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A square root of Hurwitz numbers

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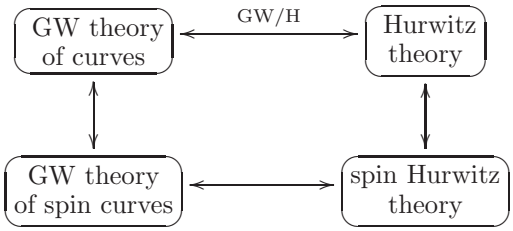
Abstract

We exhibit a generating function of spin Hurwitz numbers analogous to (disconnected) double Hurwitz numbers that is a tau function of the two-component BKP (2-BKP) hierarchy and is a square root of a tau function of the two-component KP (2-KP) hierarchy defined by related Hurwitz numbers.

1 Introduction

The Gromov-Witten theory of Kähler surfaces (with smooth canonical divisor) is entirely determined by the GW theory of spin curves (see [15, 10, 18]). The (dimension zero) sum formula for spin curves (Theorem 1.1 of [16]) indicates that the spin Hurwitz theory is to the GW theory of spin curves what the Hurwitz theory is to the GW theory of curves. For this reason, it would be interesting to find some connections between the following theories:



(1.1)

In the top arrow is the celebrated GW/H correspondence developed in [21, 22]. This paper aims to find a connection in the right arrow and also to address a question on a correspondence in the bottom arrow analogous to the GW/H correspondence.

In [20], A. Okounkov showed that a generating function of (disconnected) double Hurwitz numbers is a tau function of the 2-Toda lattice hierarchy. The key idea is to write the generating function in terms of Schur functions that are special tau functions of the KP hierarchy.

Following the same idea, we will compare the generating functions of Hurwitz numbers and spin Hurwitz numbers defined below. The Hurwitz generating function can be written via Schur functions and the spin Hurwitz generating function via Schur Q-functions that are tau functions of the BKP hierarchy.

Given partitions $\mu, \nu, \eta_2 = (2, 1^{d-2})$ and $\eta_3 = (3, 1^{d-3})$ of $d > 0$ and integers $r_2, r_3 \geq 0$, the Hurwitz number of $\mathbb{P}^1$ is the weighted sum

$$H_d^0(\mu, \nu, \eta_2^{r_2}, \eta_3^{r_3}) = \sum_f \frac{1}{|\mathrm{Aut}(f)|} \tag{1.2}$$

of possibly disconnected ramified covers $f : C \to \mathbb{P}^1$ with ramification profiles μ over $0 \in \mathbb{P}^1$, ν over $\infty \in \mathbb{P}^1$ and η_2 and η_3 over r_2 and r_3 other fixed points of $\mathbb{P}^1$. Here

η_2 denotes the simple ramification and the automorphism group $\mathrm{Aut}(f)$ of f consists of automorphisms ς of the domain curve C satisfying $f \circ \varsigma = f$. The domain Euler characteristic $\chi(C)$ is related to the partition lengths $\ell(\mu)$ and $\ell(\nu)$ and the numbers r_2 and r_3 by the Riemann-Hurwitz formula

$$\chi(C) = \ell(\mu) + \ell(\nu) - r_2 - 2r_3.$$

In particular, this implies that the Hurwitz number vanishes whenever r_2 does not have the same parity of $\ell(\mu) + \ell(\nu)$.

The spin Hurwitz numbers of the spin curve $(\mathbb{P}^1, \mathcal{O}(-1))$ also count ramified covers of $\mathbb{P}^1$, but with only odd ramifications and with sign induced from $\mathcal{O}(-1)$.

Specifically, suppose ρ and σ are odd partitions of d (i.e., all parts in ρ and σ are odd) and consider possibly disconnected ramified covers $f : C \to \mathbb{P}^1$ with ramification profiles ρ over $0 \in \mathbb{P}^1$, σ over $\infty \in \mathbb{P}^1$ and η_3 over other r fixed points of $\mathbb{P}^1$. The Riemann-Hurwitz formula in this case provides the Euler characteristic $\chi(C)$ of the domain as

$$\chi(C) = \ell(\rho) + \ell(\sigma) - 2r. \tag{1.3}$$

Since the ramification divisor R_f of $f : C \to \mathbb{P}^1$ is even, the twisted pull-back bundle

$$N_f = f^*\mathcal{O}(-1) \otimes \mathcal{O}_C(\tfrac{1}{2}R_f)$$

is a square root of the canonical bundle of C (or a theta characteristic on C). Given odd partitions ρ, σ of d and the number r, the spin Hurwitz number of $(\mathbb{P}^1, \mathcal{O}(-1))$ is defined to be

$$H_d^{0,+}(\rho, \sigma, \eta_3^r) = \sum_f \frac{(-1)^{h^0(N_f)}}{|\mathrm{Aut}(f)|}, \tag{1.4}$$

where the superscript + denotes the parity of the spin curve $(\mathbb{P}^1, \mathcal{O}(-1))$.

Let $p = (p_1, p_2, \cdots)$ and $p' = (p'_1, p'_2, \cdots)$ be two sets of variables where p_n and p'_n are power-sum symmetric functions. For a partition $\mu = (\mu_1, \mu_2, \cdots)$, let $p_\mu = \prod p_{\mu_i}$. Let $\mathrm{P}(d)$ be the set of partitions of d. Now introduce a generating function of the Hurwitz numbers (1.2) by

$$\begin{aligned}&\Phi(p, p', b, q)\\ &= 1 + \sum_{d>0} q^d \sum_{\mu,\nu \in \mathrm{P}(d)} p_\mu p'_\nu \sum_{s=0}^{\infty} \frac{b^s}{s!} \sum_{r_i \geq 0, \sum r_i = s} \frac{s!}{r_1! r_2! r_3!} \Big(\frac{d^2+d}{2}\Big)^{r_1} H_d^0(\mu, \nu, \eta_2^{r_2}, \eta_3^{r_3}).\end{aligned}$$

Under the restriction $p_2 = p_4 = \cdots = 0$ and $p'_2 = p'_4 = \cdots = 0$, the function $\Phi(p, p', b, q)$ specializes to the function $\Phi(p_{_B}, p'_{_B}, b, q)$ where

$$p_{_B} = (p_1, 0, p_3, 0, \cdots) \qquad \text{and} \qquad p'_{_B} = (p'_1, 0, p'_3, 0, \cdots).$$

Let $\mathrm{OP}(d)$ denote the set of odd partitions of d and introduce a generating function of the spin Hurwitz numbers (1.4) by

$$\begin{aligned}&\Phi_{_B}(p_{_B}, p'_{_B}, b, q)\\ &= 1 + \sum_{d>0} q^d \sum_{\rho,\sigma \in \mathrm{OP}(d)} p_\rho p'_\sigma \sum_{s=0}^{\infty} \frac{b^s}{s!} \sum_{r=0}^{s} \binom{s}{r} d^{2(s-r)} 2^{-\frac{\chi(C)}{2}} H_d^{0,+}(\rho, \sigma, \eta_3^r),\end{aligned}$$

where $\chi(C)$ is the domain Euler characteristic in (1.3).

Theorem 1.1. *The function $\Phi(p,p',b,q)$ is a tau function of the two-component KP (2-KP) hierarchy and the function $\Phi_B(p_B,p'_B,b,q)$ is a tau function of the two-component BKP (2-BKP) hierarchy such that*

$$\Phi_B^2(p_B,p'_B,b,q)=\Phi(p_B,p'_B,b,q). \tag{1.5}$$

The proof of this theorem is based on the reduction of the KP hierarchy to the BKP hierarchy. Both hierarchies are formulated by a single tau function such that the square of the BKP tau function is a KP tau function (cf. Proposition 4 of [2] and Proposition 1 of [28]).

In Section 2 we express the generating functions Φ and Φ_B via symmetric functions. In Section 3 the famous Boson-Fermion correspondence converts those expressions to vacuum expectations of corresponding operators. We use the vacuum expectations to prove Theorem 1.1 in Section 4. A discussion on a conjectural spin curve analog of the GW/H correspondence is presented in Section 5.

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2 Symmetric functions

With transition matrices between linear bases of algebras relevant to our case, we express the Hurwitz generating function Φ via Schur functions and shifted symmetric power sums, whereas we express the spin Hurwitz generating function Φ_B via Schur Q-functions and odd power-sum symmetric functions.

2.1 Hurwitz numbers

2.1.1 Central characters of the symmetric group

Irreducible representations and conjugacy classes of the symmetric group $\mathsf{S}(d)$ on d letters are indexed by partitions of d. Let C_μ denote the conjugacy class indexed by $\mu\in\mathrm{P}(d)$. The order of the centralizer of any element of C_μ is

$$z_\mu=\prod_k \mu(k)!k^{\mu(k)}, \tag{2.1}$$

where $\mu(k)$ is the number of parts in μ equal to k.

Let π^λ be the irreducible representation indexed by $\lambda\in\mathrm{P}(d)$ and χ^λ be its character. The class sum $\sum_{g\in C_\mu} g$ acts on π^λ as multiplication by constant. This constant is the central character

$$\mathbf{f}_\mu^\lambda=\frac{|C_\mu|}{\dim\pi^\lambda}\,\chi_\mu^\lambda, \tag{2.2}$$

where $|C_\mu|=d!/z_\mu$ and χ_μ^λ is the value of the character χ^λ on any element of C_μ. The character formula for the Hurwitz number (1.2) is given as

$$H_d^0(\mu,\nu,\eta_2^{r_2},\eta_3^{r_3})=\sum_{\lambda\in\mathrm{P}(d)}\frac{\chi_\mu^\lambda}{z_\mu}\frac{\chi_\nu^\lambda}{z_\nu}\left(\mathbf{f}_{\eta_2}^\lambda\right)^{r_2}\left(\mathbf{f}_{\eta_3}^\lambda\right)^{r_3}. \tag{2.3}$$

2.1.2 Shifted symmetric power sums

The shifted symmetric power sum $\mathbf{p}_n$ of degree n is defined by

$$\mathbf{p}_n(\lambda) = \sum_{i=1}^{\infty} \Big((\lambda_i - i + \tfrac{1}{2})^n - (-i + \tfrac{1}{2})^n \Big).$$

We denote by Λ^* the algebra generated by $\mathbf{p}_1, \mathbf{p}_2, \cdots$. From the central characters of the symmetric group, Kerov and Olshanski [9] obtained functions in Λ^* as follows. For any partition $\mu = (\mu_1, \cdots, \mu_\ell)$, let $|\mu| = \sum \mu_i$ and set

$$\mu \cup (1^k) = (\mu_1, \cdots, \mu_\ell, 1, \cdots, 1) \in \mathrm{P}(|\mu| + k).$$

Define a function $\mathbf{f}_\mu$ on the set of partitions by $\mathbf{f}_\mu(\lambda) = 0$ if $|\lambda| < |\mu|$ and

$$\mathbf{f}_\mu(\lambda) = \binom{\mu(1) + k}{\mu(1)} \mathbf{f}^\lambda_{\mu \cup (1^k)} \quad \text{if} \quad k = |\lambda| - |\mu| \geq 0,$$

where $\mu(1)$ is the number of parts in μ equal to 1 as introduced in (2.1). Then

$$\mathbf{f}_\mu = \frac{1}{\prod \mu_i} \mathbf{p}_\mu + \cdots,$$

where $\mathbf{p}_\mu = \prod \mathbf{p}_{\mu_i}$ and the dots denote the lower degree terms (see [9] and also [21]). This implies $\{\mathbf{f}_\mu\}$ and $\{\mathbf{p}_\mu\}$ are mutually triangular linear bases of Λ^*. Observe that for $\eta_3 = (3, 1^{d-3})$ in the character formula (2.3),

$$\mathbf{f}_{(3)}(\lambda) = \mathbf{f}^\lambda_{\eta_3}.$$

The lemma below is a consequence of the Wassermann formula [27] (see also [6]). For a formal series $A(t)$, let

$$[t^k]\{A(t)\} = \text{the coefficient of } t^k \text{ in } A(t).$$

Lemma 2.1. *Let* $\mathbf{f}_2 = \mathbf{f}_{(2)}$ *and* $\mathbf{f}_3 = \mathbf{f}_{(3)}$. *We have:*

(a) $\mathbf{f}_2 = \frac{1}{2}\mathbf{p}_2$.

(b) $\mathbf{f}_3 = \frac{1}{3}\mathbf{p}_3 - \frac{1}{2}\mathbf{p}_1^2 + \frac{5}{12}\mathbf{p}_1$.

Proof. (a) is a well-known fact (cf. [20]), which also follows by the same argument as for (b). Consider the function $\mathbf{p}_3^\#$ on the set of partitions defined by

$$\mathbf{p}_3^\#(\lambda) = \begin{cases} d^{\downarrow 3} \dfrac{\chi^\lambda_{(3,1^{d-3})}}{\dim \pi^\lambda}, & d := |\lambda| \geq 3 \\ 0, & d < 3, \end{cases}$$

where $d^{\downarrow 3} = d(d-1)(d-2)$. From (3.2) of [6], one has

$$\begin{aligned} \mathbf{p}_3^\# &= [t^4]\Big\{ -\tfrac{1}{3} \prod_{j=1}^{3} \big(1 - (j - \tfrac{1}{2})t\big) \exp\Big(\sum_{j=1}^{\infty} \frac{\mathbf{p}_j t^j}{j} \big(1 - (1-3t)^{-j}\big)\Big)\Big\} \\ &= \mathbf{p}_3 - \tfrac{3}{2}\mathbf{p}_1^2 + \tfrac{5}{4}\mathbf{p}_1. \end{aligned}$$

This together with (2.2) proves (b). ∎

2.1.3 Schur functions

The Schur functions s_λ and the monomials p_μ are related by

$$s_\lambda(p) = \sum_{\mu \in \mathrm{P}(d)} \frac{\chi^\lambda_\mu}{z_\mu} p_\mu, \tag{2.4}$$

where $\lambda \in \mathrm{P}(d)$ (see page 114 of [17]). Now by (2.3) and (2.4) one can write the generating function $\Phi(p, p', b, q)$ of Hurwitz numbers as

$$\Phi(p, p', b, q) = 1 + \sum_{d>0} \sum_{\lambda \in \mathrm{P}(d)} q^d e^{b(\frac{1}{2}(d+d^2)+\mathbf{f}_2(\lambda)+\mathbf{f}_3(\lambda))} s_\lambda(p) s_\lambda(p'). \tag{2.5}$$

2.2 Spin Hurwitz numbers

2.2.1 Central characters of the Sergeev group

The Sergeev group $\mathsf{C}(d)$ is the semidirect product $\mathsf{Cliff}(d) \rtimes \mathsf{S}(d)$, where $\mathsf{Cliff}(d)$ is the Clifford group generated by $\xi_1, \cdots, \xi_d$ and a central element ϵ subject to the relations

$$\xi_i^2 = 1, \qquad \epsilon^2 = 1, \qquad \xi_i \xi_j = \epsilon \xi_j \xi_i \quad (i \neq j)$$

and the symmetric group $\mathsf{S}(d)$ acts on $\mathsf{Cliff}(d)$ by permuting the ξ_i's.

Let $\mathrm{SP}(d)$ denote the set of strict partitions $\lambda = (\lambda_1 > \lambda_2 > \cdots)$ of d. A spin $\mathsf{C}(d)$-supermodule is a supermodule over the group algebra $\mathbb{C}[\mathsf{C}(d)]$ on which ϵ acts as multiplication by -1 such that (i) the irreducible spin $\mathsf{C}(d)$-supermodules V^λ are indexed by strict partitions $\lambda \in \mathrm{SP}(d)$, (ii) the character ζ^λ of V^λ is determined by the values ζ^λ_ρ on the conjugacy classes C_ρ indexed by odd partitions $\rho \in \mathrm{OP}(d)$, and (iii) for the conjugacy class C_ρ indexed by $\rho \in \mathrm{OP}(d)$,

$$|C_\rho| = \frac{|\mathsf{C}(d)|}{2^{\ell(\rho)+1} z_\rho} = \frac{2^{|\rho|-\ell(\rho)}|\rho|!}{z_\rho},$$

where z_ρ is introduced in (2.1) (see [7, 24] and also [4]). The class sum of C_ρ acts on V^λ as multiplication by a constant, which is the central character of V^λ given by

$$f^\lambda_\rho = \frac{|C_\rho|}{\dim V^\lambda} \zeta^\lambda_\rho. \tag{2.6}$$

For $\lambda \in \mathrm{SP}(d)$, let

$$\delta(\lambda) = \begin{cases} 0, & \text{if } \ell(\lambda) \text{ is even,} \\ 1, & \text{if } \ell(\lambda) \text{ is odd.} \end{cases}$$

From the Gunningham formula [3, 14], one has the character formula for the spin Hurwitz number (1.4):

$$H_d^{0,+}(\rho, \sigma, \eta_3^r) = 2^{\frac{-\ell(\rho)-\ell(\sigma)-2r}{2}} \sum_{\lambda \in \mathrm{SP}(d)} 2^{-\delta(\lambda)} \frac{\zeta^\lambda_\rho}{z_\rho} \frac{\zeta^\lambda_\sigma}{z_\sigma} \big(f^\lambda_{\eta_3}\big)^r. \tag{2.7}$$

2.2.2 Odd power-sum symmetric functions

Denote by Γ the algebra generated by $p_1, p_3, \cdots$. In the same manner as for $\mathbf{f}_\mu$, one can obtain functions f_ρ in Γ from the central characters of the Sergeev group. For any odd partition ρ, let $f_\rho(\lambda) = 0$ if $|\rho| > |\lambda|$ and

$$f_\rho(\lambda) = \binom{\rho(1)+k}{\rho(1)} f^\lambda_{\rho\cup(1^k)} \quad \text{if} \quad k = |\lambda| - |\rho| \geq 0,$$

where $\rho(1)$ is the number of parts in ρ equal to 1 as in (2.1). Then

$$f_\rho = \frac{1}{z_\rho} p_\rho + \cdots,$$

where the dots denote the lower degree terms (see Section 6 of [4]). It follows that $\{f_\rho\}$ and $\{p_\rho : \rho \text{ is odd}\}$ are mutually triangular linear bases of Γ.

The lemma below follows from Proposition 3.4 of [5].

Lemma 2.2. *Let $f_3 = f_{(3)}$. We have*

$$f_3 = \tfrac{1}{3} p_3 - p_1^2 + \tfrac{2}{3} p_1.$$

Proof. Consider the function $p_3^\#$ on the set of strict partitions defined by

$$p_3^\#(\lambda) = \begin{cases} d^{\downarrow 3} \dfrac{X^\lambda_{(3,1^{d-3})}}{X^\lambda_{(1^d)}}, & d := |\lambda| \geq 3, \\ 0, & d < 3, \end{cases}$$

where X^λ_μ corresponds to $X^\lambda_\mu(-1)$ in [17] such that

$$X^\lambda_{(3,1^{d-3})} = 2^{\frac{\ell(\lambda)-\delta(\lambda)-2(d-2)}{2}} \zeta^\lambda_{(3,1^{d-3})}, \qquad X^\lambda_{(1^d)} = 2^{\frac{\ell(\lambda)-\delta(\lambda)-2d}{2}} \dim V^\lambda$$

(see Proposition 3.3 of [4]). By (3.5) of [5], one has:

$$\begin{aligned} p_3^\# &= [t^4]\Big\{ -\tfrac{1}{6}(1-\tfrac{3}{2}t) \prod_{j=1}^{2} (1-jt) \exp\Big(\sum_{j=1}^{\infty} \frac{2p_{2j-1} t^{2j-1}}{2j-1} \big(1-(1-3t)^{-(2j-1)}\big)\Big)\Big\} \\ &= p_3 - 3p_1^2 + 2p_1. \end{aligned}$$

This together with (2.6) shows the lemma. □

2.2.3 Schur Q-functions

The algebra Γ has another linear basis formed by Schur Q-functions $\{Q_\lambda : \lambda \text{ is strict}\}$. For $\lambda \in \mathrm{SP}(d)$,

$$Q_\lambda(p_{_B}) = \sum_{\rho \in \mathrm{OP}(d)} 2^{\frac{\ell(\lambda)-\delta(\lambda)}{2}} \frac{\zeta^\lambda_\rho}{z_\rho} p_\rho$$

(see Corollary 3 of [7]). Thus, for $\frac{1}{2}p_{_B} = (\frac{1}{2}p_1, 0, \frac{1}{2}p_3, \cdots)$,

$$Q_\lambda(\tfrac{1}{2}p_{_B}) = \sum_{\rho \in \mathrm{OP}(d)} 2^{\frac{\ell(\lambda)-\delta(\lambda)-2\ell(\rho)}{2}} \frac{\zeta^\lambda_\rho}{z_\rho} p_\rho. \tag{2.8}$$

By (2.7) and (2.8), one can write the generating function $\Phi_{_B}(p_{_B}, p'_{_B}, b, q)$ of spin Hurwitz numbers as

$$\Phi_{_B}(p_{_B}, p'_{_B}, b, q) = 1 + \sum_{d>0} \sum_{\lambda \in \mathrm{SP}(d)} q^d e^{b(d^2 + f_3(\lambda))} 2^{-\ell(\lambda)} Q_\lambda(\tfrac{1}{2}p_{_B}) Q_\lambda(\tfrac{1}{2}p'_{_B}). \tag{2.9}$$

3 The operator formalism

The fermion operator formalism is a handy tool for handling various generating functions. We employ the formalism to express the generating functions (2.5) and (2.9) as vacuum expectations. For more detailed discussions of the formalism, we refer to Chapter 14 of [8], Appendix of [19] and Section 2 of [21].

3.1 Hurwitz numbers

3.1.1 The infinite wedge space

Let V be a vector space with basis $\{\underline{k}\}$ indexed by the half-integers $k \in \mathbb{Z} + \frac{1}{2}$. The infinite wedge space (or fermion Fock space) is the vector space

$$\Lambda^{\frac{\infty}{2}} V$$

with basis $\{v_S\}$ where $S = (s_1 > s_2 > s_3 > \cdots)$ with $s_i - s_{i+1} = 1$ for $i \gg 0$ and

$$v_S = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots .$$

Denote by Λ_0 the subspace of $\Lambda^{\frac{\infty}{2}} V$ generated by vectors v_λ indexed by partitions $\lambda = (\lambda_1, \lambda_2, \cdots)$. These are defined by

$$v_\lambda = \underline{\lambda_1 - \tfrac{1}{2}} \wedge \underline{\lambda_2 - \tfrac{3}{2}} \wedge \cdots \wedge \underline{\lambda_i - i + \tfrac{1}{2}} \wedge \cdots .$$

The vector indexed by the empty partition $\emptyset = (0, 0, \cdots)$ is the vacuum vector

$$v_\emptyset = \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \cdots .$$

Let $(\cdot\,,\cdot)$ be the inner product on $\Lambda^{\frac{\infty}{2}} V$ for which $\{v_S\}$ is an orthonormal basis. The vacuum expectation of an operator A on $\Lambda^{\frac{\infty}{2}} V$ is defined as

$$\langle A \rangle := (A v_\emptyset, v_\emptyset).$$

3.1.2 Charged fermions

For each $k \in \mathbb{Z} + \frac{1}{2}$, the operator ψ_k on $\Lambda^{\frac{\infty}{2}} V$ is defined by

$$\psi_k \, v = \underline{k} \wedge v,$$

and the operator ψ_k^* is the adjoint of ψ_k. These are charged fermions and satisfy the canonical anti-commutative relations:

$$\psi_k \psi_\ell^* + \psi_\ell^* \psi_k = \delta_{k,\ell}, \qquad \psi_k \psi_\ell + \psi_\ell \psi_k = \psi_k^* \psi_\ell^* + \psi_\ell^* \psi_k^* = 0. \tag{3.1}$$

Infinite sums of the quadratics $\psi_j \psi_j^*$ make sense as operators on $\Lambda^{\frac{\infty}{2}} V$ if we write them in terms of normal ordering defined by

$$: \psi_k \psi_\ell^* : \stackrel{def}{=} \psi_k \psi_\ell^* - \langle \psi_k \psi_\ell^* \rangle = \begin{cases} \psi_k \psi_\ell^*, & \ell > 0, \\ -\psi_\ell^* \psi_k, & \ell < 0. \end{cases}$$

3.1.3 Operators related to shifted symmetric power sums

For any integer $n \geq 0$, define

$$E_n = \sum_{k \in \mathbb{Z}+\frac{1}{2}} k^n : \psi_k \psi_k^* : .$$

One of the salient features of the half-integer infinite wedge lies in the relation between the operators E_n and the shifted symmetric power sums $\mathbf{p}_n$:

$$E_n v_\lambda = \mathbf{p}_n(\lambda) v_\lambda \tag{3.2}$$

(cf. Section 2 of [21]).

The operators E_1 and E_0 are the energy and charge operator. As $\mathbf{p}_1(\lambda) = |\lambda|$, $E_1 v_\lambda = |\lambda| v_\lambda$. On the other hand, $E_0 v_\lambda = 0$ and hence the subspace Λ_0 is the 0-eigenspace of E_0 (cf. Appendix of [19]).

To express the generating function (2.5) of Hurwitz numbers as a vacuum expectation, we consider the following operator

$$F = \tfrac{1}{3} E_3 + \tfrac{1}{2} E_2 + \tfrac{11}{12} E_1.$$

By Lemma 2.1 and (3.2), for every vector v_λ with $|\lambda| = d$ one has

$$F v_\lambda = \Big(\tfrac{1}{2}\big(d + d^2\big) + \mathbf{f}_2(\lambda) + \mathbf{f}_3(\lambda) \Big) v_\lambda. \tag{3.3}$$

As shown in [20], it is convenient to use the operator $q^{E_1} = e^{E_1 \ln q}$ because it can be written as

$$q^{E_1} = \sum_{d \geq 0} P_d \, q^d,$$

where P_d is the orthogonal projection onto the d-eigenspace of E_1.

3.1.4 The Boson-Fermion correspondence

Introduce a set $t = (t_1, t_2, \cdots)$ of Miwa variables $t_n = p_n/n$ and let $\alpha^*(t)$ denote the adjoint of the operator

$$\alpha(t) = \sum_{n>0} t_n \sum_{k \in \mathbb{Z}+\frac{1}{2}} : \psi_{k-n} \psi_k^* : .$$

Then from the remarkable Boson-Fermion correspondence (cf. §14.10 of [8]), one has

$$e^{\alpha^*(t)} v_\emptyset = \sum_\lambda s_\lambda(p) v_\lambda,$$

where the sum is over all partitions λ. This together with (2.5) and (3.3) gives

$$\Phi(p, p', b, q) = \left\langle e^{\alpha(t)} q^{E_1} e^{bF} e^{\alpha^*(t')} \right\rangle, \tag{3.4}$$

where $t' = (t'_1, t'_2, \cdots)$ is another set of Miwa variables $t'_n = p'_n/n$.

3.2 Spin Hurwitz numbers

3.2.1 Neutral fermions

There is an involution w on the charged fermions defined by

$$w(\psi_k) = (-1)^{k+\frac{1}{2}}\psi^*_{-k-1}, \qquad w(\psi^*_k) = (-1)^{k+\frac{1}{2}}\psi_{-k-1}.$$

The neutral fermions are defined as ± 1-eigenvectors of the involution.

Definition 3.1. *For $m \in \mathbb{Z}$ and $i = \sqrt{-1}$, let*

$$\phi_m = \frac{1}{\sqrt{2}}\Big(\psi_{m-\frac{1}{2}} + (-1)^m\psi^*_{-m-\frac{1}{2}}\Big), \qquad \hat{\phi}_m = \frac{i}{\sqrt{2}}\Big(\psi_{m-\frac{1}{2}} - (-1)^m\psi^*_{-m-\frac{1}{2}}\Big).$$

By the canonical anti-commutative relations (3.1) for charged fermions, the neutral fermions also satisfy the canonical anti-commutative relations:

$$\phi_n\phi_m + \phi_m\phi_n = \hat{\phi}_m\hat{\phi}_n + \hat{\phi}_n\hat{\phi}_m = (-1)^m\delta_{m,-n}, \quad \phi_m\hat{\phi}_n + \hat{\phi}_n\phi_m = 0. \tag{3.5}$$

With the relations (3.1), the charged fermions ψ_i and ψ^*_j generate a Clifford algebra, denoted Cl. The neutral fermions ϕ_m and $\hat{\phi}_m$ respectively generate isomorphic subalgebras. The isomorphism is given by the involution on the Clifford algebra Cl induced from the map $\phi_m \leftrightarrow \hat{\phi}_m$, which we denote by

$$X \ \mapsto \ \hat{X}. \tag{3.6}$$

Let Cl_B be the subalgebra of Cl generated by ϕ_m's. There is a decomposition

$$Cl_B = Cl^0_B \oplus Cl^1_B,$$

where Cl^p_B is spanned by all products of the form $\phi_{m_1}\cdots\phi_{m_s}$ with $s \equiv p \pmod 2$.

Recall that the subspace Λ_0 defined in Section 3.1.1 has an orthonormal basis $\{v_\lambda\}$. Its neutral analog is the subspace

$$\mathrm{span}\{Xv_\emptyset : X \in Cl^0_B\}. \tag{3.7}$$

As $\phi_m v_\emptyset = 0$ for $m < 0$, this subspace is also spanned by vectors v^B_λ indexed by strict partitions $\lambda = (\lambda_1 > \cdots > \lambda_\ell)$, which are defined by

$$v^B_\lambda = \begin{cases} \phi_{\lambda_1}\cdots\phi_{\lambda_\ell}v_\emptyset & \text{if } \ell \text{ is even,} \\ \sqrt{2}\phi_{\lambda_1}\cdots\phi_{\lambda_\ell}\phi_0 v_\emptyset & \text{if } \ell \text{ is odd.} \end{cases}$$

As $\phi^*_m = (-1)^m\phi_{-m}$ and $\phi_0^2 = \frac{1}{2}$, $\{v^B_\lambda\}$ is an orthonormal basis of the subspace (3.7).

3.2.2 Operators related to odd power-sum symmetric functions

For any odd integer $n \geq 1$, define a neutral analog of the operator E_n by

$$E^B_n = \sum_{m>0}(-1)^m m^n \phi_m\phi_{-m}.$$

From the canonical anti-commutative relations (3.5), it is easy to see that

$$(-1)^m\phi_m\phi_{-m}v^B_\lambda = \begin{cases} v^B_\lambda & \text{if } m = \lambda_j \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

This implies v_λ^B is an eigenvector of E_n^B with eigenvalue $p_n(\lambda)$, that is,

$$E_n^B v_\lambda^B = p_n(\lambda) v_\lambda^B. \tag{3.8}$$

As $p_1(\lambda) = |\lambda|$, the operator E_1^B plays the same role as the energy operator E_1.

The following lemma together with (3.2) and (3.8) makes a connection between odd power-sum symmetric functions and shifted symmetric power sums via neutral and charged fermions.

Lemma 3.2. *For any odd integer $n \geq 1$,*

$$E_n^B + \hat{E}_n^B = \sum_{i=0}^{n} \binom{n}{i} \frac{E_{n-i}}{2^i},$$

where the operator $\hat{E}_n^B$ is defined by the isomorphism (3.6) in an obvious way.

Proof. For $m > 0$, let $k = m - \frac{1}{2}$. Then by Definition 3.1, we have

$$(-1)^m \big(\phi_m \phi_{-m} + \hat{\phi}_m \hat{\phi}_{-m}\big) = \psi_k \psi_k^* + \psi_{-k-1}^* \psi_{-k-1}.$$

From this, we obtain

$$\begin{aligned} E_n^B + \hat{E}_n^B &= \sum_{k>0} \big(k + \tfrac{1}{2}\big)^n \psi_k \psi_k^* + \sum_{k>0} \big(k + \tfrac{1}{2}\big)^n \psi_{-k-1}^* \psi_{-k-1} \\ &= \sum_{k>0} \big(k + \tfrac{1}{2}\big)^n \psi_k \psi_k^* + \sum_{\ell<-1} \big(-\ell - \tfrac{1}{2}\big)^n \psi_\ell^* \psi_\ell \\ &= \sum_{k \in \mathbb{Z}+\frac{1}{2}} \big(k + \tfrac{1}{2}\big)^n : \psi_k \psi_k^* : = \sum_{i=0}^{n} \binom{n}{i} \frac{E_{n-i}}{2^i}, \end{aligned}$$

where the third equality follows because n is odd and $-\psi_\ell^* \psi_\ell = : \psi_\ell \psi_\ell^* :$ for $\ell < 0$. □

To express the generating function (2.9) of spin Hurwitz numbers as a vacuum expectation, consider the following operator

$$F^B = \tfrac{1}{3} E_3^B + \tfrac{2}{3} E_1^B.$$

It follows from Lemma 2.2 that for every vector v_λ^B with $|\lambda| = d$,

$$F^B v_\lambda^B = \big(d^2 + f_3(\lambda)\big) v_\lambda^B. \tag{3.9}$$

3.2.3 The neutral Boson-Fermion correspondence

Let $\beta^*(t)$ be the adjoint of the operator

$$\beta(t) = \frac{1}{2} \sum_{n \geq 0} t_{2n+1} \sum_{m \in \mathbb{Z}} (-1)^{m+1} \phi_m \phi_{-m-2n-1}.$$

Then from the neutral Boson-Fermion correspondence, one has

$$e^{\beta^*(t)} v_\emptyset = \sum_\lambda 2^{-\frac{\ell(\lambda)}{2}} Q_\lambda\big(\tfrac{1}{2} p_B\big) v_\lambda^B,$$

where the sums are over all strict partitions λ (see [28]). This together with (2.9) and (3.9) yields:

$$\Phi_B(p_B, p'_B, b, q) = \Big\langle e^{\beta(t)} q^{E_1^B} e^{bF^B} e^{\beta^*(t')} \Big\rangle = \Big\langle e^{\hat{\beta}(t)} q^{\hat{E}_1^B} e^{b\hat{F}^B} e^{\hat{\beta}^*(t')} \Big\rangle, \tag{3.10}$$

where the second vacuum expectation is given by the isomorphism (3.6).

4 Square root

Integrable hierarchies of KP type (including the 2-Toda lattice hierarchy) have fermionic forms of Hirota equations and fermionic formulas for tau functions (see [1] and also Appendix of [23]). We apply those in the proof of Theorem 1.1 below.

Proof of Theorem 1.1: The operators $q^{E_1}e^{bF}$ and $q^{E_1^B}e^{bF^B}$ satisfy the following fermionic forms of Hirota equations:

$$\Big[q^{E_1}e^{bF}\otimes q^{E_1}e^{bF}, \sum_{k\in\mathbb{Z}+\frac{1}{2}}\psi_k\otimes\psi_k^*\Big]=0, \tag{4.1}$$

$$\Big[q^{E_1^B}e^{bF^B}\otimes q^{E_1^B}e^{bF^B}, \sum_{m\in\mathbb{Z}}(-1)^m\phi_m\otimes\phi_{-m}\Big]=0. \tag{4.2}$$

One can obtain these commutation relations from: for $n\geq 0$,

$$E_n\psi_k=\psi_k(E_n+k^n),\qquad E_n\psi_k^*=\psi_k^*(E_n-k^n),$$

and for odd $n\geq 1$,

$$E_n^B\phi_{\pm m}=\phi_{\pm m}(E_n^B\pm m^n).$$

It follows from (4.1) that the sequence

$$\tau_n(t,t')=\Big(e^{\alpha(t)}q^{E_1}e^{bF}e^{\alpha^*(t')}v_n,v_n\Big)\quad(n\in\mathbb{Z})$$

is a sequence of tau functions of the 2-Toda lattice hierarchy where v_n is the vacuum vector in the n-eigenspace of the charge operator E_0 defined by

$$v_n=\underline{n-\tfrac{1}{2}}\wedge\underline{n-\tfrac{3}{2}}\wedge\underline{n-\tfrac{5}{2}}\wedge\cdots$$

(see Appendix of [19]). The Hurwitz generating function $\Phi(p,p',b,q)=\tau_0(t,t')$ by (3.4) and $\tau_0(t,t')$ is a tau function of the 2-KP hierarchy (see Theorem 1.12 of [26]).

It also follows from (4.2) that the vacuum expectation in (3.10),

$$\Phi_B(p_{_B},p'_{_B},b,q)=\Big\langle e^{\beta(t)}q^{E_1^B}e^{bF^B}e^{\beta^*(t')}\Big\rangle,$$

is a tau function of the 2-BKP hierarchy in time variables t and t' (see [25]).

Now it remains to prove (1.5). By (3.4) we have

$$\Phi(p_{_B},p'_{_B},b,q)=\Big\langle e^{\alpha_B(t)}q^{E_1}e^{bF}e^{\alpha_{_B}^*(t')}\Big\rangle,$$

where $\alpha_B(t)=\alpha(t_1,0,t_3,0,\cdots)$. By definition, one has

$$\beta(t)+\hat{\beta}(t)=\alpha_B(t).$$

Since Λ_0 is the kernel of the charge operator E_0, Lemma 3.2 shows that

$$\big(E_1^B+\hat{E}_1^B\big)|_{\Lambda_0}=E_1,\quad \big(F^B+\hat{F}^B\big)|_{\Lambda_0}=F.$$

Therefore noting $\langle Z\rangle\langle\hat{W}\rangle=\langle Z\hat{W}\rangle$ for $Z,W\in Cl_B^0$ (see Lemma 1 of [28]), we conclude

$$\begin{aligned}\Phi_B^2(p_{_B},p'_{_B},b,q)&=\Big\langle e^{\beta(t)}q^{E_1^B}e^{bF^B}e^{\beta^*(t')}e^{\hat{\beta}(t)}q^{\hat{E}_1^B}e^{b\hat{F}^B}e^{\hat{\beta}^*(t')}\Big\rangle\\&=\Big\langle e^{\beta(t)+\hat{\beta}(t)}q^{E_1^B+\hat{E}_1^B}e^{b(F^B+\hat{F}^B)}e^{\beta^*(t')+\hat{\beta}^*(t')}\Big\rangle=\Phi(p_{_B},p'_{_B},b,q),\end{aligned}$$

where the second equality follows since $Z\hat{W}=\hat{W}Z$ for $Z,W\in Cl_B^0$. This completes the proof of Theorem 1.1.

5 Conjectural spin GW/H correspondence

In [21], the GW/H correspondence was defined by the two linear bases $\{\mathbf{p}_\mu\}$ and $\{\mathbf{f}_\mu\}$ of the algebra Λ^*. Let $\mathrm{P}_{\mathbb{Q}}$ denote the vector space over $\mathbb{Q}$ with basis the set of partitions (i.e., every vector in $\mathrm{P}_{\mathbb{Q}}$ is a formal linear combination of partitions). As $\{\mathbf{f}_\mu\}$ is a linear basis of Λ^*, there is a linear isomorphism

$$\varphi : \mathrm{P}_{\mathbb{Q}} \to \Lambda^*$$

given by $\mu \mapsto \mathbf{f}_\mu$. Using this isomorphism, one can extend the character formula for Hurwitz numbers of the curve of genus h to the following multilinear form on $\mathrm{P}_{\mathbb{Q}}$:

$$H_d^h(v_1, \cdots, v_n) \;=\; \sum_{\lambda \in \mathrm{P}(d)} \Big(\frac{\dim \pi^\lambda}{d!}\Big)^{2-2h} \prod_{i=1}^n \varphi(v_i)(\lambda).$$

The GW/H correspondence is then the following equality:

$$\Big\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \Big\rangle_d^{\bullet\, h} = H_d^h \left(\varphi^{-1}\Big(\frac{\mathbf{p}_{k_1+1}^{\mathrm{reg}}}{(k_1+1)!}\Big), \cdots, \varphi^{-1}\Big(\frac{\mathbf{p}_{k_n+1}^{\mathrm{reg}}}{(k_n+1)!}\Big)\right).$$

Here the left-hand side is the degree d descendent GW invariants of the curve of genus h, ω is the Poincaré dual of the point class, $\bullet$ denotes the disconnected theory and $\mathbf{p}_n^{\mathrm{reg}}$ is the regularized shifted symmetric power sum defined by

$$\mathbf{p}_n^{\mathrm{reg}} = \mathbf{p}_n + (1 - 2^{-n})\zeta(-n),$$

where ζ is the Riemann zeta function [21].

One may ask whether interplays between the following functions yield connections between theories in (1.1):

$$\begin{array}{ccc} \{\mathbf{p}_\mu\} & \xleftrightarrow{\ \mathrm{GW/H}\ } & \{\mathbf{f}_\mu\} \\ \updownarrow & & \updownarrow \text{ Theorem 1.1} \\ \{p_\rho\} & \longleftrightarrow & \{f_\rho\} \end{array}$$

We will use the two linear bases $\{p_\rho : \rho \text{ is odd}\}$ and $\{f_\rho\}$ of the algebra Γ to describe a conjectural spin curve analog of the GW/H correspondence. Let $\mathrm{OP}_{\mathbb{Q}}$ denote the vector space over $\mathbb{Q}$ with basis the set of odd partitions. There is a linear isomorphism $\varphi_{_B} : \mathrm{OP}_{\mathbb{Q}} \to \Gamma$ given by

$$\varphi_{_B}(\rho) = 2^{\frac{\ell(\rho)-|\rho|}{2}} f_\rho.$$

Using this isomorphism, one can also extend the Gunningham formula [3, 14] for spin Hurwitz numbers of the spin curve of genus h and parity p to the following multilinear form on $\mathrm{OP}_{\mathbb{Q}}$:

$$\begin{aligned} &H_d^{h,p}(w_1, \cdots, w_n) \\ &= 2^{(d+1)(1-h)} \sum_{\lambda \in \mathrm{SP(d)}} (-1)^{p\delta(\lambda)} \Big(2^{\frac{1-\delta(\lambda)}{2}} \frac{\dim V^\lambda}{|\mathsf{C}(d)|}\Big)^{2-2h} \prod_{i=1}^n \varphi_{_B}(w_i)(\lambda). \end{aligned}$$

Then the Maulik-Pandharipande formulae ((8) and (9) of [18]), which were proved in [11, 13, 12], show that for degree $d = 1, 2$, the descendent GW invariants of the spin curve of genus h and parity p are given by

$$\begin{aligned} &\Big\langle \tau_{k_1}(\omega)\cdots\tau_{k_n}(\omega)\Big\rangle_d^{\bullet\, h,p} \\ &= H_d^{h,p}\left(\varphi_B^{-1}\Big(\frac{(-1)^{k_1}k_1!}{2^{k_1}(2k_1+1)!}\ p_{2k_1+1}\Big),\cdots,\varphi_B^{-1}\Big(\frac{(-1)^{k_n}k_n!}{2^{k_n}(2k_n+1)!}\ p_{2k_n+1}\Big)\right). \end{aligned} \tag{5.1}$$

Question 5.1. *Does the equality (5.1) hold for $d \geq 3$?*

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