v, \bar{v}) \geq C|v|_h^2, \quad \text{for every } v \in \mathcal{E}_q.$$

Proof. (2) \implies (1) is obvious. For (2) \implies (1), let

$$(2.9) \quad C = \inf_{q \in X} \sup_{u \in T_q X \setminus \{0\}} \inf_{v \in \mathcal{E}_q \setminus \{0\}} \frac{R^\mathcal{E}(u, \bar{u}, v, \bar{v})}{|u|_\omega^2 |v|_h^2}.$$

We claim $C > 0$. Indeed, if $C \leq 0$, by the compactness of X , there exists some $q \in X$ such that

$$\sup_{u \in T_q X \setminus \{0\}} \inf_{v \in \mathcal{E}_q \setminus \{0\}} \frac{R^\mathcal{E}(u, \bar{u}, v, \bar{v})}{|u|_\omega^2 |v|_h^2} \leq 0.$$

Hence, for any unit vector $u \in T_q X$,

$$\inf_{v \in \mathcal{E}_q \setminus \{0\}} \frac{R^\mathcal{E}(u, \bar{u}, v, \bar{v})}{|v|_h^2} \leq 0,$$

which contradicts to the fact that (\mathcal{E}, h) is uniformly RC-positive at point $q \in X$. It is obvious that (2.8) follows from the definition (2.9) of the constant C . \square

Proposition 2.10. *If (\mathcal{E}, h_1) is uniformly RC-positive and (\mathcal{F}, h_2) is Griffiths semi-positive, then $(\mathcal{E} \otimes \mathcal{F}, h_1 \otimes h_2)$ is uniformly RC-positive.*

Proof. It follows from the curvature formula $R^{\mathcal{E} \otimes \mathcal{F}} = R^\mathcal{E} \otimes \text{Id}_\mathcal{F} + \text{Id}_\mathcal{E} \otimes R^\mathcal{F}$. \square

The following results can be deduced by similar methods as in [Yang18, Theorem 3.5].

Corollary 2.11. *Let (\mathcal{E}, h) be a Hermitian vector bundle over a compact complex manifold X .*

- (1) *If (\mathcal{E}, h) is uniformly RC-positive, then (\mathcal{E}, h) is RC-positive.*
- (2) *(\mathcal{E}, h) is uniformly RC-positive if and only if (\mathcal{E}^*, h^*) is uniformly RC-negative;*
- (3) *If (\mathcal{E}, h) is uniformly RC-negative, every subbundle \mathcal{S} of \mathcal{E} is uniformly RC-negative;*
- (4) *If (\mathcal{E}, h) is uniformly RC-positive, every quotient bundle \mathcal{Q} of \mathcal{E} is uniformly RC-positive;*
- (5) *If (\mathcal{E}, h) is uniformly RC-positive, every line subbundle \mathcal{L} of \mathcal{E}^* is not pseudo-effective.*

The uniform RC-positivity has some important functorial properties.

Proposition 2.12. *If (\mathcal{E}, h) is uniformly RC-positive, then $(\mathcal{E}^{\otimes m}, h^{\otimes m})$ are uniformly RC-positive for $m \in \mathbb{N}^+$. Similarly, $\text{Sym}^{\otimes k} \mathcal{E}$ ($k \in \mathbb{N}^+$) and $\Lambda^p \mathcal{E}$ ($1 \leq p \leq \text{rk}(\mathcal{E})$) are all uniformly RC-positive.*

Proof. Fix a smooth metric on the compact complex manifold X . For any $q \in X$, we choose a unit vector $u \in T_q X$ such that

$$(2.10) \quad \inf_{v \in \mathcal{E}_q \setminus \{0\}} \frac{R^\mathcal{E}(u, \bar{u}, v, \bar{v})}{|v|_h^2} \geq C > 0.$$

Let $\{e_1, \dots, e_r\}$ be a local unitary frame of \mathcal{E} at point q with respect to h . Then for any local vector $v = \sum_{i_1, \dots, i_m} v^{i_1 \dots i_m} e_{i_1} \otimes \dots \otimes e_{i_m} \in \Gamma(X, \mathcal{E}^{\otimes m})$, by using the tensor product curvature formula of $(\mathcal{E}^{\otimes m}, h^{\otimes m})$, we have

$$\begin{aligned} R^{\mathcal{E}^{\otimes m}}(u, \bar{u}, v, \bar{v}) &= \sum_{i_2, \dots, i_m} R^\mathcal{E} \left(u, \bar{u}, \sum_{i_1} v^{i_1 i_2 \dots i_m} e_{i_1}, \overline{\sum_{j_1} v^{j_1 i_2 \dots i_m} e_{j_1}} \right) + \dots \\ &\quad + \sum_{i_1, i_2, \dots, i_{m-1}} R^\mathcal{E} \left(u, \bar{u}, \sum_{i_m} v^{i_1 \dots i_{m-1} i_m} e_{i_m}, \overline{\sum_{j_m} v^{i_1 \dots i_{m-1} j_m} e_{j_m}} \right) \\ &\geq C \left(\sum_{i_2, \dots, i_m} \sum_{i_1} |v^{i_1 i_2 \dots i_m}|^2 + \dots + \sum_{i_1, \dots, i_{m-1}} \sum_{i_m} |v^{i_1 i_2 \dots i_m}|^2 \right) \\ &\geq mC |v|^2, \end{aligned}$$

where the first inequality follows from (2.10). Hence $(\mathcal{E}^{\otimes m}, h^{\otimes m})$ is uniformly RC-positive. Similarly, we can show $\text{Sym}^{\otimes k} \mathcal{E}$ ($k \in \mathbb{N}^+$) and $\Lambda^p \mathcal{E}$ ($1 \leq p \leq \text{rk}(\mathcal{E})$) are all uniformly RC-positive. \square

Remark 2.13. It is not hard to see that all Schur powers of a uniformly RC-positive vector bundle are uniformly RC-positive.

3. Vanishing theorems and rational connectedness of compact Kähler manifolds

In this section, we derive vanishing theorems for uniformly RC-positive vector bundles and prove Theorem 1.2 and Corollary 1.3.

Corollary 3.1. *If (\mathcal{E}, h) is a uniformly RC-positive vector bundle over a compact complex manifold X . Then*

$$(3.1) \quad H^0(X, (\mathcal{E}^*)^{\otimes m}) = 0, \quad \text{for every } m \geq 1.$$

Proof. It follows from Proposition 2.12, Corollary 2.11 and Theorem 2.4. \square

The following special case is of particular interest.

Corollary 3.2. *Let X be a compact complex manifold. If there exists a smooth Hermitian metric h such that (T_X, h) is uniformly RC-positive, then $H_{\bar{\partial}}^{p,0}(X) = 0$ for $1 \leq p \leq \dim X$ and*

$$(3.2) \quad H^0(X, (T_X^*)^{\otimes m}) = 0, \quad \text{for all } m \geq 1.$$

Proof. By Corollary 2.11 and Proposition 2.12, we know $\Lambda^p T_X$ and $T_X^{\otimes m}$ are all uniformly RC-positive. Hence by Corollary 3.1, $H_{\bar{\partial}}^{p,0}(X) \cong H^0(X, \Lambda^p T_X^*) = 0$. \square

Theorem 3.3. *Let (\mathcal{E}, h) be a uniformly RC-positive vector bundle over a compact complex manifold X . Then for any line bundle \mathcal{L} over X , there exists a positive constant $C = C(\mathcal{L})$ such that $\mathcal{E}^{\otimes m} \otimes \mathcal{L}^{\otimes k}$ is uniformly RC-positive for all $m, k \in \mathbb{N}^+$ with $m \geq Ck$. In particular,*

$$(3.3) \quad H^0\left(X, (\mathcal{E}^*)^{\otimes m} \otimes (\mathcal{L}^*)^{\otimes k}\right) = 0.$$

Proof. We fix an arbitrary smooth Hermitian metric g on \mathcal{L} , and assume that the curvature of (\mathcal{L}, g) is bounded from below by a negative constant $-B$. Fix a point $q \in X$. Since (\mathcal{E}, h) is uniformly RC-positive, there exists a positive constant C and a unit vector $u \in T_q X$ such that

$$\inf_{v \in \mathcal{E}_q \setminus \{0\}} \frac{R^{\mathcal{E}}(u, \bar{u}, v, \bar{v})}{|v|_h^2} \geq C > 0$$

By a similar computation as in Proposition 2.12, for any $v \in \mathcal{E}_q^{\otimes m}$, we have

$$(3.4) \quad R^{(\mathcal{E}^{\otimes m}, h^{\otimes m})}(u, \bar{u}, v, \bar{v}) \geq mC|v|^2.$$

We choose a local unitary frame $e \in \Gamma(X, \mathcal{L})$ centered at q , then

$$\begin{aligned} R^{\mathcal{E}^{\otimes m} \otimes \mathcal{L}^{\otimes k}}(u, \bar{u}, v \otimes e^{\otimes k}, \overline{v \otimes e^{\otimes k}}) &= R^{(\mathcal{E}^{\otimes m}, h^{\otimes m})}(u, \bar{u}, v, \bar{v}) + k|v|^2 R^{\mathcal{L}}(u, \bar{u}) \\ &\geq mC|v|^2 - kB|v|^2. \end{aligned}$$

Hence if $m \geq kB/C + 1$, then

$$R^{\mathcal{E}^{\otimes m} \otimes \mathcal{L}^{\otimes k}}(u, \bar{u}, v \otimes e^{\otimes k}, \overline{v \otimes e^{\otimes k}}) \geq |v|^2.$$

Hence, $\mathcal{E}^{\otimes m} \otimes \mathcal{L}^{\otimes k}$ is uniformly RC-positive. By Corollary 3.1, we obtain (3.3). \square

Recall that $\{z^i\}, \{e^\alpha\}$ are the local holomorphic coordinates and holomorphic frames on X and \mathcal{E} respectively.

Lemma 3.4. *Let $\tilde{h} = e^{-f}h$ for some $f \in C^2(X, \mathbb{R})$. Then the curvature tensor \tilde{R} of (\mathcal{E}, \tilde{h}) has the expression*

$$(3.5) \quad \tilde{R}_{i\bar{j}\alpha\bar{\beta}} = e^{-f}(R_{i\bar{j}\alpha\bar{\beta}} + f_{i\bar{j}}h_{\alpha\bar{\beta}}),$$

where R is the curvature tensor of (\mathcal{E}, h) .

Proof. It follows by a standard computation. \square

Theorem 3.5. *Let (\mathcal{E}, h) be a Hermitian holomorphic vector bundle over a compact complex manifold X . Suppose (\mathcal{E}, h) is uniformly RC-semi-positive over X . If there exist two open subsets S and U of X such that*

- (1) U is strongly pseudoconvex and $\bar{S} \subset U$;
- (2) (\mathcal{E}, h) is uniformly RC-positive on $X \setminus S$.

Then there exists a smooth Hermitian metric \tilde{h} on \mathcal{E} such that (\mathcal{E}, \tilde{h}) is uniformly RC-positive over X .

Proof. Fix an arbitrary smooth Hermitian metric ω on X . We define

$$(3.6) \quad C = \inf_{q \in X \setminus S} \sup_{u \in T_q X \setminus \{0\}} \inf_{v \in \mathcal{E}_q \setminus \{0\}} \frac{R(u, \bar{u}, v, \bar{v})}{|u|_\omega^2 |v|_h^2}.$$

Since $X \setminus S$ is compact and (\mathcal{E}, h) is uniformly RC-positive over $X \setminus S$, it is easy to see that $C > 0$. There exists a ‘‘cut-off’’ function $f \in C^\infty(X, \mathbb{R})$ such that

- (1) over X , we have

$$(3.7) \quad (\sqrt{-1}\partial\bar{\partial}f)(u, \bar{u}) \geq -\frac{C}{2}|u|_\omega^2;$$

- (2) over \bar{S} , we have

$$(3.8) \quad (\sqrt{-1}\partial\bar{\partial}f)(u, \bar{u}) \geq C_1|u|_\omega^2$$

for some positive constant C_1 .

Indeed, since U is strongly pseudoconvex, there exists a smooth strictly plurisubharmonic function $\varphi \in \text{Psh}(U)$. In particular, there exists a positive constant \tilde{C}_1 such that $(\sqrt{-1}\partial\bar{\partial}\varphi)(u, \bar{u}) \geq \tilde{C}_1|u|_\omega^2$ over the compact set \bar{S} . Next, we can extend the smooth function $\varphi|_{\bar{S}}$ to X and get a new function $\Phi \in C^\infty(X)$. It is obvious that, there exists a positive constant \tilde{C} such that $(\sqrt{-1}\partial\bar{\partial}\Phi)(u, \bar{u}) \geq -\tilde{C}|u|_\omega^2$ over X . Now we define $f = \frac{C}{2C}\Phi$, then f satisfies (3.7) and (3.8) with $C_1 = \frac{C\tilde{C}_1}{2C}$.

We define a new smooth Hermitian metric $\tilde{h} = e^{-f}h$ on \mathcal{E} . By formula (3.5), the curvature tensor \tilde{R} of (\mathcal{E}, \tilde{h}) satisfies

$$(3.9) \quad \tilde{R}(u, \bar{u}, v, \bar{v}) = e^{-f} (R(u, \bar{u}, v, \bar{v}) + (\partial\bar{\partial}f)(u, \bar{u}) \cdot |v|_h^2).$$

We claim that (\mathcal{E}, \tilde{h}) is uniformly RC-positive over X . Indeed, for a point $q \in S$, since (\mathcal{E}, h) is uniformly RC-semi-positive at q , there exists a unit vector $u \in T_q X$ such that $R(u, \bar{u}, v, \bar{v}) \geq 0$. By estimates (3.8), we have

$$\begin{aligned} \tilde{R}(u, \bar{u}, v, \bar{v}) &= e^{-f} (R(u, \bar{u}, v, \bar{v}) + (\partial\bar{\partial}f)(u, \bar{u}) \cdot |v|_h^2) \\ &\geq e^{-f} (\partial\bar{\partial}f)(u, \bar{u}) \cdot |v|_h^2 \\ &\geq C_1 e^{-f} |v|_h^2. \end{aligned}$$

for all $v \in \mathcal{E}_q$. For a point $q \in X \setminus S$, since (\mathcal{E}, h) is uniformly RC-positive over $X \setminus S$, by formula (3.6) and Proposition 2.9, there exists some unit vector $u \in T_q X$ such that

$$(3.10) \quad R(u, \bar{u}, v, \bar{v}) \geq C |v|_h^2$$

for every $v \in \mathcal{E}_q$. On the other hand, by estimate (3.7), we have

$$(3.11) \quad \tilde{R}(u, \bar{u}, v, \bar{v}) = e^{-f} (R(u, \bar{u}, v, \bar{v}) + (\partial\bar{\partial}f)(u, \bar{u}) \cdot |v|_h^2) \geq \frac{C}{2} e^{-f} |v|_h^2.$$

Hence, we conclude (\mathcal{E}, \tilde{h}) is uniformly RC-positive over X . \square

The proof of Theorem 1.2. By Theorem 3.5, T_X has a uniformly RC-positive metric $\tilde{h} = e^{-f} \cdot h$. By Proposition 2.12, we know $\Lambda^2 T_X$ is uniformly RC-positive. By Corollary 2.11 and Corollary 2.5, X is a projective manifold. On the other hand, by Theorem 3.3, for any line bundle \mathcal{L} over X , there exists a positive constant $C = C(\mathcal{L})$ such that

$$(3.12) \quad H^0(X, (T_X^*)^{\otimes m} \otimes \mathcal{L}^{\otimes k}) = 0$$

for all $m, k \in \mathbb{N}^+$ with $m \geq Ck$. Therefore, by a celebrated theorem of Campana-Demailly-Peternell [CDP14, Theorem 1.1], X is rationally connected. Indeed, we deduce from (3.12) that for each $1 \leq p \leq \dim X$, any invertible sheaf $\mathcal{F} \subset \Omega_X^p$ is not pseudo-effective. Otherwise, if there exists a pseudo-effective invertible sheaf $\mathcal{F} \subset \Omega_X^p$, then there exists a very ample line bundle A such that $H^0(X, \mathcal{F}^{\otimes \ell} \otimes A) \neq 0$ for all $\ell \geq 0$, and so $H^0(X, \text{Sym}^{\otimes \ell} \Omega_X^p \otimes A) \neq 0$. Since for some large m , $\text{Sym}^{\otimes \ell} \Omega_X^p \subset (T_X^*)^{\otimes m}$, we get a contradiction to (3.12). In particular, when $p = n$, K_X is not pseudo-effective. Thanks to [BDPP13], X is uniruled. Let $\pi : X \dashrightarrow Z$ be the associated MRC fibration of X . After possibly resolving the singularities of π and Z , we may assume that π is a proper morphism and Z is smooth. By a result of Graber, Harris and Starr [GHS03, Corollary 1.4], it follows that the target Z of the MRC fibration is either a point or a positive dimensional variety which is not uniruled. Suppose X is not rationally connected, then $\dim Z \geq 1$. Hence Z is not uniruled, by [BDPP13] again, K_Z is pseudo-effective. Since $K_Z = \Omega_Z^{\dim Z} \subset \Omega_X^{\dim Z}$ is pseudo-effective, we get a contradiction. Hence X is rationally connected. \square

If $S = \emptyset$, we have

Corollary 3.6. *Let X be a compact Kähler manifold. Suppose X has a Hermitian metric h such that (T_X, h) is uniformly RC-positive, then X is a projective and rationally connected manifold.*

Remark 3.7. Corollary 3.6 can also be proved by using Proposition 2.12 and [Yang18, Theorem 1.3]. The proof here is slightly simpler.

By using rational connectedness, one has

Corollary 3.8. *Let X be a compact Kähler manifold. If X admits a smooth Hermitian metric ω such that (T_X, ω) is uniformly RC-positive, then there is no non-constant holomorphic map from X to a compact complex manifold Y where Y lies in one of the following*

- (1) Y is Kobayashi hyperbolic;
- (2) Y has nef cotangent bundle;
- (3) Y has a Hermitian metric with non-positive holomorphic sectional curvature;
- (4) Y contains no rational curve.

4. RC-positive metrics on rationally connected manifolds

In this section, we will discuss general theory for uniformly RC-positive vector bundles and prove Theorem 1.6. Let's recall that a line bundle \mathcal{L} is uniformly RC-positive if and only if it has a smooth Hermitian metric h such that its curvature $-\sqrt{-1}\partial\bar{\partial}\log h$ has at least one positive eigenvalue everywhere. In [Yang17, Theorem 1.4], we obtained an equivalent characterization for uniformly RC-positive line bundles.

Theorem 4.1. *Let \mathcal{L} be a holomorphic line bundle over a compact complex manifold X . The following statements are equivalent.*

- (1) \mathcal{L} is uniformly RC-positive;
- (2) the dual line bundle \mathcal{L}^* is not pseudo-effective.

Two key ingredients in the proof of Theorem 4.1 are a conformal (exponential) perturbation method and an integration criterion for pseudo-effectiveness over compact complex manifolds(e.g. [Lam99]). We refer to [Yang17a, Yang17] and the references therein.

Corollary 4.2. *Let X be a projective manifold. If \mathcal{L} is uniformly RC-semi-positive, then the dual line bundle \mathcal{L}^* is not big.*

Proof. Let \mathcal{A} be an ample line bundle over X . We argue by contradiction. Suppose \mathcal{L}^* is big, then there exists a large number $k \in \mathbb{Z}_+$ such that $\mathcal{L}^{*k} \otimes \mathcal{A}^*$ is big. By Theorem 4.1, $\mathcal{L}^k \otimes \mathcal{A}$ can not be uniformly RC-positive which is absurd. \square

The following concepts are generalizations of uniformly RC-positivity for line bundles.

Definition 4.3. Let \mathcal{L} be a line bundle over a compact complex manifold X .

- (1) \mathcal{L} is called *q-positive*, if there exists a smooth Hermitian metric h on \mathcal{L} such that the Chern curvature $R^{(\mathcal{L}, h)} = -\sqrt{-1}\partial\bar{\partial}\log h$ has at least $(\dim X - q)$ positive eigenvalues at every point on X .
- (2) \mathcal{L} is called *q-ample*, if for any coherent sheaf \mathcal{F} on X there exists a positive integer $m_0 = m_0(X, \mathcal{L}, \mathcal{F}) > 0$ such that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^m) = 0 \quad \text{for } i > q, m \geq m_0.$$

Note that, *the nontrivial range is* $0 \leq q \leq \dim X - 1$. By the celebrated Cartan–Serre–Grothendieck criterion for ampleness and Kodaira embedding theorem, one has

Theorem 4.4 (Cartan–Serre–Grothendieck, Kodaira). *Let $\mathcal{L} \rightarrow X$ be a holomorphic line bundle over a projective manifold X . Then the following statements are equivalent*

- (1) \mathcal{L} is 0-ample;
- (2) \mathcal{L} is 0-positive.

As a weak dual to the Cartan–Serre–Grothendieck–Kodaira Theorem 4.4, we established in [Yang17] that

Theorem 4.5. *Let $\mathcal{L} \rightarrow X$ be a holomorphic line bundle over a projective manifold X . Then the following statements are equivalent*

- (1) \mathcal{L} is $(\dim X - 1)$ -ample;
- (2) \mathcal{L} is $(\dim X - 1)$ -positive;
- (3) \mathcal{L} is uniformly RC-positive.

For related results on q -ampleness and q -positivity, we refer to [AG62, DPS93, Dem11, Tot13, Ott12, Bro12, Mat13, Yang17] and the references therein.

Let \mathcal{E} be a vector bundle over X and $\mathbb{P}(\mathcal{E}^*)$ be the projective bundle of \mathcal{E} . The points of the projective bundle $\mathbb{P}(\mathcal{E}^*)$ of $\mathcal{E} \rightarrow X$ can be identified with the hyperplanes of \mathcal{E} . Note that every hyperplane V in \mathcal{E}_z corresponds bijectively to the line of linear forms in \mathcal{E}_z^* which vanish on V . Let $\pi : \mathbb{P}(\mathcal{E}^*) \rightarrow X$ be the natural projection. There is a tautological hyperplane subbundle S of $\pi^*\mathcal{E}$ over $\mathbb{P}(\mathcal{E}^*)$ such that

$$S_{[\xi]} = \xi^{-1}(0) \subset \mathcal{E}_z$$

for all $\xi \in \mathcal{E}_z^* \setminus \{0\}$. The quotient line bundle $\pi^*\mathcal{E}/S$ is denoted $\mathcal{O}_{\mathcal{E}}(1)$ and is called the tautological line bundle associated to $\mathcal{E} \rightarrow X$. Hence there is an exact sequence of vector bundles over $\mathbb{P}(\mathcal{E}^*)$

$$(4.1) \quad 0 \rightarrow S \rightarrow \pi^*\mathcal{E} \rightarrow \mathcal{O}_{\mathcal{E}}(1) \rightarrow 0.$$

A holomorphic vector bundle $\mathcal{E} \rightarrow X$ is called ample if the line bundle $\mathcal{O}_{\mathcal{E}}(1)$ is ample over $\mathbb{P}(\mathcal{E}^*)$.

Proposition 4.6. *Let (\mathcal{E}, h) be RC-positive vector bundle over a compact complex manifold X . Then*

- (1) *the tautological line bundle $\mathcal{O}_{\mathcal{E}}(1)$ is $(\dim X - 1)$ -positive over $\mathbb{P}(\mathcal{E}^*)$;*
- (2) *$\mathcal{O}_{\mathcal{E}^*}(-1)$ is uniformly RC-positive over $\mathbb{P}(\mathcal{E})$;*
- (3) *$\mathcal{O}_{\mathcal{E}^*}(1)$ is not pseudo-effective over $\mathbb{P}(\mathcal{E})$.*

Proof. It follows from the curvature formulas of $\mathcal{O}_{\mathcal{E}^*}(-1)$ and $\mathcal{O}_{\mathcal{E}}(1)$ induced by (\mathcal{E}, h) (e.g. [Yang18, Proposition 4.1], [Yang18a, Proposition 2.6]) and Theorem 4.1. \square

Theorem 4.7. *Let \mathcal{E} be a holomorphic vector bundle over a projective manifold X . Then the following statements are equivalent:*

- (1) *$\mathcal{O}_{\mathcal{E}}(1)$ is $(\dim X - 1)$ -ample;*
- (2) *$\mathcal{O}_{\mathcal{E}^*}(1)$ is not pseudo-effective;*
- (3) *$\mathcal{O}_{\mathcal{E}^*}(-1)$ is uniformly RC-positive.*

Proof. If $\mathcal{O}_{\mathcal{E}}(1)$ is $(\dim X - 1)$ -ample, we deduce that $\mathcal{O}_{\mathcal{E}^*}(1)$ is not pseudo-effective. Otherwise, if $\mathcal{O}_{\mathcal{E}^*}(1)$ is pseudo-effective, it is well-known that there exists an ample line bundle \mathcal{L} on $\mathbb{P}(\mathcal{E})$ such that

$$(4.2) \quad H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}^*}(m) \otimes \mathcal{L}) \neq 0 \quad \text{for all } m \geq 0.$$

More precisely, \mathcal{L} can be chosen in such a way: fix a very ample line bundle \mathcal{H} over $\mathbb{P}(\mathcal{E})$, and if \mathcal{L} is an ample line bundle such that

$$\mathcal{L} \otimes K_{\mathbb{P}(\mathcal{E})}^{-1} \otimes \mathcal{H}^{-(\dim \mathbb{P}(\mathcal{E})+1)}$$

is ample, then (4.2) holds. We can choose \mathcal{L} in a special form

$$(4.3) \quad \mathcal{L} = \mathcal{O}_{\mathcal{E}^*}(m_0) \otimes \pi^*(\mathcal{A}^{\otimes m_1})$$

where \mathcal{A} is an ample line bundle over X , and m_0, m_1 are two large positive integers with $m_1 \gg m_0$ and $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$ is the projection. For reader's convenience, we write down the construction explicitly. Indeed, for large k , $\mathcal{E}^* \otimes \mathcal{A}^{\otimes k}$ is an ample vector bundle over X , i.e. $\mathcal{O}_{\mathcal{E}^*}(1) \otimes \pi^*(\mathcal{A}^{\otimes k})$ is an ample line bundle over $\mathbb{P}(\mathcal{E})$. There exists a large k_1 such that $\mathcal{H} = \mathcal{O}_{\mathcal{E}^*}(k_1) \otimes \pi^*(\mathcal{A}^{\otimes(kk_1)})$ is a very ample line bundle over $\mathbb{P}(\mathcal{E})$. On the other hand,

$$K_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_{\mathcal{E}^*}(-r) \otimes \pi^*(K_X) \otimes \pi^*(\det \mathcal{E}^*)$$

where r is the rank of \mathcal{E} . There exists a positive number k_2 such that

$$\mathcal{A}^{\otimes k_2} \otimes K_X^{-1} \otimes \det \mathcal{E}$$

is ample over X . Now we can take

$$m_0 = k_1(n+r) - r + 1 \quad \text{and} \quad m_1 = k_2 + kk_1(n+r) + k$$

and so

$$\mathcal{L} \otimes K_{\mathbb{P}(\mathcal{E})}^{-1} \otimes \mathcal{H}^{-(n+r)} = \mathcal{O}_{\mathcal{E}^*}(1) \otimes \pi^*(\mathcal{A}^{\otimes k}) \otimes \pi^* \left(\mathcal{A}^{\otimes k_2} \otimes K_X^{-1} \otimes \det \mathcal{E} \right)$$

is ample over $\mathbb{P}(\mathcal{E})$.

Therefore, by the Le Potier isomorphism, (4.2) is equivalent to

$$(4.4) \quad H^0(X, \text{Sym}^{\otimes k} \mathcal{E}^* \otimes \mathcal{A}^{\otimes m_1}) \neq 0$$

for large k . Applying the Serre duality on X and the Le Potier isomorphism again, we obtain

$$H^n(\mathbb{P}(\mathcal{E}^*), \mathcal{O}_{\mathcal{E}}(k) \otimes \pi_1^*((\mathcal{A}^*)^{\otimes m_1}) \otimes \Omega_{\mathbb{P}(\mathcal{E}^*)}^n) \neq 0,$$

for large k where $\pi_1 : \mathbb{P}(\mathcal{E}^*) \rightarrow X$ is the projection. Let $\mathcal{F} = \pi_1^*((\mathcal{A}^*)^{\otimes m_1}) \otimes \Omega_{\mathbb{P}(\mathcal{E}^*)}^n$, and we know $\mathcal{O}_{\mathcal{E}}(1)$ can not be $(n-1)$ -ample. This is a contradiction. Hence $\mathcal{O}_{\mathcal{E}^*}(1)$ is not pseudo-effective. The proof of (2) \implies (1) is similar. The equivalence of (2) and (3) follows from Theorem 4.1. \square

On the other hand, we have

Proposition 4.8. *Let X be a projective manifold. Suppose $\mathcal{O}_{\mathcal{E}}(1)$ is $(\dim X - 1)$ -ample, then the restriction of $\mathcal{O}_{\mathcal{E}}(1)$ to every smooth submanifold $Y \subset \mathbb{P}(\mathcal{E}^*)$ with $\dim Y = \dim X$ is $(\dim X - 1)$ -positive.*

Proof. Let $f : Y \rightarrow \mathbb{P}(\mathcal{E}^*)$ be the inclusion map. Using the projection formula and the Leray spectral sequence, one has

$$H^i(Y, \mathcal{F} \otimes (f^* \mathcal{O}_{\mathcal{E}}(m))) = H^i(\mathbb{P}(\mathcal{E}^*), f_*(\mathcal{F}) \otimes \mathcal{O}_{\mathcal{E}}(m))$$

Hence, if $\mathcal{O}_{\mathcal{E}}(1) \rightarrow \mathbb{P}(\mathcal{E}^*)$ is $(\dim X - 1)$ -ample, $f^*(\mathcal{O}_{\mathcal{E}}(1)) \rightarrow Y$ is also $(\dim X - 1)$ -ample. On the other hand, since $\dim Y = \dim X$, by Theorem 4.5, the $(\dim X - 1)$ -ample line bundle $f^*(\mathcal{O}_{\mathcal{E}}(1))$ over Y is $(\dim X - 1)$ -positive. \square

As motivated by these properties, we propose the following conjecture.

Conjecture 4.9. *Let \mathcal{E} be a holomorphic vector bundle over a projective manifold X . Then the following statements are equivalent.*

- (1) $\mathcal{O}_{\mathcal{E}^*}(-1)$ is RC-positive;
- (2) $\mathcal{O}_{\mathcal{E}}(1)$ is $(\dim X - 1)$ -ample;
- (3) $\mathcal{O}_{\mathcal{E}}(1)$ is $(\dim X - 1)$ -positive;
- (4) \mathcal{E} is RC-positive.

Note that, the implications (4) \implies (3) \implies (2) \iff (1) in Conjecture 4.9 are known by Proposition 4.6, [AG62] and Theorem 4.7. On the other hand, when $\text{rank}(\mathcal{E}) = 1$ or $\dim X = 1$, Conjecture 4.9 is true by Theorem 4.5 and [CF90]. Note also that Conjecture 4.9 is also analogous to a conjecture of Griffiths.

Conjecture 4.10. *Let \mathcal{E} be a vector bundle over a projective manifold X . Then the following statements are equivalent.*

- (1) $\mathcal{O}_{\mathcal{E}}(1)$ is positive;
- (2) \mathcal{E} is Griffiths positive.

It is easy to see that if \mathcal{E} is Griffiths positive, then so is $\mathcal{O}_{\mathcal{E}}(1)$.

As an application of the vector bundle theory discussed above, we obtain a differential geometric characterization of rationally connected manifolds, and Theorem 1.6 is also a special case of it.

Theorem 4.11. *Let X be a projective manifold. Then the following statements are equivalent*

- (1) X is rationally connected;
- (2) the line bundle $\mathcal{O}_{\Lambda^p T_X^*}(-1)$ is uniformly RC-positive for every $1 \leq p \leq \dim X$.

Proof. If X is rationally connected, then by a variant of [CDP14, Theorem 1.1] (e.g. [Cam16, Proposition 1.4]), for any ample line bundle \mathcal{L} on X , there exists a positive integer m_0 such that

$$H^0(X, \text{Sym}^{\otimes m}(\Lambda^p T_X^*) \otimes \mathcal{L}^{\otimes k}) = 0$$

for $m \geq m_0 k$ and all $1 \leq p \leq \dim X$. We claim $\mathcal{O}_{\Lambda^p T_X^*}(-1)$ is RC-positive. Otherwise, by Theorem 4.7, $\mathcal{O}_{\Lambda^p T_X^*}(1)$ is pseudo-effective. Hence, by using a similar proof as in Theorem 4.7, we can find an ample line bundle \mathcal{A} over X such that (4.4) holds for $\mathcal{E} = \Lambda^p T_X$, that is

$$H^0(X, \text{Sym}^{\otimes m}(\Lambda^p T_X^*) \otimes \mathcal{A}^{\otimes m_1}) \neq 0$$

for all large m . This is a contradiction.

On the other hand, if $\mathcal{O}_{\Lambda^p T_X^*}(-1)$ is uniformly RC-positive over $\mathbb{P}(\Lambda^p T_X)$, then by the Le Potier isomorphism and Theorem 3.3, for any coherent sheaf of the form $\mathcal{F} = \pi^*(\mathcal{L}^{\otimes k})$ over $\mathbb{P}(\Lambda^p T_X)$, we have

$$(4.5) \quad H^0(X, \text{Sym}^{\otimes m}(\Lambda^p T_X^*) \otimes \mathcal{L}^{\otimes k}) \cong H^0(\mathbb{P}(\Lambda^p T_X), \mathcal{O}_{\Lambda^p T_X^*}(m) \otimes \mathcal{F}) = 0$$

for large m . Therefore, by [CDP14, Theorem 1.1] or [Cam16, Proposition 1.4], X is rationally connected. \square

By Theorem 4.11, the following conjecture is a special case of Conjecture 4.9.

Conjecture 4.12. *Let X be a projective manifold. If X is rationally connected, then T_X is RC-positive.*

More generally,

Problem 4.13. The following statements are equivalent on a projective manifold X .

- (1) X is rationally connected;
- (2) T_X is RC-positive;
- (3) T_X is uniformly RC-positive.

In [Yang18, Corollary 1.5], we obtain the following result.

Corollary 4.14. *Let X be a compact Kähler manifold. If there exist a Hermitian metric ω on X and a (possibly different) Hermitian metric h on T_X such that*

$$(4.6) \quad \mathrm{tr}_\omega R^{(T_X, h)} \in \Gamma(X, \mathrm{End}(T_X))$$

is positive definite, then X is projective and rationally connected.

In particular, by the celebrated Calabi-Yau theorem ([Yau78]), one gets the classical result of Campana([Cam92]) and Kollár-Miyaoka-Mori ([KMM92]) that Fano manifolds are rationally connected. We propose a conjecture converse to Corollary 4.14, which is also analogous to the classical fact that a compact complex manifold is Fano if and only if it has a Hermitian metric with positive Chern-Ricci curvature.

Problem 4.15. The following statements are equivalent on a projective manifold X .

- (1) X is rationally connected;
- (2) there exist a Hermitian metric ω on X and a (possibly different) Hermitian metric h on T_X such that $\mathrm{tr}_\omega R^{(T_X, h)}$ is positive definite.

Remark 4.16. Problem 4.15 is also known to J.-P. Demailly [Dem]. A positive solution to Problem 4.15 gives an affirmative answer to Conjecture 4.12.

5. Compact Kähler manifolds with nonnegative holomorphic sectional curvature

A compact Kähler manifold (X, ω) has positive (resp. nonnegative) holomorphic sectional curvature, if for any nonzero vector $\xi = (\xi^1, \dots, \xi^n)$,

$$R_{i\bar{j}k\bar{\ell}} \xi^i \bar{\xi}^j \xi^k \bar{\xi}^\ell > 0 \quad (\text{resp. } \geq 0)$$

at each point of X . The negativity and non-positivity of the holomorphic sectional curvature can be defined in a similar way.

Theorem 5.1. *Let (X, ω) be a compact Kähler manifold with positive (resp. nonnegative) holomorphic sectional curvature, then (T_X, ω) is uniformly RC-positive (resp. uniformly RC-semi-positive).*

Proof. Let κ be the positive holomorphic sectional curvature of (X, ω) , i.e.

$$(5.1) \quad \kappa = \inf_{q \in X} \inf_{U \in T_q X \setminus \{0\}} \frac{R(U, \bar{U}, U, \bar{U})}{|U|^4}.$$

For any point $q \in X$, let $e \in T_q X$ be a unit vector such that

$$(5.2) \quad R(e, \bar{e}, e, \bar{e}) = \inf_{U \in T_q X \setminus \{0\}} \frac{R(U, \bar{U}, U, \bar{U})}{|U|^4}.$$

We know $R(e, \bar{e}, e, \bar{e}) \geq \kappa > 0$. On the other hand, by [Yang18, Lemma 6.1], for any unit vector $W \in T_q X$, we have

$$(5.3) \quad 2R(e, \bar{e}, W, \bar{W}) \geq (1 + |\langle W, e \rangle|^2)R(e, \bar{e}, e, \bar{e}) \geq \kappa.$$

Hence, for any vector $v \in T_q X$, we obtain

$$R(e, \bar{e}, v, \bar{v}) \geq \frac{\kappa}{2} |v|^2,$$

and so (T_X, ω) is uniformly RC-positive. \square

Remark 5.2. If a compact Kähler manifold (X, ω) has negative (resp. nonpositive) holomorphic sectional curvature, then (T_X, ω) is uniformly RC-negative (resp. uniformly RC-nonpositive). There are many Kähler and non-Kähler complex manifolds which have uniformly RC-positive tangent bundles, for instances,

- compact Kähler manifold with non-negative holomorphic bisectional curvature and positive first Chern class ([Mok88]);
- Hopf manifold $\mathbb{S}^1 \times \mathbb{S}^{2n+1}$ ([LY17, formula (6.4)]).

As an application of Theorem 5.1 and Theorem 1.2, we obtain

Theorem 5.3. *Let (X, ω) be a compact Kähler manifold with nonnegative holomorphic sectional curvature. If there exist two open subsets S and U of X such that U is strongly pseudoconvex, $\bar{S} \subset U$ and (X, ω) has positive holomorphic sectional curvature on $X \setminus S$. Then X has a uniformly RC-positive Hermitian metric. In particular, X is a projective and rationally connected manifold.*

Remark 5.4. It is not hard to see that Theorem 5.3 can also hold under certain weaker conditions. For related topics on holomorphic sectional curvature, we refer to [HW15, ACH15, Yang16, Liu16, AHZ16, YZ16, AH17, Mat18] and the references therein.

6. RC-positive Finsler vector bundles over complex manifolds

Let \mathcal{E} be a holomorphic vector bundle over a complex manifold X with complex rank r . Let z^1, \dots, z^n be the local holomorphic coordinates on X and w^1, \dots, w^r be the holomorphic coordinates on the fiber of \mathcal{E} . Let $\mathcal{E} \setminus \{0\}$ be the complement of the zero section of \mathcal{E} .

Definition 6.1. A pseudoconvex complex Finsler metric \mathfrak{F} on \mathcal{E} is a continuous function $\mathfrak{F} : \mathcal{E} \rightarrow [0, +\infty)$ satisfying

- (1) \mathfrak{F} is smooth on $\mathcal{E} \setminus \{0\}$;
- (2) $\mathfrak{F}(z, w) > 0$ for all $w \neq 0$;
- (3) $\mathfrak{F}(z, \lambda w) = |\lambda|^2 \mathfrak{F}(z, w)$ for all $\lambda \in \mathbb{C}$;
- (4) The $(r \times r)$ Hermitian matrix $\left(\frac{\partial^2 \mathfrak{F}}{\partial w^\alpha \partial \bar{w}^\beta} \right)$ is positive definite over $\mathcal{E} \setminus \{0\}$.

Let \mathfrak{F} be a pseudoconvex complex Finsler metric on $\mathcal{E} \rightarrow X$. It is well-known that

$$\left(h_{\alpha\bar{\beta}} \right) = \left(\frac{\partial^2 \mathfrak{F}}{\partial w^\alpha \partial \bar{w}^\beta} \right)$$

defines a smooth Hermitian metric on the holomorphic vector bundle $\pi^*\mathcal{E} \rightarrow \mathbb{P}(\mathcal{E}^*)$ where $\pi : \mathbb{P}(\mathcal{E}^*) \rightarrow X$ is the projection. Note that, in general \mathfrak{F} does not give a Hermitian metric on $\mathcal{E} \rightarrow X$.

Definition 6.2. Let \mathfrak{F} be a pseudoconvex complex Finsler metric on $\mathcal{E} \rightarrow X$. $(\mathcal{E}, \mathfrak{F})$ is called an RC-positive (resp. a uniformly RC-positive) Finsler vector bundle if the induced Hermitian vector bundle $(\pi^*\mathcal{E}, h)$ is RC-positive (resp. uniformly RC-positive) over $\mathbb{P}(\mathcal{E}^*)$.

Remark 6.3. If (\mathcal{E}, h) is a Hermitian holomorphic vector bundle, then it induces a pseudoconvex Finsler metric \mathfrak{F} on \mathcal{E} . Moreover, if (\mathcal{E}, h) is RC-positive (resp. uniformly RC-positive), then $(\mathcal{E}, \mathfrak{F})$ is RC-positive (resp. uniformly RC-positive).

Theorem 6.4. *Let X be a compact complex manifold. Suppose $(\mathcal{E}, \mathfrak{F})$ is an RC-positive Finsler vector bundle. Then we have*

$$H^0(X, \text{Sym}^{\otimes m} \mathcal{E}^*) = 0, \quad \text{for all } m \geq 1.$$

Moreover, for any line bundle $\mathcal{L} \rightarrow X$, there exists a positive constant $c_{\mathcal{L}}$ such that

$$H^0(X, \text{Sym}^{\otimes m} \mathcal{E}^* \otimes \mathcal{L}^{*\otimes k}) = 0,$$

for all positive integers m, k with $m \geq c_{\mathcal{L}}k$.

Proof. If $(\mathcal{E}, \mathfrak{F})$ is an RC-positive Finsler bundle, then the induced Hermitian vector bundle $(\pi^*\mathcal{E}, h)$ is RC-positive. Let $\pi : \mathbb{P}(\mathcal{E}^*) \rightarrow X$ be the natural projection. Let $\mathcal{F} = \pi^*(\mathcal{E})$ and $Y = \mathbb{P}(\mathcal{E}^*)$. By Proposition 4.6, $\mathcal{O}_{\mathcal{F}^*}(-1)$ is an RC-positive line bundle over the projective bundle $\tilde{\pi} : \mathbb{P}(\mathcal{F}) \rightarrow Y$. Hence, by Theorem 3.3, we have

$$(6.1) \quad H^0\left(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathcal{F}^*}(m) \otimes \tilde{\pi}^*\left(\pi^*(\mathcal{L}^{*\otimes k})\right)\right) = 0,$$

for all positive integers m, k with $m \geq c_{\mathcal{L}}k$. By the Le Potier isomorphism, we have

$$\begin{aligned} H^0(X, \text{Sym}^{\otimes m} \mathcal{E}^* \otimes \mathcal{L}^{\otimes k}) &\cong H^0(\mathbb{P}(\mathcal{E}^*), \pi^*(\text{Sym}^{\otimes m} \mathcal{E}^*) \otimes \pi^*(\mathcal{L}^{\otimes k})) \\ &\cong H^0(\mathbb{P}(\mathcal{E}^*), \text{Sym}^{\otimes m} \pi^*(\mathcal{E}^*) \otimes \pi^*(\mathcal{L}^{\otimes k})) \\ &= H^0(Y, \text{Sym}^{\otimes m} \mathcal{F}^* \otimes \pi^*(\mathcal{L}^{*\otimes k})) \\ &\cong H^0\left(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathcal{F}^*}(m) \otimes \tilde{\pi}^*\left(\pi^*(\mathcal{L}^{*\otimes k})\right)\right) \\ &= 0. \end{aligned}$$

If we take \mathcal{L} to be a trivial line bundle, then there exists a large positive integer m such that $H^0(X, \text{Sym}^{\otimes m} \mathcal{E}^*) = 0$. It is easy to see that $H^0(X, \mathcal{E}^*) = 0$ and so $H^0(X, \text{Sym}^{\otimes m} \mathcal{E}^*) = 0$ for all $m \geq 1$. \square

Theorem 6.5. *Let X be a compact complex manifold. Suppose $(\mathcal{E}, \mathfrak{F})$ is a uniformly RC-positive Finsler vector bundle. Then we have*

$$H^0(X, (\mathcal{E}^*)^{\otimes m}) = 0, \quad \text{for all } m \geq 1.$$

Moreover, for any line bundle $\mathcal{L} \rightarrow X$, there exists a positive constant $c_{\mathcal{L}}$ such that

$$H^0(X, (\mathcal{E}^*)^{\otimes m} \otimes \mathcal{L}^{\otimes k}) = 0,$$

for all positive integers m, k with $m \geq c_{\mathcal{L}}k$.

Proof. The induced Hermitian vector bundle $(\pi^*\mathcal{E}, h)$ is uniformly RC-positive. By Proposition 2.12, $\pi^*(\mathcal{E}^{\otimes m}) \cong (\pi^*(\mathcal{E}))^{\otimes m}$ is uniformly RC-positive for all $m \geq 1$. On the other hand,

$$H^0\left(X, (\mathcal{E}^*)^{\otimes m} \otimes \mathcal{L}^{\otimes k}\right) \cong H^0\left(\mathbb{P}(\mathcal{E}^*), (\pi^*(\mathcal{E}^*))^{\otimes m} \otimes \pi^*(\mathcal{L}^{\otimes k})\right).$$

Hence, Theorem 6.5 follows from Theorem 3.3. \square

As an application of Theorem 6.4 and [Yang18, Theorem 1.4], we obtain

Theorem 6.6. *Let X be a compact Kähler manifold of complex dimension n . Suppose that for every $1 \leq p \leq n$, there exists a pseudoconvex Finsler metric \mathfrak{F}_p on $\Lambda^p T_X$ such that $(\Lambda^p T_X, \mathfrak{F}_p)$ is RC-positive, then X is projective and rationally connected.*

Similarly, as an application of Theorem 6.5 and Corollary 1.3, we obtain

Theorem 6.7. *Let X be a compact Kähler manifold. If X admits a pseudoconvex Finsler metric \mathfrak{F} such that (T_X, \mathfrak{F}) is uniformly RC-positive, then X is projective and rationally connected.*

Remark 6.8. It is well-known that there exists a one-to-one correspondence between the set of Hermitian metrics on $\mathcal{O}_{\mathcal{E}^*}(-1)$ and the set of Finsler metrics on \mathcal{E}^* or \mathcal{E} . We can also define a Finsler vector bundle $(\mathcal{E}, \mathfrak{F})$ to be RC-positive, if the induced Hermitian metric on $\mathcal{O}_{\mathcal{E}^*}(-1)$ is RC-positive.

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